Birational invariants : cohomology, algebraic cycles and Hodge theory cohomologie

René Mboro

NNT : 2017SACLX049

THÈSE DE DOCTORAT
DE L'UNIVERSITE PARIS-SACLAY

préparée à

L'École Polytechnique

ÉCOLE DOCTORALE N°574

École doctorale de mathématiques Hadamard (EDMH)

Spécialité de doctorat : Mathématiques fondamentales

par

René MBORO

Invariants birationnels : cohomologie, cycles algébriques et théorie de Hodge

Thèse présentée et soutenue à Palaiseau, le 06 octobre 2017

Après avis des rapporteurs : Marcello BERNARDARA (Université de Toulouse)
Mingmin SHEN (University of Amsterdam)

Composition du jury :

M. Olivier DEBARRE	École Normale Supérieure	Président du jury
M. Marcello BERNARDARA	Univ. de Toulouse	Rapporteur
M. Charles FAVRE	École Polytechnique	Examinateur
M. Javier FRESÁN	École Polytechnique	Examinateur
M. Nicolas PERRIN	Univ. de Versailles St-Quentin en Yvelines	Examinateur
Mme Claire VOISIN	Collège de France	Directrice de thèse

NNT : 2017SACLX049

THÈSE DE DOCTORAT

de

l'Université Paris-Saclay

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Ecole polytechnique

Laboratoire d'accueil : Centre de mathématiques Laurent Schwartz, UMR 7640 CNRS

Spécialité de doctorat : Mathématiques fondamentales

René MBORO

Invariants birationnels : cohomologie, cycles algébriques et théorie de Hodge

Date de soutenance : 06 Octobre 2017

Après avis des rapporteurs :
Marcello BERNARDARA (Université de Toulouse)
Mingmin SHEN (University of Amsterdam)

Jury de soutenance :

Marcello BERNARDARA	(Univ. de Toulouse)	Rapporteur
Olivier DEBARRE	(École Normale Supérieure)	Président de jury
Charles FAVRE	(École Polytechnique)	Examinateur
Javier Fresán	(École Polytechnique)	Examinateur
Nicolas PERRIN	(Univ. de Versailles St-Quentin en Yvelines)	Examinateur
Claire Voisin	(Collège de France)	Directrice de thèse

Remerciements

Je tiens tout d'abord à remercier chaleureusement ma directrice de thèse Claire Voisin, qui m'a guidé de la préparation du mémoire de M2, que j'abordai sans avoir suivi de cours de géométrie algébrique, jusqu'à la complétion des travaux présentés dans ce manuscrit, qui constituent mon introduction à un thème de recherche à l'esthétique et l'intérêt certains. Cette thèse n'aurait jamais vu le jour sans les excellentes questions et suggestions qu'elle m'a présentées, sans sa disponibilité et ses explications qui, en s'attachant à faire ressortir les idées essentielles de théorèmes ou de notions, simplifient la lecture des ouvrages où ceux-ci sont exposés, sans sa patience et ses remarques éperonnantes, sans le grand soin - pas toujours plaisant - qu'elle a pris à la correction des premières versions de chacun de mes travaux et l'accueil enthousiasmé que recevaient la présentation des pistes sur lesquelles j'étais engagées lorsque celles-ci lui semblaient prometteuses. C'est un grand honneur pour moi d'avoir été son élève et je suis assurément encore loin d'avoir mis à profit tout ce que j'ai reçu d'elle.

Je voudrais également exprimer ma reconnaissance à Alena Pirutka pour son inestimable aide; elle a été ma tutrice notamment pendant une partie de la très cruciale première année et a toujours su prendre de son temps pour me dispenser des «crash »(souvent) «courses »sur divers sujets, me proposer des références, essayer de trouver des réponses à mes questions et participer à la correction de mon travail.

Je remercie Marcello Bernardara et Mingmin Shen d'avoir accepté de rapporter cette thèse. Les remarques de Marcello Bernardara, nées de sa lecture minutieuse d'une précédente version de cette thèse, ont contribué à en améliorer la présentation et la version actuelle de la prépublication ([54]) dont est issue la matière du premier chapitre doit beaucoup à Mingmin Shen qui m'avait signalé, à la suite d'un exposé à Amsterdam, une insuffisance dans la preuve d'un énoncé important (le relèvement de la torsion) présentée dans la première (réputée) version. Olivier Debarre, Charles Favre, Javier Fresán et Nicolas Perrin ont accepté de faire partie de mon jury: c'est pour moi un honneur. Je remercie particulièrement Charles Favre de son affabilité; il a consenti à passer de long moments à m'aider à comprendre des points de certains ouvrages et articles, à m'écouter lui exposer diverses questions et me faire bonnes suggestions. Je le remercie aussi de l'attention dont il entoure les doctorants.

Bien d'autres mathématiciens se sont montrés disponibles pour des discussions qui m'ont été profitables et pour des remarques sur mon travail; qu'il me soit permis de citer Erwan Brugallé, Jean-Louis Colliot-Thélène, François Charles, Bruno Kahn, Christophe Margerin, Claude Sabbah. Merci tout particulièrement à Lie Fu pour toute l'aide qu'il m'a prodiguée pendant mon séjour à l'IAS. Je suis également reconnaissant à Anna Cadoret de m'avoir offert l'opportunité de donner un exposé -mon premier en tant que doctorant- au séminaire «autour des cycles algébriques ».

Au moment d'écrire ces lignes, j'ai une pensée toute particulière pour mes professeurs de classes préparatoires (Lycée Pierre Corneille), auprès desquels j'ai beaucoup appris et qui m'ont aidé, bien au-delà de leur devoir, à m'entraîner pour les concours: merci en particulier à Thomas Chénel pour le temps qu'il a consacré à répondre à mes questions sur des exercices de Physique et à Stéphane Legros pour son extrême (sans exagérer) disponibilité. Sans leur aide mon parcours aurait probablement été tout autre.

Mes remerciements vont également à l'ensemble des membres du CMLS, laboratoire au sein duquel j'ai effectué la majeure partie de mon travail de thèse, pour l'atmosphère conviviale et propice à l'épanouissement mathématique qu'ils réussissent à créer. Je remercie ses directeurs successifs Yvan Martel et Charles Favre pour leur propension à accéder aux demandes de mission qui leur sont soumises, la cellule informatique Jean-Luc Bellon, David Delavennat et Danh Pham-Kim pour leur serviabilité et tout particulièrement Marine Amier, Pascale Fuseau (dont les soins m'accompagnent depuis le «cycle ingénieur»alors qu'elle étaient affectée à la DRE) et Carole Juppin pour leur remarquable efficacité administrative qui n'a d'égale que leur gentillesse. Parce qu'elle déploie une accortise de la même envergure que nos secrétaires du CMLS, je voudrais remercier chaleureusement Livia Kürsteiner de l'ETH-ITS, qui s'est occupé des aspects administratifs et pratiques de mon séjour -qui fut réellement profitable- à Zürich en Mai dernier.

Je voudrais remercier cordialement pour avoir gardé un oeil attentif à mon parcours et pour divers conseils et pour leur amabilité: Eric Legrix (le «comprom»2010), Frank Pacard et Jean Souphanplixa (de la «scola rouje», grâce à qui j'avais eu le privilège indécent d'entrer en possession du polycopié (l'épaisse édition que reçut la 2008) du cours de Pierre Colmez pendant mon stage de première année, document qui n'a malheureusement plus eu cours à partir de la 2010).

L'heure est venue de remercier mes camarades de thèse et d'avant, du CMLS (ou anciennement du CMLS) et d'ailleurs, doctorants et post-docs pour tous les moments agréables et toutes les discussions mathématiques et non mathématiques que j'ai pu partager avec eux. Merci à mes très aimables co-bureaux du CMLS, Xianglong Duan, Fabio Gironella et Benoît Loisel (qui m'a entre autres transmis la science des démarches à suivre pour la soutenance), de Jussieu, Valentin Hernandez, Lucas Kaufmann Sacchetto, Long Mai, Ramanujan Santharoubane et Duc-Viet Vu et de l'ETH-ITS, Ivan Bazhov et Ulrike Reiss mais également à Emiliano Ambrosi, Yakine Bahri, Aymeric Baradat, Corinne Bedussa, Valeria Bertini, Emma Brakkee, Clémentine Courtès, Nguyen-Bac Dang, Vincent De Daruvar, Lorenzo Fantini, Louis Ioos, Jacek Jendrej, Matthieu Kochersperger, Thomas Krämer, Hsueh-Yung Lin, Giancarlo Lucchini Arteche, Matilde Manzaroli, Nicolas Martin, Thomas Megarbane, Thomas Morzadec, Iván Moyano García, Rita Rodriguez, Sybille Rosset, Adrien Sauvaget, Stefan Schreieder, Zakarias Sjöström Dyrefelt, Juanyong Wang, Cong Xue, Timofey Zakrevskiy, Tatiana Zolotareva et bien d'autres encore. Une mention spéciale pour Laurent Schadeck dont la conversation en M2, en stimulant ma curiosité mathématique, a assurément contribué à me pousser vers la géométrie algébrique. Je remercie aussi mes camarades non mathématiciens de la promotion 2010 (Loïc Herviou pour un tuyau pour les entretiens de post-doc et bien d'autres), d'autres promotions (Matthieu Portail...), de classes préparatoires (Laura Lemasson...) et de plus loin encore...

On ne finirait pas de remercier en pareille occasion et il est à craindre, si je ne la retiens, que ma faconde après avoir rempli la page et le revers n'aille faire de cette innocente section un moyen d'augmenter déloyalement la taille du présent manuscrit. Aussi m'empresserai-je de remercier, en plus des membres de l'église Roc Séculaire Tabernacle (et en particulier mon père spirituel Robert Ndaye) pour leur bienveillance et leurs prières et l'atmosphère spirituelle qu'ils créent souvent, et des membres des CC plâtal, ma famille (au sens large comprenant les amis de la famille dont l'intensité de l'amitié tient lieu de sang) pour son soutien et sa bienveillance constants. Je tiens à remercier tout particulièrement ma Maman (qui en plus me tient souvent lieu de secrétaire, conseillère...) et mes deux grandes sœurs «ya»Frédérique et «ya»Christelle, mes petits frères et petites sœurs (Sandra, Grâce, Bernard et David), Tonton Grégoire, Tonton Sita, Tonton Nganga...

Qu'il me soit permis de dédier ce manuscrit à la mémoire de mon papa René Mboro (sans les troisième et second prénoms) qui aurait assurément été très fier d'assister à ma soutenance. À l'aspect du bas de page, aux traits aussi menaçants que le grondement d'un confluent tumultueux, je vais m'arrêter, pour éviter de glisser vers des remerciements fleuve, sur le Rocher des Âges, le Rocher de mon Salut, le Seigneur Jésus-Christ.

Table des matières

Chapter 0

Introduction

Soit k un corps algébriquement clos. Deux notions d'applications entre variétés algébriques sur k sont classiquement considérées: d'une part, les morphismes -d'espaces annelés sous-jacents- $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ et d'autre part, les applications rationnelles $X \dashrightarrow Y$, qui sont des morphismes définis sur un ouvert non vide de X. Celles-ci donnent lieu à deux relations d'équivalences classiques: "être isomorphes" et "être birationnellement équivalents".

On peut considérer une troisième relation d'équivalence sur les variétés algébriques: l'équivalence birationnelle stable. On dit qu'une variété X est stablement birationnelle à une variété Y s'il existe des entiers positifs r et s tels que $X \times \mathbb{P}^r_k$ est birationnelle à $Y \times \mathbb{P}^s_k$. L'importance de cette dernière relation d'équivalence est soulignée notamment par le fait suivant dû à Larsen et Lunts ([49]). Notons $K_0(\mathcal{V}_k)$ l'anneau de Grothendieck des variétés; son groupe abélien sous-jacent est le groupe abélien libre engendré par les classes d'isomorphismes de variétés modulo les relations de la forme $[X-Y] = [X]-[Y]$ où Y est une sous-variété fermée de X. Le produit est donné par le produit cartésien. Notons $I \subset K_0(\mathcal{V}_k)$ l'idéal principal engendré par la classe $\mathbb{L}$ de $\mathbb{A}^1$. Nous avons le théorème:

Théorème 0.0.1. *(Larsen et Lunts, [49]) Si la caractéristique de k est nulle, il existe un isomorphisme d'anneaux*

$$K_0(\mathcal{V}_k)/I \simeq \mathbb{Z}[SB]$$

où $\mathbb{Z}[SB]$ désigne l'algèbre du monoïde (multiplicatif) des classes d'équivalence birationnelle stable de variétés.

Ceci nous amène au problème de la caractérisation des variétés *rationnelles* i.e. des variétés projectives lisses sur k birationnellement équivalentes à l'espace projectif. Une question apparentée est la caractérisation des variétés *stablement rationnelles* i.e. des variétés projectives lisses sur k stablement birationnelles à l'espace projectif.

Pour les variétés lisses projectives de dimension 1, i.e. les courbes, il existe un critère très simple, savoir l'annulation du genre géométrique, ce dernier étant défini comme la dimension du k-espace vectoriel $H^0(C, \Omega_{C/k})$ des sections globales du faisceau des différentielles algébriques de la courbe C.

Dans le cas des courbes, le fait remarquable suivant, en rapport avec le problème de rationalité, fut prouvé par Lüroth ([51]) en 1876:

Théorème 0.0.2. *(Lüroth) Soit C une courbe lisse sur k. Supposons qu'il existe une application rationnelle dominante $\mathbb{P}^1 \dashrightarrow C$ alors C est rationnelle.*

Rappelons ici la notion de variété *unirationnelle*: on dit d'une k-variété projective X qu'elle est unirationnelle si elle admet une paramétrisation unirationnelle i.e. s'il existe une application rationnelle dominante $\mathbb{P}^N \dashrightarrow X$ pour un $N \in \mathbb{N}^*$ que l'on peut prendre égal à la dimension de X. Toute variété stablement rationnelle est unirationnelle. Une façon de le voir est d'utiliser la traduction algébrique de ces notions, qui les réduit à des propriétés du corps de fonctions $k(X)$ de X:

- la rationalité d'une variété X de dimension n se traduit par l'existence d'un k-isomorphisme de corps $k(X) \simeq k(X_1, \ldots, X_n)$;

- la rationalité stable, par l'existence d'un entier $s \geq 0$ et d'un k-isomorphisme de corps $k(X)(X_1, \ldots, X_s) \simeq k(Y_1, \ldots, Y_{n+s})$;
- l'unirationalité par l'existence d'un entier $N > 0$ et d'une k-inclusion $k(X) \subset k(X_1, \ldots, X_N)$.

Un isomorphisme $k(X)(X_1, \ldots, X_s) \simeq k(Y_1, \ldots, Y_{n+s})$ entraîne évidemment une inclusion $k(X) \subset k(Y_1, \ldots, Y_{n+s})$. Le théorème de Lüroth affirme qu'en dimension 1, les variétés unirationnelles sont rationnelles. La recherche pour les surfaces, d'un énoncé semblable à celui obtenu par Lüroth fut l'une des motivations du travail de Castelnuovo et Enriques sur les surfaces; déterminer si une surface S pour laquelle il existe une application rationnelle dominante $\mathbb{P}^2 \dashrightarrow S$ est rationnelle. En 1893, Castelnuovo (voir [94]) trouva le critère de rationalité pour les surfaces définies sur le corps des complexes: l'annulation des k-espaces vectoriels $H^0(S, \Omega_{S/k})$ et $H^0(S, (\wedge^2\Omega_{S/k})^{\otimes 2})$ des sections globales des fibrés vectoriels associés au faisceau des différentielles algébriques $\Omega_{S/k}$ et de son tenseur $(\wedge^2\Omega_{S/k})^{\otimes 2}$. Ce critère fut étendu plus tard par Zariski aux surfaces définies sur d'autres corps algébriquement clos. En l'utilisant, on obtient le

Théorème 0.0.3. *(Castelnuovo) Une surface unirationnelle définie sur un corps algébriquement clos de caractéristique* 0 *est rationnelle.*

Lorsque la caractéristique de k est positive, le théorème précédent reste vrai à condition d'ajouter une condition de séparabilité pour la (une) paramétrisation unirationnelle de S. Autrement, des contre-exemples, comme les surfaces $K3$ supersingulières qui sont unirationnelles ([50]) et pourtant loin d'être rationnelles, existent. Ces résultats appellent naturellement la question de l'existence d'un énoncé semblable pour les variétés de dimension ≥ 3:

Problème 0.0.4 (Problème de Lüroth)**.** *Les variétés unirationnelles sont-elles rationnelles?*

Ce problème admet la variation suivante:

Problème 0.0.5 (Problème de Lüroth stable)**.** *Les variétés unirationnelles sont-elles stablement rationnelles?*

Les travaux de Iskovskikh et Manin ([44]), prouvèrent en 1971 que certains solides quartiques complexes fournissaient une réponse négative au problème de Lüroth. Clemens et Griffiths ([15]), eux, prouvèrent en 1972, que les solides cubiques complexes, qui sont connus pour être unirationnels, ne sont pas rationnels donnant ainsi un autre contre-exemple au problème de Lüroth. Artin et Mumford ([3]) enfin fournirent pour certains solides doubles quartiques une réponse négative au problème de Lüroth stable. La production de ces contre-exemples et d'autres qui apparurent par la suite, nécéssite l'introduction et l'étude d'invariants birationnels pour la définition desquels nous introduisons maintenant quelques outils.

0.0.1 Théorème de résolution des singularités

Sur un corps K de caractéristique 0, par les travaux de Hironaka ([40]), et, à la condition que les variétés impliquées soient de dimension au plus 3, sur un corps K de caractéristique positive pour lequel il existe un sous-corps parfait K_0 tel que Ω_{K/K_0} est un K-espace vectoriel de dimension finie, par les travaux d'Abhyankar ([1]), Cossart et Piltant ([24],[25]), nous disposons du théorème suivant qui permet de mieux comprendre les applications rationnelles:

Théorème 0.0.6. *Soit $\phi : X \dashrightarrow Y$ une application rationnelle entre variétés projectives lisses sur k. Alors il existe un diagramme:*

$$\begin{tikzcd} \widetilde{\Gamma}_X \arrow[d, "\tau_X"'] \arrow[dr, "\tilde{\phi}"] & \\ X \arrow[r, dotted, "\phi"] & Y \end{tikzcd}$$

où $\tau_X : \widetilde{\Gamma}_X \to X$ est la composition d'éclatements le long de centres lisses et $\tilde{\phi}$ un morphisme.

Associons à toute variété projective lisse X, un groupe abélien $P(X)$. Supposons que cette association jouisse des fonctorialités suivantes:

1. l'existence d'un morphisme de groupes $f^* : P(Y) \to P(X)$ pour tout morphisme $f : X \to Y$;

2. l'existence d'un morphisme de groupes $f_* : P(X) \to P(Y)$ pour tout morphisme $f : X \to Y$;

3. de l'égalité $f_* f^* = deg(f)\mathrm{id}_{P(Y)}$ pour tout morphisme génériquement fini f.

Si P est associé à un invariant birationnel, il faut en particulier que l'on ait un isomorphisme $P(\widetilde{X}) \overset{\tau_*}{\simeq} P(X)$ pour tout éclatement $\tau : \widetilde{X} \to X$ le long d'un centre lisse.

Proposition 0.0.7. *(voir [27]) Si les fonctorialités ci-dessus sont vérifiées par $X \mapsto P(X)$, alors $P(X)$ est un invariant birationnel (resp. stable) si et seulement si $P(X)$ est invariant par éclatement le long d'un centre lisse (resp. et par produit avec l'espace projectif).*

Preuve. En effet, considérons une application birationnelle $\phi : X \dashrightarrow Y$ et une résolution des indéterminées telle que donnée par 0.0.6; l'invariance par éclatement fournit un isomorphisme $P(\widetilde{\Gamma}_X) \overset{\tau_{X,*}}{\simeq} P(X)$ et la troisième fonctorialité demandée indique que $\tilde{\phi}^*$ est injectif et fait de $P(Y)$ un facteur direct de $P(\widetilde{\Gamma}_X) \overset{\tau_{X,*}}{\simeq} P(X)$.

Considérons l'application birationnelle inverse $\tilde{\phi}^{-1} : Y \dashrightarrow \widetilde{\Gamma}_X$. Par application du théorème 0.0.6 à $\tilde{\phi}^{-1}$, nous obtenons un diagramme:

$$\begin{tikzcd} \widetilde{\Gamma}_X \arrow[d, "\tau_X"'] \arrow[dr, "\tilde{\phi}"] & \widetilde{\Gamma}_Y \arrow[l, "\widetilde{\tilde{\phi}^{-1}}"'] \arrow[d, "\tau_Y"] \\ X \arrow[r, dotted, "\phi"] & Y \end{tikzcd}$$

Nous avons la relation $\tilde{\phi} \circ \widetilde{\tilde{\phi}^{-1}} = \tau_Y$ puisqu'elle est valable sur un ouvert sur lequel $\tilde{\phi}^{-1}$ est défini (et les variétés sont séparées). Par la troisième fonctorialité $\widetilde{\tilde{\phi}^{-1}}_*$ est surjectif. De plus, $\tau_Y^* = \widetilde{\tilde{\phi}^{-1}}^* \circ \tilde{\phi}^*$ et en composant par $\widetilde{\tilde{\phi}^{-1}}_*$ on obtient $\widetilde{\tilde{\phi}^{-1}}_* \circ \tau_Y^* = \tilde{\phi}^*$. Comme τ_Y^* est un isomorphisme (l'inverse de $\tau_{Y,*}$ par la troisième fonctorialité), $\tilde{\phi}^*$ est aussi surjectif; c'est donc un isomorphisme. □

De plus, si $P(\cdot)$ est un invariant birationnel (stable) nul pour l'espace projectif, alors la fonctorialité (3) permet d'affirmer que $P(X)$ est de torsion pour toute variété unirationnelle X.

0.1 L'anneau de Chow et les correspondances

Soit X une k-variété de dimension n i.e. un schéma intègre, séparé, de type fini sur k de dimension n. Un cycle de dimension i (ou de codimension $n-i$) est une somme formelle de sous-variétés fermées (intègres) de dimension i (ou de codimension $n-i$) de X. On note $\mathcal{Z}_i(X)$ (resp. $\mathcal{Z}^{n-i}(X)$) le groupe abélien libre engendré par les sous-variétés intègres de dimension i de X.

On note $\mathcal{Z}_i(X)_{rat} \subset \mathcal{Z}_i(X)$ le sous-groupe des cycles de dimension i rationnellement triviaux; il est engendré par les éléments de la forme $pr_{2,V}^{-1}(0) - pr_{2,V}^{-1}(\infty)$, pour $V \subset X \times \mathbb{P}^1$ une sous-variété fermée de dimension $(i+1)$ telle que $pr_{2,V} : V \to \mathbb{P}^1$ est dominant. Deux cycles sont *rationnellement équivalents* si leur différence appartient au sous-groupe $\mathcal{Z}_i(X)_{rat}$.

Définition 0.1.1. *On définit le groupe de Chow de dimension i, $\mathrm{CH}_i(X)$, comme le quotient $\mathcal{Z}_i(X)/\mathcal{Z}_i(X)_{rat}$.*

On définit aussi le sous-groupe $\mathcal{Z}_i(X)_{alg} \subset \mathcal{Z}_i(X)$ des cycles algébriquement triviaux comme le sous-groupe engendré par les cycles de la forme $pr_{2,V}^{-1}(x_1) - pr_{2,V}^{-1}(x_2)$, pour $V \subset X \times C$ une sous-variété fermée de dimension $(i+1)$ de $X \times C$, C étant une courbe projective lisse, telle que $pr_{2,V} : V \to C$ est dominant et $x_1, x_2 \in C(k)$ sont des points de C. On a l'inclusion évidente $\mathcal{Z}_i(X)_{rat} \subset \mathcal{Z}_i(X)_{alg}$. Le sous-groupe $\mathcal{Z}_i(X)_{alg}$ définit l'*équivalence algébrique* sur les cycles qui est, par l'inclusion ci-dessus, plus grossière que l'équivalence rationnelle. L'équivalence algébrique est, en général, strictement plus grossière que l'équivalence rationnelle comme le montre l'exemple d'une courbe C de genre strictement positif: par définition de l'équivalence algébrique, pour une telle courbe $\mathcal{Z}_0(C)/\mathcal{Z}_0(C)_{alg} = \mathbb{Z}x$ où $x \in C(k)$ tandis que $\mathcal{Z}_0(C)_{alg}/\mathcal{Z}_0(C)_{rat}$ est égal à la jacobienne de C qui est une variété abélienne non réduite à 0.

Si de plus X est lisse, on peut définir (voir [30]) un produit d'intersection $\cdot : \mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X)$ qui munit la somme directe

$$\mathrm{CH}^*(X) = \oplus_{i=0}^n \mathrm{CH}^i(X)$$

d'une structure d'anneau.

On dispose d'une fonctorialité pour ces groupes; d'un morphisme $f_* : \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y)$ pour les morphismes propres $f : X \to Y$, d'un morphisme $f^* : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X)$ pour les morphismes plats $f : X \to Y$ ou pour les morphismes entre variétés lisses.

Les correspondances $\Gamma \in \mathrm{CH}^i(X \times Y)$, cycles algébriques sur les produits de variétés lisses X, Y jouent un rôle particulier dans la théorie des cycles algébriques entre autres parce qu'ils induisent des morphismes de groupes de Chow:

$$\Gamma_* : \mathrm{CH}^k(X) \to \mathrm{CH}^{k+i-dim_k(X)}(Y) \quad z \mapsto pr_{Y,*}(\Gamma \cdot pr_X^* z)$$

et

$$\Gamma^* : \mathrm{CH}^k(Y) \to \mathrm{CH}^{k+i-dim_k(Y)}(X) \quad z \mapsto pr_{X,*}(\Gamma \cdot pr_Y^* z)$$

où $pr_X : X \times Y \to X$ et $pr_Y : X \times Y \to Y$ sont les projections canoniques.

L'anneau de Chow est un invariant algébrique associé à une variété algébrique. On dispose, pour le morphisme birationnel élémentaire qu'est l'éclatement le long d'un lieu lisse et irréductible, des formules de Manin:

Proposition 0.1.2. *Soit X une variété projective lisse et $Z \subset X$ une sous-variété lisse de X de codimension $r \geq 2$. En notant $\widetilde{X}_Z$ l'éclaté de X le long de Z, nous avons la formule suivante:*

$$\mathrm{CH}_l(\widetilde{X}_Z) \simeq \bigoplus_{0 \leq k \leq r-2} \mathrm{CH}_{l-r+k+1}(Z) \oplus \mathrm{CH}_l(X). \tag{1}$$

Présentons ici, suivant la discussion de 0.0.1, un premier invariant birationnel: $\mathrm{CH}_0(X)$. Il s'agit même d'un invariant birationnel stable puisque $\mathrm{CH}_0(X\times\mathbb{P}^r)\simeq \mathrm{CH}_0(X)$ pour toute variété X. En particulier, si X est une variété rationnelle ou stablement rationnelle, alors $\mathrm{CH}_0(X)=\mathrm{CH}_0(\mathbb{P}^{dim_k(X)})=\mathbb{Z}$.

Plus généralement, il suffit que tout couple de point (x_1,x_2) d'une variété X soit contenu dans une courbe rationnelle i.e. l'image d'un morphisme non constant $\mathbb{P}^1\to X$ pour avoir $\mathrm{CH}_0(X)=\mathbb{Z}$. On dit dans ce dernier cas que la variété est X *rationnellement connexe*. Il existe alors une famille $g:\mathcal{C}\to M$ de courbes paramétrées par une variété M, dont les composantes des fibres géométriques consistent en des courbes rationnelles et une application d'évaluation $ev:\mathcal{C}\to X$ telle que $ev^{(2)}:\mathcal{C}\times_M\mathcal{C}\to X\times X$ est dominante. Lorsque le morphisme $ev^{(2)}$ est de plus séparable, on dit que X est *séparablement rationnellement connexe*. Une variété unirationnelle est rationnellement connexe; il suffit, pour le voir, de prendre pour un couple de points $(x_1,x_2)\in X^2$ contenus dans l'image d'une paramétrisation unirationnelle de X, la courbe rationnelle donnée par l'image de la droite joignant un couple $(p_1,p_2)\in(P^n)^2$, préimage de (x_1,x_2).

Mais il est, en général, difficile de calculer les groupes de Chow. Les exemples suivants illustrent la complexité de leur comportement en général.

Exemple 0.1.3. *1. Nous avons $\mathrm{CH}_0(\mathbb{P}^n_k)\simeq\mathbb{Z}[x]$ où $x\in\mathbb{P}^n(k)$ est un point k-rationnel (n'importe lequel) de $\mathbb{P}^n$.*

2. Soit C une courbe lisse propre sur k (k étant algébriquement clos), alors, d'après le théorème d'Abel, le groupe des 0-cycles s'insère dans la suite exacte:

$$0\to J(C)\to \mathrm{CH}_0(C)\overset{deg}{\to}\mathbb{Z}\to 0$$

où $J(C)$ est la Jacobienne de la courbe C. Le groupe des zéro cycle a donc la dimension d'une variété (groupe) algébrique.

3. Le théorème suivant est dû à Mumford:

Théorème 0.1.4. *(Mumford [59], 1968) Soit S une surface projective lisse sur $\mathbb{C}$ telle que $H^0(S,\wedge^2\Omega_S)\neq 0$. Alors pour tout entier m, la fibre générale de l'application naturelle*

$$\alpha_m: S(k)^{(m)}\times S(k)^{(m)}\to \mathrm{CH}_0(S)_{alg}$$

donnée par $(s_1+\cdots+s_m,t_1+\cdots+t_m)\mapsto\sum_i s_i-\sum_i t_i$ est dénombrable; on dit que $\mathrm{CH}_0(S)$ est de dimension infinie.

En particulier, pour une surface satisfaisant les hypothèses du théorème, le groupe $\mathrm{CH}_0(S)$ n'a pas la "dimension d'un groupe algébrique".

0.2 Cohomologie à coefficents constants

Pour analyser les groupes de Chow des variétés projectives, on utilise alors les groupes de cohomologie à coefficients constants, qui présentent l'avantage d'être, dans notre contexte, des modules de types finis sur l'anneau des coefficients et d'offrir plus d'outils pour leur calcul (théorème des sections hyperplanes de Lefschetz, théorème des coefficients universels, dualité de Poincaré...).

0.2.1 L'application classe de cycle

Cas $k = \mathbb{C}$.

Lorsque X est une variété projective lisse sur le corps de complexes, l'ensemble des points complexes, $X(\mathbb{C})$, est muni d'une structure de variété différentielle (en fait holomorphe) compacte; on peut donc utiliser la *cohomologie de Betti* de la variété $X(\mathbb{C})$ à coefficients $\mathbb{Z}, \mathbb{Q}$ ou $\mathbb{C}$ comme théorie cohomologique.

On dispose d'une application *classe de cycle*:

$$[\ ] : \mathrm{CH}^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{Z})$$

qui associe à une sous variété (intègre) $Z \subset X$, le dual de Poincaré du poussé en avant de la classe fondamentale d'une désingularization $\widetilde{Z}$ -qui est une variété projective lisse- de Z. L'application classe de cycle induit même un morphisme d'anneaux

$$(\oplus_{i=0}^{dim(X)} \mathrm{CH}^i(X), \cdot) \to (\oplus_{i=0}^{2dim(X)} H^{2i}(X(\mathbb{C}), \mathbb{Z}), \cup)$$

de l'anneau de Chow (muni du produit d'intersection) vers l'anneau de cohomologie (muni du cup produit).

L'application classe de cycle permet de définir *l'équivalence homologique* sur les cycles algébriques: les cycles homologiquement triviaux de codimension k sont les éléments de $\mathrm{CH}^k(X)_{hom} := Ker([\ ])$.

On a l'inclusion $\mathrm{CH}^k(X)_{alg} \subset \mathrm{CH}^k(X)_{hom}$. Cette inclusion est stricte en général comme le montre le théorème suivant dû à Griffiths:

Théorème 0.2.1. *(Griffiths [32], 1969) Soit $X \subset \mathbb{P}^4_{\mathbb{C}}$ une hypersurface quintique générale. Alors X admet une classe de courbe homologiquement triviale non algébriquement triviale modulo torsion.*

Définition 0.2.2. *Soit X une variété projective lisse sur $\mathbb{C}$. On appelle groupe de Griffiths des cycles de codimension k, le quotient* $\mathrm{Griff}^k(X) := \mathrm{CH}^k(X)_{hom}/\mathrm{CH}^k(X)_{alg}$.

Le théorème 0.2.1 de Griffiths affirme que pour X une hypersurface quintique générale de $\mathbb{P}^4$, $\mathrm{Griff}^2(X) \otimes \mathbb{Q} \neq 0$. On dispose d'une amélioration de ce résultat due à Clemens ([14]) affirmant que pour une telle hypersurface X, $\mathrm{Griff}^2(X) \otimes \mathbb{Q}$ est un $\mathbb{Q}$-espace vectoriel de dimension infinie.

Le résultat suivant permettra de montrer que certains invariants $P(X)$ sont des invariants birationnels pour X projective lisse.

Fait 0.2.3. *(1) Par une généralisation du théorème d'Abel, on sait néanmoins que le groupe de Griffiths* $\mathrm{Griff}^1(X)$ *des cycles de codimension* 1 *est nul pour toute variété projective lisse X.*

(2) On a aussi par construction $\mathrm{Griff}_0(X) = 0$ *pour toute variété projective lisse X.*

Cas $k \neq \mathbb{C}$.

Lorsque k est une corps quelconque algébriquement clos, les travaux de Grothendieck et ses collaborateurs permettent de définir la cohomologie étale à coefficients de torsion $H^i_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z})$, m étant premier à la caractéristique de k ou ℓ-adique $H^i_{\text{ét}}(X, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z})$-adique, avec ℓ un nombre premier différent de la caractéristique de k. On dispose encore, dans ce contexte d'une application classe de cycles:

$$[\ ] : \mathrm{CH}^k(X) \to H^{2k}_{\text{ét}}(X, A)$$

où A est l'un des anneaux $\mathbb{Z}/m\mathbb{Z}$ ou $\mathbb{Z}_\ell$, qui induit un morphisme d'anneaux

$$(\oplus_{i=0}^{dim(X)}\mathrm{CH}_i(X),\cdot) \to (\oplus_{i=0}^{2dim(X)}H^i_{\text{ét}}(X,A),\cup).$$

On définit dans ce contexte, l'équivalence homologique à coefficients dans A sur les cycles algébriques, comme avec la cohomologie de Betti, grâce au noyau $Ker([\ ]) \subset \mathrm{CH}^k(X)\otimes A$. Il n'y a pas, dans ce cas, d'anneau des coefficients A naturel et la relation d'équivalence définie dépend du choix de l'anneau, comme l'illustre la situation suivante: soit $\gamma \in \mathrm{CH}^k(X)$ un cycle de t-torsion non trivial sur une variété projective lisse X, pour t un nombre premier. Alors pour tout ℓ différent de t, $[\gamma]=0$ dans $H^{2k}_{\text{ét}}(X,\mathbb{Z}_\ell)$ car t est inversible dans $\mathbb{Z}_\ell$ mais en général $[\gamma]\neq 0$ dans $H^{2k}_{\text{ét}}(X,\mathbb{Z}_t)$. Cependant, en utilisant les groupes de cohomologie $H^i_{\text{ét}}(X,\mathbb{Q}_\ell) := H^i_{\text{ét}}(X,\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$, une des *conjectures standards* (conjecture D) sur les cycles algébriques formulées par Grothendieck ([33]), prédit que l'équivalence homologique ℓ-adique modulo torsion sur les cycles algébriques de dépend pas du choix de ℓ (différent de la caractéristique de k).

Toutefois, étant donné que pour les variétés "presque rationnelles", comme les variétés unirationnelles, les invariants birationnels utiles à l'étude des questions de rationalité, comme le problème de Lüroth (stable), sont "de torsion", on utilisera plutôt les coefficients $\mathbb{Z}_\ell$ et on choisira, comme on est amené à le faire dans le chapître 2, le nombre premier ℓ adapté à l'étude que l'on se propose de mener.

On dispose encore, pour les groupes de cohomologie, de formules de Manin pour l'éclatement, valables pour la cohomologie étale (où A peut être pris égal à $\mathbb{Z}/m\mathbb{Z}$ ou $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell\ldots$) et pour la cohomologie de Betti (où A peut être pris égal à $\mathbb{Z}$, $\mathbb{Q}\ldots$):

Proposition 0.2.4. *Soit X une variété projective lisse et $Z\subset X$ une sous-variété lisse de X de codimension $r\geq 2$. En notant $\widetilde{X}_Z$ l'éclaté de X le long de Z, nous avons, en notation générique, la formule suivante:*

$$H^k_{\cdot}(\widetilde{X}_Z,A)\simeq \bigoplus_{0\leq i\leq r-2} H^{k-2i-2}_{\cdot}(Z,A)\oplus H^k_{\cdot}(X,A). \tag{2}$$

De plus, $[\ ]$ est compatible avec les décompositions données par (1) et (2).

Les formules de Manin pour les groupes de Chow et de cohomologie et leur compatibilité avec l'application classe de cycles permettent de définir un certain nombre d'invariants birationnels nuls pour l'espace projectif.

Procédé 0.2.5. *(1) Notons, pour une variété projective lisse X, $P_*(X)$ un sous-quotient fonctoriel du groupe de Chow de dimension $*$. Supposons que pour un certain entier m et pour toute variété X et tout entier $i\leq m$, $P_i(X)$ est nul. Alors les formules de Manin assurent que le groupe $P_{m+1}(X)$ est un invariant birationnel.*

(2) De même, si $P^*(X)$ est un sous-quotient fonctoriel du groupe de Chow de X de codimension $*$, et si pour un certain entier m, $P^i(X)$ est nul pour toute variété projective lisse X et tout entier $i\leq m$, alors $P^{m+1}(X)$ est un invariant birationnel.

(3) De façon semblable, pour un sous-quotient fonctoriel $P^*(X)$ (tel que la torsion ou les classes de Hodge modulo les classes algébriques) de $H^*(X)$, si pour un certain entier m, $P^i(X)$ est nul pour toute variété projective lisse X et tout entier $i\leq m$, alors $P^{m+2}(X)$ est un invariant birationnel.

Considérons un premier exemple qui correspond à (3): posons $P^*(X) := Tors(H^*(X,\mathbb{Z}))$. Le groupe de cohomologie $H^1(X(\mathbb{C}),\mathbb{Z})\simeq \mathrm{Hom}(H_1(X),\mathbb{Z})$ (resp. $H^1_{\text{ét}}(X,\mathbb{Z}_\ell)\simeq \varprojlim_n \mathrm{Hom}_{conts}(\pi_1(X,x),\mathbb{Z}/\ell^n\mathbb{Z})$

if $k \neq \mathbb{C}$ and $\ell \neq car(k)$) est sans torsion pour toute variété projective lisse. Il en résulte d'après (3) que le groupe:

$$\mathrm{Tors}(H^3(X(\mathbb{C}), \mathbb{Z})) \text{ (resp. } \mathrm{Tors}(H^3_{\text{ét}}(X, \mathbb{Z}_\ell))) \tag{3}$$

sous-groupe de $H^3(X(\mathbb{C}), \mathbb{Z})$ (resp. de $H^3_{\text{ét}}(X, \mathbb{Z}_\ell)$) est un invariant birationnel, qui est évidemment nul pour l'espace projectif, donc pour toute variété stablement rationnelle. C'est l'invariant utilisé par Artin et Mumford pour produire une réponse négative au problème de Lüroth stable: ils prouvent qu'il existe une classe de torsion non nulle dans la cohomologie de degré 3 de certains solides quartiques doubles désingularisés, ce qui montre que ces solides, connus pour être unirationnels, ne sont pas stablement rationnels.

Partant du constat que sur l'espace projectif, équivalences algébrique et homologique coïncident, on peut sur ce modèle exhiber des invariants birationnels stables nuls pour l'espace projectif: les formules de Manin nous donnent la décomposition suivante pour les groupes de Griffiths de l'éclatement $\widetilde{X}_Z$ d'une variété projective lisse le long d'une sous-variété lisse Z:

$$\mathrm{Griff}^k(\widetilde{X}_Z) \simeq \bigoplus_{0 \leq i \leq r-2} \mathrm{Griff}^{k-i-1}(Z) \oplus \mathrm{Griff}^k(X)$$

et $\mathrm{Griff}^k(X \times \mathbb{P}^r) \simeq \bigoplus_{0 \leq i \leq r-1} \mathrm{Griff}^{dim(X)+r+2-k-i}(X)$

1. Étant donné que $\mathrm{Griff}^1(Y) = 0$ pour toute variété projective lisse Y par le théorème d'Abel, on obtient que le groupe $\mathrm{Griff}^2(Y)$ est un invariant birationnel stable nul pour l'espace projectif.

2. Étant donné que $\mathrm{Griff}_0(Y) = 0$ pour toute variété projective lisse Y, le groupe $\mathrm{Griff}_1(X)$ est un invariant birationnel stable nul pour l'espace projectif.

La question suivante est posée par Voisin dans ([92]):

Problème 0.2.6. *Le groupe* $\mathrm{Griff}_1(X)$ *est-il nul pour toute variété* X *séparablement rationnellement connexe?*

Cette question suggère que $\mathrm{Griff}_1(X)$ est un invariant conjecturalement inutile pour tester les différences entre les classes de variétés presque rationnelles.

Tian et Zong ([76]) ont répondu par l'affirmative à cette question pour les variétés intersections complètes générales de l'espace projectif qui sont de Fano (donc rationnellement connexes) *d'indice au moins* 2 sur un corps algébriquement clos de caractéristique 0. Dans la section 1, nous étudions la question 0.2.6 pour les variétés des droites $F(X)$ des hypersurfaces X de l'espace projectif et présentons des bornes portant sur le degré de X et sa dimension pour assurer une réponse positive à la question 0.2.6 pour $F(X)$:

Théorème 0.2.7. *Soient* k *un corps algébriquement clos de caractéristique* 0 *et* $X \subset \mathbb{P}^{n+1}_k$ *une hypersurface de degré* $d > 2$ *lisse et suffisamment générale pour que sa variété des droites* $F(X)$ *soit lisse et connexe. Alors,* $\mathrm{Griff}_1(F(X)) = 0$ *dès que* $\frac{d(d+1)}{2} < n$.

Notons que la borne présentée ici demande que $F(X)$ soit Fano d'indice au moins 3 (contre 2 pour les hypersurfaces d'après [76]) pour assurer l'annulation du groupe $\mathrm{Griff}_1(F(X))$.

0.2.2 Classes de Hodge et classes de Tate

Cas $k = \mathbb{C}$.

La variété $X(\mathbb{C})$ est également munie d'une structure de variété kählérienne compacte. On rappelle le théorème classique:

Théorème 0.2.8. *(Décomposition de Hodge) Soit Y une variété kählérienne compacte. Alors, pour tout entier $k \geq 0$, l'espace vectoriel complexe $H^k(Y(\mathbb{C}), \mathbb{C})$ admet une décomposition en sous-espaces vectoriels complexes*

$$H^k(Y(\mathbb{C}), \mathbb{C}) = \bigoplus_{\substack{p+q=k \\ p,q \geq 0}} H^{p,q}(Y), \tag{4}$$

où les classes de $H^{p,q}(Y)$ peuvent être représentées par des formes différentielles de type (p, q) i.e. dont l'expression dans des coordonnées holomorphes locales est une combinaison de formes de type $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. Cette décomposition satisfait la symétrie $H^{p,q}(Y) = \overline{H^{q,p}(Y)}$ et on dispose des isomorphismes $H^{p,q}(Y) \simeq H^q(Y, \Omega^p_Y)$.

Il est bien connu que, sous le morphisme de changement de coefficients induit par l'inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$, l'image de l'application classe de cycle $\mathrm{CH}^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{C})$ est contenue dans le sous-espace vectoriel $H^{k,k}(X)$.

Définition 0.2.9. *Soit Y une variété kählérienne compacte. Étant donné un entier $k \geq 0$, on définit le sous-groupe $Hdg^{2k}(Y, \mathbb{Z})$ (resp. $Hdg^{2k}(Y, \mathbb{Q})$) de $H^{2k}(Y(\mathbb{C}), \mathbb{Z})$ (resp. $H^{2k}(Y(\mathbb{C}), \mathbb{Q})$) comme le sous-groupe des classes $\alpha \in H^{2k}(Y, \mathbb{Z})$ (resp. $H^{2k}(Y(\mathbb{C}), \mathbb{Q})$), telles que $\alpha \otimes 1 \in H^{2k}(Y, \mathbb{C})$ appartient au sous-espace vectoriel $H^{k,k}(Y)$. On appelle $Hdg^{2k}(Y, \mathbb{Z})$ (resp. $Hdg^{2k}(Y, \mathbb{Q})$) le groupe des* classes de Hodge *(resp.* classes de Hodge rationnelles*).*

On a donc l'inclusion $Im([\ ] : \mathrm{CH}^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{Z})) \subset Hdg^{2k}(X, \mathbb{Z})$. Considérons la question suivante:
"**Conjecture de Hodge entière**": *Soit X une variété projective lisse sur $\mathbb{C}$. Alors toute classe de Hodge est la classe d'un cycle algébrique i.e.*

$$Im([\ ] : \mathrm{CH}^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{Z})) = Hdg^{2k}(X, \mathbb{Z}).$$

La conjecture de Hodge entière est vraie dans les degrés suivants:

1. en degré 0 et $2dim(X)$ trivialement;

2. en degré 2 par le théorème de Lefschetz sur les classes $(1, 1)$.

En général, la conjecture est fausse, comme le montrent les contre-exemples suivants, qui prouvent aussi que les degré 0, $2dim(X)$ et 2 sont les seuls degrés dans lesquels la conjecture peut être vraie en général:

- par des méthodes de K-théorie topologique Atiyah et Hirzebruch ([4]) ont exhibé une variété admettant une classe de Hodge de torsion de degré 4 qui n'est pas algébrique.

- Kollàr ([46]), montre que sur une hypersurface de $\mathbb{P}^4$ très générale de degré très divisible, il existe une classe de Hodge de degré 4 d'ordre infini qui n'est pas algébrique -dont un multiple non trivial est algébrique-.

Revenons au principe 0.2.5 et formons, pour une variété Y, les quotients

$$Z^{2i}(Y) := Hdg^{2i}(Y,\mathbb{Z})/H^{2i}(Y,\mathbb{Z})_{alg}$$

où $H^{2i}(Y,\mathbb{Z})_{alg}$ désigne l'image de l'application classe de cycle. Comme $Z^{2dim(Y)}(Y) = 0$ et $Z^2(Y) = 0$ pour toute variété projective lisse, le procédé 0.2.5 nous indique que les groupes $Z^{2dim(X)-2}(Y)$ et $Z^4(Y)$ sont des invariants birationnels stables, nuls pour toute variété stablement rationnelle.

Concernant le groupe $Z^4(X)$, Voisin ([85]) obtient par des méthodes de variation de structure de Hodge, le résultat suivant en dimension 3:

Théorème 0.2.10. *([85, Théorème 2]) Soit X une variété complexe projective lisse de dimension 3, soit uniréglée soit satisfaisant $K_X \simeq \mathcal{O}_X$ et $H^2(X,\mathcal{O}_X) = 0$. Alors $Z^4(X) = 0$.*

Ce résultat indique qu'en dimension 3, l'invariant $Z^4(X)$ n'est pas pertinent pour la distinction entre différentes classes de variétés "presque rationnelles". Mais en dimension supérieure, les travaux de Colliot-Thélène-Ojanguren et Colliot-Thélène-Voisin permettent d'obtenir le résultat suivant:

Théorème 0.2.11. *([19], [23, Théorème 5.6]) Il existe des variétés complexes unirationnelles, qui sont des fibrés en quadriques de dimension 6 qui vérifient $Z^4(X) \neq 0$. En particulier ces variétés ne sont pas stablement rationnelles.*

Cas $k \neq \mathbb{C}$.

Lorsque k est un corps arbitraire algébriquement clos, une variété projective X est toujours définie sur un corps $k_X \subset k$ d'engendrement fini sur son sous-corps premier; on peut prendre pour générateurs les coefficients du nombre fini d'équations d'un système d'équations définissant X dans un plongement projectif donné. Par la théorie du schéma de Hilbert, une sous-variété $Z \subset X$ se déforme sur une sous-variété $Z' \subset X$ définie sur une extension finie k'_X de k_X. La classe de Z dans $H^{2k}_{\text{ét}}(X,\mathbb{Z}_\ell(k))$ ($\ell \neq car(k)$) est donc invariante sous l'action du groupe de Galois $Gal(k|k'_X)$. Donc l'image de l'application classe de cycles est contenue dans $\bigcup_U H^{2k}_{\text{ét}}(X,\mathbb{Z}_\ell(k))^U$ où U parcourt le système de sous-groupes ouverts de $Gal(k|k_X)$. Le groupe $\bigcup_U H^{2k}_{\text{ét}}(X,\mathbb{Z}_\ell(k))^U$ est le groupe des *classes de Tate*. Nous avons la conjecture de Tate:

Conjecture 0.2.12. *(Conjecture de Tate) Soient X une variété projective lisse sur un corps algébriquement clos k et ℓ un nombre premier différent de la caractéristique de k. Alors toute classe de Tate est la classe d'un cycle algébrique i.e.*

$$Im([\ ] : \mathrm{CH}^k(X) \otimes \mathbb{Q}_\ell \to H^{2k}_{\acute{e}t}(X,\mathbb{Q}_\ell(k))) = \bigcup_U H^{2k}_{\acute{e}t}(X,\mathbb{Q}_\ell(k))^U.$$

La conjecture de Tate est, encore une fois, trivialement vérifié en degré 0 et $2dim(X)$ mais, contrairement au cas complexe (théorème de Lefschetz sur les classes $(1,1)$), il n'existe pas de preuve, sauf sur certaines variétés, de cette conjecture en degré 2.

Par ailleurs, considérons la question suivante:
"**Conjecture de Tate entière**": *Soient X une variété projective lisse sur un corps algébriquement clos k et ℓ un nombre premier différent de la caractéristique de k. Alors toute classe de Tate est la classe d'un cycle algébrique i.e.*

$$Im([\ ] : \mathrm{CH}^k(X) \otimes \mathbb{Z}_\ell \to H^{2k}_{\text{ét}}(X,\mathbb{Z}_\ell(k))) = \bigcup_U H^{2k}_{\text{ét}}(X,\mathbb{Z}_\ell(k))^U.$$

De même que pour la conjecture de Hodge entière, il existe des contre-exemples dus à Totaro ([77]). Il est à noter cependant que pour le cas des classes de Tate de degré 2, la conjecture rationnelle entraîne la formulation entière.

Nous pouvons encore former, pour $\ell \neq car(k)$, les quotients:

$$Z_\ell^{2k}(X) := \bigcup_U H^{2k}_{\text{ét}}(X, \mathbb{Z}_\ell(k))^U / H^{2k}_{\text{ét}}(X, \mathbb{Z}_\ell(k))_{alg}$$

où $H^{2k}_{\text{ét}}(X, \mathbb{Z}_\ell(k))_{alg}$ désigne l'image de l'application classe de cycles. Le $\mathbb{Z}_\ell$-module $Z_\ell^{2dim(X)-2}(X)$ est un invariant birationnel stable nul pour l'espace projectif.

Sur la clôture algébrique des corps finis, on dispose du théorème conjectural suivant, dû à Schoen ([70]):

Théorème 0.2.13. *([70]) Soit X une variété projective lisse sur un corps fini de caractéristique $p > 0$. Si la conjecture de Tate pour les classes de degré 2 sur les surfaces définies sur les corps finis de caractéristique p est vraie, alors, la conjecture de Tate entière est vraie pour les classes de degré $2dim(X) - 2$ sur X i.e. $Z_\ell^{2dim(X)-2}(X) = 0$.*

Ce résultat indique que sur la clôture algébrique des corps finis, l'invariant $Z_\ell^{2dim(X)-2}(X)$ est trivial. Voisin ([88]) en déduit l'énoncé suivant:

Théorème 0.2.14. *([88, Théorème 1.6]) Soit Y une variété projective lisse sur $\mathbb{C}$ et rationnellement connexe. Supposons la conjecture de Tate vraie pour les diviseurs (degré 2) sur les surfaces définies sur les corps finis. Alors $Z^{2dim(Y)-2}(Y)$ est nul.*

L'invariant birationnel $Z^{2dim(X)-2}(X)$ n'est donc, si la conjecture de Tate est vraie en degré 2, pas utile pour tester les différences entre les classes de variétés “presque rationnelles”.

0.2.3 Jacobiennes intermédiaires

Soit X une variété projective lisse sur $\mathbb{C}$ de dimension n. Grâce aux travaux de Griffiths ([31]), il est encore possible d'étudier une partie des cycles homologiquement triviaux grâce à la cohomologie. Généralisant les applications et variétés de Picard et d'Albanese, Griffiths introduit, dans [31], des tores complexes construits en utilisant les structures de Hodge sur la cohomologie de X et des applications d'Abel-jacobi

$$\Phi_X^k : \mathrm{CH}^k(X)_{hom} \to J^{2k-1}(X)$$

où $J^{2k-1}(X)$ est défini par

$$J^{2k-1}(X) := \frac{H^{2k-1}(X(\mathbb{C}), \mathbb{C})}{\oplus_{p+q=2k-1, p\geq k} H^{p,q}(X) \oplus H^{2k-1}(X(\mathbb{C}), \mathbb{Z})}.$$

Le théorème d'Abel généralisé fournit l'isomorphisme $\mathrm{CH}^1(X)_{hom} \simeq J^1(X)(\mathbb{C}) = Pic^0(X)(\mathbb{C})$.

L'application d'Abel-Jacobi $\Phi_X^{dim(X)} : \mathrm{CH}_0(X)_{hom} \to J^{2dim(X)-1}(X) = \mathrm{Alb}(X)$ est l'application d'Albanese; celle-ci est surjective. Son noyau peut être “de dimension infinie” comme le montre le théorème de Mumford rappelé dans la série d'exemples 0.1.3. Nous disposons toutefois du théorème dû à Roitman:

Théorème 0.2.15. *([67]) Soit X une variété projective lisse sur un corps k algébriquement clos. Alors l'application d'Albanese induit un isomorphisme entre les groupes de torsion première à la caractéristique de k de $\mathrm{CH}_0(X)_{hom}$ et $\mathrm{Alb}(X)(k)$. En d'autres termes, en caractéristique 0 (resp. en caractéristique $p > 0$), le groupe $\mathrm{CH}_0(X)_{tors,AJ} := Ker(\Phi^{2dim(X)-1}_{X|\mathrm{CH}_0(X)_{tors}})$ est nul (resp. de torsion p-primaire) pour toute variété projective lisse.*

Une des conséquences d'un résultat profond de K-théorie algébrique de Bloch portant sur les cycles de codimension 2 des variétés projectives est le théorème suivant:

Théorème 0.2.16. *([11], [21, Corollaire 5]) Soit X une variété complexe projective lisse. Alors l'application d'Abel-Jacobi $\Phi^2_X : \mathrm{CH}^2(X)_{hom} \to J^3(X)$ est injective sur le sous-groupe des cycles de codimension 2 de torsion et homologiquement triviaux. En d'autres termes le groupe $\mathrm{CH}^2(X)_{tors,AJ} := Ker(\Phi^2_{X|\mathrm{CH}^2(X)_{tors,hom}})$ est nul pour toute variété complexe projective lisse.*

Comme les résultats énoncés dans les théorèmes 0.2.15 et 0.2.16 (auquel il faut ajouter le théorème d'Abel généralisé) sont valables pour toute variété projective lisse, le principe 0.2.5 de construction d'invariants birationnels stables montre, grâce aux formules de Manin, que le groupe

$$\mathrm{CH}_1(X)_{tors,AJ} := Ker(\Phi^{dim(X)-1}_{X|\mathrm{CH}_1(X)_{tors,hom}}) \tag{5}$$

des 1-cycles de torsion contenus dans le noyau de l'application d'Abel-Jacobi et le groupe

$$\mathrm{CH}^3(X)_{tors,AJ} := Ker(\Phi^3_{X|\mathrm{CH}^3(X)_{tors,hom}}) \tag{6}$$

des cycles de torsion de codimension 3 contenus dans le noyau de l'application d'Abel-Jacobi sont des invariants birationnels stables ([87]), nuls pour les variétés stablement rationnelles.

Les hypersurfaces lisses de l'espace projectif complexe forment une classe de variétés naturelles sur lesquelles il est intéressant de tester les invariants introduits jusqu'ici. Les quadriques étant trivialement rationnelles, les cubiques sont le premier cas non trivial, il s'avère même assez subtil:

En dimension 1, une cubique lisse est une courbe elliptique et n'est donc pas rationnelle;

En dimension 2, toute surface cubique (lisse) est rationnelle et s'obtient comme éclatement du plan projectif $\mathbb{P}^2$ en 6 points en position générale;

En dimension 3, Clemens et Griffiths prouvent ([15]) qu'aucune hypersurface cubique (lisse) n'est rationnelle. L'invariant qu'ils utilisent porte sur la jacobienne intermédiaire $J^3(X)$ des cubiques de dimension 3. Commençons par rappeler les faits généraux suivants:

Choisissons une polarisation $\mathcal{O}_X(1) \in Pic(X)$ sur une variété projective complexe X. La structure de variété kählérienne compacte dont elle munit $X(\mathbb{C})$ non seulement permet de munir les groupes de cohomologie $H^k(X(\mathbb{C}), \mathbb{C})$ d'une décomposition de Hodge (0.2.8) mais permet aussi de définir sur $H^k(X(\mathbb{C}), \mathbb{Z})$ une forme bilinéaire $(-1)^k$-symétrique $\langle \cdot, \cdot \rangle_k$ par la formule:

$$\langle \alpha, \beta \rangle_k := \int_X c_1(\mathcal{O}_X(1))^{n-k} \cup \alpha \cup \beta$$

pour toutes classes $\alpha, \beta \in H^k(X(\mathbb{C}), \mathbb{Z})$. La décomposition de Hodge est orthogonale pour la forme hermitienne $(\alpha, \beta) \mapsto i^k \langle \alpha, \overline{\beta} \rangle_k$ induite sur $H^k(X(\mathbb{C}), \mathbb{C})$.

Notons $H^i(X(\mathbb{C}), \mathbb{Q})_{prim} := Ker(\cup c_1(\mathcal{O}_X(1))^{n-i+1} : H^i(X(\mathbb{C}), \mathbb{Q}) \to H^{2n-i+2}(X(\mathbb{C}), \mathbb{Q}))$ la cohomologie primitive de X et, pour $p+q=i$, $H^{p,q}(X)_{prim} := H^{p,q}(X) \cap H^i(X(\mathbb{C}), \mathbb{C})_{prim}$. Nous disposons du théorème:

Théorème 0.2.17. *(1) Les groupes de cohomologie admettent la décomposition:*

$$H^k(X(\mathbb{C}), \mathbb{Q}) = \bigoplus_{2r \leq k} c_1(\mathcal{O}_X(1))^r \cup H^{k-2r}(X(\mathbb{C}), \mathbb{Q})_{prim}$$

et celle-ci est orthogonale pour $\langle \cdot, \cdot \rangle_k$ et induit sur le sous-espace $c_1(\mathcal{O}_X(1))^r \cup H^{k-2r}(X(\mathbb{C}), \mathbb{Q})_{prim}$ la forme $(-1)^r \langle \cdot, \cdot \rangle_{k-2r}$.

(2) De plus, $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k} \langle \cdot, \cdot \rangle_k$ est définie positive sur $H^{p,q}(X(\mathbb{C}), \mathbb{Q})_{prim}$.

Cette forme vérifie donc, sur la décomposition de Hodge de $H^k(X(\mathbb{C}),\mathbb{C})_{prim}$, les relations dites de Hodge-Riemann:

1. $\langle\alpha,\beta\rangle_k = 0$ pour tout $\alpha \in H^{p,q}(X)_{prim}$ et $\beta \in H^{p',q'}(X)_{prim}$ avec $(p,q) \neq (p',q')$;
2. $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k}\langle\alpha,\overline{\alpha}\rangle_k > 0$ pour tout $\alpha \in H^{p,q}(X)_{prim}$.

Lorsque $k = 2s-1$ est impair et que $H^k(X(\mathbb{C}),\mathbb{Z})_{prim}$ est de niveau 1 i.e. $H^k(X(\mathbb{C}),\mathbb{C})_{prim}$ admet une décomposition de Hodge de la forme $H^k(X(\mathbb{C}),\mathbb{C})_{prim} = H^{s-1,s}(X)_{prim} \oplus H^{s,s-1}(X)_{prim}$, les relations de Hodge-Riemann permettent de munir l'espace vectoriel $H^{s-1,s}(X)_{prim}$ de la forme hermitienne définie positive $-i\langle\cdot,\cdot\rangle_k$; on obtient ainsi, par le théorème d'Appell-Humbert, une polarisation sur le tore $H^{s-1,s}(X)_{prim}/H^{2k-1}(X(\mathbb{C}),\mathbb{Z})_{prim/Tors}$ qui est donc, dans ce cas, une variété abélienne.

Lorsque la dimension n de X est impaire et que $H^n(X(\mathbb{C}),\mathbb{Q})$ est réduit à $H^n(X(\mathbb{C}),\mathbb{Q})_{prim}$ et est de niveau 1, la dualité de Poincaré se traduit par le fait que la forme d'intersection $\langle\cdot,\cdot\rangle_n$ sur $H^n(X(\mathbb{C}),\mathbb{Z})_{/Tors}$ est unimodulaire i.e. $(J^n(X),\langle\cdot,\cdot\rangle_n)$ est une variété abélienne principalement polarisée. En dehors du cas des courbes, un des premiers cas où une telle particularité apparaît est celui des variétés séparablement rationnellement connexes de dimension 3. En effet, la connexité rationnelle séparable d'une variété X peut se traduire de la façon suivante:

Théorème 0.2.18. *([47, Théorème 3.7, Chap. IV]) Soit X une variété projective lisse sur k. Alors X est séparablement rationnellement connexe si et seulement s'il existe un morphisme $f : \mathbb{P}^1 \to X$ tel que le fibré f^*T_X est ample, T_X désignant le fibré tangent de X. Les déformations d'un tel morphisme permettent alors de recouvrir un ensemble dense de X par des courbes rationnelle satisfaisant la même condition de positivité du tiré en arrière de T_X.*

Lorsque cette condition est satisfaite, d'une part la courbe rationnelle considérée est mobile dans X et d'autre part, $f^*\Omega_X$ et donc tous les $(f^*\Omega_X)^{\otimes r}$ sont des fibrés se décomposant en fibrés en droites de degré strictement négatif. On obtient donc le corollaire suivant:

Corollaire 0.2.19. *([47, Corolllaire 3.8, Chap. IV]) Soit X une variété séparablement rationnellement connexe sur k. Alors $H^0(X,(\Omega_X)^{\otimes r}) = 0$ pour tout $r > 0$.*

En particulier, pour un solide X séparablement rationnellement connexe, ce corollaire fournit l'annulation de $H^1(X(\mathbb{C}),\mathbb{Z})$ et de $H^{3,0}(X)$ (donc de $H^{0,3}(X)$ également) si bien que $H^3(X(\mathbb{C}),\mathbb{Z})$ est de niveau 1 et est réduit à $H^3(X(\mathbb{C}),\mathbb{Z})_{prim}$ car $H^5(X(\mathbb{C}),\mathbb{Z}) \simeq H_1(X(\mathbb{C})) = 0$ puisque X est simplement connexe ([71]); la jacobienne intermédiaire $(J^3(X),\langle\cdot,\cdot\rangle_3)$ est une variété abélienne principalement polarisée. Rappelons le fait suivant:

Proposition 0.2.20. *([15, Corollary 3.26]) Soit X un solide complexe rationnel. Alors $(J^3(X),\langle\cdot,\cdot\rangle_3)$ est isomorphe à une somme de jacobiennes $\oplus_i(J^1(C_i),\langle\cdot,\cdot\rangle_1)$ de courbes (connexes) projective lisses C_i.*

Preuve. Étant donné un diagramme de résolution des indéterminées d'une application birationnelle $\varphi : \mathbb{P}^3 \dashrightarrow X$:

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[d, "\tau"'] \arrow[dr, "\phi"] & \\ \mathbb{P}^3 \arrow[r, dashed, "\varphi"] & X \end{tikzcd}$$

où $\tau : \widetilde{\Gamma} \to \mathbb{P}^3$ consiste en une série d'éclatements de courbes (connexes lisses) C_i et de points et ϕ est le morphisme birationnel résultant. L'éclatement des points laissant invariant le troisième groupe de cohomologie, les formules de Manin fournissent l'isomorphisme $(J^3(\widetilde{\Gamma}), \langle\cdot,\cdot\rangle_{\widetilde{\Gamma}}) \simeq \oplus_{i\in I}(J^1(C_i), \langle\cdot,\cdot\rangle_{C_i})$. Le morphisme ϕ étant de degré 1, nous avons l'identité $\phi_*\phi^* = id_{J^3(X)}$ i.e. $(J^3(X), \langle\cdot,\cdot\rangle_X)$ est un facteur direct de $(J^3(\widetilde{\Gamma}), \langle\cdot,\cdot\rangle_{\widetilde{\Gamma}})$. Or la décomposition de $(J^3(\widetilde{\Gamma}), \langle\cdot,\cdot\rangle_{\widetilde{\Gamma}})$ en somme de variétés abéliennes principalement polarisées est unique à permutation des facteurs près (théorème de Poincaré) et la jacobienne d'une courbe connexe et lisse $J^1(C_i)$ est irréductible comme variété principalement polarisée, donc $(J^3(X), \langle\cdot,\cdot\rangle_X)$ s'identifie à une somme partielle $\oplus_{i\in J}(J^1(C_i), \langle\cdot,\cdot\rangle_{C_i})$ $(J \subset I)$. □

Clemens et Griffiths montrent dans [15] que la jacobienne intermédiaire d'un solide cubique n'est pas isomorphe à une somme de jacobienne de courbes -en tant que variété abélienne principalement polarisée-, ce qui montre son irrationalité. Le critère de Clemens-Griffiths (0.2.20) est aussi l'invariant birationnel utilisé plus tard, en 1985, par Beauville, Colliot-Thélène, Sansuc et Swinnerton-Dyer ([9]), pour exhiber un exemple de variété stablement rationnelle non rationnelle. Le résultat de Clemens et Griffiths sur les solides cubiques laisse néanmoins ouverte la question:

Problème 0.2.21. *Le solide cubique très général est-il stablement rationnel?*

Les invariants que nous avons introduits jusqu'ici une permettent pas de répondre à cette question puisqu'ils s'annulent tous pour un solide cubique.

Plus généralement le problème suivant reste ouvert:

Problème 0.2.22. *Existe-t-il des hypersurfaces cubiques lisses non (stablement) rationnelles de dimension > 3?*

Quelles informations avons-nous sur les invariants birationnels introduits jusqu'ici pour les cubiques $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, $n \geq 3$?

Les travaux de Bloch et Srinivas ([13]) permettent d'affirmer, sous l'hypothèse que $\mathrm{CH}_0(X) = \mathbb{Z}$, que $\mathrm{Griff}^2(X) = 0$. Les travaux de Shen ([72]) sur les 1-cycles sur les cubiques permettent d'affirmer que $\mathrm{Griff}_1(X) = 0$.

La conjecture de Hodge entière est trivialement vérifiée pour les 1-cycles sur les cubiques de sorte que $Z^{2n-2}(X) = 0$. Concernant $Z^4(X)$, le seul cas dans lequel la structure de Hodge de $H^4(X(\mathbb{C}), \mathbb{Z})$ est non triviale (non donnée par le théorème des sections hyperplanes de Lefschetz) est pour $n = 4$. La conjecture de Hodge entière a, dans ce cas été établie par Voisin ([86]), de sorte que $Z^4(X) = 0$ pour toute hypersurface cubique.

Concernant l'invariant birationnel stable $\mathrm{CH}_1(X)_{tors,AJ}$, nous expliquons, dans la section 1, qu'une conséquence des travaux de Shen ([72],[73]) est:

Proposition 0.2.23. *Soit $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ une hypersurface cubique lisse. Alors $\mathrm{CH}_1(X)_{tors,AJ} = 0$.*

D'après le théorème de Roitman (0.2.15), $\mathrm{CH}^3(X)_{tors,AJ} = \mathrm{CH}_0(X)_{tors,AJ}$ est nul pour un solide cubique. D'après la proposition précédente, $\mathrm{CH}^3(X)_{tors,AJ} = 0$ pour les cubiques de dimension 4. Pour les cubiques de dimension 5, nous présentons en section 1, le résultat suivant faisant le lien entre la géométrie de la cubique et celle de sa variété des droites:

Théorème 0.2.24. *Soit $X \subset \mathbb{P}^6_{\mathbb{C}}$ une hypersurface cubique lisse. Notons $F(X) \subset G(2,7)$ sa variété des droites et $P \subset F(X) \times X$ la correspondance donnée par le fibré en $\mathbb{P}^1$ universel au-dessus de $F(X)$. Alors l'action P_* de la correspondance induit un morphisme surjectif*

$$P_* : \mathrm{CH}_1(F(X))_{tors,AJ} \to \mathrm{CH}^3(X)_{tors,AJ}.$$

Mais l'annulation de $\mathrm{CH}_1(Y)_{tors,AJ}$ pour les variétés de Fano Y n'est pas connue.

Cas $k \neq \mathbb{C}$.

Lorsque $k \neq \mathbb{C}$, on ne dispose pas, en dehors des variété de Picard et d'Albanese, en général d'une variété grâce à laquelle les cycles homologue à 0 puissent être étudiés. Il existe néanmoins une application dite d'"Abel-Jacobi ℓ-adique", à laquelle nous ne nous intéresserons pas, définie en utilisant la suite spectrale de Hochschild-Serre pour la cohomologie étale permettant d'étudier les cycles au-delà l'équivalence homologique.

Pour les cycles de codimension 2, cependant, Murre prouve dans [63], grâce à des méthodes de K-théorie algébrique, qu'il existe une variété abélienne $J^3(X)$, appelée représentant algébrique de $\mathrm{CH}^2(X)_{alg}$, vérifiant $dim(J^3(X)) \leq dim_{\mathbb{Q}_\ell} H^3(X, \mathbb{Q}_\ell)$ et un morphisme de groupes régulier $\Phi_X^2 : \mathrm{CH}^2(X)_{alg} \to J^3(X)(k)$ qui est universel pour les morphismes réguliers. On dit qu'un morphisme de groupes $\rho : \mathrm{CH}^2(X)_{alg} \to A(k)$ à valeurs dans une variété abélienne A est régulier si pour toute famille, $\mathcal{Z} \subset T \times X$, de cycles de codimension 2 de X paramétrée par une k-variété quasi-projective lisse T, l'application $T(k) \to J^3(X)(k)$, composition de $T(k) \to \mathrm{CH}^2(X)_{alg}$, $t \mapsto \mathcal{Z}_t - \mathcal{Z}_{t_0}$ suivie de ρ est induite par un morphisme de variétés algébriques $T \to A$. Il est donc possible d'analyser les cycles algébriquement triviaux grâce à cette variété abélienne. Mais en général, il n'existe pas de lien entre une polarisation de $J^3(X)$ et un produit d'intersection sur la cohomologie de X.

Pour les variétés X de dimension $2m+1$ qui sont des fibrés en quadriques ordinaires au-dessus de $\mathbb{P}_k^2$ ($car(k) \neq 2$), i.e. tels que la courbe discriminante dans $\mathbb{P}^2$ est lisse, Beauville ([7]) montre qu'il existe un représentant algébrique de $\mathrm{CH}^{m+1}(X)_{alg}$, une variété abélienne principalement polarisée $(J^{2m+1}(X), \theta)$ et un morphisme régulier universel $\Phi_X^{m+1} : \mathrm{CH}^{n+1}(X)_{alg} \to J^{2m+1}(X)$. De plus l'accouplement de Weil associé à la polarisation principale θ est directement lié, comme dans le cas complexe, au produit d'intersection sur $H^{2m+1}(X, \mathbb{Z}_\ell)$. Les solides cubiques entrent dans cette catégorie et Mumford ([60]) montre que le représentant algébrique de $\mathrm{CH}^2(X)_{alg}$ muni de sa polarisation n'est pas isomorphe à une somme de jacobiennes de courbes connexes et lisses, ce qui devrait être le cas si X était rationnel. Il étend ainsi aux corps de caractéristique plus grande que 3, le résultat de non rationlité des solides cubiques. Dans ce cas aussi, la question de la rationalité stable des solides cubiques reste ouverte.

0.3 Décomposition de la diagonale

Nous avons vu que le groupe des 0-cycles était un invariant birationnel stable grâce à l'existence d'une résolution de singularités et la discussion 0.0.1. En l'absence de théorème de résolution de singularités en dimension > 4 en caractéristique positive, donnons, suivant [23], une autre preuve de cette invariance:

Proposition 0.3.1. *Soit $\phi : X \dashrightarrow Y$ une application birationnelle entre variétés projectives lisses sur un corps L (pas forcément algébriquement clos). Alors $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$.*

Preuve. Considérons la fermeture $\Gamma_\phi \subset X \times Y$ du graphe de ϕ et la fermeture $\Gamma_{\phi^{-1}} \subset Y \times X$ du graphe de ϕ^{-1}. Le point essentiel est que nous avons une égalité de cycles

$$\Gamma_{\phi^{-1}} \circ \Gamma_\phi = \Delta_X + Z \text{ dans } \mathrm{CH}^n(X \times X) \tag{7}$$

où Z est un cycle supporté sur $D \times X$, pour $D \subset X$ un fermé algébrique strict. Cette égalité est prouvée dans [23, Lemme 3.5] et repose sur le fait que (en référence à la définition de la composition des correspondances) les schémas $p_{13}(p_{12}^{-1}(\Gamma_(\phi^{-1}) \cap p_{23}^{-1}(\Gamma_\phi))$, où les p_{ij} sont les projections de $X \times Y \times X$ sur les différents produits de deux de ses facteurs, et Δ_X coïncident sur un ouvert $U \times X$.

Il suffit alors de laisser agir (7) sur les 0-cycles de X, en notant que Z_* est nul sur $\mathrm{CH}_0(X)$, pour voir que $\Gamma_{\phi^{-1},*} \circ \Gamma_{\phi,*} = id_{\mathrm{CH}_0(X)}$ i.e. $\Gamma_{\phi,*} : \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ est injective. De même, on a une égalité

$$\Gamma_\phi \circ \Gamma_{\phi^{-1}} = \Delta_Y + Z' \text{ dans } \mathrm{CH}^n(Y \times Y)$$

où Z' est supporté sur $D' \times Y$, pour un fermé strict $D' \subset Y$, grâce à laquelle on obtient la surjectivité de $\Gamma_{\phi,*} : \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$. □

Ainsi, si X est une variété rationnelle projective lisse sur k de dimension n, $\mathrm{CH}_0(X) \overset{deg}{\simeq} \mathbb{Z}$ mais il y a plus: notons $\phi : \mathbb{P}^n_k \dashrightarrow X$ une application birationnelle; par changement de base, on obtient une application birationnelle $\phi_L : \mathbb{P}^n_L \dashrightarrow X_L$ pour toute extension de corps L/k. L'invariance birationnelle du groupe des 0-cycles donne alors $\mathrm{CH}_0(X_L) \overset{deg}{\simeq} \mathbb{Z}$. On a la définition suivante:

Définition 0.3.2. *Soit X une variété projective lisse sur k de dimension n. On dit que X est à* CH_0 *universellement trivial si* $\mathrm{CH}_0(X_L) \overset{deg}{\simeq} \mathbb{Z}$ *pour toute extension de corps L/k.*

Par l'invariance birationnelle de CH_0 et du fait que $\mathrm{CH}_0(X_L \times_L \mathbb{P}^s_L) \simeq \mathrm{CH}_0(X_L)$, la propriété "être à CH_0 universellement trivial" est une propriété invariante par équivalence birationnelle stable. Les variétés stablement rationnelles sont donc à CH_0 universellement trivial. Comme remarqué par Auel, Colliot-Thélène et Parimala ([5]), cette propriété peut être reformulée de la façon suivante. Introduisons la définition suivante, inspirée des travaux de Bloch et Srinivas ([13]):

Définition 0.3.3. *On dit que X admet une décomposition de Chow de la diagonale si on a l'égalité:*

$$\Delta_X = X \times x + Z \text{ dans } \mathrm{CH}^n(X \times X) \tag{8}$$

où Z est un cycle supporté sur $D \times X$ pour D un fermé algébrique strict de X.

Il est montré dans *loc. cit*, que les deux propriétés sont en fait équivalentes pour X projective lisse. Esquissons l'argument:

Appliquons la propriété CH_0 universellement trivial à l'extension $k(X)/k$ et intéressons-nous au 0-cycle fourni par le point $k(X)$-rationnel, δ_X, de $X_{k(X)}$ qu'est la diagonale, obtenu en prenant la limite sur les ouverts de X du morphisme diagonal $X \to X \times X$ ($V \to V \times V \to V \times X$). Comme $\mathrm{CH}_0(X_{k(X)}) \simeq \mathbb{Z}$ et que δ_X et $x_{k(X)} := x \times_k k(X)$, où $x \in X(k)$, sont des 0-cycles de $X_{k(X)}$, de degré 1, ces derniers sont rationnellement équivalents sur $k(X)$ i.e.

$$\delta_X - x_{k(X)} = 0 \text{ dans } \mathrm{CH}_0(X_{k(X)}).$$

Maintenant, $\mathrm{CH}_0(X_{k(X)}) = \mathrm{CH}^n(X_{k(X)})$ et nous disposons de la formule de Bloch $\mathrm{CH}^n(X_{k(X)}) = \varinjlim_{U \subset X,\ ouvert} \mathrm{CH}^n(U \times X)$; elle nous permet donc d'affirmer qu'il existe un ouvert non vide U_0 tel que $(\Delta_X - X \times x)_{|U_0 \times X} = 0$ dans $\mathrm{CH}^n(U_0 \times X)$. Nous obtenons alors l'écriture:

$$\Delta_X = X \times x + Z \text{ dans } \mathrm{CH}^n(X \times X)$$

où Z est un cycle supporté sur $D \times X$ pour D un fermé algébrique strict ($D = X \backslash U_0$).

Réciproquement si une variété X admet une décomposition de Chow de la diagonale (8), alors, étant donné une extension L/k, par changement de base, on obtient

$$\Delta_{X_L} = X_L \times_L x_L + Z_L \text{ dans } \mathrm{CH}^n(X_L \times_L X_L).$$

En faisant agir cette identité sur un 0-cycle $z \in \mathrm{CH}_0(X_L)$, on obtient $z = deg(z)x_L$ dans $\mathrm{CH}_0(X_L)$ i.e. $\mathrm{CH}_0(X_L) \simeq \mathbb{Z}x_L$.

"Être à CH_0 universellement trivial" est donc équivalent au fait d'admettre une d'ecomposition de Chow de la diagonale.

Dans une série d'articles entamée autour de 2012, Voisin a étudié cette propriété, elle a notamment montré ([91]) en utilisant une méthode de dégérescence, que la condition nécessaire pour la rationalité stable donnée par cette propriété est strictement plus forte que l'annulation des invariants d'Artin-Mumford (torsion dans le troisième groupe de cohomologie) et qu'elle permet de conclure à la non rationalité (stable) de certains solides dont l'irrationalité ne peut être détectée par le critère de Clemens-Griffiths (jacobienne intermédiaire modulo les jacobiennes de courbes), en exhibant une famille d'exemples de solides n'admettant pas de décomposition de la diagonale, donc non stablement rationnels, mais pour lesquelles les invariants d'Artin-Mumford et de Clemens-Griffiths ne permettent pas de détecter la non rationalité (stable). La méthode de Voisin, améliorée notamment par Colliot-Thélène et Pirutka ([20]), a permis récemment (2016) à Hassett, Pirutka et Tschinkel ([39]) de montrer que la rationalité stable n'est pas une propriété invariante par déformation.

Soit X une variété à CH_0 universellement trivial ou, de façon équivalente admettant une décomposition de la diagonale. Considérons donc une écriture de la forme (8) et introduisons une désingularisation $\widetilde{D}$, s'il en existe une, de D et choisissons un cycle $\widetilde{Z} \in \mathrm{CH}^{n-1}(\widetilde{D} \times X)$ tel que $(j, id_X)_*(\widetilde{Z}) = Z$ dans $\mathrm{CH}^n(X \times X)$, où $j : \widetilde{D} \to X$ est la composition de la résolution des singularités de D suivie de l'injection de D dans X. Le cycle $\widetilde{Z}$ existe à condition de choisir D convenablement.

Soit γ un cycle de codimension $d < n$ ou une classe de cohomologie de degré $k < 2n$. Alors, l'action de l'identité $(8)^*$ sur γ donne:

$$\gamma = j_* \widetilde{Z}^* \gamma \tag{9}$$

et $\widetilde{Z}^*\gamma$ est de codimension $d-1$ sur $\widetilde{D}$ ou de degré $k-2$. De même, si γ un cycle de dimension $d > 0$, l'action de l'identité $(8)_*$, sur γ donne:

$$\gamma = \widetilde{Z}_* j^* \gamma \tag{10}$$

et $j^*\gamma$ est de dimension $d-1$ sur $\widetilde{D}$. Ces deux remarques (et la fonctorialité des groupes de Chow, de cohomologie, de l'application classe de cycles...) montrent que la décomposition de Chow de la diagonale assure la nullité des invariants birationnels stables obtenus grâce au procédé 0.2.5. En effet, (9) et (10) montrent que l'identité se factorise par les groupes en dimension (resp. codimension, degré) "un de moins" dont l'annulation pour toute variété nous avait permis d'obtenir l'invariant birationnel.

Par exemple lorsque $k = \mathbb{C}$, (9) montre que l'identité sur $\mathrm{CH}^3(X)_{tors,AJ}$ se factorise à travers $\mathrm{CH}^2(\widetilde{D})_{tors,AJ}$ qui est nul d'après 0.2.16.

L'existence d'une décomposition de la diagonale contrôle donc les invariants définis dans la section précédente.

Nous avons remarqué plus haut que la plupart des invariants de la section précédente sont nuls pour les hypersurfaces cubiques et ne permettent donc pas de détecter une éventuelle non rationalité stable des cubiques. Voisin a étudié dans [90] la décomposition de Chow de la diagonale pour les hypersurfaces cubiques. Elle obtient notamment en dimension 3, un critère, une condition nécessaire et suffisante, pour l'existence d'une décomposition de Chow de la diagonale qui lui permet de prouver le fait suivant:

Théorème 0.3.4. *([90, Théorème 1.7]) Les solides cubiques admettant une décomposition de Chow de la diagonale sont paramétrés, dans l'espace des modules des solides cubiques par au moins une union dénombrable non vide de sous-variétés de codimension ≤ 3 de l'espace des modules des solides cubiques.*

En particulier, on peut encore espérer prouver que la cubique complexe très générale n'est pas stablement rationnelle grâce à la décomposition de la diagonale.

La condition obtenue par Voisin se résume au fait que la *classe minimale*, qui est une classe de Hodge entière de degré $2g-2$ (égal à 8 dans notre cas) sur la jacobienne intermédiaire, une variété abélienne principalement polarisée de dimension g, est algébrique.

Dans la section 2, on présente une adaptation à la caractéristique positive d'une partie des résultats obtenus par Voisin sur $\mathbb{C}$ ([90]) sur les cubiques de dimension 3 permettant d'obtenir une condition analogue à celle de Voisin. On obtient alors, comme conséquence du théorème 0.2.13 de Schoen, le résultat suivant:

Théorème 0.3.5. *Soit $k = \overline{\mathbb{F}_p}$ avec $p > 2$. Supposons la conjecture de Tate vraie pour les classes de degré 2 sur les surfaces définies sur les corps finis de caractéristique p. Alors toute hypersurface cubique lisse de $\mathbb{P}^4_k$ admet une décomposition de Chow de la diagonale.*

Ce résultat nous dit que si la conjecture de Tate est vraie, la décomposition de la diagonale n'est pas un invariant assez fin pour détecter l'éventuelle non rationalité stable des solides cubiques sur la clôture algébrique des corps finis.

Pour une variété X, être à CH_0 universellement trivial peut se réécrire sous la forme suivante: il existe $x \in X(k)$, tel que, en notant $i_x : x \hookrightarrow X$ l'inclusion, $i_{x_L,*} : \mathrm{CH}_0(x_L) \to \mathrm{CH}_0(X_L)$ est surjectif pour toute extension L/k i.e. $i_{x,*}$ est universellement surjectif. De même, pour un fermé $i_Y : Y \hookrightarrow X$, on dit que $i_{Y,*} : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ est universellement surjectif si pour toute extension L/k, $i_{Y_L,*} : \mathrm{CH}_0(Y_L) \to \mathrm{CH}_0(X_L)$ est surjectif. On dit alors que $\mathrm{CH}_0(X)$ est universellement supporté sur Y. Voisin introduit, dans [90], la définition suivante:

Définition 0.3.6. *([90, Définition 1.2]) La dimension CH_0 essentielle d'une variété X est l'entier minimal $k \geq 0$ tel qu'il existe un fermé $Y \subset X$ de dimension k tel que $\mathrm{CH}_0(X)$ est universellement supporté sur Y.*

Les variétés à CH_0 universellement trivial i.e. celles admettant une décomposition de Chow de la diagonale sont exactement les variétés dont la dimension CH_0 essentielle est nulle. Dans [17], Colliot-Thélène prouve le fait suivant:

Proposition 0.3.7. *([17, Proposition 2.5 et Remarque 2.6]) Soit X une variété complexe projective lisse vérifiant $\mathrm{CH}_0(X) \simeq \mathbb{Z}$. Alors la dimension CH_0 essentielle de X est ≤ 1 si et seulement si elle est nulle (i.e. X admet une décomposition de la diagonale).*

Dans la dernière section de ce mémoire, nous nous intéressons à la différence qui existe entre variétés dont la dimension CH_0 essentielle est ≤ 2 et les variétés à dimension CH_0 essentielle nulle. Nous obtenons notamment les énoncés suivants:

Théorème 0.3.8. *Soit X une variété complexe projective lisse de dimension n vérifiant $\mathrm{CH}_0(X) \simeq \mathbb{Z}$. Supposons que la dimension CH_0 essentielle de X est ≤ 2. Si:*

1. $Tors(H^2(X(\mathbb{C}), \mathbb{Z})) = 0$ *et*
2. $H^3(X(\mathbb{C}), \mathbb{Z}) = 0$

alors X est de dimension CH_0 essentielle 0 (i.e. X admet une décomposition de la diagonale).

Théorème 0.3.9. *Soit X une variété complexe projective lisse de dimension n vérifiant* $\mathrm{CH}_0(X) \simeq \mathbb{Z}$. *Supposons que la dimension* CH_0 *essentielle de X est ≤ 2.*

S'il existe un schéma projectif lisse Y de dimension $(n-1)$ et un morphisme $j : Y \to X$ tel que

$$j_* : Pic^0(Y) \to \mathrm{CH}^2(X)_{alg}$$

est universellement surjectif,

alors X est de dimension CH_0 *essentielle* 0 *(i.e. X admet une décomposition de la diagonale).*

Chapter 1

Remarks on the CH_2 of cubic hypersurfaces

Résumé: Dans ce chapitre, nous présentons deux façons de ramener les problèmes portant sur les 2-cycles d'une hypersurface cubique lissse X sur un corps algébrquement clos de caractéristique $\neq 2$, en problèmes portant sur les 1-cycles de sa variété des droites $F(X)$. La première méthode utilise les droites osculatrices de X et le théorème de Tsen-Lang. Elle permet de montrer que $\mathrm{CH}_2(X)$ est engendré par $\mathrm{CH}_1(F(X))$, via l'action de la correspondance donnée par le fibré en $\mathbb{P}^1$ universel au-dessus de $F(X)$. La seconde méthode repose sur une extension aux cycles de dimension supérieure d'une formule d'inversion dévoloppée par Shen ([72], [73]) pour les 1-cycles de X. Cette formule d'inversion permet de relever les 2-cycles de torsion de X en 1-cycles de torsion de $F(X)$. Pour les hypersurfaces cubiques X de dimension 5, cela permet de montrer que l'invariant birationnel de X, $\mathrm{CH}^3(X)_{tors,AJ}$ est contrôlé par l'invariant birationnel $\mathrm{CH}_1(F(X))_{tors,AJ}$ de $F(X)$.

Abstract: In this chapter, we present two approaches to reduce problems on 2-cycles on a smooth cubic hypersurface X over an algebraically closed field of characteristic $\neq 2$, to problems on 1-cycles on its variety of lines $F(X)$. The first one relies on osculating lines of X and Tsen-Lang theorem. It allows to prove that $\mathrm{CH}_2(X)$ is generated, via the action of the universal $\mathbb{P}^1$-bundle over $F(X)$, by $\mathrm{CH}_1(F(X))$. The second approach consists of an extension to subvarieties of X of higher dimension of an inversion formula developed by Shen ([72], [73]) in the case of 1-cycles of X. This inversion formula allows to lift torsion cycles in $\mathrm{CH}_2(X)$ to torsion cycles in $\mathrm{CH}_1(F(X))$. For complex cubic 5-folds, it allows to prove that the birational invariant provided by the group $\mathrm{CH}^3(X)_{tors,AJ}$ of homologically trivial, torsion codimension 3 cycles annihilated by the Abel-Jacobi morphism is controlled by the group $\mathrm{CH}_1(F(X))_{tors,AJ}$ which is a birational invariant of $F(X)$.

1.0 Introduction

Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree $d \geq 2$. Let $F_r(X) \subset G(r+1, n+2)$ be the variety of $\mathbb{P}^r$'s contained in X and $P_r = \mathbb{P}(\mathcal{E}_{r+1|F_r(X)}) \subset F_r(X) \times X$ be the universal $\mathbb{P}^r$-bundle. One has the incidence correspondence

$$p_r : P_r \to F_r(X),\ q_r : P_r \to X.$$

We will be particularly interested in this chapter in the cases $r = 1$ and $r = 2$, $d = 3$. It is known (see for example [29], [83]) that if X is covered by projective spaces of dimension $1 \leq r < \frac{n}{2}$, that is q_r is surjective, then $\mathrm{CH}_i(X)_{\mathbb{Q}} \simeq \mathbb{Q}$ for $i < r$ and for $\frac{n}{2} > i \geq r$, there is an inversion formula implying that

$$P_{r,*} : \mathrm{CH}_{i-r}(F_r(X))_{hom,\mathbb{Q}} \to \mathrm{CH}_i(X)_{hom,\mathbb{Q}}$$

is surjective. We recall briefly how it works: Up to taking a desingularization and general hyperplane sections of $F_r(X)$, we can assume that $F_r(X)$ is smooth and q_r is generically finite of degree $N > 0$.

Let $H_X = c_1(O_X(1)) \in \mathrm{CH}^1(X)$ and $h = q_r^*H_X \in \mathrm{CH}^1(P_r)$. Given a cycle $\Gamma \in \mathrm{CH}_i(X)_{hom}$, we have $N\Gamma = q_{r*}q_r^*\Gamma$ and we can write $q_r^*\Gamma = \sum_{j=0}^r h^j \cdot p_r^*\gamma_j$ where $\gamma_j \in \mathrm{CH}_{i+j-r}(F_r(X))_{hom}$. Now, $dq_{r*}(h^j \cdot p_r^*\gamma_j) = dH_X^j \cdot q_{r*}(p_r^*\gamma_j) = 0$ for $j > 0$ since

$$dH_X \cdot = i_X^* i_{X,*} : \mathrm{CH}_l(X)_{hom} \to \mathrm{CH}_{l-1}(X)_{hom},$$

where i_X is the inclusion of X into $\mathbb{P}^{n+1}$, factors through $\mathrm{CH}_l(\mathbb{P}^{n+1}_{\mathbb{C}})_{hom}$ hence is zero. So we get

$$dN\Gamma = q_{r*}p_r^*(d\gamma_0)$$

which gives $\mathrm{CH}_i(X)_{hom,\mathbb{Q}} = 0$ for $i < r$ since in this range $\mathrm{CH}_{i-r}(F(X)) = 0$, and more generally the desired surjectivity. Working a little more, this method gives, in the case of 2-cycles on cubic fivefolds, the following result (which is a precision of [29], [65]):

Proposition 1.0.1. *Let X be a smooth cubic fivefold. Then the kernel of the Abel-Jacobi map $\mathrm{CH}_2(X)_{AJ} := \mathrm{Ker}\,(\Phi_X : \mathrm{CH}_2(X)_{hom} \to J^5(X))$ is of 18-torsion.*

Proof. For cubic hypersurfaces of dimension ≥ 3, after taking hyperplane sections of $F_1(X)$, the degree of the generically finite morphism $P_1 \to X$ is 6. If $\Gamma \in \mathrm{CH}_2(X)_{AJ}$, we can use the fact that $3H_X \cdot \Gamma = 0$ in $\mathrm{CH}^4(X)$ and thus (using the notation P, p, q in this case)

$$3h \cdot q^*\Gamma = 3(h \cdot p^*\gamma_0 + h^2 \cdot p^*\gamma_1) = 0 \tag{1.1}$$

in $\mathrm{CH}^4(P)$. As $h^2 = h \cdot p^*l - p^*c_2$ in $\mathrm{CH}^2(P)$, where l and c_2 are the natural Chern classes on $F_1(X)$ restricted from the Grassmannian, we deduce from (1.1):

$$3\gamma_0 = -3l \cdot \gamma_1 \text{ in } \mathrm{CH}^3(F_1(X)).$$

Combining this with the previous argument then gives $18\Gamma = q_*p^*(3\gamma_1 \cdot l^3)$ where $3\gamma_1$ is a codimension 2-cycle homologous to 0 and Abel-Jacobi equivalent to 0 on $F_1(X)$. Finally we conclude using [13, Theorem 1 (i)] and the fact that $F_1(X)$ is rationally connected, which implies that $\mathrm{CH}^2(F_1(X))_{AJ} = 0$. □

The denominators appearing in the above argument do not allow to understand 2-torsion cycles. On the other hand, as smooth cubic hypersurfaces admit a degree 2 unirational parametrization ([15]), all functorial birational invariants are 2-torsion so that, for functorial birational invariants constructed using torsion cycles, the above method gives no interesting information. Our aim in this chapter is to give inversion formulas with integral coefficients, allowing in some cases to also control the torsion of the group of cycles, which is especially important for those hypersurfaces in view of rationality problems.

In this chapter, we present two approaches to study the surjectivity of the map P_{1*} on cycles with integral coefficient for cubic hypersurfaces. The first one is presented in the first section and uses the osculating lines of X; it gives the following result:

Theorem 1.0.1. *Let $X \subset \mathbb{P}^{n+1}_k$, with $n \geq 2^i + 1$ be a smooth cubic hypersurface over an algebraically closed field k of characteristic not equal to 2, containing a linear subspace of dimension $i < \frac{n}{2}$. Assuming resolution of singularities in dimension $\leq i$, $P_{1,*} : \mathrm{CH}_{i-1}(F_1(X)) \to \mathrm{CH}_i(X)$ is surjective.*

In the case where $i = 2$, the theorem associates to any 2-cycle a 1-cycle on $F_1(X)$. As, for $i = 2$, the condition to apply the theorem is $dim_k(X) \geq 5$, $F_1(X)$ is a smooth Fano variety hence separably rationally connected in characteristic 0. By work of Tian and Zong ([76]), $\mathrm{CH}_1(F_1(X))$ is then generated by classes of rational curves. A direct consequence is the following:

Corollary 1.0.2. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over an algebraically closed field k of characteristic 0. If $n \geq 5$, then $\mathrm{CH}_2(X)$ is generated by cycle classes of rational surfaces.*

Remark 1.0.3. This result is true for a different reason also in dimension 4, see Proposition 1.2.1.

In the second section, we study 1-cycles on $F_1(X)$ in order to prove that, in some cases, we can take as generators of $\mathrm{CH}_1(F_1(X))$ only the "lines" i.e. the rational curves of degree 1, of $F_1(X)$. We obtain the following result:

Theorem 1.0.4. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree d over an algebraically closed field k of characteristic 0. If $\frac{d(d+1)}{2} < n$ and $F_1(X)$ is smooth then $\mathrm{Griff}_1(F_1(X)) = 0$. Moreover, $\mathrm{CH}_1(F_1(X))$ is generated by lines.*

This theorem has the following consequence in the case of cubic hypersurfaces:

Corollary 1.0.5. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over an algebraically closed field k of characteristic 0. If $n \geq 7$, then $\mathrm{CH}_2(X)$ is generated by classes of planes $\mathbb{P}^2 \subset X$ and therefore $\mathrm{CH}_2(X)_{hom} = \mathrm{CH}_2(X)_{alg}$. If $n \geq 9$, then $\mathrm{CH}_2(X) \simeq \mathbb{Z}$.*

Remark 1.0.6. Some of the results of the first two sections had already been obtained by Pan ([66]) in characteristic 0 but with weaker bounds. For example for cubic hypersurfaces, he proves the surjectivity of $P_{1,*} : \mathrm{CH}_1(F_1(X)) \to \mathrm{CH}_2(X)$ for $n \geq 17$, the fact that $\mathrm{CH}_1(F_1(X))_{hom} = \mathrm{CH}_1(F_1(X))_{alg}$ for $n \geq 13$ and that $\mathrm{CH}_2(X) = \mathbb{Z}$ for $n \geq 18$ *(see [66, Theorem 1.2 and Proposition 2.2]).*

The last section is devoted to a second approach to the integral coefficient problem; it consists of a generalization of a formula developed by Shen ([72], see also [73]) in the case of 1-cycles of cubic hypersurfaces. Let us introduce some notations. Let us denote $Y^{[2]}$ the Hilbert scheme of length 2 subschemes of any variety Y. For a smooth cubic hypersurface X, let us denote $i_{P_2} : P_2 \hookrightarrow X^{[2]}$ the subscheme parametrizing length 2 subschemes supported on a line of X. The variety P_2 admits, by definition a projection $p_{P_2} : P_2 \to F_1(X)$ associating to a length 2 subscheme, the line it is supported on. We prove the following:

Theorem 1.0.7. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over a field k, and Σ a smooth subvariety of X of dimension d. Then, there is an integer m_Σ such that:*

$$(2deg(\Sigma) - 3)\Sigma + P_{1,*}[(p_{P_2,*} i_{P_2}^* \Sigma^{[2]}) \cdot c_1(\mathcal{O}_{F_1(X)}(1))^{d-1}] = m_\Sigma H_X^{n-d}$$

where $\mathcal{O}_{F_1(X)}(1)$ is the Plücker line bundle.

This inversion formula is more powerful than the first approach as it will allow us to lift, modulo $\mathbb{Z} \cdot H_X^{n-2}$, torsion 2-cycles on X to torsion 1-cycles on $F_1(X)$. The application we have in mind is the study of certain birational invariants of X. When $k = \mathbb{C}$, it was observed in [87] that the group $\mathrm{CH}^3_{tors,AJ}$ of homologically trivial torsion codimension 3 cycles annihilated by the Abel-Jacobi map is a birational invariant of smooth projective varieties which is trivial for stably rational varieties and more generally for varieties admitting a Chow-theoretic decomposition of the diagonal. This is a consequence of the deep result due to Bloch ([11], [21]) that the group $\mathrm{CH}^2(Y)_{tors,AJ}$ of homologically trivial torsion codimension 2 cycles annihilated by the Abel-Jacobi map is 0 for any smooth projective variety. For cubic hypersurfaces, as already mentioned, it follows from the existence of a unirational parametrization of degree 2 that $\mathrm{CH}^3(X)_{tors,AJ}$ is a 2-torsion group. Although we have not been able to compute this group, we obtain the following:

Theorem 1.0.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface, with $n \geq 5$. Then for any $\Gamma \in \mathrm{CH}_2(X)_{tors}$, there are a homologically trivial cycle $\gamma \in \mathrm{CH}_1(F_1(X))_{tors,hom}$ and an odd integer m such that $P_*(\gamma) = m\Gamma$.*

Moreover, when $n = 5$, starting from a cycle $\Gamma \in \mathrm{CH}_2(X)_{tors,AJ} = \mathrm{CH}^3(X)_{tors,AJ}$, we can find a $\gamma \in \mathrm{CH}_1(F(X))_{tors,AJ}$ such that $P_*(\gamma) = \Gamma$. In particular, if the 2-torsion part of $\mathrm{CH}_1(F_1(X))_{tors,AJ}$ is 0 then $\mathrm{CH}^3(X)_{tors,AJ} = 0$.

As a consequence of a theorem of Roitman ([67]) asserting that torsion 0-cycles of any smooth projective variety Y inject in $\mathrm{Alb}(Y)$, the group $\mathrm{CH}_1(F_1(X))_{tors,AJ}$ is a stable birational invariant of the variety $F_1(X)$ which is trivial for stably rational varieties or even for varieties admitting a Chow theoretic decomposition of the diagonal.

The group $\mathrm{CH}^3(X)_{tors,AJ}$ has a quotient which has an interpretation in terms of unramified cohomology. We recall that, for a smooth complex projective variety Y and an abelian group A, the degree i unramified cohomology group $H^i_{nr}(Y, A)$ of Y with coefficients in A can be defined (see [12]) as the group of global sections $H^0(Y, \mathcal{H}^i(A))$, $\mathcal{H}^i(A)$ being the sheaf associated to the presheaf $U \mapsto H^i(U(\mathbb{C}), A)$, where this last group is the Betti cohomology of the complex variety $U(\mathbb{C})$. The groups $H^i_{nr}(Y, A)$ provide stable birational invariants (see [19]) of Y, which vanish for projective space i.e. these groups are invariants under the relation:

$$Y \sim Z \text{ if } Y \times \mathbb{P}^r \text{ is birationally equivalent to } Z \times \mathbb{P}^s \text{ for some } r,\ s.$$

Unramified cohomology group with coefficients in $\mathbb{Z}/m\mathbb{Z}$ or $\mathbb{Q}/\mathbb{Z}$ has been used in the study of Lüroth problem, that is the study of unirational varieties which are not rational, to provide examples of unirational varieties which are not stably rational (see [3],[19]). In the case of smooth cubic hypersurfaces $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, since there is a unirational parametrization of degree 2 of X (see [15]) and there is an action of correspondences on unramified cohomology groups compatible with composition of correspondences (see [23, Appendice]), the groups $H^i_{nr}(X, \mathbb{Q}/\mathbb{Z})$, $i \geq 1$, are 2-torsion groups. It is known that $H^1_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$ for $n \geq 2$ since this group is isomorphic to the torsion in the Picard group of X (see [16, Proposition 4.2.1]).

Since for cubic hypersurfaces of dimension at least 2, $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z})$ is equal to the Brauer group $Br(X)$ (see [16, Proposition 4.2.3]), we have $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$.

As for $H^3_{nr}(Y, \mathbb{Q}/\mathbb{Z})$, it was reinterpreted in [23, Theorem 1.1] for rationally connected varieties Y as the torsion in the group $Z^4 := Hdg^4(Y)/H^4(Y, \mathbb{Z})_{alg}$, quotient of degree 4 Hodge classes by the subgroup of $H^4(Y, \mathbb{Z})$ generated by classes of codimension 2 algebraic cycles, i.e. $H^3_{nr}(Y, \mathbb{Q}/\mathbb{Z})$ measures the failure of the integral Hodge conjecture in degree 4. For cubic hypersurfaces $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, by Lefschetz hyperplane theorem, the only non trivial case where the integral Hodge conjecture could fail in degree 4 is for cubic 4-folds but it was proved to hold by Voisin in [86].

The group $H^4_{nr}(Y, \mathbb{Q}/\mathbb{Z})$ was reinterpreted in [87, Corollary 0.3] for rationally connected varieties Y as the group $\mathrm{CH}^3(Y)_{tors,AJ}/alg$ of homologically trivial torsion codimension 3 cycles annihilated by Abel-Jacobi map (or torsion codimension 3 cycles annihilated by Deligne cycle map) modulo algebraic equivalence. For dimension reason $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$ for cubic hypersurfaces of dimension ≤ 3. For cubic 4-folds, since $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{CH}^3(X)_{tors,AJ}/alg \simeq \mathrm{CH}_1(X)_{tors,AJ}/alg \subset \mathrm{Griff}_1(X)$, the work of Shen ([72]) proves that $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$. The vanishing of $\mathrm{CH}^3(X)_{tors,AJ} \simeq \mathrm{CH}_1(X)_{tors,AJ}$ for cubic 4-flods follows also essentially from the work of Shen (see Proposition 1.3.4). For a cubic 5-fold X, by the choice of a $\mathbb{P}^2 \subset X$ to project from, we see that X is birational to a quadric bundle over $\mathbb{P}^3_{\mathbb{C}}$ so that by work of Kahn and Sujatha ([45, Theorem 3]), $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$. Hence, for a cubic hypersurface $\mathrm{CH}^3(X)_{tors,AJ} \subset \mathrm{CH}^3(X)_{alg}$.

1.1 First formula

Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree $d \geq 2$ and dimension $n \geq 3$ over an algebraically closed field k. Let us denote $F(X) \subset G(2, n+2)$ its variety of lines and $P \subset F(X) \times X$ the correspondence given by the universal $\mathbb{P}^1$-bundle, and

$$p : P \to F(X),\ q : P \to X$$

the two projections. For a general hypersurface of degree $d \leq 2n-2$, $F(X)$ is a smooth connected variety ([47, Theorem 4.3, Chap. V]).

Let us denote $Q = \{([l], x) \in \mathbb{P}(\mathcal{E}_2),\ l \subset X \text{ or } l \cap X = \{x\}\}$ the correspondence associated to the family of osculating lines of X, and

$$\pi : Q \to X,\ \varphi : Q \to G(2, n+2)$$

the two projections. We have $P \subset Q$.

We have the following easy lemma:

Lemma 1.1.1. *The fiber of $\pi : Q \to X$ (resp. $q : P \to X$) over any point x in the image of π (resp. of q) is isomorphic to an intersection of hypersurfaces of type $(2, 3, \ldots, d-1)$ in $P(T_{X,x})$ (resp. of type $(2, 3, \ldots, d)$). Moreover, for X general, Q is a local complete intersection subscheme of $\mathbb{P}(\mathcal{E}_2)$ of dimension $2n - d + 1$. If $char(k) = 0$, then Q is smooth for X general.*

Proof. By definition Q is the set of $([l], x)$ in $\mathbb{P}(\mathcal{E}_2)$ over $G(2, n+2)$ where the restriction of the equation defining X is 0 or proportional to λ_x^d, where λ_x is the linear form defining x in l. Let $x \in X$ and $\mathcal{P}$ a hyperplane not containing x. There is an isomorphism $P(T_{\mathbb{P}^{n+1},x}) \to \mathcal{P}$ given by $[v] \mapsto l_{(x,v)} \cap \mathcal{P}$, where $l_{(x,v)}$ is the line of $\mathbb{P}^{n+1}$ determined by (x, v). We can assume that $x = [1, 0, \ldots, 0]$ and $\mathcal{P} = \{X_0 = 0\}$. Let l be a line through x and $[0, Y_1, \ldots, Y_{n+1}] \in \mathcal{P}$ the point associated to l. Then, denoting f an equation defining X, since $x \in X$, we can write $f(X_0, \ldots, X_{n+1}) = \sum_{i=0}^{d-1} X_0^i f_{d-i}(X_1, \ldots, X_{n+1})$, where f_i is a homogeneous polynomial of degree i. The general point of l has coordinates $(\mu, \lambda Y_1, \ldots, \lambda Y_{n+1})$ where $\lambda = \lambda_x$ and μ form a basis of linear forms on l. The restriction of f to l thus writes $\sum_{i=0}^{d-1} \mu^i \lambda^{d-i} f_{d-i}(Y_1, \ldots, Y_{n+1})$. Thus the line l is osculating if and only if $f_j(Y_1, \ldots, Y_{n+1}) = 0$, $\forall j < d$. The first equation f_1 is the differential of f at x and its vanishing hyeperplane is $P(T_{X,x})$, so we proved that $\pi^{-1}(x)$ is isomorphic to an intersection of hypersurfaces of type $(2, 3, \ldots, d-1)$ in $P(T_{X,x})$. We show likewise that the fiber $q^{-1}(x)$ is isomorphic to an intersection of hypersurfaces of type $(2, 3, \ldots, d)$.

On the projective bundle $p_G : \mathbb{P}(\mathcal{E}_2) \to G(2, n+2)$, we have the exact sequence:

$$0 \to \Omega_{\mathbb{P}(\mathcal{E}_2)/G(2,n+2)}(1) \to p_G^* \mathcal{E}_2 \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1) \to 0 \tag{1.2}$$

The last morphism being the evaluation morphism, we see that $\Omega_{\mathbb{P}(\mathcal{E}_2)/G(2,n+2)}(1)_{([l],x)}$ is the ideal sheaf of x in l. Taking the symmetric power of the dual of (1.2) yields the exact sequence:

$$0 \to \mathrm{Sym}^d(\Omega_{\mathbb{P}(\mathcal{E}_2)/G(2,n+2)}(1)) \to p_G^* \mathrm{Sym}^d \mathcal{E}_2 \to p_G^* \mathrm{Sym}^{d-1} \mathcal{E}_2 \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1) \to 0$$

where the first morphism is the d-th symmetric power of the (first) inclusion in (1.2).

Now, let f be an equation defining X; it gives rise to a section σ_f of $p_G^* \mathrm{Sym}^d \mathcal{E}_2$. Let $\overline{\sigma_f}$ be the section of $p_G^* \mathrm{Sym}^{d-1} \mathcal{E}_2 \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1)$ induced by σ_f. Then the zero locus of $\overline{\sigma_f}$ is exactly the locus of $([l], x)$ where the restriction to l of the equation defining X is 0 or equal to the linear form induced by x on l to the power d. So Q is the zero locus in $\mathbb{P}(\mathcal{E}_2)$ of a section of the vector bundle $p_G^* \mathrm{Sym}^{d-1} \mathcal{E}_2 \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1)$. As this vector bundle is globally generated, the zero locus of a general section is a local complete intersection (even regular if $char(k) = 0$) subscheme of $\mathbb{P}(\mathcal{E}_2)$ of dimension $2n - d + 1$. $\square$

Theorem 1.1.2. *Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree d and let $P \subset F(X) \times X$ be the incidence correspondence. Assume $\sum_{i=1}^{d-1} i^r \leq n$ with $r > 0$ and, if $r > 3$ and $char(k) > 0$, assume resolution of singularities of varieties of dimension r. Then for any cycle $\Gamma \in \mathrm{CH}_r(X)$ there is a $\gamma \in \mathrm{CH}_{r-1}(F(X))$ such that*

$$d\Gamma + P_*(\gamma) \in \mathbb{Z} \cdot H_X^{n-r}$$

where $H_X = c_1(\mathcal{O}_X(1))$.

Proof. Let $\Sigma \subset X$ be an integral subvariety of dimension $r > 0$. By Tsen-Lang theorem ([48], [75, Theorem 2.10]), the function field $k(\Sigma)$ of Σ is C_r. As the fibers of $\pi : Q \to X$ are isomorphic to intersection of hypersurfaces of type $(2, 3, \ldots, d-1)$ and $\sum_{i=1}^{d-1} i^r \leq n$, the restriction $\pi_\Sigma : Q_{|\Sigma} \to \Sigma$ admits a rational section $\sigma : \Sigma \dashrightarrow Q$.

Case 1: The rational section σ is actually a rational section of $P_{|\Sigma} \to \Sigma$. This means that for any $x \in \Sigma$, the line $p \circ \sigma(x)$ is contained in X. We have the following diagram of resolution of indeterminacies:

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[d, "\tau"] \arrow[dr, "\tilde{\sigma}"] & & \\ \Sigma \arrow[r, dotted] & P \arrow[r, "p"] & F(X) \end{tikzcd}$$

Let us denote $\mathbb{P}_{\widetilde{\Sigma}}$ the pull-back via $p \circ \tilde{\sigma}$ of the $\mathbb{P}^1$-bundle on $F(X)$, $f : \mathbb{P}_{\widetilde{\Sigma}} \to X$ the projection on X (which is the restriction of q) and $p_\Sigma : \mathbb{P}_{\widetilde{\Sigma}} \to \widetilde{\Sigma}$ the projective bundle. The line bundle $\tau^*\mathcal{O}_X(1)_{|\Sigma}$ gives rise to a section $\eta : \widetilde{\Sigma} \to \mathbb{P}_{\widetilde{\Sigma}}$ (given by $s \mapsto (p \circ \tilde{\sigma}(s), \tau(s))$) of p_Σ. We have the decomposition $\mathrm{Pic}(\mathbb{P}_{\widetilde{\Sigma}}) \simeq \mathbb{Z} \cdot f^*H_X \oplus p_\Sigma^*\mathrm{Pic}(\widetilde{\Sigma})$ so that we can write

$$\eta(\widetilde{\Sigma}) = f^*H_X + p_\Sigma^* D \tag{1.3}$$

for D a divisor on $\widetilde{\Sigma}$. We apply f_* to that equality: we have $f_*\eta(\widetilde{\Sigma}) = \tau_*(\widetilde{\Sigma}) = \Sigma$ in $\mathrm{CH}_r(X)$. Projection formula yields $f_*f^*H_X = H_X \cdot f_*(1)$. Finally, we see that $f_*p_\Sigma^*D = P_*(p_*\tilde{\sigma}_*(D))$. So, we get

$$\Sigma = H_X \cdot f_*(1) + P_*(p_*\tilde{\sigma}_*(D)).$$

Remembering that $dH_X \cdot f_*(1) = i_X^* i_{X,*} f_*(1) \in \mathbb{Z} \cdot H_X^{n-r}$, we are done for this case.

Case 2: The rational section σ is not a rational section of $P_{|\Sigma} \to \Sigma$. This means that for the general point $x \in \Sigma$, the line $\varphi \circ \sigma(x)$ is not contained in X, hence intersects X at x with multiplicity d. We have the following diagram of resolution of indeterminacies:

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[d, "\tau"] \arrow[dr, "\tilde{\sigma}"] & & \\ \Sigma \arrow[r, dotted] & Q \arrow[r, "\varphi"] & G(2, n+2) \end{tikzcd} .$$

Let again $\mathbb{P}_{\widetilde{\Sigma}}$ be the pull-back via $\varphi \circ \tilde{\sigma}$ of the $\mathbb{P}^1$-bundle on $G(2, n+2)$ and let $f : \mathbb{P}_{\widetilde{\Sigma}} \to \mathbb{P}^{n+1}$ be the natural morphism. Let $\widetilde{\Sigma}_1$ be the locus in $\widetilde{\Sigma}$ consisting of $x \in \widetilde{\Sigma}$ such that the line $\varphi \circ \tilde{\sigma}(x)$ is contained in X. We have an equality of r-cycles

$$(f_*\mathbb{P}_{\widetilde{\Sigma}})_{|X} = d\Sigma + R \tag{1.4}$$

in $\mathrm{CH}_r(X)$, where the residual cycle R is supported on the r-dimensional locus $\mathbb{P}_{\widetilde{\Sigma}_1}$, or rather its image in X. It is clear that R is a cycle in the image of P_* so that (1.4) proves the result in this case. $\square$

In the case of smooth cubic hypersurfaces of dimension ≥ 3, $F(X)$ is always smooth and connected ([2, Corollary 1.12, Theorem 1.16]). We have the following result which is essentially Theorem 1.0.1 of the introduction:

Theorem 1.1.3. *Let $X \subset \mathbb{P}_k^{n+1}$, with $n \geq 3$ and $char(k) > 2$, be a smooth cubic hypersurface containing a linear space of dimension $d \geq 1$. Then, for $1 \leq i \leq d$ and $2i \neq n$,*

$$P_* : \mathrm{CH}_{i-1}(F(X)) \to \mathrm{CH}_i(X)/\mathbb{Z} \cdot H_X^{n-i}$$

is surjective on $2\mathrm{CH}_i(X)/\mathbb{Z} \cdot H_X^{n-i}$.

If moreover, $n \geq 2^r + 1$ for some $r > 0$ and resolution of singularities holds of k-varieties of dimension r, then $P_* : \mathrm{CH}_{i-1}(F(X)) \to \mathrm{CH}_i(X)/\mathbb{Z} \cdot H_X^{n-i}$ is surjective for any $i \neq \frac{n}{2}$, $1 \leq i \leq r$.

Proof. According to [15, Appendix B], X admits a unirational parametrization of degree 2 constructed as follows: for a general line Δ in X, consider the projective bundle $P(T_{X|\Delta})$ over Δ and the rational map $\varphi : P(T_{X|\Delta}) \dashrightarrow X$ which to a point $x \in \Delta$ and a nonzero vector $v \in T_{X,x}$ associates the residual point to x (x has multiplicity 2) in the intersection $X \cap l_{(x,v)}$ of X with the line of $\mathbb{P}^{n+1}$ determined by (x, v). The indeterminacy locus Z corresponds to the (x, v) such that $l_{(x,v)} \subset X$. It has codimension 2 for general lines. Indeed, if Δ is general, it is generally contained in the locus where the fibers of the projection $q : P \to X$ are complete (since P has dimension $2n - d$) intersection of type $(2, 3)$ in the projectivized tangent spaces so that the general fiber of $Z \to \Delta$ has dimension $n - 3$. Choosing a sufficiently general Δ, we can also assume that Z is smooth. Then, blowing-up $P(T_{X|\Delta})$ along Z yields the resolution of indeterminacies; let us denote $\tau : \widetilde{P(T_{X|\Delta})} \to P(T_{X|\Delta})$ that blow-up, E the exceptional divisor and $\tilde{\varphi} : \widetilde{P(T_{X|\Delta})} \to X$ the resulting degree 2 morphism. For $1 \leq i \leq d$, by the formulas for blowing-up, we have the decomposition

$$\mathrm{CH}_i(\widetilde{P(T_{X|\Delta})}) = \tau^*\mathrm{CH}_i(P(T_{X|\Delta})) \oplus j_{E,*}\tau_{|E}^*\mathrm{CH}_{i-1}(Z) \oplus j_{E,*}(j_E^*\tilde{\varphi}^*H_X) \cdot \tau_{|E}^*\mathrm{CH}_i(Z).$$

As $\tau_{|E}$ is flat, we can see that $\tilde{\varphi}_* j_{E,*}\tau_{|E}^*(\cdot) = \tilde{\varphi}_* j_{E,*}[\tau_{|E}^{-1}(\cdot)]$ identifies with the composition of the morphism $\mathrm{CH}_*(Z) \to \mathrm{CH}_*(F(X))$ (induced by the restriction of natural morphism $P(T_X) \to G(2, n+2)$) followed by the action P_*.

So let $\Gamma \in \mathrm{CH}_i(X)$, with $2i \neq n$, be a cycle on X. As X contains a linear space of dimension i and $H_{ét}^{2(n-i)}(X, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$ ($\forall \ell \neq char(k)$) by Lefschetz hyperplane section theorem ($n \neq 2i$), for any $\mathcal{P} \simeq \mathbb{P}^i \subset X$, $\Gamma - deg(\Gamma)[\mathcal{P}]$ is homologically trivial and $\tilde{\varphi}_*\tilde{\varphi}^*(\Gamma - deg(\Gamma)[\mathcal{P}]) = 2(\Gamma - deg(\Gamma)[\mathcal{P}])$. As $P(T_{X|\Delta})$ is a projective bundle over $\mathbb{P}^1$, $\mathrm{CH}_*(P(T_{X|\Delta}))_{hom} = 0$ so, from the above discussion, we conclude that there are a $(i-1)$-cycle $\gamma \in \mathrm{CH}_{i-1}(F(X))_{hom}$ and a i-cycle $D_\Gamma \in \mathrm{CH}_i(F(X))_{hom}$ such that

$$2(\Gamma - deg(\Gamma)[\mathcal{P}]) = P_*\gamma + H_X \cdot P_* D_\Gamma.$$

It remains to deal with the term $H_X \cdot P_* D_\Gamma$. For this, let $j : Y \hookrightarrow X$ be a hyperplane section with one ordinary double point p_0 as singularity. Then $H_X \cdot P_* D_\Gamma = j_* j^* P_* D_\Gamma$.

We have $Y \subset \mathbb{P}^n$ and if we choose coordinates in which $p_0 = [0 : \cdots : 0 : 1]$, the equation of Y has the following form: $F(X_0, \cdots, X_n) = X_n Q(X_0, \cdots, X_{n-1}) + T(X_0, \cdots, X_{n-1})$ where $Q(X_0, \cdots, X_{n-1})$ is a quadratic homogeneous polynomial and $T(X_0, \cdots, X_{n-1})$ is a degree 3 homogeneous polynomial. The linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ centered at p_0 induces a birational map $Y \dashrightarrow \mathbb{P}^{n-1} \simeq [p_0]$ where $[p_0]$ denotes the scheme parametrizing lines of $\mathbb{P}^n$ passing through p_0. The indeterminacies of the inverse map $\mathbb{P}^{n-1} \dashrightarrow Y$ are resolved by blowing-up $\mathbb{P}^{n-1}$ along the complete intersection $F_{p_0}(Y) =$

$\{Q=0\}\cap\{T=0\}$ of type $(2,3)$. The variety of lines of Y passing through p_0 is isomorphic to $F_{p_0}(Y)$ and we have the following diagram:

$$\begin{tikzcd} \widetilde{\mathbb{P}^{n-1}}^{F_{p_0}(Y)} \arrow[d, "\chi"] \arrow[dr, "q"] & \\ \mathbb{P}^{n-1} \arrow[r, dashed] & Y \end{tikzcd}$$

By projection formula, $(j\circ q)_*(j\circ q)^*P_*D_\Gamma = P_*D_\Gamma\cdot j_*q_*1 = P_*D_\Gamma\cdot[Y] = P_*D_\Gamma\cdot H_X$ and $(j\circ q)^*P_*D_\Gamma$ is a homologically trivial cycle on $\widetilde{\mathbb{P}^{n-1}}^{F_{p_0}(Y)}$. But since the ideal $\mathrm{CH}_*(\mathbb{P}^{n-1})_{hom}$ of homologically trivial cycles on $\mathbb{P}^{n-1}$ is 0, from the decomposition of the Chow groups of a blow-up, we get that $(j\circ q)^*P_*D_\Gamma$ can be written $j_{E_{F_{p_0}(Y)},*}\chi^*_{|E_{F_{p_0}(Y)}}w$ for a cycle w on $F_{p_0}(Y)$ so that $H_X\cdot P_*D_\Gamma = j_*q_*j_{E_{F_{p_0}(Y)},*}\chi^*_{|E_{F_{p_0}(Y)}}w$ which can be written $P_*i_{F_{p_0}(Y),*}w$ where $i_{F_{p_0}(Y)}: F_{p_0}(Y)\hookrightarrow F(X)$ is the inclusion. Finally, $\mathcal{P}$ is in $Im(P_*)$ so we have: $2\Gamma = 2deg(\Gamma)\mathcal{P} + P_*(\gamma + i_{F_{p_0}(Y),*}w)$ which proves that $2\mathrm{CH}_i(X)$ is in the image of P_*.

When $n\geq 2^r+1$, we can also apply Theorem 1.1.2; we get, for a cycle $\Gamma\in\mathrm{CH}_i(X)$, a cycle $\gamma'\in\mathrm{CH}_{i-1}(F(X))$ such that $3\Gamma + P_*\gamma'\in\mathbb{Z}\cdot H_X^{n-i}$ in $\mathrm{CH}_i(X)$ so that, putting the two steps together, we get $(3-2)\Gamma - 2deg(\Gamma)[\mathcal{P}] + P_*(\gamma'-\gamma-i_{F_{p_0}(Y),*}w)\in\mathbb{Z}\cdot H_X^{n-i}$ in $\mathrm{CH}_i(X)$.

□

Proposition 1.1.1. *Let $X\subset\mathbb{P}^{n+1}_k$, with $n\geq 4$ and $char(k)>2$, be a smooth cubic hypersurface. Then $H_X^{n-2}\in Im(P_*:\mathrm{CH}_1(F(X))\to\mathrm{CH}_2(X))$. In particular, by Theorem 1.1.3, for $n\geq 5$, $P_*:\mathrm{CH}_1(F(X))\to\mathrm{CH}_2(X)$ is surjective.*

Proof. Since, according to [61, Lemma 1.4], any smooth cubic threefold contains some lines of second type (lines whose normal bundle contains a copy of $\mathcal{O}_{\mathbb{P}^1}(-1)$), X contains lines of second type. Let $l_0\subset X$ be a line of second type. According to [15, Lemma 6.7], there is a (unique) $\mathbb{P}^{n-1}\subset\mathbb{P}^{n+1}$ tangent to X along l_0. So, when $n\geq 4$, we can choose a $P_0\simeq\mathbb{P}^3\subset\mathbb{P}^{n+1}$ tangent to X along l_0. Then $S:=P_0\cap X$ is a cubic surface singular along l_0 which is ruled by lines of X. Indeed, for any $x\in S\backslash l_0$, $span(x,l_0)\cap S$ is a plane cubic containing l_0 with multiplicity 2; so that the residual curve is a line passing through x. So, we can write $S=q(p^{-1}(D))$ for a closed subscheme of pure dimension 1, $D\subset F(X)$ so, in $\mathrm{CH}_2(X)$, we have $H_X^{n-2}=[S]=P_*([D])$. □

Here is one consequence of this proposition:

Corollary 1.1.4. *Let $\pi:\mathcal{X}\to B$ be a family of complex cubic hypersurfaces of dimension $n\geq 5$ i.e. π is a smooth projective morphism of connected quasi-projective complex varieties with n-dimensional cubic hypersurfaces as fibers. Then, the specialization map*

$$\mathrm{CH}_2(X_{\overline{\eta}})/alg\to\mathrm{CH}_2(X_t)/alg$$

where $X_{\overline{\eta}}$ is the geometric generic fiber and $X_t:=\pi^{-1}(t)$ for $t\in B(k)$ any closed point, is surjective.

Proof. The statement follows from Proposition 1.1.1 and the following property, essentially written in the proof of [88, Lemma 2.1]

Proposition 1.1.2. *([88, Lemma 2.1]). Let $\pi:\mathcal{Y}\to B$ be a smooth projective morphism with rationally connected fibers. Then for any $t\in B(t)$, the specialization map $\mathrm{CH}_1(Y_{\overline{\eta}})/alg\to\mathrm{CH}_1(Y_t)/alg$ is surjective.*

Proof. We just recall briefly the proof: by attaching sufficiently very free rational curves to it (so that the resulting curve is smoothable), any curve $C \subset Y_t$ is algebraically equivalent to a (non effective) sum of curves $C_i \subset Y_t$ such that $H^1(C_i, N_{C_i/Y_t}) = 0$. Then the morphism of deformation of each (C_i, Y_t) to B is smooth. So we have a curve $C_{i,\eta} \subset Y_{K_i}$ where K_i is a finite extension of the function field of B, which is sent by specialization in the fiber Y_t, to C_i. □

Applying this proposition to the relative variety of lines $F(\mathcal{X}) \to B$, yields a surjective map: $\mathrm{CH}_1(F(X_{\overline{\eta}}))/alg \to \mathrm{CH}_1(F(X_t))/alg$. The universal $\mathbb{P}^1$-bundle $\mathcal{P} \subset F(\mathcal{X}) \times_B \mathcal{X}$ gives the surjective maps $\mathcal{P}_{t,*} : \mathrm{CH}_1(F(X_t))/alg \to \mathrm{CH}_2(X_t)/alg$ and $\mathcal{P}_{\overline{\eta},*} : \mathrm{CH}_1(F(X_{\overline{\eta}}))/alg \to \mathrm{CH}_2(X_{\overline{\eta}})/alg$ and they commute ([30, 20.3]). □

1.2 One-cycles on the variety of lines of a Fano hypersurface in $\mathbb{P}^n$

Throughout this section, k will designate an algebraically closed field. According to [47, Theorem 4.3, Chap. V], for a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \leq 2n-2$, the variety of lines $F(X)$ is smooth, connected of dimension $2n-d-1$. In the case of cubic hypersurfaces of dimension $n \geq 3$, we even know, by work of Altman and Kleiman ([2, Corollary 1.12, Theorem 1.16], see also [6]) that for any smooth hypersurface X, $F(X)$ is smooth and connected.

We recall that, for a smooth hypersurface $X \subset \mathbb{P}^{n+1}_k$ of degree d, when $F(X)$ has the expected dimension $2n-d-1$, it is the zero-locus in $G(2,n+2)$ of a regular section of $\mathrm{Sym}^d(\mathcal{E}_2)$, where $\mathcal{E}_2$ is the rank 2 quotient bundle on $G(2,n+2)$ and its dualizing sheaf, given by adjunction formula ([37, Theorem III 7.11]), is $-((n+2) - \frac{d(d+1)}{2}))$ times the Plücker line bundle on $G(2,n+2)$ restricted to $F(X)$. In particular, when $F(X)$ is smooth, connected and $\frac{d(d+1)}{2} < (n+2)$, $F(X)$ is Fano so rationally connected.

From now, we assume that the condition $d(d+1) < 2(n+2)$ holds and that $X \subset \mathbb{P}^{n+1}_k$ is a smooth hypersurface such that $F(X)$ is smooth and connected. Then the following theorem applies to $F(X)$ if $char(k) = 0$ or, when $char(k) > 0$, if $F(X)$ is, moreover separably rationally connected:

Theorem 1.2.1. *([76, Theorem 1.3]). Let Y be a smooth proper and separably rationally connected variety over an algebraically closed field. Then every 1-cycle is rationally equivalent to a $\mathbb{Z}$-linear combination of cycle classes of rational curves. That is, the Chow group $\mathrm{CH}_1(Y)$ is generated by rational curves.*

Corollary 1.2.2. *When $char(k) = 0$ and X is a smooth cubic hypersurface of dimension ≥ 5, $F(X)$ is separably rationally connected; then Proposition 1.1.3 together with Theorem 1.2.1 yields that $\mathrm{CH}_2(X)$ is generated by classes of rational surfaces. In positive characteristic, the same is true for smooth cubic hypersurfaces X whose variety of lines $F(X)$ is separably rationally connected.*

Remark 1.2.3. When $k = \mathbb{C}$ and X is a smooth cubic hypersurface of dimension 5, the group of 1-cycles modulo algeraic equivalence, $\mathrm{CH}_1(F(X))/alg$ is finitely generated. Indeed, according to [47, Theorem 5.7, Chap. II], any rational curve is algebraically equivalent to a sum of rational curves of anti-canonical degree at most $dim_k(F(X))+1$. As there are finitely many irreducible varieties parametrizing rational curves of bounded degree, $\mathrm{CH}_1(F(X))/alg$, is finitely generated. So, by the surjectivity of P_*, $\mathrm{CH}_2(X)/alg$ is finitely generated. So $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{CH}_2(X)_{tors,AJ}/alg \subset \mathrm{CH}_2(X)/alg$ is finitely generated and being a functorial birational invariant of a cubic hypersurface, 2-torsion. So by this geometric method, we are just able to prove the finiteness of the group $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z})$. By more algebraic methods, Kahn and Sujatha ([45]) prove the vanishing of that group.

Actually, by completely different methods using a variant of [86, Theorem 18], the first item of Corollary 1.2.2 turns out to be true for cubic 4-folds also in characteristic 0.

Proposition 1.2.1. *Let $X \subset \mathbb{P}^5_{\mathbb{C}}$ be a smooth cubic hypersurface. Then $\mathrm{CH}_2(X)$ is generated by classes of rational surfaces.*

Proof. As X has trivial CH_0 group (since X is Fano), the work of Bloch and Srinivas ([13, Theorem 1]) allows us to conclude, since moreover $H^3(X, \mathbb{Z}) = 0$ (hence $J^3(X) = 0$), that homological and rational equivalences coincide on $\mathrm{CH}^2(X) = \mathrm{CH}_2(X)$. So we just have to prove the statement of the proposition at the level of cohomology. The statement at the level of cohomology follows essentially from the proof given by Voisin in [86, Theorem 18] of the integral Hodge conjecture for cubic 4-folds. What Voisin actually proves in *loc. cit*, is that any degree 4 Hodge class is, modulo the complete intersection class (a multiple of the class of a cubic surface, which is rational), the sum of classes of surfaces fibered in elliptic quintic curves over a rational curve. The rational curves of the bases are related to the fact that she works with a Lefschetz pencil (of cubic 3-folds) of X and the elliptic quintic curves as fibers are related to the fact the Abel-Jacobi map from the family normal elliptic quintic curves of a smooth cubic 3-fold yields a parametrization of the intermediate Jacobian of the cubic 3-fold with rationally connected (general) fiber, according to [52]and [43]. Here is the change we make: instead of using the parametrization of the intermediate Jacobian of a cubic 3-fold given by [52]and [43] (which use elliptic quintics), we can replace this parametrization by the parametrization of the intermediate Jacobian of a cubic 3-fold given by [35, Theorem 9.2] which uses, instead of elliptic quintics on the cubic 3-fold, rational curves of degree 4 on the cubic 3-fold. The proof of Voisin then shows that any degree 4 Hodge class is homologically equivalent to the class of a combination of rational surfaces swept-out by a family of rational curves of degree 4 in X parameterized by a rational curve. □

1.2.1 One-cycles modulo algebraic equivalence

In this section, we apply the methods of [76, Theorem 6.2], using a coarse parametrization of rational curves lying on $F(X)$, to study 1-cycles on varieties $F(X)$. Our goal is to prove:

Theorem 1.2.4. *Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree d over an algebraically closed field of characteristic 0, with $\frac{d(d+1)}{2} < n$, such that $F(X)$ is smooth, connected. Then every rational curve on $F(X)$ is algebraically equivalent to an integral sum of lines. In particular, any 1-cycle on $F(X)$ is algebraically equivalent to an integral sum of lines and thus $\mathrm{CH}_1(F(X))_{hom} = \mathrm{CH}_1(F(X))_{alg}$.*

We start with some preparation. Let V be a $(n+2)$-dimensional k-vector space and $X \subset \mathbb{P}(V) \simeq \mathbb{P}^{n+1}_k$ a smooth hypersurface of degree d. A morphism $r : \mathbb{P}^1 \to G(2, V)$ such that $r^*\mathcal{O}_{G(2,V)}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(e)$, with $e \geq 1$, is associated to the datum of a globally generated rank 2 vector bundle on $\mathbb{P}^1$, which is a quotient of the trivial bundle $V \otimes \mathcal{O}_{\mathbb{P}^1}$ i.e. to an exact sequence

$$V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \to 0$$

with $a, b \geq 0$ and $a + b = e$. So a natural parameter space for those morphisms is

$$\mathbb{P} := \mathbb{P}(Hom(V^*, H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)))).$$

Given $[P_0, \ldots, P_{n+1}, Q_0, \ldots, Q_{n+1}] \in \mathbb{P}$, where the P_i's are in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and the Q_i's are in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$, the points in the image in $\mathbb{P}(V)$ of $Im(\mathbb{P}^1 \to G(2, n+2))$ under the correspondence given by the universal $\mathbb{P}^1$-bundle are of the form $[P_0(Y_0, Y_1)\lambda + Q_0(Y_0, Y_1)\mu, \ldots, P_{n+1}(Y_0, Y_1)\lambda + Q_{n+1}(Y_0, Y_1)\mu]$ where $\mathrm{Span}(Y_0, Y_1) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Let $\Pi_{i=0}^{n+1} X_i^{\alpha_i} \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$ be a mononial with $\sum_{i=0}^{n+1} \alpha_i = d$. Then the induced equation on the image in $\mathbb{P}^{n+1}$ of the morphism $\mathbb{P}^1 \to G(2, n+2)$ associated to $[P_0, \ldots, P_{n+1}, Q_0, \ldots, Q_{n+1}]$ has the following form:

$$\sum_{k=0}^{d} (\sum_{\substack{0 \le l_0 \le \alpha_0, \ldots, 0 \le l_{n+1} \le \alpha_{n+1} \\ \sum_i l_i = k}} \Pi_{i=0}^{n+1} \binom{\alpha_i}{l_i} P_i^{\alpha_i - l_i} Q_i^{l_i}) \lambda^{d-k} \mu^k$$

so that, denoting $\mathbb{P}_X$, the closed subset of $\mathbb{P}$ parametrizing the $[P_0, \ldots, P_{n+1}, Q_0, \ldots, Q_{n+1}]$ whose image in $\mathbb{P}^{n+1}$ is contained in the hypersurface X of degree d, is defined by $\sum_{k=0}^{d}(a(d-k) + bk + 1) =$ homogeneous polynomials of degree d on $\mathbb{P}$.

The closed subset $B \subset \mathbb{P}$ parametrizing the $M \in \mathbb{P}(Hom(V^*, H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))))$ whose rank is ≤ 2 has dimension $2(e+n)+3$. Now, we have the following lemma:

Lemma 1.2.5. *([36]). Let Y be a subscheme of a projective space $\mathbb{P}^N$ defined by M homogeneous polynomials. Let Z be a closed subset of Y with dimension $< N - M - 1$. Then $Y \backslash Z$ is connected.*

The closed subset $B \cap \mathbb{P}_X$ of $\mathbb{P}_X$ has dimension

$$dim_k(F(X)) + 2(a+1) + 2(b+1) - 1 = 2n - d - 1 + 2e + 3 = 2(e+n) - d + 2$$

since it parametrizes (generically) a point of $F(X)$ and over that point 2 polynomials in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and 2 polynomials in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$. Applying Lemma 1.2.5 with $Y = \mathbb{P}_X$ and $Z = B \cap \mathbb{P}_X$, so that $N = (n+2)(e+2) - 1$ and $M = \sum_{k=0}^{d}(a(d-k) + bk + 1) = d + 1 + e\frac{d(d+1)}{2}$, yields the following condition for the connectedness of $\mathbb{P}_X \backslash (B \cap \mathbb{P}_X)$:

$$e(n - \frac{d(d+1)}{2}) > 1 \tag{1.5}$$

Proof of Theorem 1.2.4. We proceed by induction on the degree of the considered rational curve, following the arguments of [76, Theorem 6.2].

Let $D \subset \mathbb{P}$ be the closed subset parametrizing $2(n+2)$-tuples $[P_0, \ldots, P_{n+1}, Q_0, \ldots, Q_{n+1}]$ that have a common non constant factor. Assume $e \ge 2$. Let $p \in \mathbb{P}_X \backslash (\mathbb{P}_X \cap (B \cup D))$ be a point parametrizing a degree e morphism $\mathbb{P}^1 \to F(X)$ generically injective. As $e \ge 2$, $\mathbb{P}_X \backslash (\mathbb{P}_X \cap B)$ is connected; so there is a connected curve γ in $\mathbb{P}_X \backslash (\mathbb{P}_X \cap B)$ connecting p to a point $q = [P_{0,q}, \ldots, P_{n+1,q}, Q_{0,q}, \ldots, Q_{n+1,q}]$ of $\mathbb{P}_X \cap D \backslash (\mathbb{P}_X \cap B)$. Factorizing out the common factor of $(P_{i,q}, Q_{i,q})_{i=0\ldots n+1}$, we get a $(P'_{i,q}, Q'_{i,q})_{i=0\ldots n+1}$ which parametrizes a morphism $\mathbb{P}^1 \to F(X)$ of degree $< e$ (finite onto its image), since $q \notin B$. So, approching q from points of γ outside D and using standard bend-and-break construction, we get from q a morphism from a connected curve whose components are isomorphic to $\mathbb{P}^1$ to $F(X)$ such that the restriction to each component yields a rational curve of degree $< e$ (or a contraction). So the rational curve parametrized by p is algebraically equivalent to a sum of rational curve each of which has degree $< e$. We conclude by induction on e that the rational curve parametrized by p is algebraically equivalent to a sum of lines. □

1.2.2 One-cycles modulo rational equivalence

From now on, we will assume that $X \subset \mathbb{P}_k^{n+1}$ is a smooth hypersurface of degree $d > 2$, with $d(d+1)/2 < n$, and that $char(k) = 0$. The following is proved in [26, Proposition 6.2]:

Proposition 1.2.2. *Assume $char(k) = 0$ and $X \subset \mathbb{P}_k^{n+1}$ is a smooth hypersurface of degree $d > 2$, with $d(d+1)/2 < n$. Then, $F(X)$ is chain connected by lines.*

Proceeding as in [76], we get the following result:

Theorem 1.2.6. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree $d > 2$ over an algebraically closed field of characteristic 0, with $\frac{d(d+1)}{2} < n$, such that $F(X)$ is smooth and connected. Then $\mathrm{CH}_1(F(X))$ is generated by lines i.e. any 1-cycle is rationally equivalent to a $\mathbb{Z}$-linear combination of lines.*

Proof. Let γ be a 1-cycle on $F(X)$. According to Theorem 1.2.4, there is a $\mathbb{Z}$-linear combination of lines $\sum_i m_i l_i$ such that $\gamma - \sum_i m_i l_i$ is algebraically equivalent to 0. Then, using [76, Proposition 3.1], we know there is a positive integer N such that for every 1-cycle C on $F(X)$, NC is rationally equivalent to a $\mathbb{Z}$-linear combination of lines. As the group $\mathrm{CH}_1(X)_{alg}$ of 1-cycles algebraically equivalent to 0 is divisible ([7, Lemme 0.1.1]), we conclude that $\gamma - \sum_i m_i l_i$ is rationally equivalent to a $\mathbb{Z}$-linear combination of lines. □

This provides us the following results for cubic hypersurfaces (cf. Proposition 1.0.5):

Corollary 1.2.7. *Let k be an algebraically closed field of characteristic 0 and X a smooth cubic hypersurface. We have the following properties:*

(i) if $dim_k(X) \geq 7$, then, $\mathrm{CH}_2(X)$ is generated (over $\mathbb{Z}$) by cycle classes of planes contained in X and $\mathrm{CH}_2(X)_{hom} = \mathrm{CH}_2(X)_{alg}$;

(ii) if $dim_k(X) \geq 9$, then, $\mathrm{CH}_2(X) \simeq \mathbb{Z}$

Proof. Item (ii) is an application of Proposition 1.1.1 and Theorem 1.2.6.

(iii) The variety of lines $F(F(X))$ of $F(X)$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E}_{3|F_2(X)})$ over $F_2(X) \subset G(3, n+2)$, where $\mathcal{E}_3$ is the rank 3 quotient bundle on $G(3, n+2)$ and $F_2(X)$ is the variety on planes of X, since a line in $F(X)$ correspond to the lines of $\mathbb{P}^{n+1}$ contained in a plane $\mathbb{P}^2 \simeq P \subset \mathbb{P}^{n+1}$ passing through a given point of P. Now, when $n \geq 9$, according to [26, Proposition 6.1], $\mathrm{CH}_0(F_2(X)) \simeq \mathbb{Z}$ so that $\mathrm{CH}_0(F(F(X))) \simeq \mathbb{Z}$. □

1.3 Inversion formula

Let $X \subset \mathbb{P}_k^{n+1}$, where $n \geq 3$, be a smooth cubic hypersurface over a field k. Let as before $F(X) \subset G(2, n+2)$ be the variety of lines of X and $P \subset F(X) \times X$ the correspondence given by the universal $\mathbb{P}^1$-bundle over $F(X)$. The variety $F(X)$ is smooth, connected of dimension $2n-4$ ([2, Corollary 1.12, Theorem 1.16]).

1.3.1 Inversion formula

In this section, adapting constructions and arguments developed in [73] (see also [72]), we establish an inversion formula for a smooth subvariety Σ of X.

For subvarieties Σ in general position, this formula express the class of Σ in $\mathrm{CH}_{dim(\Sigma)}(X)$ in terms of the class of the subscheme of $F(X)$ consisting of the lines of X bisecant to Σ.

First of all, the lines of $\mathbb{P}_k^{n+1}$ bisecant to any subvariety Σ are naturally in relation with the punctual Hilbert scheme $Hilb_2(\Sigma)$, that we shall denote $\Sigma^{[2]}$, via the morphism

$$\varphi : \Sigma^{[2]} \to G(2, n+2) \tag{1.6}$$

which associates to a length 2 subscheme of Σ the line it determines.

We recall that for any smooth variety Y, $Y^{[2]}$ is smooth and is obtained as the quotient of the blow-up $\widetilde{Y\times Y}$ of $Y\times Y$ along the diagonal Δ_Y, by its natural involution. Let us denote $q:\widetilde{Y\times Y}\to Y^{[2]}$ the quotient morphism, $\tau:\widetilde{Y\times Y}\to Y\times Y$ the blow-up and $j_{E_Y}:E_Y\hookrightarrow\widetilde{Y\times Y}$ the exceptional divisor of τ. As the involution acts trivially on E_Y, $q_{|E_Y}$ is an isomorphism onto its image and q is a double cover of $Y^{[2]}$ ramified along $q(E_Y)$. So let us denote $\delta_Y\in\mathrm{CH}^1(Y^{[2]})$ a divisor satisfying $[q(E_Y)]=2\delta_Y$.

For a subvariety Σ of X in general position, the relation between lines of X bisecant to Σ and Σ rests on the existence of a residual map:

$$r:\Sigma^{[2]}\dashrightarrow X \tag{1.7}$$

associating to a length 2 subscheme of Σ, $x+y$, the point $z\in X$ residual to $x+y$ in the intersection of $l_{(x+y)}\cap X$, $l_{(x+y)}$ being the line determined by $x+y$. The map (1.7) is not defined on length 2 subschemes whose associated line is contained in X.

Let us denote P_2 the subscheme of $X^{[2]}$ of length 2 subschemes of X, whose associated line is contained in X and let us denote $i_{P_2}:P_2\hookrightarrow X^{[2]}$ the embedding. We can see that P_2 admits a structure $p_{P_2}:P_2\to F(X)$ of $\mathbb{P}^2$-bundle over $F(X)$ as P_2 is the symmetric product of P over $F(X)$. In particular P_2 is a smooth subvariety of $X^{[2]}$ of codimension 2.

Now, for any smooth subvariety $\Sigma\subset X$, we prove the following inversion formula:

Theorem 1.3.1. *Let $X\subset\mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface and $\Sigma\subset X$ a smooth subvariety of dimension $1\leq d\leq n$. Then, the following equality holds in $\mathrm{CH}_d(X)$:*

$$(2deg(\Sigma)-3)[\Sigma]+P_*[(p_{P_2,*}i^*_{P_2}\Sigma^{[2]})\cdot c_1(\mathcal{O}_{F(X)}(1))^{d-1}]=m(\Sigma)H_X^{n-d} \tag{1.8}$$

where $m(\Sigma)$ is an integer, $H_X=c_1(\mathcal{O}_X(1))$ and $\mathcal{O}_{F(X)}(1)$ is the Plücker line bundle on $F(X)$.

Let us start with an analysis of the geometry of (1.7) for X. The indeterminacies of

$$r:X^{[2]}\dashrightarrow X$$

are resolved by blowing up $X^{[2]}$ along P_2. Let us denote $\chi:\widetilde{X^{[2]}}\to X^{[2]}$ this blow-up morphism and E_{P_2} the exceptional divisor. The variety $\widetilde{X^{[2]}}$ is naturally a subvariety of $X^{[2]}\times X$ and, as such, can be regarded also as a correspondence between $X^{[2]}$ and X. For a subvariety Σ, the bisecant lines (of $\mathbb{P}^{n+1}$ at least) have to do with $\Sigma^{[2]}$, which lives in $X^{[2]}$. Thus, in order to get a relation between the scheme of bisecant lines of a subvariety in general position and that subvariety, being able to compute the action of the correspondence $\widetilde{X^{[2]}}\subset X^{[2]}\times X$ is of special interest.

We recall that we have a morphism $\varphi:X^{[2]}\to G(2,n+2)$ (1.6) from which we get, by pulling back objects from $G(2,n+2)$, a diagram:

$$\begin{tikzcd}
\mathbb{P}(\varphi^*\mathcal{E}_2) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}^{n+1} \\
X^{[2]} &
\end{tikzcd}$$

We have the following proposition:

Proposition 1.3.1. *(i) There is an embedding* $\sigma : \widetilde{X \times X} \to \mathbb{P}(\varphi^*\mathcal{E}_2)$ *given by* $(x, y) \mapsto (l_{(x,y)}, x)$ *if* $(x, y) \in \widetilde{X \times X} \backslash E$ *and* $(x, v) \mapsto (l_{(x,v)}, x)$ *if* $(x, v) \in E \simeq P(T_X)$.

(ii) The class of $\sigma(\widetilde{X \times X})$ *in* $\mathrm{Pic}(\mathbb{P}(\varphi^*\mathcal{E}_2))$ *is:*

$$\sigma(\widetilde{X \times X}) = 2f^*H - \pi^*(q_*\tau^*pr_1^*H_X - 2\delta_X) \tag{1.9}$$

where $H = c_1(\mathcal{O}_{\mathbb{P}_k^{n+1}}(1))$, H_X *is the restriction of* H *to* X *and* $pr_1 : X \times X \to X$ *the first projection.*

(iii) We have an inclusion of divisors $\sigma(\widetilde{X \times X}) \subset f^*(X)$ *and the residual divisor to* $\sigma(\widetilde{X \times X})$ *in* $f^*(X)$ *is isomorphic to* $\widetilde{X^{[2]}}$ *and* $\pi_{|\widetilde{X^{[2]}}} = \chi$ *so that the class of* $\widetilde{X^{[2]}}$ *in* $\mathrm{Pic}(\mathbb{P}(\varphi^*\mathcal{E}_2))$ *is:*

$$\widetilde{X^{[2]}} = f^*H + \pi^*(q_*\tau^*pr_1^*H_X - 2\delta_X) \tag{1.10}$$

Proof. As for any point $p \in \widetilde{X \times X}$, the point $pr_1(\tau(p))$ lies on the line $\varphi(q(p))$ (defined over $k(p)$), the evaluation morphism $q^*\varphi^*\mathcal{E}_2 \to \tau^*pr_1^*\mathcal{O}_X(1)$, where $\mathcal{O}_X(1) \simeq \mathcal{O}_{\mathbb{P}_k^{n+1}}(1)_{|X}$, is surjective i.e. gives rise to a section σ' of the projective bundle $\pi' : \mathbb{P}(q^*\varphi^*\mathcal{E}_2) \to \widetilde{X \times X}$. Let us denote $q' : \mathbb{P}(q^*\varphi^*\mathcal{E}_2) \to \mathbb{P}(\varphi^*\mathcal{E}_2)$ the morphism obtained from q by base change; it is also a ramified double cover. The composition $\sigma := q' \circ \sigma' : \widetilde{X \times X} \to \mathbb{P}(\varphi^*\mathcal{E}_2)$ is an isomorphism onto its image and we have the inclusion of divisors $\sigma(\widetilde{X \times X}) \subset f^{-1}(X)$. Let us denote R the residual scheme to $\sigma(\widetilde{X \times X})$ in $f^{-1}(X)$. We need to prove that $\pi_{|R}$ is the blow-up of $\Sigma^{[2]}$ along P_2 i.e. $R \simeq \widetilde{X^{[2]}}$.

As σ' is the section of the projective bundle $\mathbb{P}(q^*\varphi^*\mathcal{E}_2)$ given by $\tau^*pr_1^*\mathcal{O}_X(1)$, its class in $\mathrm{CH}^1(\mathbb{P}(q^*\varphi^*\mathcal{E}_2))$ is given by $c_1(\pi'^*\mathcal{K}^\vee \otimes \mathcal{O}_{\mathbb{P}(q^*\varphi^*\mathcal{E}_2)}(1))$, where $\mathcal{K}$ is a line bundle defined by the exact sequence:

$$0 \to \mathcal{K} \to q^*\varphi^*\mathcal{E}_2 \to \tau^*pr_1^*\mathcal{O}_X(1) \to 0 \tag{1.11}$$

and $\mathcal{O}_{\mathbb{P}(q^*\varphi^*\mathcal{E}_2)}(1) \simeq q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)$. We have:

$$\begin{array}{rcl}
[\sigma(\widetilde{X \times X})] & = & q'_*\sigma'_*(\widetilde{X \times X}) \\
& = & q'_*c_1(\pi'^*\mathcal{K}^\vee \otimes \mathcal{O}_{\mathbb{P}(q^*\varphi^*\mathcal{E}_2)}(1)) \\
& = & q'_*\pi'^*c_1(\mathcal{K}^\vee) + q'_*c_1(q'^*f^*\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \\
& = & \pi^*q_*c_1(\mathcal{K}^\vee) + c_1(f^*\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \cdot q'_*(1) \text{ since } q \text{ and } \pi \text{ are proper and flat} \\
& = & \pi^*q_*c_1(\mathcal{K}^\vee) + 2c_1(f^*\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \text{ since } q' \text{ is a double cover} \\
& = & \pi^*q_*[c_1(\tau^*pr_1^*\mathcal{O}_X(1)) - c_1(q^*\varphi^*\mathcal{E}_2)] + 2f^*H \text{ using (1.11)} \\
& = & \pi^*[q_*\tau^*pr_1^*c_1(\mathcal{O}_X(1)) - 2c_1(\varphi^*\mathcal{E}_2)] + 2f^*H \text{ since } q \text{ is a double cover}
\end{array}$$

As a linear form on $\mathbb{P}^1$ is determined by its value on a length 2 subscheme, the evaluation morphism yields an isomorphism of sheaves:

$$\varphi^*\mathcal{E}_2 \simeq q_*\tau^*pr_1^*\mathcal{O}_X(1), \tag{1.12}$$

so that, using Grothendieck-Riemann-Roch theorem for q, we have the equality $c_1(\varphi^*\mathcal{E}_2) = q_*c_1(\tau^*pr_1^*\mathcal{O}_X(1)) - \delta_X$. We end the computation of $[\sigma(\widetilde{X \times X})]$ as follows:

$$\begin{array}{rcl}
[\sigma(\widetilde{X \times X})] & = & 2f^*H + \pi^*[q_*\tau^*pr_1^*c_1(\mathcal{O}_X(1)) - 2c_1(\varphi^*\mathcal{E}_2)] \\
& = & 2f^*H + \pi^*[(1-2)q_*\tau^*pr_1^*c_1(\mathcal{O}_X(1)) + 2\delta_X]
\end{array}$$

Now, we have $R = [f^{-1}(X)] - [\sigma(\widetilde{X \times X})] = 3f^*H - [\sigma(\widetilde{X \times X})] = f^*H + \pi^*(q_*\tau^*pr_1^*H_X - 2\delta_X)$ so that by projection formula $\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R) \simeq \varphi^*\mathcal{E}_2 \otimes \mathcal{O}_{X^{[2]}}(q_*\tau^*pr_1^*H_X - 2\delta_X)$. Letting $s_R \in$

$|\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R)|$ be a section whose zero locus is equal to R, we can consider s_R as a section of the rank 2-vector bundle $\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R)$. Then the zero locus of this section corresponds to length 2 subschemes whose associated line is contained in X that is to P_2. So the class P_2 in $\mathrm{CH}^2(X^{[2]})$ is $c_2(\pi^*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R))$.

Let $U \simeq Spec(A)$ be an affine open subset of $X^{[2]}$ such that $\mathbb{P}(\varphi^*\mathcal{E}_2)_{|U} \simeq \mathbb{P}^1_A$. Denoting $[Y_0 : Y_1]$ the homogeneous (relative) coordinates on $\mathbb{P}^1_A$, the equation s_R of $R_{|U} \subset \mathbb{P}^1_A$, is of the form $f_0Y_0 + f_1Y_1 = 0$, where $f_0, f_1 \in A$, since $R \in |\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(1) \otimes \pi^*\mathcal{O}_{X^{[2]}}(q_*\tau^*pr_1^*H_X - 2\delta_X)|$. Then the section s_R of $(\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R))_{|U}$ is (f_0, f_1). As P_2 is the zero locus of s_R, the ideal of $P_2 \cap U$ in U is generated by (f_0, f_1) and as P_2 is smooth of codimension 2, (f_0, f_1) is a regular sequence in A. As (f_0, f_1) is a regular sequence, the equation $f_0Y_0 + f_1Y_1 = 0$ tells exactly that R is the blow-up of $X^{[2]}$ along P_2 i.e. $R \simeq \widetilde{X^{[2]}}$. □

The divisors $\widetilde{X^{[2]}}$, $\sigma(\widetilde{X \times X})$ and $f^*X = [f^{-1}(X)]$ can be considered as correspondences from $X^{[2]}$ to X. The following fiber square:

$$\begin{tikzcd} f^{-1}(X) \arrow[r, "f'"] \arrow[d, hook, "i'_X"] & X \arrow[d, hook, "i_X"] \\ \mathbb{P}(\varphi^*\mathcal{E}_2) \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}^{n+1} \\ X^{[2]} & \end{tikzcd}$$

yields the following easy lemma:

Lemma 1.3.2. *The action $[f^*(X)]_* : \mathrm{CH}^*(X^{[2]}) \to \mathrm{CH}^*(X)$ factors through $\mathrm{CH}^*(\mathbb{P}^{n+1})$ i.e. for any $z \in \mathrm{CH}^i(X^{[2]})$, there is an integer $m_z \in \mathbb{Z}$ such that $[f^{-1}(X)]_*z = m_zH_X^{n+i-2d}$.*

By Lemma 1.3.2, $\left[\widetilde{X^{[2]}}\right]_* + [\sigma(\widetilde{X \times X})]_* : \mathrm{CH}^*(X^{[2]}) \to \mathrm{CH}^*(X)$ factors through $\mathrm{CH}^*(\mathbb{P}^{n+1}_k)$. As $[\sigma(\widetilde{X \times X})]$ is tautological, we can compute the action of $\left[\widetilde{X^{[2]}}\right]$ modulo cycles coming from $\mathbb{P}^{n+1}_k$. We now have to find a suitable relation on which we can use the action of $\left[\widetilde{X^{[2]}}\right]$.

Lemma 1.3.3. *We have the following equality in $\mathrm{CH}^1(\widetilde{X^{[2]}})$:*

$$(f^*H)_{|\widetilde{X^{[2]}}} = 2\pi^*_{|\widetilde{X^{[2]}}}q_*\tau^*pr_1^*H_X - 3\pi^*_{|\widetilde{X^{[2]}}}\delta_X - E_{P_2}. \tag{1.13}$$

Proof. Using that $\pi_{|\widetilde{X^{[2]}}}$ is a blow-up, we have $K_{\widetilde{X^{[2]}}} = \pi^*_{|\widetilde{X^{[2]}}}K_{X^{[2]}} + E_{P_2}$. Secondly, adjunction formula gives $K_{\widetilde{X^{[2]}}} = (K_{\mathbb{P}(\varphi^*\mathcal{E}_2)} + \widetilde{X^{[2]}})_{|\widetilde{X^{[2]}}}$. As $K_{\mathbb{P}(\varphi^*\mathcal{E}_2)} = \pi^*K_{X^{[2]}} + K_{\mathbb{P}(\varphi^*\mathcal{E}_2)/X^{[2]}}$, clearing $\pi^*K_{X^{[2]}}$, we get

$$E_{P_2} = (K_{\mathbb{P}(\varphi^*\mathcal{E}_2)/X^{[2]}} + \widetilde{X^{[2]}})_{|\widetilde{X^{[2]}}} \tag{1.14}$$

Using formulas for projective bundle, we have

$$K_{\mathbb{P}(\varphi^*\mathcal{E}_2)/X^{[2]}} = -2c_1(\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(1)) + \pi^*c_1(\varphi^*\mathcal{E}_2) = -2f^*H + \pi^*(q_*\tau^*pr_1^*c_1(\mathcal{O}_X(1)) - \delta_X).$$

Then, (1.14) yields

$$E_{P_2} = (-f^*H + 2\pi^*q_*\tau^*pr_1^*H_X - 3\pi^*\delta_X)_{|\widetilde{X^{[2]}}}$$

□

Proof of Theorem 1.3.1. Let $i_\Sigma : \Sigma \hookrightarrow X$ be a smooth subvariety of X of dimension d. Then we have the description of $\Sigma^{[2]}$ as the quotient of the blow-up $\widetilde{\Sigma \times \Sigma}$ of $\Sigma \times \Sigma$ along the diagonal by the involution. The class of $\widetilde{\Sigma \times \Sigma}$, which is the strict transform of $\Sigma \times \Sigma$ under τ, in $\mathrm{CH}_{2d}(\widetilde{X \times X})$ is given by the excess formula ([30, Theorem 6.7 and Corollary 4.2.1]):

$$\widetilde{\Sigma \times \Sigma} = \tau^*(\Sigma \times \Sigma) - j_{E_X,*}\{c(\tau^*_{|E_X} T_X)(1 + E_{X|E_X})^{-1} \cdot \tau^*_{|E_X} i_{\Sigma,*}(c(T_\Sigma)^{-1})\}_{2d} \text{ in } \mathrm{CH}_{2d}(\widetilde{X \times X}) \quad (1.15)$$

We recall from (1.12) that $c_1(\varphi^*\mathcal{E}_2) = q_*\tau^* pr_1^* H_X - \delta_X$. Intersecting (1.13) with $\pi^*_{|\widetilde{\Sigma^{[2]}}}(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1})$ and projecting to X, we get in $\mathrm{CH}_d(X)$:

$$\begin{aligned} H_X \cdot \left[\widetilde{X^{[2]}}\right]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) = \left[\widetilde{X^{[2]}}\right]_* &\left[(2q_*\tau^* pr_1^* H_X - 3\delta_X) \cdot (\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1})\right] \\ &- f'_{|\widetilde{X^{[2]}},*}(E_{P_2} \cdot \pi^*_{|\widetilde{X^{[2]}}}(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1})) \end{aligned} \quad (1.16)$$

To simplify this expression, we use the following lemma:

Lemma 1.3.4. *We have the following formulas (by induction):*
(i) for $k \geq 1$, $(q_*\tau^* pr_1^* H_X)^k = \sum_{j=0}^{k-1} \binom{k-1}{j} q_*\tau^*(pr_1^* H_X^{k-j} \cdot pr_2^* H_X^j)$;
(ii) for $k, k' \geq 0$ and $m \geq 1$,

$$q_*\tau^*(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot \delta_X^m = q_* j_{E_X,*}[\tau^*_{|E_X} i^*_{\Delta_X}(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot (E_{X|E_X})^{m-1}]$$

where $j_{E_X} : E_X \hookrightarrow \widetilde{X \times X}$ is the inclusion of the exceptional divisor, $i_{\Delta_X} : \Delta_X \hookrightarrow X \times X$ is the inclusion of the diagonal (so that $i^*_{\Delta_X}(pr_1^* H_X^k \cdot pr_2^* H_X^{k'})$ is the hyperplane section $H_X^{k+k'}$ on $X \simeq \Delta_X$) and $c_1(\mathcal{O}_{E_X}(-1)) \simeq E_{X|E_X}$ is the tautological line bundle of the projective bundle $\tau_{|E_X} : E_X \to \Delta_X$.
(iii) it follows that for $m \geq 2$,

$$\begin{aligned} c_1(\varphi^*\mathcal{E}_2)^m = &\sum_{l=0}^{m-1} \binom{m-1}{l} q_*\tau^*(pr_1^* H_X^{m-l} \cdot pr_2^* H_X^l)) + (-1)^m \delta_X^m \\ + &\sum_{k=1}^{m-1} \sum_{l=0}^{m-1-k} (-1)^k \binom{m}{k} \binom{m-1-k}{l} q_* j_{E_X,*}(\tau^*_{|E_X} i^*_{\Delta_X}(pr_1^* H_X^{m-k-l} \cdot pr_2^* H_X^l) \cdot E_{X|E_X}^{k-1}) \end{aligned}$$

Proof. (i) The proof goes by induction starting from the following computation:

$$\begin{aligned} (q_*\tau^* pr_1^* H_X)^2 &= q_*(q^* q_*\tau^* pr_1^* H_X \cdot \tau^* pr_1^* H_X) \text{ by projection formula} \\ &= q_*([q^{-1}(q_*\tau^* pr_1^* H_X)] \cdot \tau^* pr_1^* H_X) \text{ since } q \text{ is flat} \\ &= q_*(\tau^*(pr_1^* H_X + pr_2^* H_X) \cdot \tau^* pr_1^* H_X) \\ &= q_*\tau^*(pr_1^* H_X^2 + pr_1^* H_X \cdot pr_2^* H_X) \end{aligned}$$

(ii) We have:

$$\begin{aligned} q_*\tau^*(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot \delta_X^m &= q_*(\tau^*(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot (q^*\delta_X)^m) \text{ by projection formula} \\ &= q_*(\tau^*(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot E_X^m) \\ &= q_*(\tau^*(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot E_X^{m-1} \cdot j_{E_X,*}(1)) \\ &= q_* j_{E_X,*}[j_{E_X}^* \tau^*(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot j_{E_X}^* E_X^{m-1}] \text{ by projection formula} \\ &= q_* j_{E_X,*}[\tau^*_{|E_X} i^*_{\Delta_X}(pr_1^* H_X^k \cdot pr_2^* H_X^{k'}) \cdot (E_{X|E_X})^{m-1}] \\ &\ \text{using the fiber square given by the blow} - \text{up} \end{aligned}$$

(iii) Finally we have:

$$c_1(\varphi^*\mathcal{E}_2)^m = (q_*\tau^* pr_1^* H_X - \delta_X)^m = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (q_*\tau^* pr_1^* H_X)^{m-k} \cdot \delta_X^k$$

and use the previous computations to obtain the result. □

In order to establish (1.8), let us now compute the different terms of (1.16) modulo cycles coming from $\mathbb{P}^{n+1}$, using the correspondence $\sigma(\widetilde{X \times X})$. We recall that by construction, $[\sigma(\widetilde{X \times X})]_*(\cdot) = f'_{|\sigma(\widetilde{X\times X}),*} q^*(\cdot)$ and we have:

$$\begin{aligned} q^*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) \quad = \quad & \widetilde{\Sigma \times \Sigma} \cdot [\textstyle\sum_{l=0}^{d-2} \binom{d-2}{l} \tau^*(pr_1^* H_X^{d-1-l} \cdot pr_2^* H_X^l + pr_1^* H_X^l \cdot pr_2^* H_X^{d-1-l}) \\ & +(-1)^{d-1} E_X^{d-1} + \textstyle\sum_{k=1}^{d-2} \sum_{l=0}^{d-2-k} (-1)^k \binom{d-1}{k} \binom{d-2-k}{l} j_{E_X,*}(\tau^*_{|E_X} H_X^{d-1-k} \cdot E_{X|E_X}^{k-1})] \end{aligned}$$

The different terms are computed using the equalities:

(i) for $m, m' \geq 0$, $\tau^*(\Sigma \times \Sigma) \cdot \tau^*(pr_1^* H_X^m \cdot pr_2^* H_X^{m'}) = \tau^*((\Sigma \cap H_X^m) \times (\Sigma \cap H_X^{m'}))$ and its image in X under $f'_{|\sigma(\widetilde{X\times X}),*}$ is supported on $(\Sigma \cap H_X^m)$.

(ii) for $m \geq 0$, $\tau^*(\Sigma \times \Sigma) \cdot j_{E_X,*}(E_{X|E_X}^l \cdot \tau^*_{|E_X} H_X^m) = j_{E_X,*}(E_{X|E_X}^l \cdot \tau^*_{|E_X}(\Sigma^2 \cdot H_X^m))$ and its image in X under $f'_{|\sigma(\widetilde{X\times X}),*}$ is supported on $\Sigma^2 \cap H_X^m$.

(iii) for $m > 0$, $\tau^*(\Sigma \times \Sigma) \cdot E_X^m = j_{E_X,*}(E_{X|E_X}^{m-1} \cdot \tau^*_{|E_X} \Sigma^2)$ and its image in X under $f'_{|\sigma(\widetilde{X\times X}),*}$ is supported on Σ^2.

(iv) $j_{E_X,*}\{c(\tau^*_{|E_X} T_X)(1 + E_{X|E_X})^{-1} \cdot \tau^*_{|E_X} i_{\Sigma,*}(c(T_\Sigma)^{-1})\}_{2d} \cdot \tau^*(pr_1^* H_X^m \cdot pr_2^* H_X^{m'})$
$= j_{E_X,*}(\sum_{i=0}^{min(n-d,d)} (-1)^{n-i-1-d} E_{X|E_X}^{n-i-1-d} \cdot \tau^*_{|E_X} c_i(N_{\Sigma/X}))) \cdot \tau^*(pr_1^* H_X^m \cdot pr_2^* H_X^{m'})$
$= 2 j_{E_X,*} \sum_{i=0}^{min(n-d,d)} (-1)^{n-i-1-d} E_{X|E_X}^{n-i-1-d} \cdot \tau^*_{|E_X}(c_i(N_{\Sigma/X}) \cdot H_X^{m+m'})$ and its image in X under $f'_{|\sigma(\widetilde{X\times X}),*}$ is supported on $\cup_i (c_i(N_{\Sigma/X}) \cap H_X^{m+m'})$.

(v) $j_{E_X,*}\{c(\tau^*_{|E_X} T_X)(1 + E_{X|E_X})^{-1} \cdot \tau^*_{|E_X} i_{\Sigma,*}(c(T_\Sigma)^{-1})\}_{2d} \cdot E_X^m$
$= j_{E_X,*}(\sum_{i=0}^{min(n-d,d)} (-1)^{n-i-1-d} E_{X|E_X}^{m+n-i-1-d} \cdot \tau^*_{|E_X} c_i(N_{\Sigma/X}))$ and its image in X under $f'_{|\sigma(\widetilde{X\times X}),*}$ is supported on $\cup_i c_i(N_{\Sigma/X})$.

(vi)$j_{E_X,*}\{c(\tau^*_{|E_X} T_X)(1 + E_{X|E_X})^{-1} \cdot \tau^*_{|E_X} i_{\Sigma,*}(c(T_\Sigma)^{-1})\}_{2d} \cdot j_{E_X,*}(E_{X|E_X}^{k-1} \cdot \tau^*_{|E_X} H_X^m)$
$= j_{E_X,*} \sum_{i=0}^{min(n-d,d)} (-1)^{n-i-1-d} E_{X|E_X}^{m+n-i-1-d} \cdot \tau^*_{|E_X}(c_i(N_{\Sigma/X}) \cdot H_X^m)$ and its image in X under $f'_{|\sigma(\widetilde{X\times X}),*}$ is supported on $\cup_i (c_i(N_{\Sigma/X}) \cap H_X^m)$.

With these formulas, we can see that:

(1) We have $[\sigma(\widetilde{X \times X})]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) = 0$ as its support in X is the union of subvarieties of dimension $\leq d$ whereas $q^*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1})$ has dimension $d+1$. So $\left[\widetilde{X^{[2]}}\right]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) \in \mathbb{Z} \cdot H_X^{n-d-1}$.

(2) We have

$$[\sigma(\widetilde{X \times X})]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^d) = f_{|\sigma(\widetilde{X\times X}),*}[\tau^*(\Sigma \times (\Sigma \cdot H_X^d)) - j_{E_X,*}((-1)^{n-1}E_{X|E_X}^{n-1} \cdot \tau_{|E_X}^* c_0(N_{\Sigma/X})]$$

since all the other terms are supported on $\bigcup_{k,j,i,m}(\Sigma \cap H_X^k) \cup (\Sigma^2 \cap H_X^j) \cup (c_i(N_{\Sigma/X}) \cap H_X^m)$ with $k > 0$, $j \geq 0$ and $m > 0$ if $i = 0$ and $m \geq 0$ else, which is a union of subschemes of dimension $< d$. So $[\sigma(\widetilde{X \times X})]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^d) = deg(\Sigma)\Sigma - \Sigma$ in $\mathrm{CH}_d(X)$. Hence $\left[\widetilde{X^{[2]}}\right]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^d) = -(deg(\Sigma)\Sigma - \Sigma) \bmod \mathbb{Z} \cdot H_X^{n-d}$.

(3) Likewise $[\sigma(\widetilde{X \times X})]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1} \cdot \delta_X) = f_{|\sigma(\widetilde{X\times X}),*}((-1)^n E_{X|E_X}^{n-1} \tau_{|E_X}^* c_0(N_{\Sigma/X}))$ so that $\left[\widetilde{X^{[2]}}\right]_*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) \cdot \delta_X) = \Sigma \bmod \mathbb{Z} \cdot H_X^{n-d}$.

(4) For the last term, we have

$$f'_{|\widetilde{X^{[2]}},*}(E_{P_2} \cdot \pi^*_{|\widetilde{X^{[2]}}}(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1})) = P_*[p_{P_2,*} i_{P_2}^*(\Sigma^{[2]} \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1})].$$

□

1.3.2 A digression on the Hilbert square of subvarieties

Assume $k = \mathbb{C}$. On one hand, as any smooth cubic hypersurface X admits a unirational parametrization of degree 2, any functorial birational invariant of X is 2-torsion and as the coefficient appearing with Σ in the inversion formula of Theorem 1.3.1, is odd, the formula will be useful to study birational invariants obtained as functorial subquotient of Chow groups. On the other hand, in the inversion formula, the operation $\Sigma \mapsto \Sigma^{[2]}$ plays a key role. So let us look at some properties of this operation.

Proposition 1.3.2. *Let Y be a smooth projective k-variety. Let V, V' be smooth subvarieties of Y of dimension $d < dim(Y)$ and $N > 0$ an integer such that $N(i_{V,*}(c(V)^{-1}) - i_{V',*}(c(V')^{-1})) = 0$ in $\mathrm{CH}_*(Y)$ (resp. $\mathrm{CH}_*(Y)/alg$), where i_V (resp. $i_{V'}$) is the inclusion of V (resp. of V') in Y.*

(i) Then $2N(V^{[2]} - V'^{[2]}) = 0$ in $\mathrm{CH}_{2d}(Y^{[2]})$ (resp. $\mathrm{CH}_{2d}(Y^{[2]})/alg$).

(ii) Moreover if the groups $\mathrm{CH}_i(Y)$ are torsion-free for $i \leq 2d$, then $V^{[2]} = V'^{[2]}$ in $\mathrm{CH}_{2d}(Y^{[2]})$.

Proof. (i) Let us denote $\tau : \widetilde{Y \times Y} \to Y \times Y$ the blow-up of $Y \times Y$ along the diagonal Δ_X and $q : \widetilde{Y \times Y} \to Y^{[2]}$ the quotient by the involution. For a smooth subvariety $V \subset Y$, we have $q_*(\widetilde{V \times V}^{\Delta_V}) = 2V^{[2]}$ in $\mathrm{CH}_{2d}(Y^{[2]})$, where $\widetilde{V \times V}^{\Delta_V}$ is the blow-up of $V \times V$ along its diagonal Δ_V, i.e. the proper transform of $V \times V$ under τ. We recall ([30, Theorem 6.7 and Corollary 4.2.1]) that we have

$$\widetilde{V \times V}^{\Delta_V} = \tau^*(V \times V) - j_{E_Y,*}\{c(\tau_{|E_Y}^* T_Y)c(E_{Y|E_Y})^{-1} \cdot \tau_{|E_Y}^* i_{V,*}(c(T_V)^{-1})\}_{2d} \text{ in } \mathrm{CH}_{2d}(\widetilde{Y \times Y}) \quad (1.17)$$

where $j_{E_Y} : E_Y \hookrightarrow \widetilde{Y \times Y}$ is the exceptional divisor of τ and for an element $z \in \mathrm{CH}_*(\widetilde{Y \times Y})$, $\{z\}_k$ is the part of dimension k of z.

Now, if $N(V-V')=0$ in $\mathrm{CH}_d(Y)$ (resp. $\mathrm{CH}_d(Y)/alg$), V and V' being smooth subvarieties of Y, then

$$N(V\times V)=Npr_1^*V\cdot pr_2^*V=pr_1^*(NV)\cdot pr_2^*V=pr_1^*(NV')\cdot pr_2^*V=pr_1^*V'\cdot pr_2^*(NV)=N(V'\times V')$$

in $\mathrm{CH}_{2d}(Y\times Y)$ (resp. $\mathrm{CH}_{2d}(Y\times Y)/alg$). So we see that the hypothesis yields

$$\begin{aligned}
2N(V^{[2]}-V'^{[2]}) &= Nq_*(\widetilde{V\times V}^{\Delta_V}-\widetilde{V'\times V'}^{\Delta_{V'}})\\
&= \tau^*N[(V\times V)-(V'\times V')]\\
&\quad -j_{E_Y,*}\{c(\tau^*_{|E_Y}T_Y)c(E_{Y|E_Y})^{-1}\cdot\tau^*_{|E_Y}N((i_{V,*}(c(T_V)^{-1})-i_{V',*}(c(T_{V'})^{-1}))\}_{2d}\\
&= 0 \ \text{ in } \mathrm{CH}_{2d}(Y^{[2]}) \text{ (resp. } \mathrm{CH}_{2d}(Y^{[2]})/alg).
\end{aligned}$$

(ii) As $\mathrm{CH}_{*\leq d}(Y)$ is assumed to be torsion-free, $V=V'$ in $\mathrm{CH}_d(Y)$. Then, by [58, Proposition 1.4], $V^{(2)}=V'^{(2)}$ in $\mathrm{CH}_{2d}(Y^{(2)})$, where, for a variety Z, $Z^{(2)}$ is the symmetric product of Z. We have the localization exact sequence

$$\mathrm{CH}_{2d}(E_Y)\to\mathrm{CH}_{2d}(Y^{[2]})\to\mathrm{CH}_{2d}(Y^{[2]}\backslash E_Y)\to 0$$

and since $Y^{[2]}\backslash E_Y\simeq Y^{(2)}\backslash\Delta_Y$, $V^{[2]}-V'^{[2]}$ can be written $q_*j_{E_Y,*}\gamma$ for a $2d$-cycle $\gamma\in\mathrm{CH}_{2d}(E_Y)$. According to item (i), $2(V^{[2]}-V'^{[2]})=0$ so that $q_*j_{E,*}(2\gamma)=0$. As, q is flat, $q^*q_*j_{E,*}(2\gamma)=[q^{-1}q_*j_{E,*}(2\gamma)]=j_{E,*}(2\gamma)$ and by the decomposition of the Chow groups of the blow-up $\widetilde{Y\times Y}$, $2\gamma=0$ in $\mathrm{CH}_{2d}(E_Y)$. So, by the decomposition of the Chow groups of projective bundle and torsion-freeness of $\mathrm{CH}_{*\leq d}(Y)$, $\gamma=0$ i.e. $V^{[2]}-V'^{[2]}=0$ in $\mathrm{CH}_{2d}(Y^{[2]})$. $\square$

Unfortunately, in general, for a smooth subvariety V of a smooth projective variety Y, one cannot expect the class of $V^{[2]}$ in $\mathrm{CH}_*(Y^{[2]})$ to be determined by $(i_{V,*}(c(T_V)^{-1}))$ as the following example, which was communicated to the author by Voisin, shows.

Let S be an abelian surface and x, $y\in S$ be two distinct 2-torsion points. For any sufficiently ample linear system $\mathcal{L}$ on S, there exists a curve $C_x\in|\mathcal{L}|$ not containing y, resp. $C_y\in|\mathcal{L}|$ not containing x, which is smooth away from x, resp. y, and has an ordinary double point at x, resp. y. Let $\tau:\widetilde{S}\to S$ be the blow-up of S at x and y and E_x,E_y the corresponding exceptional divisors. The normalization $\widetilde{C_x}$ (resp. $\widetilde{C_y}$) of C_x (resp. C_y) is the strict transform of C_x (resp. C_y) under τ and its class in $\mathrm{Pic}(\widetilde{S})$ is $\tau^*c_1(\mathcal{L})-2E_x$ (resp. $\tau^*c_1(\mathcal{L})-2E_y$).

Let $\mathcal{N}\in\mathrm{Pic}(S)$ be sufficiently ample on S so that the line bundle $\tau^*_{|\widetilde{C_x}}\mathcal{N}_{|C_x}$ is very ample (once its degree on $\widetilde{C_x}$ is large enough) on $\widetilde{C_x}$ and $\tau^*_{|\widetilde{C_y}}\mathcal{N}_{|C_y}$ is very ample on $\widetilde{C_y}$. We can pick a meromorphic function $f_x:\widetilde{C_x}\to\mathbb{P}^1$ in $|\tau^*_{|\widetilde{C_x}}\mathcal{N}_{|C_x}|$ such that, denoting x_1 and x_2 the points lying over the node x, $f_x(x_1)\neq f_x(x_2)$. Likewise, we can pick a meromorphic function $f_y:\widetilde{C_y}\to\mathbb{P}^1$ in $|\tau^*_{|\widetilde{C_y}}\mathcal{N}_{|C_y}|$ such that $f_y(y_1)\neq f_y(y_2)$, where y_1,y_2 are the points lying over the node y.

Let $X=S\times\mathbb{P}^1$ be the trivial projective bundle over S. By construction the morphisms $(\tau_{|\widetilde{C_x}},f_x):\widetilde{C_x}\to X$ and $(\tau_{|\widetilde{C_y}},f_y):\widetilde{C_y}\to X$ are embeddings so $D_x=(\tau_{|\widetilde{C_x}},f_x)(\widetilde{C_x})$ and $D_y=(\tau_{|\widetilde{C_y}},f_y)(\widetilde{C_y})$ are smooth curves on X.

Proposition 1.3.3. *In this situation, we have* $i_{D_x,*}(c(T_{D_x})^{-1})=i_{D_y,*}(c(T_{D_y})^{-1})$ *in* $\mathrm{CH}_*(X)$ *but* $D_x^{[2]}\neq D_y^{[2]}$ *in* $\mathrm{CH}_2(X^{[2]})$.

Proof. We have the decomposition $\mathrm{CH}_1(X) \simeq pr_1^*\mathrm{CH}_0(S) \oplus pr_1^*\mathrm{CH}_1(S)$. The projection on $\mathrm{CH}_1(S)$ is given by $pr_{1,*}$; we have $pr_{1,*}D_x = \tau_{|\widetilde{C_x},*}\widetilde{C_x} = C_x$ and $pr_{1,*}D_y = \tau_{|\widetilde{C_y},*}\widetilde{C_y} = C_y$ and $C_x, C_y \in |\mathcal{L}|$. As the Chern classes of the trivial bundle are trivial the projection on $\mathrm{CH}_0(S)$ is given by the composition of the intersection with $pr_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ followed by $pr_{1,*}$. We have $f_x^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \tau^*_{|\widetilde{C_x}}\mathcal{N}_{|C_x}$ and using projection formula and $C_x \in |\mathcal{L}|$, we get $pr_{1,*}(D_x \cdot pr_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))) = c_1(\mathcal{L}) \cdot c_1(\mathcal{N})$ in $\mathrm{CH}_0(S)$. Likewise, we have $pr_{1,*}(D_y \cdot pr_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))) = c_1(\mathcal{L}) \cdot c_1(\mathcal{N})$. So $D_x = D_y$ in $\mathrm{CH}_1(X)$.

By adjunction, we have $K_{\widetilde{C_x}} = (K_{\widetilde{S}} + \widetilde{C_x})_{|\widetilde{C_x}} = (\tau^*(c_1(\mathcal{L}) + K_S) + E_y - E_x)_{|\widetilde{C_x}}$ so that in $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(S)$,

$$\begin{aligned} i_{D_x,*}K_{D_x} &= (\tau \circ i_{\widetilde{C_x}})_* i^*_{\widetilde{C_x}}(\tau^*c_1(\mathcal{L}) + E_y - E_x) \\ &= \tau_* i_{\widetilde{C_x},*} i^*_{\widetilde{C_x}}(\tau^*c_1(\mathcal{L}) + E_y - E_x) \\ &= \tau_*((\tau^*c_1(\mathcal{L}) + E_y - E_x) \cdot \widetilde{C_x}) \\ &= \tau_*(\tau^*(c_1(\mathcal{L})^2 + 2E_x^2) \\ &= c_1(\mathcal{L})^2 - 2x \end{aligned}$$

Likewise $i_{D_y,*}K_{D_y} = c_1(\mathcal{L})^2 - 2y$. As $2x = 2y$ in $\mathrm{CH}_0(S)$, $i_{D_x,*}K_{D_x} = i_{D_y,*}K_{D_y}$ in $\mathrm{CH}_0(X)$. So $i_{D_x,*}(c(T_{D_x})^{-1}) = i_{D_y,*}(c(T_{D_y})^{-1})$.

The variety of lines of X, with respect to a very ample line bundle of the form $pr_1^*\mathcal{L}' \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, is isomorphic to S since any morphism from a projective space to a abelian variety is constant. Let us denote $P_2 = \mathbb{P}(\mathrm{Sym}^2\mathcal{E}) \simeq S \times \mathbb{P}^2$; it parametrizes the length 2 subschemes of X contained in a line of X. So $D_x^{[2]} \cap P_2$ parametrizes length 2 subschemes of D_x such that the associated line is contained in X. But by construction, since $pr_{1,|D_x} : D_x \to C_x$ is an isomorphism above $C_x \backslash \{x\}$, the only length 2 subscheme whose associated line is contained in X is $\{x_1 + x_2\}$ whose associated line is $\mathbb{P}(\mathcal{E}_x)$. So, denoting $i_{P_2} : P_2 \hookrightarrow X^{[2]}$ the natural inclusion and $\pi_1 : P_2 \to S$ the first projection, we have $\pi_{1,*}(i^*_{P_2}D_x^{[2]}) = x$ in $\mathrm{CH}_0(S)$.

Likewise $\pi_{1,*}(i^*_{P_2}D_y^{[2]}) = y$ in $\mathrm{CH}_0(S)$. So $\pi_{1,*}i^*_{P_2}(D_x^{[2]} - D_y^{[2]}) = x - y \neq 0$ in $\mathrm{CH}_0(S)$, in particular $D_x^{[2]} - D_y^{[2]}$ is a nonzero 2-torsion element in $\mathrm{CH}_2(X^{[2]})$. □

1.3.3 Application of the inversion formula

Using the results of the previous sections, we get the following:

Theorem 1.3.5. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, with $n \geq 5$, be a smooth cubic hypersurface. For any $\Gamma \in \mathrm{CH}_2(X)$ of t-torsion (hence homologically trivial), there is a homologically trivial $2t$-torsion 1-cycle $\gamma \in \mathrm{CH}_1(F(X))$ and an odd integer m such that $m\Gamma = P_*\gamma$ in $\mathrm{CH}_2(X)$.*

Proof. Let $\Gamma \in \mathrm{CH}_2(X)$ be a cycle annihilated by $t \in \mathbb{Z}_{>0}$. Using Proposition 1.1.1, we can find a 1-cycle α in $F(X)$ such that $P_*(\alpha) = \Gamma$. As Γ is a torsion cycle and $\mathrm{CH}_0(X) = \mathbb{Z}$, $\Gamma \cdot H^2 = 0$ and since $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{2|F(X)})}(1)) = q^*H$, we get $deg(\alpha \cdot c_1(\mathcal{O}_{F(X)}(1))) = deg(q_*[p^*(\alpha \cdot c_1(\mathcal{O}_{F(X)}(1)) \cdot q^*H]) = 0$, where $\mathcal{O}_{F(X)}(1) = det(\mathcal{E}_{2|F(X)})$ is the Plücker line bundle, which implies that $\alpha \cdot c_1(\mathcal{O}_{F(X)}(1)) = 0$ in $\mathrm{CH}_0(F(X))$ since $F(X)$ is rationally connected. As $\mathrm{Pic}(F(X)) \simeq \mathbb{Z}$ ([26, Corollaire 3.5]), α is numerically trivial. We have the following lemma:

Lemma 1.3.6. *Let Y be a smooth projective variety of dimension $d \geq 3$ and D a numerically trivial 1-cycle of Y. Then there are smooth curves $D_1, D_2 \subset Y$ of the same genus such that $D = D_1 - D_2$ in $\mathrm{CH}_1(Y)$.*

Postponing the proof of the lemma, we conclude as follows: let $E_1, E_2 \subset F(X)$ be two smooth curves of genus g such that $\alpha = E_1 - E_2$ in $\mathrm{CH}_1(F(X))$; they have also the same degree (α is numerically trivial) that we shall denote d. Let us denote $S_{E_1} = q(p^{-1}(E_1))$ and $S_{E_2} = q(p^{-1}(E_2))$ the associated ruled surfaces in X, we have $\Gamma = P_*\alpha = S_{E_1} - S_{E_2}$in $\mathrm{CH}_2(X)$. By transversality arguments, we can arrange that q induces an embedding of $p^{-1}(E_1)$ (resp. $p^{-1}(E_2)$) in X so that S_{E_1} (resp. S_{E_2}) is smooth and isomorphic to $p^{-1}(E_1)$ (resp. $p^{-1}(E_2)$). An easy computation then gives:

$$\begin{aligned} i_{S_{E_1},*}(c(T_{S_{E_1}})^{-1}) &= S_{E_1} + P_*(i_{E_1,*}K_{E_1} + dc_1(\mathcal{O}_{F(X)}(1)) \cdot E_1) - 2S_{E_1} \cdot H_X - 2H_X \cdot P_*(i_{E_1,*}K_{E_1}) \\ &= P_*(E_1) + (2g-2+d)P_*[l_0] - 2P_*(E_1) \cdot H_X - 2(2g-2)P_*([l_0]) \cdot H_X \end{aligned}$$

since $\mathrm{CH}_0(F(X)) \simeq \mathbb{Z} \cdot [l_0]$ for a (any) point $[l_0] \in F(X)$. Likewise

$$i_{S_{E_2},*}(c(T_{S_{E_2}})^{-1}) = P_*(E_2) + (2g-2+d)P_*[l_0] - 2P_*(E_2) \cdot H_X - 2(2g-2)P_*([l_0]) \cdot H_X$$

so that $i_{S_{E_1},*}(c(T_{S_{E_1}})^{-1}) - i_{S_{E_2},*}(c(T_{S_{E_2}})^{-1}) = (S_{E_1} - S_{E_2}) \cdot (1 - 2H_X)$ is annihilated by t in $\mathrm{CH}_*(X)$. Using Proposition 1.3.2, we get that $S_{E_1}^{[2]} - S_{E_2}^{[2]}$ is annihilated by $2t$ in $\mathrm{CH}_4(X^{[2]})$. According to [78, Theorem 2.2], since $H^*(X,\mathbb{Z})$ is torsion-free (by Lefschetz hyperplane and universal coefficient theorems), $H^*(X^{[2]},\mathbb{Z})$ is torsion-free so that $[S_{E_1}^{[2]} - S_{E_2}^{[2]}] = 0$ in $H^{4n-8}(X^{[2]},\mathbb{Z})$.

Now, Theorem 1.3.1, says that there are integers m_1, m_2 such that

$$(2d-3)S_{E_1} + P_*(p_{P_2,*}i_{P_2}^*S_{E_1}^{[2]} \cdot c_1(\mathcal{O}_{F(X)}(1))) = m_1 H_X^{n-2} \quad \text{in } \mathrm{CH}_2(X)$$

and

$$(2d-3)S_{E_2} + P_*(p_{P_2,*}i_{P_2}^*S_{E_2}^{[2]} \cdot c_1(\mathcal{O}_{F(X)}(1))) = m_2 H_X^{n-2} \quad \text{in } \mathrm{CH}_2(X)$$

in particular $(2d-3)\Gamma + P_*(p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]}) \cdot c_1(\mathcal{O}_{F(X)}(1))) \in \mathbb{Z} \cdot H_X^{n-2}$. But intersecting with H_X^2, since Γ and $p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]})$ are torsion cycles, we see that actually:

$$(2d-3)\Gamma + P_*(p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]}) \cdot c_1(\mathcal{O}_{F(X)}(1))) = 0$$

in $\mathrm{CH}_2(X)$. Moreover $p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]})$ is homologically trivial since $S_{E_1}^{[2]} - S_{E_2}^{[2]}$ is. □

Proof of Lemma 1.3.6. Using Hironaka's smoothing of cycles ([41]) and moving lemma, we can write $D = \sum_i m_i C_i$ where $(C_i)_{1 \leq i \leq N}$ is a family of smooth pairwise disjoint connected curves. We can always assume that there is a i_0 such that $m_{i_0} = 1$. Indeed, if none of the m_i is equal to 1, then we can pick 2 smooth curves $C_{N+1}, C_{N+2} \subset Y$ which are rationally equivalent such that $(C_i)_{1 \leq i \leq N+2}$ is still a family of pairwise disjoint smooth curves. Then $D = \sum_i m_i C_i + C_{N+1} - C_{N+2}$ in $\mathrm{CH}_1(Y)$.

Let $C \subset Y$ be a smooth curve intersecting C_{i_0} transversally in a unique point and disjoint from the remaining C_i and $Z = (\cup_{i=1}^N C_i) \cup C$. The subscheme Z is purely 1-dimensional and smooth away from the point $C \cap C_{i_0}$ which is an ordinary double point. In particular Z is a local complete intersection subscheme, so that the sheaf I_Z/I_Z^2 on Z is a vector bundle that we shall denote $N_{Z/Y}^\vee$. Let $\mathcal{L} \in \mathrm{Pic}(Y)$ be a very ample line bundle such that $H^1(Y, \mathcal{L} \otimes I_Z^2) = 0$ and $N_{Z/Y}^\vee \otimes \mathcal{L}_{|Z}$ is globally generated. Then, from the exact sequence

$$0 \to I_Z^2 \to I_Z \to N_{Z/Y}^\vee \to 0$$

we get a surjective morphism $H^0(Y, \mathcal{L} \otimes I_Z) \xrightarrow{\rho} H^0(Z, N_{Z/Y}^\vee \otimes \mathcal{L}_{|Z})$. According to [69, Lemma 1], for any nonzero section $s \in H^0(Y, \mathcal{L} \otimes I_Z)$, the zero scheme $V(s) \subset Y$ is singular at a point $x \in Z$ if and only if the section $\rho(s)$ of $N_{Z/Y}^\vee \otimes \mathcal{L}_{|Z}$ vanishes at x. As, $N_{Z/Y}^\vee \otimes \mathcal{L}_{|Z}$ is globally generated of rank ≥ 2, the zero locus of a generic section of $N_{Z/Y}^\vee \otimes \mathcal{L}_{|Z}$ has codimension $rank(N_{Z/Y}^\vee \otimes \mathcal{L}_{|Z}) \geq 2$ i.e. is empty. So we can find a smooth hypersurface in $|\mathcal{L}|$ containing Z. Repeating the process, we can get a smooth surface $S \subset Y$, which is complete intersection of hypersurfaces given by sections of powers of $\mathcal{L}$, containing Z. Next it is a standard fact (e.g consequence of Riemann-Roch formula) that for any divisor W on a smooth projective surface S, $deg(W \cdot (W + K_S))$ is even. Applying this fact to the divisor $D = \sum_i m_i C_i$ of S, $\int_S D \cdot (D + K_S)$ is even and since D is numerically trivial on Y and K_S is the restriction of a divisor of Y by adjunction formula (S is complete intersection in Y), $deg(D^2) \in 2\mathbb{Z}$. Let us write $deg(D^2) = 2\ell$. Let $H \in \mathrm{Pic}(S)$ be a very ample divisor coming from Y such that the line bundles $\mathcal{O}_S(H - \ell C)$ and $\mathcal{O}_S(H - \ell C + D)$ are ample. We can choose smooth connected curves $E_1 \in |H - \ell C + D|$ and $E_2 \in |H - \ell C|$; we then have $D = E_1 - E_2$ in $\mathrm{Pic}(S)$ (thus, in $\mathrm{CH}_1(Y)$ also). By adjunction formula, we have:

$$\begin{aligned}
2g(E_1) - 2 &= \int_S (H - \ell C + D) \cdot (H - \ell C + D + K_S) \\
&= \int_S (H - \ell C) \cdot (H - \ell C + K_S) + \int_S D \cdot (H - \ell C + D + K_S) + \int_S D \cdot (H - \ell C) \\
&= \int_S (H - \ell C) \cdot (H - \ell C + K_S) + \int_S D^2 - 2\ell D \cdot C \\
&\quad \text{since } D \text{ is numerically trivial on } Y \text{ and } H \text{ and } K_S \text{ come from divisors of } Y \\
&= \int_S (H - \ell C) \cdot (H - \ell C + K_S) \text{ since by construction } \int_S C \cdot D = \int_S C \cdot C_{i_0} = 1
\end{aligned}$$

and

$$2g(E_2) - 2 = \int_S (H - \ell C) \cdot (H - \ell C + K_S).$$

i.e. $g(E_1) = g(E_2)$. □

We draw the following corollary in the case of cubic 5-folds:

Corollary 1.3.7. *Let $X \subset \mathbb{P}^6_{\mathbb{C}}$ be smooth cubic hypersurface. Then $P_* : \mathrm{CH}_1(F(X))_{tors,AJ} \to \mathrm{CH}^3(X)_{tors,AJ}$ is surjective. So the birational invariant $\mathrm{CH}^3(X)_{tors,AJ}$ of X is controlled by the group $\mathrm{CH}_1(F(X))_{tors,AJ}$.*

Proof. Let $\Gamma \in \mathrm{CH}^3(X)_{tors,AJ} \simeq \mathrm{CH}_2(X)_{tors,AJ}$; by Theorem 1.3.5, there are a homologically trivial torsion cycle $\gamma \in \mathrm{CH}_1(F(X))$ and an integer d such that $(2d-3)\Gamma = P_*\gamma$. Because of the degree 2 unirational parametrization of X, $\mathrm{CH}^3(X)_{tors,AJ}$ is a 2-torsion group; in particular $(2d-3)\Gamma = \Gamma$ in $\mathrm{CH}^3(X)$ and it is equal to $P_*\gamma$. By functoriality of Abel-Jacobi maps (P_* induces morphisms of Hodge structures), denoting, for a complex variety Y, Φ_Y^{2c-1} the Abel-Jacobi map for homologically trivial cycles of codimension c, $\Phi_X^5(\Gamma) = P_*\Phi^9_{F(X)}(\gamma)$. Now, by [74], P_* is an isomorphism of abelian varieties so γ is annihilated by the Abel-Jacobi map. □

Under the assumption of the corollary, the variety $F(X)$ is Fano, hence rationally connected. Along the lines of the questions asked in [92] for the group $\mathrm{Griff}_1(Y)$, and the results proved in [88] for the group $H_2(Y,\mathbb{Z})/H_2(Y,\mathbb{Z})_{alg}$ (showing that it should be trivial for rationally connected varieties), it would be tempting to believe that the group $\mathrm{CH}_1(Y)_{tors,AJ}$ is always trivial for rationally connected varieties. For cubic hypersurfaces, we have the following result which follows essentially from the work of Shen ([72]):

Proposition 1.3.4. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, with $n \geq 3$, be a smooth cubic hypersurface. Then $\mathrm{CH}_1(X)_{tors,AJ} = 0$*

Proof. For cubic threefolds, the proposition can be obtained as a consequence of the work of Bloch and Srinivas ([13, Theorem 1]) which asserts that $\mathrm{CH}_1(X)_{hom} \simeq \mathrm{CH}_1(X)_{alg} \simeq J^3(X)(\mathbb{C})$. For cubic hypersurfaces of dimension ≥ 5, the result follows from the work of Shen ([72]) who proved that $\mathrm{CH}_1(X) \simeq \mathbb{Z}$. The only case left is the case of cubic 4-folds but the following proof works for cubic hypersurfaces of any dimension ≥ 3.

Pick $\gamma \in \mathrm{CH}_1(X)_{tors,AJ}$. It is a numerically trivial 1-cycle of X so according to Lemma 1.3.6 we can write it as $\gamma = C_1 - C_2$ where C_i are smooth connected curves on X of same genus g and same degree d. Applying the inversion formula to C_i yields:

$$(2d-3)\gamma = P_* p_{P_2,*} i^*_{P_2}(C_2^{(2)} - C_1^{(2)}) \text{ in } \mathrm{CH}_1(X). \tag{1.18}$$

Since the C_i have the same genus and $\gamma = C_1 - C_2$ is torsion, by Proposition 1.3.2, $C_2^{(2)} - C_1^{(2)} \in \mathrm{CH}_2(X^{[2]})$ is torsion. As $H^*(X^{[2]}, \mathbb{Z})$ is torsion-free ([78]), $C_2^{(2)} - C_1^{(2)}$ is homogically trivial. When $n = 3$, P_* yields an isomorphism $\mathrm{Alb}(F(X)) \simeq J^3(X)$ ([15]), so that applying Abel-Jacobi map to (1.18), we see that $p_{P_2,*} i^*_{P_2}(C_2^{(2)} - C_1^{(2)}) \in \mathrm{CH}_0(F(X))_{tors,AJ}$. When $n \geq 4$, as the Albanese variety of $F(X)$ is trivial ([10, Proposition 3], [26]), $p_{P_2,*} i^*_{P_2}(C_2^{(2)} - C_1^{(2)}) \in \mathrm{CH}_0(F(X))_{tors,AJ}$ and we conclude by Roitman theorem ([67]). $\square$

We are able to prove the vanishing of $\mathrm{CH}_1(X)_{tors,AJ}$ for cubic 4-folds but the proof presented here hinges on quite subtle arguments (inversion formula, torsion-freeness of $H^*(X^{[2]}, \mathbb{Z})$...). So $\mathrm{CH}_1(X)_{tors,AJ}$ might not be trivial for all rationally connected varieties.

Chapter 2

On the universal CH_0 group of cubic threefolds in positive characteristic

Résumé: Dans ce chapitre, nous adaptons en caractéristique > 2 deux résultats de Voisin, exposés dans [90], sur la décomposition de la diagonale des hypersurfaces cubiques complexes de dimension 3, à savoir l'équivalence entre l'existence d'une décomposition de Chow et l'existence d'une décomposition cohomologique de la diagonale pour ces hypersurfaces d'une part et l'équivalence entre l'existence d'un 1-cycle sur la jacobienne intermédiaire $J(X)$ de X à coefficients dans $\mathbb{Z}_2$ dont la classe de cohomologie est la classe minimale $\theta^4/4!$ de $J(X)$ et l'existence d'une décomposition cohomologique (donc de Chow également) de la diagonale d'autre part. En utilisant le second résultat, la conjecture de Tate pour les diviseurs sur les surfaces définies sur un corps fini, prédit, via un théorème dû à Schoen ([70]), que tout solide cubique défini sur la clôture algébrique d'un corps fini de caractéristique > 2 admet une décomposition de Chow de la diagonale.

Abstract: In this chapter, we adapt for algebraically closed fields k of characteristic greater than 2 two results of Voisin, presented in [90], on the decomposition of the diagonal of a smooth cubic hypersurface X of dimension 3 over $\mathbb{C}$, namely: the equivalence between Chow-theoretic and cohomological decompositions of the diagonal of those hypersurfaces and the equivalence between the algebraicity (with $\mathbb{Z}_2$-coefficients) of the minimal class $\theta^4/4!$ of the intermediate Jacobian $J(X)$ of X and the cohomological (hence Chow-theoretic) decomposition of the diagonal of X. Using the second result, the Tate conjecture for divisors on surfaces defined over finite fields predicts, via a theorem of Schoen ([70]), that every smooth cubic hypersurface of dimension 3 over the algebraic closure of a finite field of characteristic > 2 admits a Chow-theoretic decomposition of the diagonal.

2.0 Introduction

Let k be an algebraically closed field and X a smooth n-dimensional projective k-variety (irreducible). X is said to have universally trivial CH_0 group if for any field L containing k, $\mathrm{CH}_0(X_L) = \mathbb{Z}$. As explained in [5], X has universally trivial CH_0 group if and only if $\mathrm{CH}_0(X_{k(X)}) = \mathbb{Z}$. We recall briefly the idea of the proof; it uses, for any extension $k \subset L$, the action, by the identity, of the correspondence Δ_{X_L} on $\mathrm{CH}_0(X_L)$ and the equivalence of the two previous properties with a so called Chow-theoretic decomposition of the diagonal. Passing to the limit in the diagonal morphism $X \to X \times_k X$ over all open subsets $V \subset X$ ($V \to V \times_k V \hookrightarrow V \times_k X$) yields the diagonal point $\delta_X \in X(k(X))$ (image of the generic point of X by the diagonal morphism). When $\mathrm{CH}_0(X_{k(X)}) = \mathbb{Z}$, the diagonal point is rationally equivalent over $k(X)$ to a constant point $x_{k(X)} = x \times_k k(X)$, with $x \in X(k)$. Then, using the equalities $\mathrm{CH}_0(X_{k(X)}) = \mathrm{CH}^n(X_{k(X)}) = \varinjlim_{U \subset X \ open} CH^n(U \times_k X)$ and the localization exact sequence, one gets an equality

$$\Delta_X = X \times_k x + Z \text{ in } \mathrm{CH}^n(X \times_k X) \tag{2.1}$$

where Z is supported on $D \times_k X$ for some proper closed subset D of X i.e. a Chow theoretic decomposition of the diagonal. Now, letting L/k be an extension, a base change in (2.1) yields

$$\Delta_{X_L} = X_L \times_L x_L + Z_L \text{ in } \mathrm{CH}^n(X_L \times_L X_L)$$

so that letting both sides act on $\mathrm{CH}_0(X_L)$ and using the fact that Δ_{X_L} acts as the identity, one sees that $\mathrm{CH}_0(X_L) = \mathbb{Z}$. So, having a universally trivial CH_0 is equivalent to the existence of a Chow-theoretic decomposition of the diagonal. Projective spaces have universally trivial CH_0 so stably rational projective varieties also have universally trivial CH_0. Voisin studied in the case $k = \mathbb{C}$ the a priori weaker property of the existence of a cohomological decomposition of the diagonal

$$[\Delta_X] = [X \times_k x] + [Z] \textit{ in } H_B^{2n}(X \times_k X, \mathbb{Z}) \tag{2.2}$$

where $Z \in \mathrm{CH}^n(X \times_k X)$ is supported on $D \times_k X$ for some proper closed subset D of X and the cohomology is the Betti cohomology. In [90], she proved, in characteristic zero, that this weaker property is, in fact, equivalent to the existence of a Chow-theoretic decomposition of the diagonal for smooth cubic hypersurfaces of odd dimension (≥ 3) or of dimension 4. She also worked out necessary and sufficient conditions for the existence of a cohomological (hence Chow-theoretic) decomposition of the diagonal for cubic threefolds, among other varieties. The second section of this chapter is devoted to the proof of the equivalence between cohomological and Chow-theoretic decomposition of the diagonal of a cubic threefold in positive characteristic, greater than 2:

Theorem 2.0.1. *Let k be an algebraically closed field of characteristic greater than 2 and $X \subset \mathbb{P}_k^4$ be a smooth cubic hypersurface. Then X admits a Chow-theoretic decomposition of the diagonal (i.e. has universally trivial CH_0) if and only if it admits a cohomological decomposition of the diagonal with coefficients in $\mathbb{Z}_2$.*

Remark 2.0.2. *(1) In the Betti setting, having a cohomological decomposition of the diagonal with $\mathbb{Z}$ coefficients is equivalent to having a cohomogical decomposition of the diagonal with coefficients in $\mathbb{Z}/2\mathbb{Z}$ since $2\Delta_X$ has a Chow-theoretic decomposition (see Proposition 2.1.2). This is why in our case, only étale cohomology with $\mathbb{Z}_2$ coefficients is used. In fact, $\mathbb{Z}/2\mathbb{Z}$ would also do.*

(2) Theorem 2.0.1 could have been stated also for cubic fourfolds and odd dimensional cubic of higher dimension since the proof given by Voisin adapts to the positive characteristic setting. But since in the case of cubic threefolds we can give a shorter proof and it is the only case where we have an interesting application in sight, we state the theorem only in this case.

We also check that there is a criterion, similar to the one given in [90, Theorem 4.1] and [89, Theorem 4.9] in characteristic zero, for the existence of a cohomological decomposition of the diagonal of a cubic threefold.

Theorem 2.0.3. *Let $X \subset \mathbb{P}_k^4$ be a smooth cubic hypersurface ($k = \bar{k}$ and $char(k) > 2$). Then X admits a cohomological (hence Chow-theoretic) decomposition of the diagonal if and only if the principal polarization, θ, of the associated Prym variety, $J(X)$, satisfies the following property: there is a cycle $Z \in \mathrm{CH}_1(J(X)) \otimes \mathbb{Z}_2$ such that $[Z] = \frac{[\theta]^4}{4!}$ in $H^8(J(X), \mathbb{Z}_2)$.*

Using Theorem 2.0.3 and a theorem of C. Schoen (see [22], [70]), we get the following consequence:

Theorem 2.0.4. *On an algebraic closure of a finite field of characteristic greater than 2, assuming the Tate conjecture for divisors on surfaces, every smooth cubic hypersurface of dimension 3 has universally trivial CH_0 group.*

In the Betti setting, a key feature in the proof of the criterion was the existence of a parametrization of the intermediate Jacobian of cubic threefolds with separably rationally connected general fiber, namely the condition:

(*) there exist a smooth quasi-projective k-variety B and a correspondence $Z \in \mathrm{CH}^2(B \times_k X)$ with $Z_b \in \mathrm{CH}^2(X)$ trivial modulo algebraic equivalence for any $b \in B(k)$ such that the induced Abel-Jacobi morphism $\phi : B \to J(X)$ to the intermediate Jacobian $J(X)$ of the cubic threefold X is dominant with $\mathbb{P}^5$ as general fiber.

In Section 3, after a reminder on the definition of the intermediate Jacobian of a cubic threefold and Abel-Jacobi morphisms in the positive characteristic setting, we prove Thereom 2.0.3 and Theorem 2.0.4 under the assumption that (*) is still true in our setting.

Over the complex numbers, such a parametrization was constructed by Iliev-Markushevich and Markuschevich-Tikhomirov ([52] and [43], see also [28]) using the space of smooth normal elliptic quintics lying on the cubic hypersurface. Section 4 will be devoted to proving that we still have (*) using the space of stable normal elliptic quintics.

For a variety X, the cohomology groups $H^i(X, \mathbb{Z}_\ell)$ will be the étale cohomology groups and $H^i_B(X, \mathbb{Z}_\ell)$, if X is defined over a field $K \hookrightarrow \mathbb{C}$ of characteristic 0, will be the Betti cohomology group of $X^{an}_{\mathbb{C}}$.

Throughout this text, k will denote an algebraically closed field of characteristic > 2.

For a smooth projective k-variety X, $\langle \cdot, \cdot \rangle_X$ will denote the intersection pairing on $H^*(X, \mathbb{Z}_\ell)$ induced by the cup-product and the trace map. We will use the following standard facts about cubic hypersurfaces in $\mathbb{P}^4_k$. Let X be a smooth cubic hypersurface in $\mathbb{P}^4_k$.

1. From the exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^4_k}(-3) \to \mathcal{O}_{\mathbb{P}^4_k} \to \mathcal{O}_X \to 0$$
we have $\omega_X \simeq \mathcal{O}_X(-2)$ and $h^i(\mathcal{O}_X(k)) = 0$ for $i \in \{1, 2\}$, $k \in \mathbb{Z}$.

2. The Fano variety of lines $F(X) = \{[l] \in Gr(2, 5),\ l \subset X\}$ of X is a smooth projective surface.

3. By Lefschetz hyperplane theorem, for $\ell \neq p$ the ℓ-adic cohomology of X is
$$H^0(X, \mathbb{Z}_\ell) = \mathbb{Z}_\ell \cdot [X],\ \ H^2(X, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell \cdot [\mathcal{O}_X(1)],\ \ H^4(X, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell \cdot [l],\ \ H^6(X, \mathbb{Z}_\ell(3)) = \mathbb{Z}_\ell \cdot [x],$$
$$H^1(X, \mathbb{Z}_\ell) = 0 = H^5(X, \mathbb{Z}_\ell)$$
where $[l]$ is the class of a line $[l] \in F(X)$.

4. Using, for example, a smooth proper lifting of X to characteristic zero ([56, Section 20]), we have $H^3(X, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell^{10}$. By Grothendieck-Lefschetz theorem, $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot [\mathcal{O}_X(1)]$.

2.1 Chow-theoretic and $\mathbb{Z}_2$-cohomological decomposition of the diagonal

In this section, for a k-variety Y, $\mathrm{B}^i(Y)$ will designate the Chow group of codimension i cycle modulo algebraic equivalence. We prove in this section Theorem 2.0.1, adapting arguments of Voisin presented in [90] to the positive characteristic case. The key point is to prove that one can derive a decomposition of the diagonal modulo algebraic equivalence from a cohomological decomposition of the diagonal Δ_X of a smooth cubic hypersurface X. Then we use the following proposition to obtain a Chow-theoretic decomposition of the diagonal:

Proposition 2.1.1. *Let X be a smooth projective k-variety of dimension n. Suppose there exists $Z \in \mathrm{CH}^n(X \times_k X)$, supported on $D \times_k X$ for some proper closed subset $D \subset X$, and $x \in X(k)$ such that*

$$\Delta_X - X \times x = Z \text{ in } \mathrm{B}^n(X \times_k X).$$

Then X admits a Chow-theoretic decomposition of the diagonal.

Proof. It is proposition 2.1 of [90] since even in positive characteristic, cycles algebraically equivalent to 0 are nilpotent for the composition of self-correspondences (see [81] and [80] which use no assumptions on $char(k)$). □

We recall the following classical fact on a cubic hypersurface.

Proposition 2.1.2. *Let X be a smooth cubic hypersurface of dimension 3. Then X admits a degree 2 dominant rational map $\mathbb{P}^3 \dashrightarrow X$. It follows that $2\Delta_X$ admits a decomposition*

$$2\Delta_X = 2(X \times_k x) + Z \textit{ in } \mathrm{CH}^3(X \times_k X)$$

where $x \in X(k)$ and Z is supported on $D \times_k X$ for some divisor $D \subsetneq X$.

Sketch of proof. The first fact is classical and is presented, for example in appendix B of [15]. We recall briefly the construction of the degree 2 map from a rational variety. Let l_0 be a general line in X then the map $P(T_{X|l_0}) \dashrightarrow X$ taking a point (x, v) (with $x \in l_0$ and $v \in P(T_{X,x})$) such that the line $\langle x, v \rangle$ is tangent to X at x but not contained in X, to the other point of the intersection $X \cap \langle x, v \rangle$, is generically finite and $2:1$.

So we have a rational map $\varphi : \mathbb{P}^3_k \dashrightarrow X$ of degree 2. Since resolution of singularities for threefolds exists in $char(k) > 0$ by work of Cossart and Piltant ([24] and [25]), there is a smooth projective k-variety Γ, with a birational morphism $p : \Gamma \to \mathbb{P}^3_k$ and a degree 2 morphism $\phi : \Gamma \to X$, resolving the indeterminacies of φ. We have the following lemma:

Lemma 2.1.1. *([18, Proposition 6.3]). Let $f : Z \to Y$ be a birational morphism of smooth geometrically integral projective varieties over a field L. Then $\mathrm{CH}^0_0(Z) \simeq \mathrm{CH}^0_0(Y)$, where $\mathrm{CH}^0_0(T)$, for a proper L-variety T, is the group of 0-cycles of degree 0.*

Applying the lemma to the morphism obtained from p by base change $p_{k(\mathbb{P}^3_k)} : \Gamma_{k(\mathbb{P}^3_k)} \to \mathbb{P}^3_{k(\mathbb{P}^3_k)}$ yields $\mathrm{CH}_0(\Gamma_{k(\mathbb{P}^3_k)}) \simeq \mathbb{Z}$ i.e. Γ has universally trivial CH_0 group. Then, by base change we have the morphism $\phi_{k(X)} : \Gamma_{k(X)} \to X_{k(X)}$. The 0-cycle of $X_{k(X)}$, $\delta_X - k(X) \times_k x$ has degree 0, where $\delta_X \in X(k(X))$ is the diagonal point (the image of the generic point of X by the diagonal morphism) and $x \in X(k)$. Since $\phi_{k(X)}$ is a degree 2 proper morphism, we have $\phi_{k(X),*}\phi^*_{k(X)}(\delta_X - k(X) \times_k x) = 2(\delta_X - k(X) \times_k x)$ but since that operation factors through $\mathrm{CH}^0_0(\Gamma_{k(X)})$, which zero, we have $2(\delta_X - k(X) \times_k x) = 0$. □

For a smooth projective k-variety, the second punctual Hilbert scheme $X^{[2]}$ of X is obtained as the quotient of the blow-up $\widetilde{X \times_k X}$ of $X \times_k X$ along the diagonal by its natural involution. Let $\mu : X \times_k X \dashrightarrow X^{[2]}$ be the natural rational map and $r : \widetilde{X \times_k X} \to X^{[2]}$ be the quotient morphism. We collect some results of [90] whose proofs are essentially the same. So we just mention, when needed, the change needed or the facts required in characteristic $p > 2$:

Lemma 2.1.2. *([90, Lemma 2.3]). Let X be a smooth projective variety of dimension n. Then there exists a codimension n cycle Z in $X^{[2]}$ such that $\mu^*Z = \Delta_X$ in $\mathrm{CH}^n(X \times_k X)$.*

Corollary 2.1.3. *([90, Corollary 2.4]). Any symmetric codimension n cycle on $X \times_k X$ is rationally equivalent to $\mu^*\Gamma$ for a codimension n cycle Γ on $X^{[2]}$.*

Lemma 2.1.4. *([90, Lemma 2.5]). Let X be smooth projective k-variety of dimension n. Suppose X admits a $\mathbb{Z}_2$-cohomological decomposition of the diagonal*

$$[\Delta_X - x \times_k X] = [Z] \text{ in } H^{2n}(X \times_k X, \mathbb{Z}_2(n))$$

where Z is a cycle supported on $D \times_k X$ for some proper closed subset D of X and x k-rational point of X. Then X admits a $\mathbb{Z}_2$-cohomological decomposition of the diagonal

$$[\Delta_X - x \times_k X - X \times_k x] = [W] \text{ in } H^{2n}(X \times_k X, \mathbb{Z}_2(n)),$$

where W is a cycle supported on $D \times_k X$ and W is invariant under the natural involution of $X \times_k X$.

The following result is proved in [90, Proposition 2.6] over $\mathbb{C}$.

Proposition 2.1.3. *Let X be a smooth odd degree complete intersection of odd dimension n. If X admits a $\mathbb{Z}_2$-cohomological decomposition of the diagonal, there exists a cycle $\Gamma \in \mathrm{CH}^n(X^{[2]})$ with the following properties:*

1. *$\mu^*\Gamma = \Delta_X - x \times_k X - X \times_k x - W$ in $\mathrm{CH}^n(X \times_k X)$, with W supported on $D \times_k X$, for some closed proper subset $D \subset X$.*
2. *$[\Gamma] = 0$ in $H^{2n}(X^{[2]}, \mathbb{Z}_2(n))$.*

Sketch of proof of 2.1.3. The proof uses an analysis of the cohomology group $H^{2n}(X^{[2]}, \mathbb{Z}_2(n))$ and more precisely of the morphism $j_{E_X*} : H^{2n-2}(E_X, \mathbb{Z}_2) \to H^{2n}(X^{[2]}, \mathbb{Z}_2(n))$, where E_X is the exceptional divisor of $X^{[2]}$. This analysis is delicate for even dimensional odd degree complete intersections, but for odd dimension and odd degree complete intersections, the restriction map from the even degree cohomology of projective space to the even degree cohomology of X is surjective, so the result follows from the analysis of the cohomology of $(\mathbb{P}^N)^{[2]}$ which is Chow theoretic and works in any characteristic.

The only additional fact to check is that the cohomology of $X^{[2]}$ has no 2-torsion when X is an odd degree complete intersection in projective space. To see this, choose a smooth projective lifting of X to characteristic 0 over a discrete valuation ring $\mathfrak{X} \to Spec(R)$. As a zero dimensional length two subscheme is local complete intersection, and has trivial degree 1 coherent cohomology, $\mathrm{Hilb}_2(\mathfrak{X}/Spec(R)) \to Spec(R)$ is smooth and projective by [47, Proposition 2.15.4, Chap. I] (and smoothness of the fibers $\mathfrak{X}_{\bar\eta}^{[2]}$ and $X^{[2]}$). So, $H^r(\mathfrak{X}_{\bar\eta}^{[2]}, \mathbb{Z}_2) \simeq H^r(X^{[2]}, \mathbb{Z}_2)$ $\forall r \geq 0$ by the smooth proper base change. Now by the comparison theorem, $H^r(\mathfrak{X}_{\bar\eta}^{[2]}, \mathbb{Z}_2) \simeq H^r_B(\mathfrak{X}_{\bar\eta}^{[2]}, \mathbb{Z}_2)$ and by [78] these last groups have no 2-torsion. The rest of the proof works just like in [90]. □

For X a smooth cubic hypersurface in $\mathbb{P}^{n+1}$, we recall another description of $X^{[2]}$ used in [90]. Let $F(X)$ be the variety of lines of X and $P = \{([l], x),\ x \in l,\ l \subset X\}$ be the universal $\mathbb{P}^1$-bundle over $F(X)$ with projections $p : P \to F(X)$ and $q : P \to X$, and let $P_2 \to F(X)$ be the $\mathbb{P}^2$-bundle defined as the symmetric product of P over $F(X)$. There is a natural embedding $P_2 \overset{i_{P_2}}{\hookrightarrow} X^{[2]}$ which maps each fiber of $P_2 \to F(X)$, that is the second symmetric product of a line in X, isomorphically onto the set of subschemes of length 2 of X contained in this line. Let $p_X : P_X \to X$ be the projective bundle with fiber over $x \in X$ the set of lines in $\mathbb{P}^{n+1}$ passing through x. Note that P is naturally contained in P_X.

Proposition 2.1.4. *([90, Proposition 2.9]). In the above situation, we have the following properties:*

(i) The natural map $\Phi : X^{[2]} \dashrightarrow P_X$ which to a unordered pair of points $x,\ y \in X$ not contained in a common line of X associates the pair $([l_{x,y}], z)$, where $l_{x,y}$ is the line in $\mathbb{P}^{n+1}$ generated by x and y, and $z \in X$ is the residual point of the intersection $l_{x,y} \cap X$, is desingularized by the blow-up of $X^{[2]}$ along P_2.

(ii) The induced morphism $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ identifies $\widetilde{X^{[2]}}$ with the blow-up $\widetilde{P_X}$ of P_X along P.

(iii) The exceptional divisors of the two maps $\widetilde{X^{[2]}} \to X^{[2]}$ and $\widetilde{P_X} \to P_X$ are identified by the isomorphism $\widetilde{\Phi}' : \widetilde{X^{[2]}} \cong \widetilde{P_X}$ of (ii).

Proof of Theorem 2.0.1. Let $X \subset \mathbb{P}^4_k$ be a smooth cubic threefold. By Proposition 2.1.2, we see that there is a nonempty open subset $U_0 \subset X$, such that $(\Delta_X - X \times_k x)_{|U_0 \times_k X}$ is a 2-torsion element of $\mathrm{B}^3(U_0 \times_k X)$. The subgroup of 2-torsion elements of $\mathrm{B}^3(U_0 \times_k X)$ is a $\mathbb{Z}/2\mathbb{Z}$-module and since $\mathbb{Z}/2\mathbb{Z}$ is a quotient of the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ in $2\mathbb{Z}$, a 2-torsion element of $\mathrm{B}^3(U_0 \times_k X)$ is 0 if and only if it is 0 in $\mathrm{B}^3(U_0 \times_k X) \otimes \mathbb{Z}_{(2)}$. Since $\mathbb{Z}_2$ is the completion of the local ring $\mathbb{Z}_{(2)}$ along its maximal ideal, $\mathbb{Z}_2$ is a faithfully flat $\mathbb{Z}_{(2)}$-module. Hence, a 2-torsion element in $\mathrm{B}^3(U_0 \times_k X)$ is 0 if and only if it is 0 in $\mathrm{B}^3(U_0 \times_k X) \otimes \mathbb{Z}_2$. So in order to prove that X admit a Chow-theoretic decomposition of the diagonal, we only need to check that it is 0 in $\mathrm{B}^3(U' \times X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$, for some open subset $U' \subset X$. Once we know that, we will have that $(\Delta_X - X \times_k x)_{|U' \times_k X}$ is 0 in $B^3(U' \times_k X)$ i.e. a decomposition of the diagonal

$$\Delta_X = X \times_k x + Z \text{ in } \mathrm{B}^3(X \times_k X)$$

where Z is supported on $D \times_k X$ for some proper closed subset $D \subsetneq X$. Applying Proposition 2.1.1, this will yield the Chow-theoretic decomposition of the diagonal. So we shall work with $\mathbb{Z}_2$ coefficients and, adapting arguments of [90], show that a cohomological decomposition of the diagonal with coefficients in $\mathbb{Z}_2$ implies that $(\Delta_X - X \times_k x)_{|U \times_k X} = 0$ in $\mathrm{B}^3(U \times X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ for some nonempty open subset U of X.

Assume X admits a cohomological decomposition of the diagonal with coefficients in $\mathbb{Z}_2$. The assumptions of Proposition 2.1.3 are satisfied by X, since the cohomology of a smooth cubic hypersurface with coefficients in $\mathbb{Z}_2$ has no torsion and is algebraic in even degree. Using the notation introduced previously, there exists, by Proposition 2.1.3, a cycle $\Gamma \in \mathrm{CH}^3(X^{[2]})$ such that

$$\mu^*\Gamma = \Delta_X - x \times_k X - X \times_k x - W \text{ in } \mathrm{CH}^3(X \times_k X), \tag{2.3}$$

with W supported on $D \times_k X$, $D \subsetneq X$, and $[\Gamma] = 0$ in $H^6(X^{[2]}, \mathbb{Z}_2(3))$.

By Proposition 2.1.4, the blow-up $\sigma : \widetilde{X^{[2]}} \to X^{[2]}$ of $X^{[2]}$ along P_2 identifies via $\widetilde{\Phi}$ with the blow-up $\widetilde{P_X}$ of P_X along P. Furthermore, the exceptional divisor $E \overset{i_E}{\hookrightarrow} \widetilde{X^{[2]}}$ of $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ is also the exceptional divisor of $\sigma : \widetilde{X^{[2]}} \to X^{[2]}$, hence maps via σ to $P_2 \subset X^{[2]}$. Since $\widetilde{\Phi}$ is a blow-up of a smooth subvariety, the Chow groups of $\widetilde{X^{[2]}}$ decomposes as $\mathrm{CH}^*(\widetilde{X^{[2]}}) = \widetilde{\Phi}^*\mathrm{CH}^*(P_X) \oplus i_{E*}\mathrm{CH}^*(E)$; we have a similar decomposition for the cohomology groups $H^*(\widetilde{X^{[2]}}, \mathbb{Z}_2) = \widetilde{\Phi}^* H^*(P_X, \mathbb{Z}_2) \oplus i_{E*}H^*(E, \mathbb{Z}_2)$ and these decompositions are compatible with the cycle map. By work of Shen [72, Theorem 1.1], the group of 1-cycles of X is generated by lines of X i.e. the action of correspondence P induces a surjective morphism $P_* : \mathrm{CH}_0(F(X)) \to \mathrm{CH}_1(X)$.

So let γ be a 1-cycle homologically trivial with coefficients in $\mathbb{Z}_2$ on X; we can write it $P_*(z)$ for a $z \in \mathrm{CH}_0(F(X)) \otimes \mathbb{Z}_2$. The degree of $\gamma \cdot H$, where $H = c_1(\mathcal{O}_X(1))$, is 0 in $\mathbb{Z}_2$ since it is algebraically trivial in $\mathrm{CH}_0(X) \otimes \mathbb{Z}_2 \overset{deg}{\simeq} \mathbb{Z}_2$. But, with the above notations $\gamma \cdot H = P_*(z) \cdot H = q_*p^*(z \cdot q^*H)$ and q^*H is the relative hyperplane divisor of the projective bundle $p : P \to F(X)$; so that the degree of z is also 0 in $\mathrm{CH}_0(F(X)) \otimes \mathbb{Z}_2$ i.e. z is algebraically trivial. So $\gamma = P_*(z)$ is algebraically trivial in

$\mathrm{CH}_1(X) \otimes \mathbb{Z}_2$. So algebraic and homological equivalences with coefficients in $\mathbb{Z}_2$ coincide on $\mathrm{CH}_1(X)$. Since these relations coincide also on $\mathrm{CH}_2(X) = Pic(X)$ and $\mathrm{CH}_0(X)$, they coincide on the Chow ring of X.

Then, since P_X is a projective bundle over X, the two equivalence relations coincide also on P_X. On the other hand, $F(X)$ being a surface, algebraic and homological equivalences coincide on $F(X)$ hence also on the projective bundle P over $F(X)$. Since E is also a projective bundle over P, the two equivalence relations coincide on the blow-up $\widetilde{P}_X$ of P_X along P, which is isomorphic to $\widetilde{X}^{[2]}$.

Now $\sigma^*[\Gamma]$ is 0 in $H^6(\widetilde{X}^{[2]}, \mathbb{Z}_2(3))$ since $[\Gamma] = 0$ so that $\sigma^*\Gamma = 0$ in $B^3(\widetilde{X}^{[2]}) \otimes \mathbb{Z}_2$. We conclude that $\Gamma = 0$ in $B^3(X^{[2]}) \otimes \mathbb{Z}_2$ since $\sigma_*\sigma^* = id_{\mathrm{CH}^*(X^{[2]})}$. So (2.3) yields

$$\Delta_X = x \times_k X + X \times_k x - W' \ in\ \mathrm{B}^3(X \times_k X) \otimes \mathbb{Z}_2$$

where W' is supported on $D' \times_k X$ for some $D' \subsetneq X$. So $(\Delta_X - X \times_k x)_{|U \times_k X} = 0$ in $\mathrm{B}^3(U \times_k X) \otimes \mathbb{Z}_2$ (with $U = X \backslash D'$) as we wanted.

□

2.2 Cohomological decomposition of the diagonal

We prove in this section Theorem 2.0.3, again adapting arguments of Voisin presented in [90]. We begin by the following theorem which was proved in [90, Theorem 3.1] over $\mathbb{C}$.

Theorem 2.2.1. *Let X be a smooth projective k-variety of dimension $n > 0$ and $\ell \neq p$ a prime number.*

1. *Assume $H^*(X, \mathbb{Z}_\ell)$ has no torsion, $H^{2i}(X, \mathbb{Z}_\ell(i))$ is algebraic for $2i \neq n$, $H^{2i+1}(X, \mathbb{Z}_\ell) = 0$ for $2i + 1 \neq n$ and that X satisfies the following condition:*
   (*) There exist finitely many smooth projective varieties Z_i of dimension $n - 2$, correspondences $\Gamma_i \in \mathrm{CH}^{n-1}(Z_i \times_k X)$, and $n_i \in \mathbb{Z}_2$, such that for any $\alpha, \beta \in H^n(X, \mathbb{Z}_\ell)$,
   $$\langle \alpha, \beta \rangle_X = \sum_i n_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Z_i}. \tag{2.4}$$
   Then X admits a cohomological decomposition of the diagonal with coefficients in $\mathbb{Z}_\ell$.

2. *If $n = 3$, $char(k) \neq \ell$ and X admits a cohomogical decomposition of the diagonal with coefficients in $\mathbb{Z}_\ell$, then (*) is satisfied.*

Sketch of proof. The adaptation of the proof given in [90], to positive characteristic is straightforward; the only fact to use for the second point is the existence of (embedded) resolution of singularities in dimension 3 for algebraically closed fields of positive characteristic (see [24] and [25]). □

2.2.1 The intermediate Jacobian of a cubic threefold and Abel-morphisms

Let us denote $\mathrm{CH}^2_{alg}(Y)$ the group of codimension 2 cycles algebraically equivalent to zero on a k-variety Y. Given an abelian variety Ab over k, following [64, VIa], we shall call a (group) homomorphism $f : \mathrm{CH}^2_{alg}(Y) \to Ab$ a regular morphism if for any smooth quasi-projective k-variety T and $Z \in \mathrm{CH}^2(T \times_k Y)$ such that for any $t \in T(k)$, $Z_t \in \mathrm{CH}^2_{alg}(Y)$, the composition $T \to \mathrm{CH}^2_{alg}(Y) \xrightarrow{f} Ab$ is a morphism of algebraic varieties. We say that $\mathrm{CH}^2_{alg}(Y)$ admits an algebraic representative if there is an abelain variety $Ab(Y)$ over k and a regular morphism $\phi : \mathrm{CH}^2_{alg}(Y) \to Ab(Y)$ which is universal in the sense that any regular morphism factor as a composition of ϕ followed by a momorphism of algebraic varieties. In that case we call the morphism $\phi_Z : T \to \mathrm{CH}^2_{alg}(Y) \xrightarrow{f_Y} Ab(Y)$ the Abel-Jacobi morphism induced by Z. By [63, Theorem 1.9], $\mathrm{CH}^2_{alg}(Y)$ admits an algebraic representative when Y is a smooth projective variety over an algebraically closed field.

Now, let $X \subset \mathbb{P}^4_k$ be a smooth cubic hypersurface. The linear projection $\mathbb{P}^4_k \dashrightarrow \mathbb{P}^2_k$ centered along a general line $l \subset X$ (see for example [61, Proposition 1.25]) gives a rational map $X \dashrightarrow \mathbb{P}^2_k$ which after blowing up l, yields an ordinary conic bundle $\widetilde{X} \to \mathbb{P}^2_k$. By results of Beauville ([7, Theorem 3.1 and Proposition 3.3]), the Prym variety A associated to the conic bundle is the algebraic representative of $\mathrm{CH}^2_{alg}(\widetilde{X}) \simeq \mathrm{CH}^2_{alg}(X)$ and $\mathrm{CH}^2_{alg}(X) = A(k)$. The principally polarized abelian variety A obtained by this construction is independent of the choice of a general $[l]$. So we call $J(X) := A$ the intermediate Jacobian of X; it is a 5-dimensional abelian variety endowed with the principal polarization θ of A. By results of Beauville ([7, Remark 2.7]), we know that there is an isomorphism of $\mathbb{Z}_2$-modules with their intersection forms

$$t : (H^1(J(X), \mathbb{Z}_2), \theta) \to (H^3(\widetilde{X}, \mathbb{Z}_2), \langle , \rangle_{\widetilde{X}}) \simeq (H^3(X, \mathbb{Z}_2), \langle , \rangle_X) \tag{2.5}$$

It is known (see Murre [62, section VI], see also [15] over $\mathbb{C}$) that the Abel-Jacobi morphism associated to the universal $\mathbb{P}^1$-bundle $P \subset F(X) \times X$ over the variety of lines $F(X)$ of X, induces an isomorphism of abelian varieties $\phi_P : Alb(F(X)) \simeq J(X)$ where $Alb(F(X))$ is the Albanese variety of $F(X)$, which is defined in this setting as the dual of the Picard variety $\mathrm{Pic}^0(F(X))$.

Since $F(X)$ is the zero locus of a regular section of the vector bundle $\mathcal{E} = Sym_3(E)$ on the grassmannian $Gr(2,5)$, where E is the rank 2 quotient bundle on $Gr(2,5)$, we have the following exact sequence:

$$0 \to \wedge^4\mathcal{E}^* \to \wedge^3\mathcal{E}^* \to \wedge^2\mathcal{E}^* \to \mathcal{E}^* \to \mathcal{O}_{G(2,5)} \to \mathcal{O}_{F(X)} \to 0$$

given by the Kozsul resolution of the sheaf of ideals of $F(X)$ in $Gr(2,5)$ (the exactness follows from the fact that $F(X)$ has codimension $4 = rank(\mathcal{E})$) so, we have a quasi-isomorphism of complexes $\wedge^{\cdot}\mathcal{E}^* \simeq \mathcal{O}_{F(X)}[4]$.

Now, we have a spectral sequence $E_1^{p,q} = H^q(Gr(2,5), \wedge^{4-p}\mathcal{E}^*) \Rightarrow H^{p+q-4}(F(X), \mathcal{O}_{F(X)})$, which, according to [2, Theorem 5.1], degenerates at E_1, so that $H^1(F(X), \mathcal{O}_{F(X)}) \simeq H^3(Gr(2,5), \wedge^2\mathcal{E}^*)$. According to [2, Proposition 5.11 and Lemma 5.7], we have an isomorphism $H^3(Gr(2,5), \wedge^2\mathcal{E}^*) \simeq H^0(\mathbb{P}^4_k, T_{\mathbb{P}^4_k}(-1))$. By the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^4_k}(-1) \to H^0(\mathbb{P}^4_k, \mathcal{O}_{\mathbb{P}^4_k}(1))^\vee \otimes \mathcal{O}_{\mathbb{P}^4_k} \to T_{\mathbb{P}^4_k}(-1) \to 0,$$

we have $H^1(F(X), \mathcal{O}_{F(X)}) \simeq H^0(\mathbb{P}^4_k, \mathcal{O}_{\mathbb{P}^4_k}(1))^\vee$. Tensoring the normal bundle exact sequence of the inclusion $X \subset \mathbb{P}^4_k$ with $\Omega^3_{X/k} \simeq \mathcal{O}_{\mathbb{P}^4_k}(-2)_{|X}$ and looking at the associated long exact sequence, we have $H^0(X, \mathcal{O}_X(1)) \simeq H^1(X, \Omega^2_{X/k})$. Since $H^0(X, \mathcal{O}_X(1)) \simeq H^0(\mathbb{P}^4_k, \mathcal{O}_{\mathbb{P}^4_k}(1))$, we have $H^1(F(X), \mathcal{O}_{F(X)}) \simeq H^1(X, \Omega^2_{X/k})^\vee$. Since by [62, Theorem 8], the Picard and Albanese varieties of $F(X)$ are isomorphic, we have $T_{J(X),0} \simeq H^2(X, \Omega_{X/k})$.

In order to apply Voisin's method in [89] to analyse the existence of a decomposition of the diagonal for a cubic threefold, we need to make sure there exists, as it is the case in characteristic 0 ([52] and [43]), a parametrization of $J(X)$ with separably rationally connected generic fiber, namely condition (*) of the introduction. This parametrization will be constructed in Section 4. So let us proceed to the proof of the criterion (Theorem 2.0.3 of the introduction) for the existence of a cohomological decomposition of the diagonal assuming that (*) holds.

2.2.2 Decomposition of the diagonal for a smooth cubic threefold

In this section, we prove the following theorem which was first proved over $\mathbb{C}$ in [89, Theorem 4.9], [90, Theorem 4.1]. Item (ii) is specific to the finite field situation and is Theorem 2.0.4 of the introduction.

Theorem 2.2.2. *(i) Let $X \subset \mathbb{P}^4_k$ be a smooth cubic hypersurface ($k = \bar{k}$ and $char(k) > 2$). Then X admits a cohomological (hence Chow-theoretic by Theorem 2.0.1) decomposition of the diagonal if and only if there is a $\gamma \in \mathrm{CH}_1(J(X)) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ such that $\theta^4/4! = [\gamma]$ in $H^8(J(X), \mathbb{Z}_2(4))$.*

(ii) If $k = \overline{\mathbb{F}_p}$, $p > 2$ and the Tate conjecture is true for divisors on surfaces over finite fields, then every smooth cubic hypersurface of $\mathbb{P}^4_k$ admits a Chow-theoretic decomposition of the diagonal.

Proof. Assume $\theta^4/4! \in \mathrm{CH}_1(J(X)) \otimes \mathbb{Z}_2$. We will prove that X admits a cohomological (with coefficient in $\mathbb{Z}_2$) decompositon of the diagonal. We know that $H^*(X, \mathbb{Z}_2)$ has no torsion so applying Künneth decomposition, we can write $[\Delta_X] = \sum_{i=0}^{6} \delta_{i,6-i}$ where $\delta_{i,6-i} \in H^i(X, \mathbb{Z}_2) \otimes H^{6-i}(X, \mathbb{Z}_2)$ are the components of $[\Delta_X] \in H^6(X \times_k X, \mathbb{Z}_2)$. Since $H^1(X, \mathbb{Z}_2) = 0 = H^5(X, \mathbb{Z}_2)$ we have $\delta_{1,5} = 0 = \delta_{5,1}$. We know that $\delta_{6,0}$ is the class $X \times_k x$ for any point $x \in X(k)$ and $\delta_{0,6}$ is the class of the subvariety $x \times_k X$ which obviously does not dominate X by the first projection. Since $H^2(X, \mathbb{Z}_2)$ and $H^4(X, \mathbb{Z}_2)$ are algebraic, $\delta_{2,4}$ and $\delta_{4,2}$ are linear combinations of classes of algebraic subvarieties of $X \times_k X$ that do not dominate X by the first projection. The existence of a cohomological decomposition of the diagonal with coefficients in $\mathbb{Z}_2$ is thus equivalent to the existence a cycle $Z \subset X \times_k X$ such that the support of Z is contained in $D \times_k X$, with $D \subset X$ a proper subscheme, and $Z^* : H^3(X, \mathbb{Z}_2) \to H^3(X, \mathbb{Z}_2)$ is the identity map since in this case $\delta_{3,3} = [Z]$. We proceed as in [89, Theorem 4.9] to construct such a Z.

Let $C = \sum_i m_i C_i$ be a 1-cycle of $J(X)$ of class $\theta^4/4!$ in $H^8(J(X), \mathbb{Z}_2)$, where $C_i \subset J(X)$ are curves and $m_i \in \mathbb{Z}_2$. According to condition (*), which is Theorem 2.3.2, there are a smooth 10-dimensional quasi-projective k-variety B and a cycle $\mathcal{Z} \subset B \times_k X$, flat over B, such that the induced Abel-Jacobi morphism $\phi_{\mathcal{Z}} : B \to J(X)$ is dominant with general fiber $\mathbb{P}^5_k$. Let B' be the closure of the graph of $\phi_{\mathcal{Z}}$ in $\overline{B} \times J(X)$, where $\overline{B}$ is a compactification of B; B' is birational to the quasi-projective variety B and the projection yields a proper, surjective morphism $\phi : B' \to J(X)$ with general fiber $\mathbb{P}^5_k$. Let $W \subset J(X)$ be an open subscheme contained in the image of $\phi_{\mathcal{Z}}$ such that $\forall x \in W(k)$, $\phi^{-1}(x) \simeq \mathbb{P}^5_k$. Using Chow's moving lemma, we can assume that the generic point of each C_i is in W. Let $n_i : \widetilde{C_i} \to C_i$ be the normalization of C_i. Let $N_i : B_i \to B' \times_{C_i} \widetilde{C_i}$ be the normalization morphism. A component of B_i is proper and flat (because integral over a smooth curve) over the smooth curve $\widetilde{C_i}$ with general fiber $\mathbb{P}^5_k$ so by Tsen's Theorem ([79]), $B_i \to \widetilde{C_i}$ admits a section σ_i and $\phi \circ n'_i \circ N_i \circ \sigma_i = n_i$ (where $n'_i : B' \times_{C_i} \widetilde{C_i} \to B'$ is the projection). Let $Z_i \subset \widetilde{C_i} \times_k X$ be the cycle $(pr_B^{B \times_k J(X)} \circ n'_i \circ N_i \circ \sigma_i, id_X)^* \mathcal{Z}$. We have the easy equality:

Lemma 2.2.3. *The homomorphisms $Z_{i,*}$ and $t \circ \phi_{Z_{i,*}} : H^1(\widetilde{C_i}, \mathbb{Z}_2) \to H^3(X, \mathbb{Z}_2)$ coincide (where $t : (H^1(J(X), \mathbb{Z}_2), \theta) \to (H^3(X, \mathbb{Z}_2), \langle, \rangle_X)$ is the isomorphism).*

Let $Z \subset X \times_k X$ be the cycle $\sum_i m_i Z_i \circ {}^t Z_i$. We have

$$(Z_i \circ {}^t Z_i)^* = Z_{i,*} \circ Z_i^* = (t^{-1})^* \circ \phi_{Z_{i,*}} \circ \phi_{Z_i}^* \circ t^{-1}$$

where $(t^{-1})^*$ is the Poincaré dual of t^{-1}. But ϕ_{Z_i} is just n_i so $\phi_{Z_{i,*}} \circ \phi_{Z_i}^*$ is $n_{i,*} \circ n_i^*$ which is just $[C_i] \cup : H^1(J(X), \mathbb{Z}_2) \to H^9(J(X), \mathbb{Z}_2)$. Hence Z^* is the composite map

$$H^3(X, \mathbb{Z}_2) \overset{t^{-1}}{\simeq} H^1(J(X), \mathbb{Z}_2) \xrightarrow{(\theta^4/4!) \cup = \sum_i [C_i] \cup} H^9(J(X), \mathbb{Z}_2) \overset{(t^{-1})^*}{\simeq} H^3(X, \mathbb{Z}_2).$$

So Z^* is the identity on $H^3(X, \mathbb{Z}_2)$. On the other hand, Z does not dominate X by the first projection since it is supported on $\bigcup_i \Sigma_i \times \Sigma_i$, where Σ_i is the image in X of $\mathrm{Supp}(Z_i)$.

We prove now the second direction. Assume X admits a Chow-theoretic decomposition of the diagonal. Then X admits a cohomological decomposition of the diagonal with coefficients in $\mathbb{Z}_2$ so by Theorem 2.2.1 we have finitely many smooth projective curves Z_i and for each curve, a correspondence $\Gamma_i \in \mathrm{CH}^2(Z_i \times_k X)$ and a $n_i \in \mathbb{Z}_2$ satisfying (2.4). The Abel-Jacobi map Φ_X of X induces (after choosing a reference point in Z_i) a morphism

$$\gamma_i = \Phi_X \circ \Gamma_{i*} : Z_i \to J(X)$$

with image $Z'_i := \gamma_{i*} Z_i \in \mathrm{CH}_1(J(X))$. Now we have $\wedge^2 H^1(J(X), \mathbb{Z}_2) \simeq H^2(J(X), \mathbb{Z}_2) = H^8(J(X), \mathbb{Z}_2)^*$ and an isomorphism $(H^1(J(X), \mathbb{Z}_2), \theta) \overset{t}{\simeq} (H^3(X, \mathbb{Z}_2), \langle , \rangle_X)$ with their intersection form (2.5). For all $\alpha \in H^1(J(X), \mathbb{Z}_2)$, $\gamma_i^* \alpha = \Gamma_i^* t(\alpha)$ so for all $\alpha,\ \beta \in H^1(J(X), \mathbb{Z}_2)$,

$$\begin{aligned}\langle \textstyle\sum_i n_i [Z'_i], \alpha \cup \beta \rangle_{J(X)} &= \textstyle\sum_i n_i \langle [Z'_i] \cup \alpha, \beta \rangle_{J(X)} \\ &= \textstyle\sum_i n_i \langle \gamma_i^* \alpha, \gamma_i^* \beta \rangle_{Z_i} \\ &= \textstyle\sum_i n_i \langle \Gamma_i^* t(\alpha), \Gamma_i^* t(\beta) \rangle_{Z_i} \\ &= \langle t(\alpha), t(\beta) \rangle_X \text{ (by 2.4)} \\ &= \theta(\alpha, \beta) \\ &= \langle \tfrac{\theta^4}{4!}, \alpha \cup \beta \rangle_{J(X)}\end{aligned}$$

hence $\frac{\theta^4}{4!} = \sum_i n_i [Z'_i]$.

(ii) If the Tate conjecture is true for divisors on surfaces defined over finite fields, then the theorem of Schoen ([70]) says that the cycle map $\mathrm{CH}_1(J(X)) \otimes \mathbb{Z}_2 \to \bigcup_U H^8(J(X), \mathbb{Z}_2(4))^U$, where U runs through all open subgroups of $Gal(k/k_{def})$ (k_{def} being a finite field over which $J(X)$ is defined), is surjective. Since θ^4 is algebraic, $\theta^4/4! \in \bigcup_U H^8(J(X), \mathbb{Z}_2(4))^U$ and we conclude by point (i) of the theorem. □

2.3 Parametrization of the intermediate Jacobian of a smooth cubic threefold

The goal of this section is to prove that condition (*) still hold in the positive characteristic setting. Over $\mathbb{C}$, such a parametrization was achieved using the space of smooth normal elliptic quintics which we do not know to exist a priori in our setting. So we will construct some stable normal elliptic quintics using the lines on the cubic threefold.

2.3.1 Some facts on the Fano variety of lines.

Let X be a smooth cubic hypersurface $\mathbb{P}^4_k$. Since $\mathbb{P}^3_k$ is separably rationally connected and there is a dominant degree 2 rational map $\mathbb{P}^3_k \dashrightarrow X$ and $2 \neq char(k)$, X is separably rationally connected.

The Fano variety of lines $F(X) = \{[l] \in Gr(2,5),\ l \subset X\}$ of X is a smooth projective surface. Denote by $P \overset{p}{\to} F(X)$ the universal $\mathbb{P}^1$-bundle and by $q : P \to X$ the projection on X.

We collect some results from Murre ([61]) on the geometry of $F(X)$. Let $\mathcal{F}_0 \subset F(X)$ be the subset defined by

$$\mathcal{F}_0 = \{[l] \in F(X),\ \exists K \subset \mathbb{P}^4_k \text{ a 2−plane s.t. } K \cap X = 2l + l' \text{ as divisors in } K\}.$$

Then $\mathcal{F}_0$ is a non-singular curve ([61, Corollary 1.9]), the lines $[l] \in \mathcal{F}_0$ are said to be of the second type and those not in $\mathcal{F}_0$ are said to be of the first type. The subscheme

$$\mathcal{F}'_0 = \{[l] \in F(X),\ \exists K \subset \mathbb{P}^4_k \text{ a 2−plane and } l' \in F(X) \text{ s.t. } K \cap X = l + 2l'\}$$

has dimension at most 1 ([61, Lemma 1.11]). For $[l] \in F(X)$, let us denote by

$$\mathcal{H}(l) = \overline{\{[l'] \in F(X),\ [l] \neq [l'],\ l \cap l' \neq \emptyset\}}.$$

Then $\mathcal{H}(l)$ is a curve in $F(X)$. We have the following properties:

Proposition 2.3.1. *([61, (1.17), (1.18), (1.24), (1.25)]) (i) If $[l]$ is a line of the first type on X and $x \in l$ then there are only finitely many (in fact at most 6) lines on X through x. Moreover there is no 2-plane tangent to X in all points of l.*

(ii) There is a nonempty open subscheme $\mathcal{U}' \subset F(X)$ contained in $F(X)\backslash(\mathcal{F}_0 \cup \mathcal{F}_0')$ such that any $[l] \in \mathcal{U}'$ (l is of the first type) is contained in a smooth hyperplane section of X and $\mathcal{H}(l)$ is a smooth irreducible curve of genus 11.

By the jacobian criterion, given a hyperplane $H \subset \mathbb{P}^4_k$, $X \cap H$ is not a smooth cubic surface if and only if there is an $x \in X$ such that H is tangent to X at x. Looking at the Gauss map $\mathcal{D} : X \to (\mathbb{P}^4_k)^*$, ([15]) we see that the variety $\mathcal{D}(X) \subset (\mathbb{P}^4_k)^*$ parametrizing those hyperplanes tangent to X at some point $x \in X$ is a hypersurface in $(\mathbb{P}^4_k)^*$. So the general (parametrized by the open subscheme $(\mathbb{P}^4_k)^*\backslash\mathcal{D}(X)$) hyperplane section of X is smooth. Moreover, if $[l] \in \mathcal{U}'$, we know that $K^*_l = \{[H] \in (\mathbb{P}^4_k)^*,\ l \subset H\}$, which is a 2-plane in $(\mathbb{P}^4_k)^*$, is not contained in $\mathcal{D}(X)$, hence $\mathcal{D}(X) \cap K^*_l$ is of dimension at most 1 and $K^*_l\backslash(\mathcal{D}(X) \cap K^*_l)$ is a 2-dimensional open subscheme. So we see that in fact, for $[l] \in \mathcal{U}'$, the general hyperplane containing l gives a smooth hyperplane section of X. We have also the following properties:

Proposition 2.3.2. *(i) For $[l] \in \mathcal{U}'$, $Im(\mathcal{H}(l))$ $(= \cup_{[l'] \in \mathcal{H}(l)} l' = q(p^{-1}(\mathcal{H}(l))))$ is not contained in a fixed 2-plane. So if $[l'] \in F(X)$ is a line distinct from $[l]$ such that $l \cap l' \neq \emptyset$, $\mathcal{H}(l)$ and $\mathcal{H}(l')$ have no common component.*

(ii) Let $h_{l_0} : F(X)\backslash(\mathcal{H}(l_0) \cup \{[l_0]\}) \to K^*_{l_0}$ be the morphism defined by $[l] \mapsto [span(l, l_0)]$ for $[l_0] \in \mathcal{U}'$. Then h_{l_0} is dominant and there is an open subcheme $V_{l_0} \subset \mathcal{U}'\backslash(\mathcal{H}(l_0) \cup \{[l_0]\})$ such that for $[l] \in V_{l_0}$, $h_{l_0}([l])$ gives a smooth hyperplane section.

(iii) For $[l] \in \mathcal{U}'$, there is an open subscheme $\mathcal{U}'_{[l]} \subset \mathcal{U}'$ such that for all $[l'] \in \mathcal{U}'_{[l]}$, $\mathcal{H}(l') \cap \mathcal{H}(l)$ is finite. So there is an open subscheme $\mathcal{U} \subset \mathcal{U}'$ such that $\forall [l] \in \mathcal{U}$, $\mathcal{H}(l) \cap \mathcal{U}'$ is a non empty open subscheme of $\mathcal{H}(l)$.

Proof. (i) Suppose there exists $K \subset \mathbb{P}^4_k$ a 2-plane such that $q(p^{-1}(\mathcal{H}(l))) \subset K$. $p^{-1}(\mathcal{H}(l))$ is a smooth irreducible ruled surface over a curve of genus 11. Since X is smooth, it cannot contain a 2-plane, so $q(p^{-1}(\mathcal{H}(l))$ is contracted into a curve or a set of points in $K \cap X$. Since the ruled surface is irreducible, $q(p^{-1}(\mathcal{H}(l)))$ is an irreducible closed (q proper) subscheme of K and we have $l \subset q(p^{-1}(\mathcal{H}(l)))$ so $l = q(p^{-1}(\mathcal{H}(l)))$. But, by 2.3.1 (i), since l is of the first type, the fiber of q over a point of l is 0-dimensional. So such K does not exist.

Moreover, suppose $[l'] \in F(X)\backslash\{[l]\}$ is such that $l' \cap l \neq \emptyset$ and $\mathcal{H}(l)$ (which is irreducible) is one of the components of $\mathcal{H}(l')$, then all (except maybe the other 4 ones passing through $l \cap l'$) lines that meet l are contained in the 2-plane $span(l, l')$, which is impossible.

(ii) Since $[l_0] \in \mathcal{U}'$, the general member of $K^*_{l_0}$ gives a smooth hyperplane section which contains some lines that do not meet l_0 (i.e in $F(X)\backslash(\mathcal{H}(l_0) \cup \{[l_0]\})$), h_{l_0} is dominant and the general fiber is 0-dimensional of cardinal < 27. So $h_{l_0\ |\mathcal{U}'\backslash\mathcal{H}(l_0)}$ is still dominant. Let $\mathcal{C} = K^*_{l_0} \cap \mathcal{D}(X)$ be the closed subscheme of dimension ≤ 1 parametrizing singular hyperplane sections containing l_0. Since h_{l_0} is dominant, the closed subscheme $h^{-1}(\mathcal{C})$ has dimension ≤ 1. So the property is proved with $V_{l_0} = \mathcal{U}'\backslash(\mathcal{H}(l_0) \cup h^{-1}(\mathcal{C}))$.

(iii) For $[l_0] \in \mathcal{U}'$, according to the previous point, there is an open subscheme $V_{l_0} \subset \mathcal{U}'\backslash(\mathcal{H}(l_0) \cup \{[l_0]\})$ such that $\forall [l] \in V_{l_0}$, $h_{l_0}([l])$ is a smooth hyperplane section, in particular, $\forall [l] \in V_{l_0}$, $\mathcal{H}(l_0) \neq$

$\mathcal{H}(l)$ (otherwise, all these lines would be contained in the smooth cubic surface and k being algebraically closed, $\mathcal{H}(l_0)(k)$ is infinite). For the last statement, should they exist, take finitely many $[l_j] \in \mathcal{U}'$ such that the $\mathcal{H}(l_j)$ are the irreducible components of the divisors $(\mathcal{F}_0, \mathcal{F}'_0 ...)$ removed from $F(X)$ to obtain $\mathcal{U}'$ and set $\mathcal{U} = \cap_j V_{l_j}$ (and $\mathcal{U}' = \mathcal{U}$ if there is no such l_j). □

2.3.2 Space of normal elliptic quintics

We will construct in this section, singular normal elliptic quintic curves, namely cycles of rational curves (generically) with four components, 3 lines and a conic. We will use for this the properties of the Fano surface from 2.3.1 and 2.3.2.

Take $[l_0] \in \mathcal{U}'$. Let $\mathcal{C} \subset K^*_{l_0}$ the closed subscheme of dimension ≤ 1 parametrizing the singular hyperplane sections containing l_0. Its preimage $h_{l_0}^{-1}(\mathcal{C})$ has dimension at most 1 since there is at least one (hence a 2-dimensional open set of) smooth hyperplane section containing l_0. Let us denote by $(C_i)_{1 \leq i \leq m}$ the irreducible components of $h_{l_0}^{-1}(\mathcal{C})$ and should they exist such, let us denote by $([l^i])_{1 \leq i \leq m} \in \mathcal{U}^m$ some lines such that $\mathcal{H}(l^i) \cap C_i$ has dimension 1. We can choose $[l_1]$ in the (2-dimensional) open subscheme $\cap_{i=1}^m \mathcal{U}'_{[l^i]} \cap V_{l_0} \cap \mathcal{U}$. Then $\mathcal{H}(l_1) \cap h_{l_0}^{-1}(\mathcal{C}) = \cup_{i=1}^m \mathcal{H}(l_1) \cap C_i \subset \cup_{i=1}^m (\mathcal{H}(l_1) \cap \mathcal{H}(l^i))$, the last set is finite as finite union of finite sets ($[l_1] \in \cap_{i=1}^m \mathcal{U}'_{[l^i]}$). Since the hyperplane section $S = H_1 \cap X$ is smooth (where $H_1 = span(l_0, l_1)$) and the lines meeting l_0 and l_1 are in S, $\mathcal{H}(l_1) \cap \mathcal{H}(l_0)$ is finite.

Since $(\mathcal{H}(l_1) \cap \mathcal{U}') \backslash (h_{l_0}^{-1}(\mathcal{C}) \cup \mathcal{H}(l_0))$ is an nonempty open subscheme of $\mathcal{H}(l_1)$ and S contains 27 lines, we can choose $[l_2]$ in this open subset such that $[l_2]$ is not contained in S (in particular not contained in H_1) and $H_2 \cap X$ is a smooth cubic surface, where $H_2 = span(l_0, l_2)$ and $H_2 \neq H_1$ (l_2 is not in S). We have $\mathcal{H}(l_2) \neq \mathcal{H}(l_0)$ (because $[l_1] \in \mathcal{H}(l_2)$ and is not in $\mathcal{H}(l_0)$) and $\mathcal{H}(l_2) \neq \mathcal{H}(l_1)$ by Proposition 2.3.2 (i) so that $\mathcal{H}(l_2) \backslash (\mathcal{H}(l_0) \cup \mathcal{H}(l_1))$ is an nonempty open subscheme of $\mathcal{H}(l_2)$. Since $H_2 \cap X$ contains only finitely many lines, we can take $[l_3] \in \mathcal{H}(l_2) \backslash (\mathcal{H}(l_0) \cup \mathcal{H}(l_1))$ not in that surface and not intersecting l_2 at $l_1 \cap l_2$ (there are at most 4 other lines passing through $l_1 \cap l_2$ by Proposition 2.3.1 (i)).

Letting $H_3 = span(l_0, l_3)$, we have $H_1 \cap H_2 \not\subset H_3$ otherwise the point $l_2 \cap l_1$ would be in $H_1 \cap H_2 \subset H_3$ and the same with the point $l_2 \cap l_3$ so that ($l_2 \cap l_1 \neq l_2 \cap l_3$) we would have $l_2 \subset H_3$, i.e. $H_3 = H_2$ but $l_3 \not\subset H_2$. So $H_1 \cap H_2 \cap H_3$ is the line l_0.

Lemma 2.3.1. *A hyperplane section of a smooth hypersurface of degree ≥ 2 in projective space $\mathbb{P}^n$ has a zero-dimensional singular set. Hence a hyperplane section of a smooth cubic surface does not contain a double line.*

Indeed, the singular locus of a hyperplane section $H \cap X$ of X is the fiber over the point $[H] \in (\mathbb{P}^n)^*$ of the Gauss map of X which is given by a base point free ample linear system.

Applying the lemma, we see that the residual conic D to l_0 in $H_3 \cap S$ is the union of two (secant) distinct lines or a nondegenerate conic. We have $D \cap l_2 = \emptyset$ otherwise, a point $x \in D \cap l_2$ would be in $H_1 \cap H_2 \cap H_3$ which is known to be l_0 and $l_0 \cap l_2 = \emptyset$. As the intersection of the 2-plane $H_1 \cap H_3$ with the line l_1 in the 3-dimensional projective space H_1, $(H_1 \cap H_3) \cap l_1$ is a complete intersection point of $(H_1 \cap H_3) \cap X$ not on l_0 (since $l_0 \cap l_1 = \emptyset$) so $D \cap l_1$ is that point. In particular if D is degenerate, it is not the meeting point of the two components of D. The same is true for $D \cap l_3$ which the complete intersection point $(H_1 \cap H_3) \cap l_3$, intersection of the 2-plane $H_1 \cap H_3$ with the line l_3 in the 3-dimensional projective space H_3. So $C = l_1 \cup l_2 \cup l_3 \cup D$ is a locally complete intersection closed subset of X of pure dimension 1. It is a curve of arithmetic genus 1, with trivial dualizing sheaf, not contained in any hyperplane and whose intersection with a general hyperplane is a degree 5 effective zero cycle. It is thus a singular linearly normal elliptic quintic curve. From this construction, we see that we have at least a 6-dimensional family of such curves: $[l_0], [l_1]$ are chosen in open subsets of $F(X)$, $[l_2]$ is chosen in an open subset of the curve $\mathcal{H}(l_1)$ and $[l_3]$ in an open subset of the curve $\mathcal{H}(l_2)$.

The curve C thus constructed have the following properties:

Proposition 2.3.3. *(i) $h^i(X, \mathcal{I}_C) = 0$ $i \in \{0, 1, 3\}$ and $h^2(X, \mathcal{I}_C) = 1$;*

(ii) $h^i(X, \mathcal{I}_C(1)) = 0$ for all $0 \leq i \leq 2$;

(iii) $h^0(X, \mathcal{I}_C(2)) = 5$, $h^i(X, \mathcal{I}_C(2)) = 0$ for $i = 1, 2$;

(iv) $dim_k Ext^1(\mathcal{I}_C, \mathcal{O}_X(-2)) = 1$ futhermore the generator of $Ext^1(\mathcal{I}_C, \mathcal{O}_X(-2))$ generates $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))$ at any point of C.

Proof. (i) It follows immediately from the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0 \tag{2.6}$$

the connectedness of C, which imposes $h^0(C, \mathcal{O}_C) = 1$, Serre duality on C and the isomorphism $\omega_C \simeq \mathcal{O}_C$.

(ii) By the normalization exact sequence we have the inclusion $H^0(C, \mathcal{O}_C(l)) \hookrightarrow \oplus_i H^0(C_i, \mathcal{O}_{C_i}(l))$ for any $l \in \mathbb{Z}$, where C_i are the irreducible components of C which are either a projective line or a non-singular conic lying in a plane. So for $l = -1$, we have $h^0(\mathcal{O}_C(-1)) = 0$. Now, applying Riemann-Roch theorem to C (see [34] p.83) yields $h^0(C, \mathcal{O}_C(1)) = deg(\mathcal{O}_C(1)) = 5$ since the intersection of C with a generic hyperplane has degree 5. Moreover since C is not contained in a hyperplane, $h^0(\mathcal{I}_C(1)) = 0$. So tensoring (2.6) by $\mathcal{O}_X(1)$ and taking the long exact sequence gives the desired cancelations.

(iii) By a projective transformation, we can suppose that C is the curve whose components are: the lines $[A_0A_1]$, $[A_1A_2]$, $[A_2A_3]$ and the conic given by an equation of the form $Q = \alpha X_4^2 + \beta X_0X_3 + \gamma X_0X_4 + \delta X_3X_4$ (so that it meets A_0 and A_3) in the plane $X_1 = 0 = X_2$ where $A_0 = [1 : 0 \cdots : 0], \ldots, A_3 = [0 : \cdots : 1 : 0]$. Then we can check that the quadrics given by union of hyperplanes $Q_0 = X_0X_2$, $Q_1 = X_1X_3$, $Q_2 = X_1X_4$, $Q_3 = X_2X_4$ and the quadric Q form a basis of the space of quadrics containing C; so we have $h^0(\mathcal{I}_C(2)) = 5$. The cancelation of $h^2(X, \mathcal{I}_C(2))$ follows immediately from the long exact sequence associated to (2.6) tensorized by $\mathcal{O}_X(2)$. For $h^1(\mathcal{I}_C(2))$, we proceed as in [42, Proposition IV.1.2]: for a generic hyperplane H, letting $\Gamma := C \cap H = \{P_1, \ldots P_5\}$ where the P_i are five points whose span is equal to H and $S := H \cap X$ a smooth cubic surface, we have the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_C(1) \arrow[r] \arrow[d, "H"'] & \mathcal{O}_X(1) \arrow[r] \arrow[d, "H"'] & \mathcal{O}_C(1) \arrow[r] \arrow[d, "H"'] & 0 \\
0 \arrow[r] & \mathcal{I}_C(2) \arrow[r] \arrow[d] & \mathcal{O}_X(2) \arrow[r] \arrow[d] & \mathcal{O}_C(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_\Gamma(2) \arrow[r] \arrow[d] & \mathcal{O}_S(2) \arrow[r] \arrow[d] & \oplus_{i=1}^5 k_{P_i} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

For any P_{i_0}, since Γ spans H, there is a plane for the form $h_1 = span(P_{j_1}, P_{j_2}, P_{j_3})$ ($j_i \in \{1, \dots 5\} \backslash \{i_0\}$) that do not contains P_{i_0}. we can also choose a plane h_2 containing the last point P_k ($\{k\} = \{1, \dots 5\} \backslash \{i_0, j_1, j_2, j_3\}$) and not containing P_{i_0}, then the union $h_1 \cup h_2$ is a quadric of H that does not contain P_{i_0}. Thus $H^0(S, \mathcal{O}_S(2)) \to \oplus_{i=1}^5 k_{P_i}$ is surjective, so $h^1(S, \mathcal{I}_\Gamma(2)) = 0$. This gives the surjectivity of $H^1(X, \mathcal{I}_C(1)) \to H^1(X, \mathcal{I}_C(2))$ hence $h^1(\mathcal{I}_C(2)) = 0$.

(iv) The first terms of the local-to-global spectral sequence used to compute the groups $Ext(\mathcal{I}_C, \mathcal{O}_X(-2))$ gives

$$\begin{aligned} 0 \to H^1(X, \mathcal{O}_X(-2)) \to Ext^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \to H^0(X, \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))) \\ \to H^2(X, \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-2))). \end{aligned} \tag{2.7}$$

By [68, lemma 1], $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq det(\mathcal{N}_{C/X}) \otimes i^* \underbrace{\mathcal{O}_X(-2)}_{=\omega_X}$, where the vector bundle $\mathcal{N}_{C/X}$ on C is the normal bundle of the locally complete intersection subscheme C in X and $i : C \hookrightarrow X$ the inclusion so $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq i_*\omega_C$ the dualizing sheaf of C, hence $H^0(X, \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))) \simeq H^0(C, \omega_C)$. But $\omega_C \simeq \mathcal{O}_C$ and C is proper and connected so $H^0(C, \omega_C) \simeq k$. Next we have $H^1(X, \mathcal{O}_X(-2)) = 0$ and $\mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq \mathcal{O}_X(-2)$ ([68, lemma1]) so that $H^2(X, \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-2))) \simeq H^2(X, \mathcal{O}_X(-2))$. This last group being zero, we have $Ext^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq H^0(C, \mathcal{O}_C) \simeq k$. It is thus clear that the generator of $Ext^1(\mathcal{I}_C, \mathcal{O}_X(-2))$ generates $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))$ at any point of C. □

By the Serre construction in codimension 2 (see for example [68]), we can associate, using Proposition 2.3.3 (iv), to a locally complete intersection curve $C \subset X$ constructed as above, a rank 2 vector bundle on X.

Proposition 2.3.4. *For any curve $C \subset X$ constructed as above, there is a unique auto-dual rank 2 vector bundle $\mathcal{E}$ on X fitting in the exact sequence:*

$$0 \to \mathcal{O}_X \to \mathcal{E}(1) \to \mathcal{I}_C(2) \to 0 \tag{2.8}$$

given by any non zero element of $Ext^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq k$.

Vector bundles obtained by this method have the following properties which were proved in [52, Lemmas 2.1, 2.7, 2.8 and Proposition 2.6] for vector bundles constructed by the same method but starting from smooth normal elliptic quintic curves:

Proposition 2.3.5. *Let $\mathcal{E}$ be a rank 2 vector bundle obtained from a singular linearly normal elliptic quintic curve applying Proposition 2.3.4. Then we have:*

(i) $h^0(\mathcal{E}(-1)) = 0$, $h^0(\mathcal{E}) = 0$, $h^0(\mathcal{E}(1)) = 6$ and $h^i(\mathcal{E}(-1)) = 0 = h^i(\mathcal{E}(1))$ $\forall i \geq 1$;

(ii) $\mathcal{E}(1)$ is a stable vector bundle generated by its global sections so that the zero locus of any non zero global section s of $\mathcal{E}(1)$ is a local complete intersection curve C_s on X with trivial dualizing sheaf, whose ideal sheaf satisfies all the equalities of Proposition 2.3.3 and fits in an exact sequence $0 \to \mathcal{O}_X \to \mathcal{E}(1) \to \mathcal{I}_{C_s}(2) \to 0$;

(iii) $h^0(\mathcal{E} \otimes \mathcal{E}) = 1$, $h^1(\mathcal{E} \otimes \mathcal{E}) = 5$, $h^2(\mathcal{E} \otimes \mathcal{E}) = 0 = h^3(\mathcal{E} \otimes \mathcal{E})$;

(iv) $h^0(\mathcal{N}_{C/X}) = 10$, $h^1(\mathcal{N}_{C/X}) = 0$. In particular, the Hilbert scheme $Hilb_{5n}(X/k)$, which parametrizes 1-dimensional subschemes of degree 5 and arithmetic genus 1 in X, is smooth of dimension 10 at the point $[C]$.

(v) The morphism $\mathbb{P}^5 \simeq \mathbb{P}(H^0(\mathcal{E}(1))) \to Hilb_{5n}(X/k)$, that associates to any non-zero global section of $\mathcal{E}(1)$ the subscheme of X defined by zero locus, is injective.

Proof. (i) The computation of the dimension of these cohomology groups is immediate from (2.8) and Proposition 2.3.3 using long exact sequences with appropriate twists and Serre's duality.

(ii) We have seen that $h^1(\mathcal{E}) = 0$, $h^2(\mathcal{E}(-1)) = 0$ and $h^3(\mathcal{E}(-2)) = h^0(\mathcal{E}) = 0$ so by Mumford-Castelnuovo criterion, $\mathcal{E}(1)$ is generated by its global sections. There is no assumption on the characteristic of the base field in the arguments used in [52, Proposition 2.6] to prove the stability of $\mathcal{E}(1)$ so the proof adapts to our setting.

The proofs of items (iii) and (iv) given in [52, Lemma 2.7] adapt in positive characteristic since it only uses the fact that stability of vector bundles implies their simplicity, Grothendieck-Riemann-Roch theorem and the fact that $h^1(\mathcal{N}^*_{C/\mathbb{P}^4_k}(2)) = 0$ which is still true in our setting since the second proof of this fact given in [42, Proposition V.2.1] makes no use of the characteristic ($\neq 2$) nor of a better regularity than local complete intersection. Item (v) is [52, Lemma 2.8] whose proof uses no assumption on the charateristic of the base field. □

Define the locally closed subscheme $\mathcal{H}$ of $Hilb_{5n}(X/k)$:

$$\mathcal{H} = \left\{ \begin{array}{l} [Z] \in Hilb_{5t}(X/k),\ (i)\ Z \text{ is a locally complete intersection of pure} \\ \text{dimension } 1, (ii)\ h^1(\mathcal{I}_Z) = 0 = h^0(\mathcal{I}_Z(1)) = h^1(\mathcal{I}_Z(1))\ (hence\ h^0(\mathcal{O}_Z) = 1), \\ (iii)\ h^1(\mathcal{I}_Z(2)) = 0 = h^2(\mathcal{I}_Z(2))\ (hence\ h^0(\mathcal{I}_Z(2) = 5),\ (iv)\ \omega_Z \simeq \mathcal{O}_Z \end{array} \right\}. \tag{2.9}$$

The 6-dimensional family of singular linearly normal elliptic quintic curves constructed in the previous section is contained in $\mathcal{H}$. Moreover, the subschemes parametrized by $\mathcal{H}$ are connected (since $h^0(\mathcal{O}_Z) = 1$) local complete intersection curves with trivial dualizing sheaf and whose ideal sheaves have all the properties needed to guarantee that the coherent sheaf arising from Serre's construction in codimension 2 (see Proposition 2.3.4) is a rank 2 vector bundle on X satisfying the properties of Proposition 2.3.5. In particular, by item (iv) of Proposition 2.3.5, $\mathcal{H}$ is a smooth 10-dimensional quasi-projective k-scheme. So, using the pull-back $\mathcal{Z}$ of the universal sheaf over $Hilb_{5n}(X/k)$ on $\mathcal{H}$, we can define an Abel-Jacobi morphism $\phi_{\mathcal{Z}} : \mathcal{H} \to J(X)$. We have the following theorem which proves that the condition (*) of the introduction is satisfied by cubic threefolds:

Theorem 2.3.2. *ϕ_Γ is smooth and its general fiber is isomorphic to $\mathbb{P}^5_k$.*

Proof. To prove the smoothness of $\phi_{\mathcal{Z}}$, we just have to see that the differential of the Abel-Jacobi morphism $T\phi_{\mathcal{Z}} : T_{\mathcal{H},[C]} \simeq H^0(C, \mathcal{N}_{C/X}) \to T_{J(X),\phi_{\mathcal{Z}}([C])} \simeq H^2(X, \Omega_{X/k})$ is surjective for any $[C] \in \mathcal{H}$. To do so, we proceed as in [52, Theorem 5.6] trying the prove that the dual of this map $(T\phi_{\mathcal{Z}})^* : H^2(X, \Omega_{X/k})^* \simeq H^1(X, \Omega^2_{X/k}) \to H^0(C, \mathcal{N}_{C/X})^*$ is injective. We use the technique of "tangent bundle sequence" following [93, Section 2]. It is presented there for a smooth subvariety $C \hookrightarrow X$ and in characteristic zero but the arguments to prove the following lemma make no use, for the subvariety C, of a greater regularity than local complete intersection (so that $\mathcal{N}_{C/X}$ and $\mathcal{N}_{C/\mathbb{P}^4_k}$ are vector bundles on C and we can use Serre duality) nor of the characteristic ($\neq 2$). So $(T\phi_{\mathcal{Z}})^*$ admits the following description:

Lemma 2.3.3. *([93, Lemma 2.8]). The following diagram is commutative*

$$\begin{array}{ccc} H^0(X, \mathcal{N}_{X/\mathbb{P}^4_k} \otimes \omega_X) & \xrightarrow{R} & H^1(X, \Omega^2_{X/k}) \\ \downarrow{\scriptstyle r_C} & & \downarrow{\scriptstyle (T\phi_{\mathcal{Z}})^*} \\ H^0(C, \mathcal{N}_{X/\mathbb{P}^4_k} \otimes \omega_{X|C}) & \xrightarrow{\beta_C} & H^0(C, \mathcal{N}_{C/X})^* \end{array}$$

where r_C is the restriction map to C, β_C is part of the exact sequence

$$\cdots \to H^0(C, \mathcal{N}_{C/\mathbb{P}^4_k} \otimes \omega_X) \to H^0(C, \mathcal{N}_{X/\mathbb{P}^4_k} \otimes \omega_{X|C}) \stackrel{\beta_C}{\to} H^1(C, \mathcal{N}_{C/X} \otimes \omega_X) \simeq H^0(C, \mathcal{N}_{C/X})^*$$

from the long exact sequence arising from the short exact sequence

$$0 \to \omega_X \otimes \mathcal{N}_{C/X} \to \omega_X \otimes \mathcal{N}_{C/\mathbb{P}^4_k} \to \omega_X \otimes \mathcal{N}_{X/\mathbb{P}^4_k|C} \to 0$$

and R is the first connecting morphism in the long exact sequence associated to

$$0 \to \Omega^2_{X/k} \otimes \mathcal{N}^*_{X/\mathbb{P}^4_k} \to \Omega_{\mathbb{P}^4_k/k|X} \to \omega_X \to 0$$

coming from the exterior cube of the short exact sequence $0 \to \mathcal{N}^*_{X/\mathbb{P}^4_k} \to \Omega_{\mathbb{P}^4_k|X} \to \Omega_{X/k} \to 0.$

Now, R is an isomorphism since Bott formula for $\mathbb{P}^4_k$ imply the vanishing of $H^0(X, \Omega^3_{\mathbb{P}^4_k/k|X}(3))$ and $H^1(X, \Omega^3_{\mathbb{P}^4_k/k|X}(3))$. The kernel of r_C is $H^0(X, \mathcal{N}_{X/\mathbb{P}^4_k} \otimes \omega_X \otimes \mathcal{I}_{C/X}) = H^0(X, \mathcal{I}_{C/X}(1))$ which is 0 by assumption. As for the kernel of β_C, we have already seen in the proof of Proposition 2.3.5, that $H^1(C, \mathcal{N}^*_{C/X}(2)) = H^0(C, \mathcal{N}_{C/X}(-2))$ is 0 since the second proof given in [42, Proposition V.2.1] for the vanishing of $h^1(C, \mathcal{N}^*_{C/\mathbb{P}^4_k}(2)) = h^0(C, \mathcal{N}_{C/\mathbb{P}^4_k}(-2))$ makes no use of the characteristic $\neq 2$ nor of a greater regularity than local complete intersection and $\mathcal{N}_{C/X}(-2) \subset \mathcal{N}_{C/\mathbb{P}^4_k}(-2)$; so β_C is also injective. Hence $T\phi_{\mathcal{Z}}$ is surjective at all points of $\mathcal{H}$ i.e. $\phi_{\mathcal{Z}}$ is smooth on $\mathcal{H}$.

Since $\phi_{\mathcal{Z}}$ is smooth, any nonempty closed fiber of $\phi_{\mathcal{Z}}$ is a disjoint union of smooth k-varieties. In fact it is the union of copies of $\mathbb{P}^5_k$: using the curve C parametrized by a point $[C]$ of this fiber, we can construct a rank 2 vector bundle $\mathcal{E}$ having the properties of Proposition 2.3.5; in particular $\mathbb{P}(H^0(\mathcal{E}(1))) \simeq \mathbb{P}^5_k$ (2.3.5 (i)) and the inclusion of 2.3.5 (v) $\mathbb{P}(H^0(\mathcal{E}(1))) \hookrightarrow Hilb_{5n}(X/k)$ has values in $\mathcal{H}$ since, by 2.3.5 (ii), the curves defined by the zero locus of elements of $\mathbb{P}(H^0(\mathcal{E}(1)))$ satisfy the conditions defining $\mathcal{H}$. Since a morphism from a projective space to an abelian variety is constant, we see that $\phi_{\mathcal{Z}}(\mathbb{P}(H^0(\mathcal{E}(1)))) = \phi_{\mathcal{Z}}([C])$ i.e. $\mathbb{P}^5_k = \mathbb{P}(H^0(\mathcal{E}(1)))$ is included in the fiber $\phi_{\mathcal{Z}}^{-1}(\phi_{\mathcal{Z}}([C]))$ and $[C] \in \mathbb{P}^5_k = \mathbb{P}(H^0(\mathcal{E}(1)))$ (by exact sequence 2.8). So every nonempty closed fiber of $\phi_{\mathcal{Z}}$ is the disjoint union of a finite number of $\mathbb{P}^5_k$.

Now, let $\pi : \mathfrak{X} \to S = Spec(R)$ be a smooth projective lifting of the smooth cubic $X \subset \mathbb{P}^4_k$ to characteristic 0 over a DVR. Let us denote by $K = Frac(R)$, the generic point of S and s the closed point. We have also an abelian scheme over S, $\mathcal{J} = Pic^0(\mathcal{F}(\mathfrak{X})/S)$, the relative Picard scheme of the relative Fano surface of π, whose geometric fibers are isomorphic to the intermediate jacobian of the corresponding smooth cubic hypersurfaces. Let $\mathcal{H}(\mathfrak{X}/S)$ be the locally closed subscheme of the relative Hilbert scheme $Hilb_{5t}(\mathfrak{X}/S)$ defined as

$$\left\{\begin{array}{l} [Z] \in Hilb_{5t}(\mathfrak{X}/S),\ (i)\ Z \text{ is a locally complete intersection of pure} \\ \text{dimension } 1, (ii)\ h^1(\mathcal{I}_Z) = 0 = h^0(\mathcal{I}_Z(1)) = h^1(\mathcal{I}_Z(1))\ (hence\ h^0(\mathcal{O}_Z) = 1), \\ (iii)\ h^1(\mathcal{I}_Z(2)) = 0 = h^2(\mathcal{I}_Z(2))\ (hence\ h^0(\mathcal{I}_Z(2) = 5),\ (iv)\ \omega_Z \simeq \mathcal{O}_Z \end{array}\right\}. \tag{2.10}$$

whose fiber over s is $\mathcal{H}$ and fiber over K is the subscheme of the Hilbert scheme $Hilb_{5n}(\mathfrak{X}_\eta/K)$ defined likewise which, according to [52, Corollary 5.5], is also smooth of dimension 10. It is easy to see that a singular linearly normal elliptic quintic curve, as those constructed in 2.3.2, lifts in charateristic 0 over R so $\mathcal{H}(\mathfrak{X}/S)$ has a S-point that we can use as a reference point to define a morphism $\phi_{\mathcal{Z}_S} : \mathcal{H}(\mathfrak{X}/S) \to \mathcal{J}$ (using the universal family $\mathcal{Z}_S$ of $Hilb_{5t}(\mathfrak{X}/S)$) inducing the Abel-Jacobi morphisms over s and K.

Let $\Gamma_{\phi_{\mathcal{Z}_S}}$ be the closure of the graph of $\phi_{\mathcal{Z}_S}$ in the S-projective scheme $Hilb_{5n}(\mathfrak{X}/S) \times_S \mathcal{J}$. Then the second projection $p_S : \Gamma_{\phi_{\mathcal{Z}_S}} \to \mathcal{J}$ yields a projective morphism inducing the Abel-Jacobi morphism $\phi_{\mathcal{Z}_S}$ on a dense open subscheme of $\Gamma_{\phi_{\mathcal{Z}_S}}$ which is surjective on S. By Stein factorization theorem, there is a proper S-scheme $\mathcal{M}$ such that p_S factorizes as $\Gamma_{\phi_{\mathcal{Z}_S}} \stackrel{\Phi}{\to} \mathcal{M} \stackrel{\Psi}{\to} \mathcal{J}$ with Ψ a finite morphism and Φ a morphism with connected fibers. By work of Iliev, Markushevich and Tikhomirov ([52, Theorem 5.6] and [43, Theorem 3.2]; see also Druel [28, Théorème 1.4]), the general fiber of p_K is $\mathbb{P}^5_K$ so that $\Psi_K : \mathcal{M}_K \to \mathcal{J}_k$ is an isomorphism. So Ψ is a birational morphism. Let us denote $\sigma : \mathcal{J} \dashrightarrow \mathcal{M}$ the inverse map. Since $\mathcal{J}$ is regular, the local ring of any codimension 1 point of $\mathcal{J}$ is a DVR and since $\mathcal{M}$ is proper, σ is defined on any codimension 1 point. In particular, it is defined at the generic point of $J(X) = \mathcal{J}_s$; hence $J(X)$ is birational to a component M_0 of $\mathcal{M}_s$. Let us denote $B = \Phi_s^{-1}(M_0)$. Then $p_s : B \to J(X)$ is dominant with general fiber isomorphic to $\mathbb{P}^5$ and on a open subset of B, it is the Abel-Jacobi morphism given by the pull-back of the universal family of $Hilb_{5t}(X/k)$. $\square$

Chapter 3

Remarks on approximate decomposition of the diagonal

Résumé: Dans ce chapitre, nous étudions, pour des variétés définies sur $\mathbb{C}$ à CH_0 trivial, la différence qui existe entre avoir une dimension CH_0 essentielle égale à 2 et avoir une dimension CH_0 essentielle nulle. Nous présentons notamment des conditions suffisantes (et nécessaires) pour assurer qu'une variété à CH_0 trivial qui a une dimension CH_0 essentielle ≤ 2 a, en fait, une dimension CH_0 essentielle égale à 0.

Abstract: In this chapter, we investigate, for varieties over $\mathbb{C}$ with trivial group of 0-cycles, the gap between essential CH_0-dimension 2 and essential CH_0-dimension 0. In particular, we present sufficient (and necessary) conditions for a variety with trivial group of 0-cycles and essential CH_0-dimension ≤ 2 to have, in fact, essential CH_0-dimension 0.

3.0 Introduction

The characterization of the complex smooth projective varieties of dimension $n \geq 3$ which are rational, i.e. birationally equivalent to $\mathbb{P}^n_{\mathbb{C}}$, is a long standing problem in algebraic geometry. If such a variety X is rational then, by birational invariance of the group CH_0 of 0-cycles modulo rational equivalence, we have an isomorphism $\mathrm{CH}_0(X) = \mathbb{Z}x$ for a (any) point $x \in X(\mathbb{C})$.

However a more general class of varieties satisfy the isomorphism $\mathrm{CH}_0(X) \simeq \mathbb{Z}$. Among them, we can mention the class of rationally connected varieties, for which any pair of point is contained in a rational curve, or the (conjecturally) smaller class of unirational varieties, i.e. varieties admitting a dominant rational map from a projective space, which, in dimension ≤ 2 coincide with the class of rational varieties. In dimension ≥ 3, efficient obstructions to rationality for unirational varieties have been found since the early $1970's$: Clemens and Griffiths ([15]) proved that the intermediate Jacobian of a cubic threefold X, which is a unirational variety, is not isomorphic to a sum of Jacobian of curves which would be the case should X be rational; in a 1989 paper ([19]), Colliot-Thélène and Ojanguren studied higher degree analog of the obstruction that Artin and Mumford used to provide an example of a unirational non stably rational threefold, namely the unramified cohomology groups, which are birational invariants trivial for stably rational varieties, and used them to prove non stable rationality of some unirational quadric bundles.

In fact, the birational invariance of the CH_0 group yields more for a rational variety X; for any field extension $L/\mathbb{C}$, we have, as for the projective space, $\mathrm{CH}_0(X) = \mathbb{Z}x_L$ where $x \in X(\mathbb{C})$ and $x_L = x \times_{\mathbb{C}} Spec(L)$. The varieties satisfying this property are said to have universally trivial CH_0 group or have their 0-cycles universally supported on a point. We can reformulate this property, after ([5]), saying that there exists a Chow-theoretic decomposition of the diagonal i.e. than we can write the diagonal of X as :

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X) \tag{3.1}$$

where $x \in X(\mathbb{C})$ and Z is a cycle supported on $D \times X$ for some proper closed algebraic subset D of X. In [91], Voisin proved that there are solids (desingularization of certain quartic double solids) whose intermediate Jacobian is a Jacobian of a curve, i.e. whose irrationality cannot be detected by Clemens-Griffiths criterion, and whose unramified cohomology groups are trivial but which do not admits a Chow-theoretic decomposition of the diagonal. By doing so, she proves that the necessary condition for stable rationality given by the existence of a Chow-theoretic decompostion of the diagonal is strictly stronger than the triviality of the unramified cohomology groups; she also proves that the non-existence of a decomposition of the diagonal can help proving the stable irrationality of some solids for which the Clemens-Griffiths criterion is of no use (eg. the solids whose intermediate Jacobian has dimension ≤ 3).

For a variety X satisfying $\mathrm{CH}_0(X) = \mathbb{Z}$, we can also consider a weaker property than having the 0-cycles universally supported on a point: let us say that the CH_0 is universally supported on a subvariety $Y \subset X$ if the natural morphism $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective, i.e. for any $L/\mathbb{C}$, $\mathrm{CH}_0(Y_L) \to \mathrm{CH}_0(X_L)$ is surjective.

Remark 3.0.1. An equivalent formulation is the existence of a decomposition of the diagonal

$$\Delta_X = \Gamma_1 + \Gamma_2 \text{ in } \mathrm{CH}^n(X \times X) \tag{3.2}$$

where Γ_1 is supported on $D \times X$ for some proper closed subset $D \subset X$ and Γ_2 is supported $X \times Y$. Indeed, applying universal surjectivity to the field $L = \mathbb{C}(X)$, we see that the diagonal point $\delta_X \in \mathrm{CH}_0(X_{\mathbb{C}(X)})$ can be written $i_{L,*}z$ in $\mathrm{CH}_0(X_{\mathbb{C}(X)}) \simeq \mathrm{CH}^n(X_{\mathbb{C}(X)})$, where $z \in \mathrm{CH}_0(Y_L)$ and $i : Y \hookrightarrow X$. This implies (3.2), using Bloch identity $\mathrm{CH}^n(X_{\mathbb{C}(X)}) = \varinjlim_{U \subset X,\ open} \mathrm{CH}^n(U \times X)$ and the localization exact sequence.

Voisin made in [90] the following definition:

Definition 3.0.2. *([90, Definition 1.2]) The essential CH_0-dimension of a variety is the minimal integer k such that there is a closed subscheme $Y \subset X$ of dimension k such that the CH_0 group of X is universally supported on Y.*

In the case of cubic hypersurfaces, the following is proved in [90]:

Theorem 3.0.3. *([90, Theorem 1.4]) The essential CH_0-dimension of a very general n-dimensional cubic hypersurface over $\mathbb{C}$ is either n or 0, for $n = 4$ or n odd.*

In [17], Colliot-Thélène proved the following proposition concerning varieties of essential CH_0-dimension ≤ 1:

Theorem 3.0.4. *([17, Proposition 2.5]) Let X a smooth projective variety over $\mathbb{C}$ satisfying $\mathrm{CH}_0(X) = \mathbb{Z}$. Assume X has essential CH_0-dimension ≤ 1. Then the essential CH_0-dimension of X is 0 (i.e. X admits a Chow-theoretic decomposition of the diagonal).*

The goal of this chapter is to investigate how far a variety with trivial CH_0, which has essential CH_0 dimension ≤ 2 is from having a universally trivial CH_0 group i.e. essential CH_0 dimension 0. A first result in this direction is the following:

Theorem 3.0.5. *([90, Corollary 2.2], [5, Proposition 1.9]) Let Σ be a smooth complex projective surface. Assume $\mathrm{CH}_0(\Sigma) = \mathbb{Z}$ and $\mathrm{Tors}(H^*(\Sigma, \mathbb{Z})) = 0$. Then Σ has CH_0 universally trivial.*

We show the following generalizations:

Theorem 3.0.6. *Let X be a smooth complex projective variety of dimension n such that $\mathrm{CH}_0(X) = \mathbb{Z}$. Assume X has essential CH_0 dimension ≤ 2. If*

1. $\mathrm{Tors}(H^2(X, \mathbb{Z})) = 0$ *and*
2. $H^3(X, \mathbb{Z}) = 0$,

then X has universally trivial CH_0 group i.e. has essential CH_0 dimension 0.

In another direction, we have the following result. Let us introduce first the following condition:

(*) there is a smooth projective variety $\widetilde{Y}$ of dimension $(dim(X) - 1)$ and a morphism $j : \widetilde{Y} \to X$ such that

$$j_* : \mathrm{Pic}^0(\widetilde{Y}) \to \mathrm{CH}^2(X)_{alg}$$

is universally surjective.

In more geometric terms, the condition means that any family of algebraically trivial codimension 2 cycles factors generically through j_*. Indeed, let $\mathcal{Z} \in \mathrm{CH}^2(B \times X)$ be such a family, parametrized by a smooth projective base B. Applying condition (*) to the field $\mathbb{C}(B)$, gives that the cycle $\mathcal{Z}_{\mathbb{C}(B)} \in \mathrm{CH}^2(X_{\mathbb{C}(B)})_{alg} = \varinjlim_{U \subset B,\ open} \mathrm{CH}^2(U \times X)_{alg}$ has a pre-image $D \in \mathrm{Pic}^0(Y)(\mathbb{C}(B))$ by $j_{\mathbb{C}(B),*}$. The $\mathbb{C}(B)$-point D corresponds naturally to a rational map (thus a morphism since $\mathrm{Pic}^0(Y)$ is an abelian variety) $\mathcal{D} : B \to \mathrm{Pic}^0(Y)$. The identity $j_{\mathbb{C}(B),*}(D) = \mathcal{Z}_{\mathbb{C}(B)}$ says that the applications $B(\mathbb{C}) \to \mathrm{CH}^2(X)_{alg}$, given by $b \mapsto \mathcal{Z}_b$ and $B(\mathbb{C}) \xrightarrow{\mathcal{D}} \mathrm{Pic}^0(Y)(\mathbb{C}) \xrightarrow{j_*} \mathrm{CH}^2(X)_{alg} \simeq J^3(X)$ coincide on a dense open set of B hence everywhere since targets are abelian varieties.

Conversely, any cycle $\mathcal{Z}_L \in \mathrm{CH}^2(X_L)_{alg}$ has a model $\mathcal{Z}$ which is a family of algebraically trivial codimension 2 cycles of X parametrized by a smooth quasi-projective model B of L. The factorization of that family through j_* gives rise to a morphism $\mathcal{D} : B \to \mathrm{Pic}^0(Y)$ which, passing to the limit over the open sets of B yields a $\mathbb{C}(B) = L$ point of $\mathrm{Pic}^0(Y)$ mapped by $j_{L,*}$ to $\mathcal{Z}_L$.

Now, let us state the second theorem of the chapter:

Theorem 3.0.7. *Let X be a smooth complex projective variety of dimension n such that $\mathrm{CH}_0(X) = \mathbb{Z}$. Assume X has essential CH_0 dimension ≤ 2. If X satisfies the condition (*), then X has universally trivial CH_0 group i.e. has essential CH_0 dimension 0.*

Remark 3.0.8. We observe, conversely, that if X has universally trivial CH_0 group, then the conditions (1) of Theorem 3.0.6 and (*) are satisfied (see Lemma 3.1.1).

The key condition $(*)$ appearing in the theorem is expressed in terms of universal generation, a notion introduced by Shen in [73], where he uses universal generation of 1-cycles on cubic hypersurfaces (of dimension ≥ 3) to relate the existence of a decomposition of the diagonal for cubic 3-folds and 4-folds to the algebraicity of some cohomological classes on their Fano varieties of lines associated to the pairing in the middle cohomology of cubic hypersurfaces.

In the case of threefolds, another relation between essential CH_0-dimension and condition (*) is presented in Section 3.3. Combined with Theorem 3.0.7, it yields the following result:

Theorem 3.0.9. *The essential CH_0-dimension of a very general Fano complete intersection threefold is 0 or 3. In particular, the essential CH_0-dimension of the very general quartic threefold is equal to 3.*

The chapter is organized as follows: the first section is devoted to the proofs of Theorems 3.0.6 and 3.0.7. The second section is devoted to the analysis of condition $(*)$; we try to relate it, at least in some special case to more geometric conditions. The third second is devoted to examples for which Theorem 3.0.7 gives interesting consequences.

3.1 Main theorems

Let us begin this section by a lemma which proves that the conditions (1) of Theorem 3.0.6 and (*) are also necessary.

Lemma 3.1.1. *Let X be a smooth projective variety of dimension n whose essential CH_0 dimension is 0. Then*

1. $\mathrm{Tors}(H^2(X,\mathbb{Z})) = 0 = \mathrm{Tors}(H^3(X,\mathbb{Z}))$ *and*
2. *X satisfies $(*)$.*

Proof. Saying that X has essential CH_0-dimension 0 is equivalent to the existence of a Chow-theoretic decomposition of the diagonal of X. We have:

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X) \tag{3.3}$$

where $x \in X(\mathbb{C})$ and Z is a cycle supported on $D \times X$ for some proper closed algebraic subset D of X. We can choose D such that, denoting $\widetilde{D}$ a desingularization of D and $j : \widetilde{D} \to X$ the composition of the desingularization followed by the inclusion, the cycle Z lifts to a cycle $\widetilde{Z} \in \mathrm{CH}^{n-1}(\widetilde{D} \times X)$. Item (1) is proved in [89, Theorem 4.4]. Let us prove (2). Let $L/\mathbb{C}$ be a field extension and $\gamma \in \mathrm{CH}^2(X_L)_{alg}$. Letting both sides of the extension of (3.3) to L act on γ, we get the equality:

$$\gamma = \Delta_{X_L}^* \gamma = j_{L,*}(\widetilde{Z}_L^* \gamma) \text{ in } \mathrm{CH}^2(X_L)$$

with $\widetilde{Z}_L^* \gamma \in \mathrm{CH}^1(\widetilde{D}_L)_{alg} = \mathrm{Pic}^0(\widetilde{D}_L) = \mathrm{Pic}^0(\widetilde{D})_L$, which proves the universal surjectivity of j_*. □

Proof of Theorem 3.0.6. Let us assume that X satisfies conditions (1) and (2) of the Theorem 3.0.6. By Remark 3.0.1, the diagonal of X can be written:

$$\Delta_X = \Gamma_1 + \Gamma_2 \text{ in } \mathrm{CH}^n(X \times X) \tag{3.4}$$

where Γ_1 is supported on $D \times X$ for some proper closed subset $D \subset X$ and Γ_2 is supported $X \times \Sigma$. Let $\tau : \widetilde{\Sigma} \to \Sigma$ be a desingularization of Σ. Enlarging Σ if necessary we can find $\widetilde{\Gamma}_2 \in \mathrm{CH}^2(X \times \widetilde{\Sigma})$ such that $(id_X, i \circ \tau)_* \widetilde{\Gamma}_2 = \Gamma_2$ in $\mathrm{CH}^n(X \times X)$, where $i : \Sigma \hookrightarrow X$ is the inclusion. To get a Chow-theoretic decomposition of the diagonal, it is sufficient to prove that $\widetilde{\Gamma}_2$ (hence Γ_2 in $\mathrm{CH}^n(X \times X)$) can be decomposed as $X \times x + Z$ in $\mathrm{CH}^2(X \times \widetilde{\Sigma})$ for a cycle Z supported on $D' \times \widetilde{\Sigma}$, D' being a proper closed subset of X. We have the following proposition:

Proposition 3.1.1. *([90, Proposition 2.1]) Let Y be a smooth projective variety. If Y admits a decomposition of the diagonal modulo algebraic equivalence, that is*

$$\Delta_Y = Y \times y + Z \text{ in } \mathrm{CH}^{dim(Y)}(Y \times Y)/alg$$

with Z supported on $D \times Y$ for some proper closed algebraic subset D of Y, then Y admits a Chow-theoretic decomposition of the diagonal.

We include the proof for the sake of completeness:

Sketch of proof. The result is obtained as a consequence of a nilpotence result of Voevodsky ([80]) and Voisin ([81]) asserting that given a self-correspondence $\Gamma \in \mathrm{CH}^n(Y \times Y)$ that is algebraically trivial, there is an integer N such that $\Gamma^{\circ N} = 0$ in $\mathrm{CH}^n(Y \times Y)$.

Applying the nilpotence result to the algebraically trivial self-correpondence $(\Delta_Y - (Y \times y) + Z)$ yields the result since, writting down the different terms, using the fact that $Z \circ (Y \times y) = 0$ and $(\Delta_Y - Y \times y) \circ (\Delta_Y - Y \times y) = \Delta_Y - Y \times y$, we see that any power of $(\Delta_Y - (Y \times y) + Z)$ is of the form $\Delta_Y - (Y \times y) + Z'$ for a cycle Z' supported on $D \times Y$. □

We conclude from this proposition that in order to get the decomposition of the diagonal of X, it suffices to decompose $\widetilde{\Gamma}_2$ as $X \times x + Z$ for a cycle Z supported on $D' \times \widetilde{\Sigma}$, D' being a proper closed subset of X, in $\mathrm{CH}^2(X \times \widetilde{\Sigma})/alg$. Indeed, this will decompose Γ_2 as $X \times x + Z$, with Z supported on $D \times X$ in $\mathrm{CH}^n(X \times X)/alg$.

Now, since $\mathrm{CH}_0(X) = \mathbb{Z}$, we have $\mathrm{CH}_0(X \times \widetilde{\Sigma}) \simeq \mathrm{CH}_0(\widetilde{\Sigma})$ i.e. the group of 0-cycles of $X \times \widetilde{\Sigma}$ supported on a 2-dimensional subscheme of $X \times \widetilde{\Sigma}$. Then, by work of Bloch and Srinivas ([13, Theorem 1 (ii)]), algebraic and homological equivalences coincide on $\mathrm{CH}^2(X \times \widetilde{\Sigma})$ so that, in order to get the equality $\Gamma_2 = X \times x + Z$ in $\mathrm{CH}^2(X \times \widetilde{\Sigma})/alg$, it is sufficient to prove the corresponding cohomological decomposition, that is to prove that $[\widetilde{\Gamma}_2]$ can be written $[X \times x] + [Z]$ in $H^4(X \times \widetilde{\Sigma})$ for a cycle Z supported on $D' \times \widetilde{\Sigma}$, D' being a proper closed subset of X.

We have the Künneth exact sequence:

$$0 \to \bigoplus_{i+j=4} H^i(X,\mathbb{Z}) \otimes H^j(\widetilde{\Sigma},\mathbb{Z}) \to H^4(X \times \widetilde{\Sigma},\mathbb{Z}) \to \bigoplus_{i+j=5} \mathrm{Tor}_1(H^i(X,\mathbb{Z}), H^j(\widetilde{\Sigma},\mathbb{Z})) \to 0$$

from which, we see, using the fact that the groups $H^{*\leq 1}(*,\mathbb{Z})$ are always torsion-free and the assumption $\mathrm{Tors}(H^2(X,\mathbb{Z})) = 0 = \mathrm{Tors}(H^3(X,\mathbb{Z}))$, that $H^4(X \times \widetilde{\Sigma},\mathbb{Z})$ admits a Künneth decomposition.

Let us denote $\delta^{i,j} \in H^i(X,\mathbb{Z}) \otimes H^j(\widetilde{\Sigma},\mathbb{Z})$ the Künneth components of $[\widetilde{\Gamma}_2]$. They are Hodge classes since the projection on Künneth types are morphism of Hodge structures.

The component $\delta^{3,1}$ is 0 since by assumption (2), $H^3(X,\mathbb{Z}) = 0$.

The component $\delta^{0,4}$ is of the form $[X \times z]$ for a 0-cycle z on $\widetilde{\Sigma}$.

Let us write $\widetilde{\Sigma} = \sqcup_i \widetilde{\Sigma}_i$ where the $\widetilde{\Sigma}_i$ are smooth connected surfaces. Since $H^0(\widetilde{\Sigma},\mathbb{Z}) = \oplus_i \mathbb{Z}[\widetilde{\Sigma}_i]$, for each i, the component $\delta_i^{4,0} \in H^4(X,\mathbb{Z}) \otimes H^0(\widetilde{\Sigma}_i,\mathbb{Z})$ can be written $pr_1^*\alpha_i$ for a cohomology class $\alpha_i \in H^4(X,\mathbb{Z})$ and by projection formula $pr_{1,*}(\delta_i^{4,0} \cup [X \times x_i]) = pr_{1,*}(pr_1^*\alpha_i \cup [X \times x_i]) = \alpha_i$ for a (any) point $x_i \in \widetilde{\Sigma}_i$. Now, we have $[\widetilde{\Gamma}'_2] \cup [X \times x_i] = [\widetilde{\Gamma}'_2 \cdot (X \times x_i)] = \delta_i^{4,0} \cup [X \times x_i]$; applying $pr_{1,*}$ to these equalities yields $[pr_{1,*}(\widetilde{\Gamma}'_2 \cdot (X \times x_i))] = \alpha_i$ i.e. $\delta_i^{4,0} = pr_1^*\alpha_i$ is algebraic and supported on $pr_{1,*}(\widetilde{\Gamma}'_2 \cdot (X \times x_i)) \times \widetilde{\Sigma}_i$ and $pr_{1,*}(\widetilde{\Gamma}'_2 \cdot (X \times x_i))$ has codimension 2 in X, in particular it does not dominate X. So the component $\delta^{4,0} = \sum_i \delta_i^{4,0}$ is algebraic and represented by a cycle which does not dominate X by the first projection.

As $\mathrm{CH}_0(X) \simeq \mathbb{Z}$, by [13, Proposition 1], there is an integer $N \neq 0$ such that

$$N\Delta_X = N(X \times x) + Z \text{ in } \mathrm{CH}^n(X \times X) \tag{3.5}$$

where Z is supported on $D' \times X$ for some proper closed subset D' of X. Looking at the action on $H^1(X,\mathbb{Z})$, we see that $H^1(X,\mathbb{Z})$ is a torsion group annihilated by N but since $H^1(X,\mathbb{Z})$ is torsion-free, $H^1(X,\mathbb{Z}) = 0$. Hence, $\delta^{1,3} = 0$. Letting the correspondences of (3.5) act on the complex vector space $H^0(X,\Omega_X^2)$ we see that it is annihilated by N i.e. it is 0, so that by Lefschetz theorem on $(1,1)$ classes, $H^2(X,\mathbb{Z})$ is algebraic. So the Hodge class $\delta^{2,2}$ belongs to $H^{1,1}(X) \otimes H^{1,1}(\widetilde{\Sigma})$; it is thus algebraic and of the form $[\sum_i D_i \times C_i]$ where the D_i are divisors on X and the C_i are curves on $\widetilde{\Sigma}$, in particular it does not dominate X by the first projection. □

Proof of Theorem 3.0.7. Let us assume that X satisfies condition (*) and has essential CH_0-dimension ≤ 2. We thus have:

$$\Delta_X = \Gamma_1 + \Gamma_2 \text{ in } \mathrm{CH}^n(X \times X) \tag{3.6}$$

where Γ_1 is supported on $D \times X$ for some proper closed subset $D \subset X$ and Γ_2 is supported $X \times \Sigma$.

Let us write $\widetilde{\Sigma} = \sqcup_i \widetilde{\Sigma}_i$ where the $\widetilde{\Sigma}_i$ are smooth connected surfaces. Choose a point $\sigma_i \in \widetilde{\Sigma}_i$. For each i, the cycle $\widetilde{\Gamma}_{2,i} := \widetilde{\Gamma}_{2|\widetilde{\Sigma}_i} \in \mathrm{CH}^2(X \times \widetilde{\Sigma}_i)$ can be written as $Z_i \times \widetilde{\Sigma}_i + \widetilde{\Gamma}_{2,i,alg}$ where $Z_i \in \mathrm{CH}^2(X)$ is defined as $\widetilde{\Gamma}_{2,i,*}(\sigma_i)$ and $\widetilde{\Gamma}_{2,i,alg}$ is a family of cycles algebraically equivalent to 0 on X parametrized by $\widetilde{\Sigma}_i$. By condition (*) applied to each field $\mathbb{C}(\widetilde{\Sigma}_i)$, we get a cycle $\mathcal{Z}_i \in Pic(Y \times \widetilde{\Sigma}_i)$ such that $\mathcal{Z}_i - \widetilde{\Gamma}_{2,i,alg}$ vanishes in $\mathrm{CH}^2(X \times U_i)$ where $U_i \subset \widetilde{\Sigma}_i$ is a dense open subset. By the localization exact sequence, we conclude that the cycle $\mathcal{Z}_i - \widetilde{\Gamma}_{2,i,alg}$ is supported on $\cup_j X \times C_{i,j}$ where $\cup_j C_{i,j} = \widetilde{\Sigma}_i \backslash U_i$. Putting everything together, we conclude that $\Delta_X = Z' + Z''$ where Z' is supported on $D' \times X$ and Z'' is supported on $X \times C$ where $C = \cup_{i,j} C_{i,j}$. We thus conclude that the essential CH_0-dimension of X is ≤ 1 and the proof is concluded by applying Theorem 3.0.4. □

3.2 Universal generation of codimension 2 cycles

In this section, we discuss the relation of condition (*) to some conditions that are more intuitive geometrically.

Let X be a smooth projective complex variety satisfying $\mathrm{CH}_0(X) = \mathbb{Z}$. Then by a theorem of Roitman $H^{i,0}(X) = 0$ for any $i > 0$, so that the Hodge structure on $H^3(X, \mathbb{Z})$ has level 1 and $H^{2n-1}(X, \mathbb{Q}) = 0$. So $H^3(X, \mathbb{Z})_{prim} := Ker(c_1(\mathcal{O}_X(1))^{n-3+1} \cup : H^3(X, \mathbb{Z})_{/Tors} \to H^{2n-1}(X, \mathbb{Z})_{/Tors})$ is the whole of $H^3(X, \mathbb{Z})_{/Tors}$ so that the bilinear form defined on $H^3(X, \mathbb{Z})_{/Tors}$ using a polarization $\mathcal{O}_X(1)$, polarizes the intermediate Jacobian for codimension 2 cycles $J^3(X)$ is an abelian variety. By work of Bloch and Srinivas, we have in our setting $\mathrm{CH}^2(X)_{alg} = \mathrm{CH}^2(X)_{hom} \simeq J^3(X)(\mathbb{C})$. The condition $(*)$ is related to codimension 2-cycles. In [91, Theorem 2.1], Voisin exhibited a (birationally invariant) necessary condition for stable rationality, namely the existence of a universal codimension 2 cycle i.e. the existence of a correspondence $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)$ such that the induced Abel-Jacobi morphism $\Phi_{\mathcal{Z}} : J^3(X) \to J^3(X)$, given by $t \mapsto \rho(\mathcal{Z}_t - \mathcal{Z}_{t_0})$, where $\rho : \mathrm{CH}^2(X)_{alg} \to J^3(X)$ is the natural regular morphism in the sense of Murre ([64]), is the identity. We have the following relation with condition (*):

Proposition 3.2.1. *Let X be a smooth projective complex variety satisfying $\mathrm{CH}_0(X) = \mathbb{Z}$ and condition $(*)$. Assume moreover that $j_* : \mathrm{Pic}^0(\widetilde{Y}) \to J^3(X) \simeq \mathrm{CH}^2(X)_{alg}$ is split. Then there is a universal codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)$.*

Proof. As j_* is split, the splitting morphism gives an imbedding $s : J^3(X) \hookrightarrow \mathrm{Pic}^0(\widetilde{Y})$. Denoting $\mathcal{P}$ the Poincaré divisor on $\mathrm{Pic}^0(\widetilde{Y}) \times \widetilde{Y}$, set $\mathcal{Z} = (id_{J^3(X)}, j)_*(s, id_{\widetilde{Y}})^*\mathcal{P}$ in $\mathrm{CH}^2(J^3(X) \times X)$. Then, by construction, $\mathcal{Z}$ is a universal codimension 2 cycle. Indeed, the Abel-Jacobi morphism $\Phi_{\mathcal{Z}} : J^3(X) \to J^3(X)$ is just given by $j_* \circ s_*$ which is the identity by definition of s. □

We have this other proposition relating condition (*) to the existence of a universal codimension 2 cycle:

Proposition 3.2.2. *Let X be a smooth projective complex variety satisfying* $\mathrm{CH}_0(X) = \mathbb{Z}$. *Assume $J^3(X)$ is a direct factor of a sum of Jacobian of curves $\oplus_i J(C_i)$. Then the existence a universal codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)$ implies condition $(*)$.*

Proof. Let us denote $p : \oplus_i J(C_i) \to J^3(X)$ the projection to $J^3(X)$ and $s : J^3(X) \to \oplus_i J(C_i)$ a section. Using the morphisms

$$C_i \overset{j_i}{\hookrightarrow} J(C_i) \overset{p}{\to} J^3(X)$$

we get a correspondence $\mathcal{Y} \in \mathrm{CH}^2(\sqcup_i C_i \times X)$, defined on each $C_i \times X$ as $(p \circ j_i, id_X)^* \mathcal{Z}$, such that the induced Abel-jacobi morphism $\Phi_{\mathcal{Y}} : \sqcup_i C_i \to J^3(X)$ coincides with $\oplus_i p \circ j_i$. Let us denote $D = pr_{2,*}(\mathcal{Y}) \in \mathrm{CH}^1(X)$ and $\widetilde{D}$ a desingularization of (a divisor in the class) D. We have a morphism $j : Pic^0(\widetilde{D}) \to J^3(X)$ and Abel-Jacobi morphism $\Phi_{\mathcal{Y}} : \sqcup_i C_i \to J^3(X)$ naturally factors through the morphism j so that, by the universal property of the Albanese variety, the morphism $p : \oplus_i J(C_i) \to J^3(X)$ also factors through j.

Now, let $\mathcal{K} \in \mathrm{CH}^2(W \times X)$ be a family of codimension 2 cycles of X parametrized by a smooth quasi-projective base W. Let us consider the Abel-Jacobi morphism $\Phi_{\mathcal{K}} : W \to J^3(X)$; we have the equality $p \circ s \circ \Phi_{\mathcal{K}} = \Phi_{\mathcal{K}}$ but p factors through $j : Pic^0(\widetilde{D}) \to J^3(X)$. □

3.3 Application to threefolds

As we have seen in Remark 3.0.1, when X has dimension 3, the fact that the CH_0 group of X is universally supported on a surface Σ can be written in $\mathrm{CH}^3(X \times X)$, as

$$\Delta_X = \Gamma_1 + \Gamma_2 \tag{3.7}$$

where Γ_2 is supported $X \times \Sigma$ and Γ_1 is supported on $D \times X$ for some proper closed subset $D \subset X$, in particular D is a 2-dimensional closed subscheme of X. We thus have a symmetric situation in term of dimension of the supports of the Γ_i. We recall that in this setting, $J^3(X)$ is a principally polarized abelian variety, whose polarization is given by the unimodular intersection form $\langle \cdot, \cdot \rangle_{J^3(X)}$ induced by the intersection form on $H^3(X, \mathbb{Z})$ via the isomorphism $H_1(J^3(X), \mathbb{Z}) \simeq H^3(X, \mathbb{Z})_{/Tors}$. We have the following result:

Theorem 3.3.1. *Let X be a smooth threefold satisfying* $\mathrm{CH}_0(X) = \mathbb{Z}$ *and whose essential* CH_0*-dimension is* ≤ 2. *Assume that any endomomorphism of $J^3(X)$ is self-adjoint for $\langle \cdot, \cdot \rangle_{J^3(X)}$. Then X satisfies condition $(*)$. So that, by Theorem 3.0.7, the essential* CH_0*-dimension of X is* 0.

Proof. Let us consider the decomposition (3.7). For any $z \in \mathrm{CH}^2(X)_{alg} \simeq J^3(X)(\mathbb{C})$, letting both sides act on z, we get

$$z = \Gamma_1^*(z) + \Gamma_2^*(z).$$

By construction, denoting $\widetilde{D} \to X$ a desingularization (followed by the inclusion) of D, Γ_1^* factorizes through $j_* : \mathrm{Pic}^0(\widetilde{D}) \to J^3(X)$. Moreover, as Γ_2^* is an endomorphism of $J^3(X)$, it is self-adjoint so that $\Gamma_2^*(z) = \Gamma_{2,*}(z)$ and the last term factorizes through $\tilde{i}_* : \mathrm{Pic}^0(\widetilde{\Sigma}) \to J^3(X)$ where $\tilde{i} : \widetilde{\Sigma} \to X$ is the desingularization (followed by the inclusion) of Σ. Thus, we see that $j_* + \tilde{i}_* : \mathrm{Pic}^0(\widetilde{D} \sqcup \widetilde{\Sigma}) \to J^3(X)$ is surjective. Now, let $\mathcal{Z}$ be a family of algebraically trivial codimension 2 cycles parametrized by a smooth quasi-projective variety T. Since the identity on $\mathrm{CH}^2(X)_{alg}$ factors as $\Gamma_1^* + \Gamma_{2,*}$, the map $T \to \mathrm{CH}^2(X)_{alg}$ factors through $\mathrm{Pic}^0(\widetilde{D} \sqcup \widetilde{\Sigma})$. So X satisfies (*). We conclude applying Theorem 3.0.7. □

As an application of Theorem 3.3.1, let us state the following result:

Corollary 3.3.2. *The essential* CH_0*-dimension of a very general Fano complete intersection of dimension* 3 *is either* 0 *or* 3*. In particular a very general quartic threefold has essential* CH_0*-dimension equal to* 3*.*

Proof. Let us begin by the following lemma:

Lemma 3.3.3. *For a very general complete intersection* X *of dimension* 3*, we have* $End_{HS}(H^3(X,\mathbb{Q})) = \mathbb{Q}Id$ *where* $End_{HS}(H^3(X,\mathbb{Q}))$ *denote the space of endomorphisms of Hodge structure of* $H^3(X,\mathbb{Q})$*.*

Proof. The proof works as in [90, Lemma 5.1]. Indeed, by [8], the monodromy group of a smooth complete intersection of dimension 3 is Zariski dense in the symplectic group of $H^3(X,\mathbb{Q}) = H^3(X,\mathbb{Q})_{prim}$. By [84], the Mumford-Tate group of the Hodge structure on $H^3(X,\mathbb{Q})$ contains a finite index sub-group of the monodromy group so that the Mumford-Tate group of $H^3(X,\mathbb{Q})$ in the above cases is the symplectic group. $\square$

Let X be a very general Fano complete intersection of dimension 3, then the endomorphisms of $J^3(X)$ are all self-adjoint for $\langle\cdot,\cdot\rangle_{J^3(X)}$. Suppose X has essential CH_0-dimension < 3. Then by Theorem 3.3.1, X has essential CH_0-dimension 0. So we have the alternative.

For quartic threefolds, it was proved in ([20]) that a very general quartic threefold does not admit a Chow-theoretic decomposition of the diagonal, so its essential CH_0-dimension is 3. $\square$

Bibliographie

[1] S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Acad. Press (1966). (Cité à la page 10.)

[2] A. B. Altman, S.L. Kleiman, *Foundation of the theory of the Fano schemes*, Compositio Mathematica 34 (1977), no.1, pp.3-47. (Cité aux pages 35, 37, 40 and 60.)

[3] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) 25 (1972), 7595. (Cité aux pages 10 and 32.)

[4] M. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology 1 (1962), pp. 25-45. (Cité à la page 17.)

[5] A. Auel, J.-L. Colliot-Thélène, R. Parimala, *Universal unramified cohomology of cubic fourfolds containing a plane*, in Brauer groups and obstruction problems: moduli spaces and arithmetic, Eds. A. Auel, B. Hassett, T. Várilly-Alvarro and B. Viray, Progress in Mathematics, vol. 320, Birkhäuser, Basel, 2017, pp. 29-56. (Cité aux pages 24, 53, 71 and 72.)

[6] W. Barth, A. Van de Ven, *Fano-varieties of lines on hypersurfaces*, Archiv der Math. 31 (1978), pp. 96-104. (Cité à la page 37.)

[7] A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) 10 (1977), no. 3, pp. 309-391. (Cité aux pages 23, 40 and 60.)

[8] A. Beauville, *Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes*, Complex analysis and Algebraic Geometry, LN 1194, 195-207; Springer-Verlag (1986). (Cité à la page 78.)

[9] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles*, Ann. of Math. 121 (1985), pp. 283-318. (Cité à la page 22.)

[10] A. Beauville, R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*, C. R. Acad. Sci. Paris Sér. I Math., 301 (14):703-706, 1985. (Cité à la page 51.)

[11] S. Bloch, *Torsion algebraic cycles and a theorem of Roitman*, Comp. Math. 39 (1979), pp. 107-127. (Cité aux pages 20 and 31.)

[12] S. Bloch, A. Ogus *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup., Sér. 4, 7, pp. 181-201 (1974). (Cité à la page 32.)

[13] S. Bloch, V. Srinivas, *Remarks on orrespondences and algebraic cycles*, Amer. J. of Math. 105 (1983), pp. 1235-1253. (Cité aux pages 22, 24, 30, 38, 50 and 75.)

[14] H. Clemens, *Homological equivalence modulo algebraic equivalence is not finitely generated*, Inst. Hautes Études Sci. Publ. Math. 58 (1983), pp. 19-38. (Cité à la page 14.)

[15] H. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. 95, pp. 281-356. (Cité aux pages 10, 20, 21, 22, 30, 32, 35, 36, 51, 56, 60, 63 and 71.)

[16] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, in K-Theory and Algebraic Geometry : Connections with Quadratic Forms and Division Algebras, AMS Summer Research Institute, Santa Barbara 1992, ed. W. Jacob and A. Rosenberg, Proceedings of Symposia in Pure Mathematics 58, Part I (1995) pp. 1-64. (Cité à la page 32.)

[17] J.-L. Colliot-Thélène, CH_0-*trivialité universelle d'hypersurfaces cubiques partiellement diagonales*, preprint arXiv:1607.05673, 2016. (Cité aux pages 26 and 72.)

[18] J.-L. Colliot-Thélène, D. Coray, *L'équivalence rationnelle sur les points fermés des surfaces rationnelles fibrées en coniques*, Compositio Mathematica 39 (1979), no. 3, pp. 301-332. (Cité à la page 56.)

[19] J.-L. Colliot-Thélène, M. Ojanguren *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford*, Invent. math. 97 (1989), no. 1, pp. 141-158. (Cité aux pages 18, 32 and 71.)

[20] J.-L. Colliot-Thélène, A. Pirutka, *Hypersurfaces quartiques de dimension 3: non rationalité stable*, Ann. Sci. École Norm. Sup. (2) 49 (2016), pp. 373-399. (Cité aux pages 25 and 78.)

[21] J.-L. Colliot-Thélène, J.-J. Sansuc, C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. 50 (1983), pp. 763-801. (Cité aux pages 20 and 31.)

[22] J.-L. Colliot-Thélène, T. Szamuely, *Autour de la conjecture de Tate à coefficients $\mathbb{Z}_\ell$ pour les variétés sur les corps finis*, in The Geometry of Algebraic Cycles, ed. R. Akhtar, P. Brosnan, R. Joshua, AMS/Clay Institute Proceedings (2010), pp. 83-98. (Cité à la page 54.)

[23] J.-L. Colliot-Thélène, C. Voisin, *Cohomologie non ramifiée et conjecture de Hodge entière*, Duke Math. J. 161 (5), pp.735-801. (Cité aux pages 18, 23 and 32.)

[24] V. Cossart, O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra 320 (2008), no.3, pp. 1051-1082. (Cité aux pages 10, 56 and 59.)

[25] V. Cossart, O. Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra 321 (2009), no.7, pp. 1836-1976. (Cité aux pages 10, 56 and 59.)

[26] O. Debarre, L. Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. 312 (1998), pp. 549-574. (Cité aux pages 39, 40, 48 and 51.)

[27] P. Deligne, *Variétés unirationnelles non rationnelles*, Séminaire Bourbaki 402 (Nov. 1971), Lectures Notes in Math. 317, Springer-Verlag 1973, pp. 45-57. (Cité à la page 11.)

[28] S. Druel, *Espace des modules des faisceaux de rang 2 semi-stables de classes de Chern $c_1 = 0$, $c_2 = 2$ et $c_3 = 0$ sur la cubique de $\mathbb{P}^4$*, Int. Math. Res. Not. 19 (2000), pp. 985-1004. (Cité aux pages 55 and 69.)

[29] H. Esnault, M. Levine, E. Viehweg, *Chow groups of projective varieties of very small degree*, Duke Math. J. 87 (1997), pp. 29-58. (Cité aux pages 29 and 30.)

[30] W. Fulton, *Intersection Theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebi ete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics an d Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. (Cité aux pages 12, 37, 44 and 46.)

[31] P. Griffiths, *Periods of integrals on algebraic manifolds I, II*, I Amer. J. Math. 90 (1968), pp. 568-626; II pp.805-865. (Cité à la page 19.)

[32] P. Griffiths, *On the periods of certain rational integrals I, II*, Ann. of Math. (2) 90 (1969), 460-495; ibid. (2) 90 (1969), 496-541. (Cité à la page 14.)

[33] A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic Geometry (Internat. Colloq. Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 193-199. (Cité à la page 15.)

[34] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics 187 (1998), Springer. (Cité à la page 65.)

[35] J. Harris, M. Roth, J. Starr, *Abel-Jacobi maps associated to smooth cubic three-folds*, arXiv preprint math/0202080. (Cité à la page 38.)

[36] R. Hartshorne, *Complete Intersections and Connectedness*, American J. of Math., vol. 84, no. 3 (Jul., 1962), pp. 497-508. (Cité à la page 39.)

[37] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52 (1977), Springer. (Cité à la page 37.)

[38] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. 254 (1980), pp. 121-176. (Non cité.)

[39] B. Hassett, A. Pirutka, Y. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, arXiv:1603.09262. (Cité à la page 25.)

[40] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, I Ann. of Math. (2), 79 (1), pp. 109-203; II pp. 205-326. (Cité à la page 10.)

[41] H. Hironaka, *Smoothing of algebraic cycles of small dimensions*, Amer. J. Math. 90 (1968), pp. 1-54. (Cité à la page 49.)

[42] K. Hulek, *Projective geometry of elliptic curves*, Astérisque 137 (1986). (Cité aux pages 65, 67 and 68.)

[43] A. Iliev and D. Markushevich, *The Abel-Jacobi map for a cubic threefold and periods of Fano threefolds of degree* 14, Documenta Mathematica, 5 (2000), pp. 23-47. (Cité aux pages 38, 55, 60 and 69.)

[44] V. Iskovskikh, Y. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Math. USSR-Sb. 15 (1971), pp. 141-166. (Cité à la page 10.)

[45] B. Kahn, R. Sujatha, *Unramified cohomology of quadrics, II*, Duke Math. J. 106 (2001), pp. 449-484. (Cité aux pages 32 and 38.)

[46] J. Kollár, Lemma p. 134 in *Classification of irregular varieties*, edited by E. Ballico, F. Catanese, C. Ciliberto, Lecture Notes in Math. 1515, Springer (1990). (Cité à la page 17.)

[47] J. Kollár, *Rational curves on algebraic varieties*, volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in M athematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in M athematics]. Springer-Verlag, Berlin, 1996. (Cité aux pages 21, 33, 37 and 57.)

[48] S. Lang, *On quasi algebraic closure*, Ann. of Math. 55 (2), 1952, pp. 373-390. (Cité à la page 34.)

[49] M. Larsen, V. A. Lunts, *Motivic measures and stable birational geometry*, Mosc. Math. J. 3 (2002) no. 1, pp. 85-95, 259. (Cité à la page 9.)

[50] C. Liedtke, *Supersingular k3 surfaces are unirational*, Invent. Math. 200 (2015), pp. 979-1014. (Cité à la page 10.)

[51] J. Lüroth, *Beweis eines Satzes über rationale Curven*, Math. Ann.9 (1876), pp. 163-165. (Cité à la page 9.)

[52] D. Markushevich, A. Tikhomirov, *The Abel-Jacobi map of a moduli component of vector bundles on the cubic threefold*, J. Algebraic Geometry 10 (2001), pp. 37-62. (Cité aux pages 38, 55, 60, 66, 67 and 69.)

[53] R. Mboro, *On the universal* CH_0 *group of cubic threefolds in positive characteristic*, Manuscripta Mathematica, 154(1), pp. 147-168. (Non cité.)

[54] R. Mboro, *Remarks on the* CH_2 *of cubic hypersurfaces*, arXiv:1701.04488. (Cité à la page 5.)

[55] R. Mboro, *Remarks on approximate decompositions of the diagonal*, arXiv:1708.02422. (Non cité.)

[56] J. S. Milne, *Lectures on étale cohomology*, available at http://www.jmilne.org/math/. (Cité à la page 55.)

[57] J. S. Milne, *Abelian Varieties*, in Arithmetic geometry, ed. G. Cornell, J. Silverman, Springer (1986), pp. 103-150. (Non cité.)

[58] B. Moonen, A. Polishchuk, *Divided powers in Chow rings and integral Fourier transforms*, Adv. Math. 224 (2010), no. 5, pp. 2216-2236. (Cité à la page 47.)

[59] D. Mumford, *Rational equivalence of zero-cycles on surfaces*, J. Math. Kyoto Univ. 9 (1968), pp. 195-204. (Cité à la page 13.)

[60] D. Mumford, *Prym varieties I*, in Contributions to analysis, Acad. Press, New-York (1974). (Cité à la page 23.)

[61] J. P. Murre, *Algebraic equivalence modulo rational equivalence on a cubic threefold*, Compositio Mathematica 25 (1972), pp.161-206. (Cité aux pages 36, 60, 62 and 63.)

[62] J. P. Murre, *Some results on cubic threefolds*, in Classification of algebraic varieties and compact complex manifolds, Lecture Notes in Math. 412, pp 140-164, Springer Berlin Heidelberg, 1974. (Cité à la page 60.)

[63] J. P. Murre, *Application of algebraic K-theory to the theory of algebraic cycles*, Proc. Conf. Algebraic Geometry, Sitges 1983, Lecture Notes in Math. 1124 (1985), pp. 216-261, Springer-Verlag, Berlin, etc (Cité aux pages 23 and 60.)

[64] J. P. Murre, *Algebraic cycles and algebraic aspects of cohomology and k-theory*, in Algebraic cycles and Hodge theory, ed. A. Albano, F. Bardelli, Lecture Notes in Math. 1594 (1994), pp. 93-152. (Cité aux pages 59 and 76.)

[65] A. Otwinowska, *Remarques sur les groupes de Chow des hypersurfaces de petit degré*, C. R. Acad. Sci. Paris Sér. I Math., 329(1):51-56, 1999. (Cité à la page 30.)

[66] X. Pan, 2-*cycles on higher Fano hypersurfaces*, Math. Ann. (2016) pp. 1-28. (Cité à la page 31.)

[67] A. A. Roitman, *The torsion of the group of* 0-*cycles modulo rational equivalence*, Annals of Math. vol. 111, no. 3, 1980, pp.553-569. (Cité aux pages 19, 32 and 51.)

[68] C. Schnell, *The Serre construction in codimension two*, available at https://www.math.stonybrook.edu/∼cschnell/. (Cité à la page 66.)

[69] C. Schnell, *Hypersurfaces containing a given subvariety*, online lecture notes. (Cité à la page 49.)

[70] C. Schoen, *An integral analog of the Tate conjecture for one-dimensional cycles on varieties over finite field*, Math. Ann. 311 (1998), no. 3, 493-500. (Cité aux pages 19, 53, 54 and 62.)

[71] J.-P. Serre, *On the fundamental group of a unirational variety*, J. London Math. Soc. 34 (1959), pp. 481-484. (Cité à la page 21.)

[72] M. Shen, *On relations among* 1-*cycles on cubic hypersurfaces*, J. Algebraic Geom. 23 (2014), pp. 539-569. (Cité aux pages 22, 29, 31, 32, 40, 50 and 58.)

[73] M. Shen, *Rationality, universal generation and the integral Hodge conjecture*, arXiv:1602.07331. (Cité aux pages 22, 29, 31, 40 and 73.)

[74] I. Shimada, *On the cylinder homomorphism for a family of algebraic cycles*, Duke Math. J. 64 (1991), pp. 201-205. (Cité à la page 50.)

[75] J. Starr, *Brauer groups and Galois cohomology of function fields of varieties*, Lecture notes, 2008. (Cité à la page 34.)

[76] Z. Tian, R. Zong, *One-cycles on rationally connected varieties*, Compos. Math. 150 (2014), no. 3, pp. 396-408. (Cité aux pages 16, 30, 37, 38, 39 and 40.)

[77] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. 10 (1997), no. 2, pp. 467-493. (Cité à la page 19.)

[78] B. Totaro, *The integral cohomology of the Hilbert scheme of two points*, arXiv:1506.00968. (Cité aux pages 49, 51 and 57.)

[79] C. Tsen, *Quasi-algebraische-abgeschlossene Funktionenkörper*, J. Chinese Math., 1:81-92, 1936. (Cité à la page 61.)

[80] V. Voevodsky, *A nilpotence theorem for cycles algebraically equivalent to zero*, Internat. Math. Res. Notices, no. 4 (1995), pp. 187-198. (Cité aux pages 56 and 75.)

[81] C. Voisin, *Remarks on zero-cycles of self-products of varieties*, in Moduli of vector bundles (Proceedings of the Taniguchi congress on vector bundles), M. Maruyama Ed., Decker (1994) 265-285. (Cité aux pages 56 and 75.)

[82] C. Voisin, *Hodge theory and complex algebraic geometry I*, Cambridge Studies in Advanced Mathematics 77, Cambridge University Press, Cambridge (2003). (Non cité.)

[83] C. Voisin, *Hodge theory and complex algebraic geometry II*, Cambridge Studies in Advanced Mathematics 77, Cambridge University Press, Cambridge (2003). (Cité à la page 29.)

[84] C. Voisin, *Hodge loci*, in Handbook of moduli (Eds G. Farkas and I. Morrison), Advanced Lectures in Mathematics 25, Volume III, International Press, 507-546. (Cité à la page 78.)

[85] C. Voisin, *On integral Hodge classes on uniruled and Calabi-Yau threefolds*, Moduli Spaces and Arithmetic Geometry, Adv. Stud. Pure Math. 45, Math. Soc. Japan, Tokyo, 2006, pp. 43-73. (Cité à la page 18.)

[86] C. Voisin, *Some aspects of the Hodge conjecture*, Japan J. Math. 2 (2007), pp. 261-296. (Cité aux pages 22, 32 and 38.)

[87] C. Voisin, *Degree* 4 *unramified cohomology with finite coefficients and torsion codimension* 3 *cycles*, in Geometry and Arithmetic, (C. Faber, G. Farkas, R. de Jong Eds), Series of Congress Reports, EMS 2012, 347-368. (Cité aux pages 20, 31 and 32.)

[88] C. Voisin, *Remarks on curve classes on rationally connected varieties*, in: A Celebration of Algebraic Geometry, Clay Math. Proc. 18, 591-599, AMS, Providence, RI, 2013. (Cité aux pages 19, 36 and 50.)

[89] C. Voisin, *Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal*, J. Algebraic Geom. 22 (2013), no. 1, 141-174. (Cité aux pages 54, 60, 61 and 74.)

[90] C. Voisin, *On the universal* CH_0 *group of cubic hypersurfaces*, J. Eur. Math. Soc. (JEMS), vol. 19, Issue 6 (2017), pp. 1619-1653. (Cité aux pages 25, 26, 53, 54, 56, 57, 58, 59, 61, 72, 74 and 78.)

[91] C. Voisin, *Unirational threefolds with no universal codimension* 2*-cycle*, Invent. Math., 201(1), pp. 207-237, 2015. (Cité aux pages 25, 72 and 76.)

[92] C. Voisin, *Stable birational invariants and the Lüroth problem*, Surveys in Differential Geometry XXI (2016). (Cité aux pages 16 and 50.)

[93] G.E. Welters, *Abel-Jacobi isogenies for certain types of Fano threefolds*, Mathematical Centre Tracts 141; Math. Centrum, Amsterdam, 1981. (Cité aux pages 67 and 68.)

[94] O. Zariski, *On Castelnuovo's criterion of rationality* $p_a = P_2 = 0$ *of an algebraic surface*, Illinois Journal of Mathematics, 2: 303-315.

(Cité à la page 10.)

Titre : Invariants birationnels: cohomologie, cycles algébriques et théorie de Hodge.

Mots Clefs : invariant birationnel, cycles algébriques, décomposition de la diagonale, variété rationnelle, jacobienne intermédiaire, application d'Abel-Jacobi.

Résumé : Dans cette thèse, nous étudions certains invariants birationnels des variétés projectives lisses, en lien avec les questions de rationalité pour ces variétés. Elle se compose de trois chapitres qui peuvent être lus indépendamment.
Dans le premier chapitre, nous étudions, pour certaines familles de variétés, certains invariants birationnels stables, nuls pour l'espace projectif et apparaissant naturellement avec les formules de Manin. D'une part nous montrons que l'invariant birationnel stable $\mathrm{CH}^3(X)_{tors,AJ}$ qu'est le groupe des cycles de torsion de codimension 3 contenus dans le noyau de l'application classe de cycle de Deligne est, pour une hypersurface cubique complexe X de dimension 5, contrôlé par l'invariant birationnel stable $\mathrm{CH}_1(F(X))_{tors,AJ}$ de sa variété des droites $F(X)$ donné par le groupe des 1-cycles de torsion contenus dans le noyau de l'application classe de cycle de Deligne. D'autre part, on établit dans ce chapitre la nullité de l'invariant birationnel stable qu'est le groupe de Griffiths $\mathrm{Griff}_1(F(X))$ des 1-cycles pour la variété des droites $F(X)$ d'une hypersurface X de l'espace projectif sur un corps algébriquement clos de caractéristique 0, lorsque celle-ci est lisse et de Fano d'indice au moins 3.
Les deux derniers chapitres se concentrent sur des aspects différents d'une propriété invariante par équivalence birationnelle stable introduite récemment par Voisin: l'existence d'une décomposition de Chow de la diagonale. Dans le second chapitre, nous étendons à la caractéristique positive $p > 2$, une partie des résultats obtenus par Voisin sur la décomposition de Chow de la diagonale des hypersurfaces cubiques complexes de dimension 3.
Dans le dernier chapitre, on étudie la notion de dimension CH_0 essentielle introduite par Voisin et reliée à l'existence d'une décomposition de Chow de la diagonale en ce que dire d'une variété qu'elle est de dimension CH_0 essentielle nulle équivaut à affirmer l'existence d'une décomposition de Chow de sa diagonale. Nous présentons des conditions suffisantes (et nécessaires) pour assurer qu'une variété complexe à CH_0 trivial, dont la dimension CH_0 essentielle est ≤ 2 est de dimension CH_0 essentielle nulle.

Title : Birational invariants: cohomology, algebraic cycles and Hodge theory.

Keys words : birational invariants, algebraic cycles, decomposition of the diagonal, rational variety, intermediate Jacobian, Abel-Jacobi map.

Abstract : In this thesis, we study some birational invariants of smooth projective varieties related to rationality questions for those varieties. It consists of three parts, that can be read independently.
In the first chapter, we study, for some families of varieties, some stable birational invariants, trivial for the projective space and that appear naturally with Manin formulas. On one hand, we show that, for complex cubic 5-folds, the stable birational invariant $\mathrm{CH}^3(X)_{tors,AJ}$ given by the group of torsion codimension 3 cycles annihilated by the Deligne cycle class, is controlled by a stable birational invariant of the variety of lines $F(X)$ of X: the group $\mathrm{CH}_1(F(X))_{tors,AJ}$ of torsion 1-cycles annihilated by the Deligne cycle class. On the other hand, we show, in the chapter, that the Griffiths group of 1-cycles $\mathrm{Griff}_1(F(X))$ of the variety of lines $F(X)$ of a hypersurface X over an algebraically closed field of characteristic 0, which is a stable birational invariant of $F(X)$, is trivial when $F(X)$ is smooth and Fano of index at least 3.
The last two chapters focus on different aspects of the Chow-theoretic decomposition of the diagonal, a property which is invariant under stable birational equivalence, recently introduced by Voisin. In the second chapter, we adapt in characteristic > 2 part of the results, obtained by Voisin over $\mathbb{C}$, on the decomposition of the diagonal of cubic threefolds.
In the last chapter, we study the notion of essential CH_0-dimension introduced by Voisin and related to the decomposition of the diagonal in that having essential CH_0-dimension 0 is equivalent to admitting a Chow-theoretic decomposition of the diagonal. We give sufficient (and necessary) conditions, for a complex variety with trivial CH_0, having essential CH_0-dimension ≤ 2 to admit a Chow-theoretic decomposition of the diagonal.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France