Homogenization of Hamilton-Jacobi equations and applications to traffic flow modelling

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Homogénéisation d'équations de Hamilton-Jacobi et applications au trafic routier

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Résumé

Cette thèse contient deux contributions à l'homogénéisation en espace-temps des équations de Hamilton-Jacobi du premier ordre. Ces équations sont en lien avec la modélisation du trafic routier. Enfin, sont présentés des résultats d'homogénéisation en milieu presque périodique.

Le premier chapitre est consacré à l'homogénéisation d'un système infini d'équations différentielles couplées avec temps de retard. Ce système provient ici d'un modèle microscopique de trafic routier simple. Les conducteurs se suivent sur une route rectiligne infinie et l'on tient compte de leur temps de réaction. On suppose que la vitesse de chaque conducteur est une fonction de l'interdistance avec le conducteur qui le précède: on parle d'un modèle du type “follow-the-leader”. Grâce à un principe de comparaison strict, on montre la convergence vers un modèle macroscopique pour des temps de réaction inférieurs à une valeur critique. Dans un second temps, on exhibe un contre-exemple à l'homogénéisation pour un temps de réaction supérieur à cette valeur critique, pour des conditions initiales particulières. Pour cela, on perturbe la solution stationnaire dans laquelle les véhicules sont tous équidistants aux instants initiaux.

Le second chapitre porte sur l'homogénéisation d'une équation de Hamilton-Jacobi dont l'Hamiltonien est discontinu en espace. Le modèle de trafic associé est une route rectiligne comportant une infinité de feux tricolores. Ces feux sont supposés identiques, équidistants et le déphasage entre deux feux successifs est supposé constant. On étudie l'influence à grande échelle de ce déphasage sur le trafic. On distingue la portion de route libre, qui sera représentée par un modèle macroscopique, et les feux, qui seront modélisés par des limiteurs de flux périodiques en temps. Le cadre théorique est celui par C. Imbert et R. Monneau (2017) pour les équations de Hamilton-Jacobi sur réseaux. L'étude se décompose en l'homogénéisation théorique, où l'Hamiltonien effectif dépend du déphasage, puis l'obtention de propriétés qualitatives de cet Hamiltonien à l'aide d'observations via des simulations numériques.

Le troisième chapitre présente des résultats d'homogénéisation en milieu presque périodique. On étudie tout d'abord un problème d'évolution avec un Hamiltonien stationnaire, presque périodique en espace. À l'aide d'arguments presque périodiques, on effectue dans un second temps une nouvelle preuve du résultat d'homogénéisation du second chapitre. L'Hamiltonien est alors périodique en temps et presque périodique en espace. Sont également présentes des questions encore ouvertes, notamment dans le cas où l'Hamiltonien est presque périodique en temps-espace, et dans le cas d'un modèle de trafic où les feux sont assez proches, avec donc un modèle microscopique entre les feux.

Mots-clés:

-Equations de Hamilton-Jacobi;

-solutions de viscosité;

-échelles microscopiques et macroscopiques;

-homogénéisation;

-temps de réaction;

-feux tricolores;

-presque périodicité.

Abstract

This thesis report deals with the homogenization in space and time of some first order Hamilton-Jacobi equations. It contains two contributions. The corresponding equations are derived from traffic flow modelling. We finally present some results of almost periodic homogenization.

In the first chapter, we consider a one dimensional pursuit law with delay which is derived from traffic flow modelling. It takes the form of an infinite system of first order coupled delayed equations. Each equation describes the motion of a driver who interacts with the preceding one: such a model is referred to as a "follow-the-leader" model. We take into account the reaction time of drivers. We derive a macroscopic model, namely a Hamilton-Jacobi equation, by a homogenization process for reaction times that are below an explicit threshold. The key idea is to show, that below this threshold, a strict comparison principle holds for the infinite system. Above this threshold, we show that collisions can occur. In a second time, for well-chosen dynamics and higher reaction times, we show that there exist some microscopic pursuit laws that do not lead to the previous macroscopic model. Such a law is here derived as a perturbation of the stationary solution, for which all the vehicles are equally spaced at initial times.

The second chapter is dedicated to the homogenization of a Hamilton-Jacobi equation for traffic lights. We consider an infinite road where lights are equally spaced and with a constant phase shift between two lights. This model takes the form of a first order Hamilton-Jacobi equation with an Hamiltonian that is discontinuous in the space variable and the notion of viscosity solution is the one introduced in [52]. Each light is modelled as a time-periodic flux limiter and the traffic flow between two lights corresponds to the classical LWR model. The global Hamiltonian will be time-periodic but not periodic in space for a general phase shift. We first show that the rescaled solution converges toward the solution of the expected macroscopic model where the effective Hamiltonian depends on the phase shift. In a second time, numerical simulations are used to analyse the effect of the phase shift on the effective Hamiltonian and to reveal some properties of the effective Hamiltonian from the numerical observations.

In the third chapter, we are interested in some homogenization problems of Hamilton-Jacobi equations within the almost periodic setting which generalizes the usual periodic one. The first problem is the evolutionary version of the work [54], with the same stationary Hamiltonian. The second problem has already been solved in the second chapter but we use here almost periodic arguments for the time periodic and space almost periodic Hamiltonian. We only study the ergodicity of the associated cell problems. We finally discuss open problems, the first one concerning a space and time almost periodic Hamiltonian and the second one being a microscopic model for traffic flow modelling where the Hamiltonian is almost periodic in space.

key-words:

-viscosity solutions theory;

-Hamilton-Jacobi equations;

-microscopic and macroscopic scales;

-homogenization;

-reaction time;

-lights;

-almost-periodicity.

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Contents

0.1 Introduction

Les deux contributions de cette thèse étudient l'influence, sur de grandes échelles de temps et d'espace, de certains paramètres rencontrés dans le cadre du trafic routier. On s'intéresse notamment au temps de réaction des conducteurs et au déphasage entre des feux tricolores. Le point de vue adopté est celui de l'homogénéisation d'équations de Hamilton-Jacobi (locales ou non).

Intéressons-nous de plus près au temps de réaction des conducteurs. Si un conducteur avec un grand temps de réaction rencontre un obstacle au dernier moment, celui-ci freinera brusquement, obligeant le conducteur derrière lui à faire de même, et ainsi de suite. Une instabilité provoquée en un endroit donné peut donc se propager sur de grandes échelles de temps et d'espace. De façon analogue, des feux mal régulés peuvent causer l'apparition d'embouteillages sur de grandes échelles.

On distingue principalement deux échelles de modélisation du trafic routier. À l'échelle microscopique, on suit la position de chaque conducteur au cours du temps. On peut par exemple supposer que la vitesse de chaque conducteur dépend de l'interdistance avec le conducteur qui le précède. On parle alors d'un modèle "follow-the-leader". L'échelle macroscopique assimile le trafic à un fluide. On y rencontre le vocabulaire associé à un écoulement: densité, flux... Certains paramètres rencontrés dans les modèles microscopiques ont un effet macroscopique. Il apparaît naturel de chercher à établir des transitions entre ces deux échelles. Les modèles macroscopiques étant liés à des approches de type fluide, il est impossible de prendre en compte la variabilité des véhicules et du comportement des conducteurs ou de certaines spécificités locales de la route. Au contraire les modèles microscopiques sont plus adaptés à une représentation fine de l'environnement local, ils sont en revanche bien plus complexes à simuler. Lorsque cela est possible, une façon rigoureuse de les relier est l'homogénéisation, après une remise à l'échelle.

Ces passages entre modèles microscopiques et modèles macroscopiques, pour une dynamique type "follow the leader", ont déjà été étudiés pour les lois de conservation scalaires par exemple dans [4] et [25]. Le cadre général comporte le plus souvent des hypothèses de périodicité. Les modèles microscopiques de ce type se présentent sous la forme d'équations aux dérivées partielles ou équations différentielles non locales en espace (à cause de la présence du conducteur qui précède). Le modèle microscopique avec temps de réaction présenté au premier chapitre est aussi non local en temps. Les modèles macroscopiques prennent en général la forme de lois de conservation scalaires ou d'équations de Hamilton-Jacobi (ces deux formes étant en général équivalentes).

Cette introduction générale démarrera par une description succincte de la théorie des solutions de viscosité des équations de Hamilton-Jacobi afin d'en présenter les origines et quelques motivations. Dans un second temps, nous exposerons quelques modèles de trafic routier. Nous verrons en quoi la théorie des solutions de viscosité peut être utilisée pour les passages entre modèles microscopiques et macroscopiques via l'homogénéisation. Nous donnerons ensuite le positionnement et les contributions de la thèse.

0.1.1 Introduction à la théorie des solutions de viscosité

Cette sous-section a pour but de donner un aperçu de la théorie des solutions de viscosité à travers mon point de vue. Il est inspiré des sources citées ci-dessous et des discussions que j'ai pues avoir au cours de mes trois années de thèse. Il ne s'agit nullement d'un cours complet ou d'une partie exhaustive, on ne traite par exemple que le cas stationnaire. On pourra se référer à [24] ou à [8] pour plus de détails.

Une motivation intéressante: définir et sélectionner des solutions faibles

Les idées présentées ci-dessous sont inspirées de [9]. De nombreux autres exemples et application de la théorie des solutions de viscosité sont présents dans la littérature, voir [8] par exemple.

Considérons l'équation suivante avec conditions de Dirichlet aux bords sur le segment $I = [0; 1]$:

$$\begin{cases} |u'(x)| = 1, & \forall x \in I \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

L'analyse des équations différentielles ou aux dérivées partielles repose en grande partie sur l'étude des propriétés d'existence, d'unicité et de régularité de solutions. Cet exemple permet d'observer que certaines de ces propriétés ne vont pas être vérifiées a priori. En effet, s'il existait une solution régulière u, on aurait existence d'un réel $c \in]0; 1[$ tel que $u'(c) = 0 \neq 1$.

La forte non-linéarité de l'équation est due à la présence d'une valeur absolue. Elle empêche d'utiliser la théorie des distributions. Si on se restreint à l'ensemble des fonctions Lipschitz sur I, c'est-à-dire, $u \in W^{1,\infty}(I)$, on peut définir des solutions de (1) au sens "presque partout" par le théorème de Rademacher. Cependant, comme montré sur la figure 1, toutes les fonctions vérifiant les conditions au bord et ayant des pentes alternant entre $+1$ et -1 sont solutions. Pour ces solutions faibles, on perd donc l'unicité.

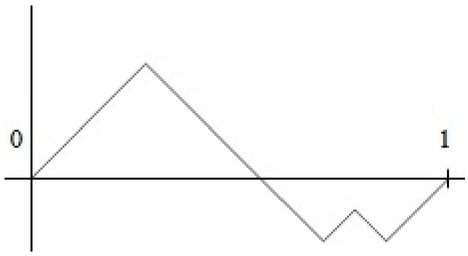

Figure 1: Exemple de solution Lipschitz.

En utilisant la méthode de la viscosité évanescente, c'est-à-dire en ajoutant un terme de la forme $-\varepsilon u''(x)$ au membre de gauche de (1) on peut obtenir une unique solution régulière, u_ε, pour chaque $\varepsilon > 0$. Une question naturelle est alors de se demander si u_ε converge lorsque ε tend vers 0. Si oui, on peut se demander si la convergence est assez forte pour que la limite soit solution de (1) en un sens qui garantirait l'unicité.

La théorie des solutions de viscosité contient la réponse à cette question, pour une large classe d'équations contenant la précédente: les équations de Hamilton-Jacobi. Elle permettra de définir un cadre précis pour obtenir l'existence et l'unicité de solutions et de définir des dérivées au sens faible pour des fonctions non régulières.

Définitions et concepts généraux

Le paragraphe 0.1.1 est largement inspirée des cours que j'ai reçus à l'ENSTA ParisTech par Hasnaa Zidani et Olivier Bokanowski (voir [16]) sur la propagation des fronts d'onde, des notes de cours de Cyril Imbert et Jérôme Droniou sur les solutions variationnelles et de viscosité pour des EDP non-linéaires (voir [27]) et du fameux "User's guide to viscosity solutions" de M. Crandall, H. Ishii, et P-L. Lions (voir [24]).

Notations. Dans le reste de la sous-section 0.1.1, on considère des équations scalaires posées sur Ω un ouvert borné de $\mathbb{R}^n$ muni de la norme euclidienne $||.||$. $S(n)$ représente l'ensemble des matrices symétriques $n \times n$ muni de son ordre partiel usuel. La solution générique des équations étudiées est notée u, son gradient par Du et sa Hessienne par D^2u.

On va maintenant voir, via la théorie des solutions de viscosité, comment donner un sens au gradient et à la Hessienne pour des fonctions non régulières. Cela permettra d'étudier toute une classe d'équations elliptiques complètement non-linéaires, qui ne possèdent pas forcément de solutions classiques.

Introduisons une définition préliminaire. Soit u une fonction localement bornée sur Ω. On définit son enveloppe semi-continue supérieure u^* par:

$$u^*(x) = \limsup_{y \to x} \; u(y) \tag{2}$$

et de façon analogue son enveloppe semi-continue inférieure u_* en remplaçant la limite supérieure par la limite inférieure. u est dite semi-continue supérieure si $u = u^*$ et semi-continue inférieure si $u = u_*$.

Ces équations elliptiques complètement non-linéaires sont de la forme:

$$F(x, u(x), Du(x), D^2u(x)) = 0, \qquad \forall x \in \Omega \tag{3}$$

où $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ vérifie les hypothèses suivantes:

$$\begin{cases} F \text{ est continue,} & \\ \forall (x,r,s,p,X) \in \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S(n), & r \leq s \Rightarrow F(x,r,p,X) \leq F(x,s,p,X) \\ \forall (x,r,p,X,Y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \times S(n), & Y \leq X \Rightarrow F(x,r,p,X) \leq F(x,s,p,Y). \end{cases} \tag{4}$$

Une telle non-linéarité F est dite propre, ce que nous supposerons par la suite dans toute la sous-section 0.1.1. La troisième hypothèse est appelée ellipticité dégénérée.

Supposons qu'il existe $u \in C^2(\Omega, \mathbb{R})$, solution classique de (3), et considérons $\varphi \in C^2(\Omega, \mathbb{R})$ telle que $u - \varphi$ ait un minimum en $x_0 \in \Omega$.

Cela implique: $Du(x_0) = D\varphi(x_0)$ et $D^2u(x_0) \geq D^2\varphi(x_0)$. Par ellipticité dégénérée de F, ceci donne: $F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$ où aucune dérivée de u n'apparaît. Cela suggère la définition suivante:

Definition 0.1.1 (Solutions de viscosité pour les équations de Hamilton-Jacobi)**.** *Supposons* (4) *et considérons $u : \Omega \to \mathbb{R}$ une fonction localement bornée.*

- *On dit que la fonction u est une sur-solution de viscosité (resp. une sous-solution)de* (3) *si u est semi-continue inférieure (resp. semi-continue supérieure) et pour tout $x_0 \in \Omega$ et toute fonction test $\varphi \in C^2(\Omega, \mathbb{R})$ telle que $u - \varphi$ atteigne un minimum local (resp. un maximum local) en x_0, alors on a:*
$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \quad (\textit{resp.} \leq 0).$$
- *Une fonction u est une solution de viscosité de* (3) *si elle est à la fois sous-solution et sur-solution.*

Une solution de viscosité est donc continue par définition et par construction une solution classique est une solution de viscosité.

Remark 0.1.2 (À propos de l'extremum local)**.** *Les hypothèses à propos du minimum (resp. maximum) peuvent être remplacée par minimum local strict (resp. maximum local strict) quitte à soustraire (resp. ajouter) un terme régulier $||x - x_0||^4$ aux fonctions test et une constante. Ce terme ne change pas les dérivées premières et secondes en x_0.*

Il existe une caractérisation en termes de sous- et sur-différentiels d'ordre 2 (voir [27] pour plus de détails). Cette caractérisation est particulièrement utile pour savoir si une fonction donnée est solution de viscosité ou non. Par exemple, elle nous aide à montrer que la fonction suivante est une solution de viscosité de (1) et qui en réalité, est unique:

$$u : \begin{cases} [0,1] \longrightarrow \mathbb{R} \\ x \in [0, \frac{1}{2}] \mapsto x \\ x \in]\frac{1}{2}, 1] \mapsto 1 - x. \end{cases} \tag{5}$$

Il est par ailleurs important de noter que par construction, la théorie des solutions de viscosité ne permettra pas de sélectionner les mêmes solutions pour résoudre $F = 0$ et $-F = 0$.

Stabilité, passages à la limite et exemple historique de la méthode de la viscosité évanescente.

On peut se demander d'où provient le mot "viscosité" utilisé dans la définition précédente. La raison historique est que pour les équations de Hamilton-Jacobi du premier ordre ($F(x, u, Du) = 0$), les solutions de viscosité ont d'abord été obtenues par la méthode de la viscosité évanescente. Le premier objectif du paragraphe 0.1.1 est d'illustrer la façon dont cette méthode a historiquement inspiré la Définition 0.1.1.

Afin d'étudier les solutions de $F(x, u, Du) = 0$, considérons l'équation suivante sur Ω pour $\varepsilon > 0$:

$$-\varepsilon \Delta u_\varepsilon(x) + F(x, u_\varepsilon(x), Du_\varepsilon(x)) = 0. \tag{6}$$

On admet que pour tout $\forall \varepsilon > 0$, il existe une unique solution $u_\varepsilon \in C^2(\Omega, \mathbb{R}) \cap W^{1,\infty}(\Omega, \mathbb{R})$ solution de (6) (voir la Section 1.4 de [9] pour plus de détails). On admet aussi que la constante de Lipschitz ne dépend pas du paramètre ε. Par le théorème d'Ascoli, on en déduit (à une sous-suite près) que $(u_\varepsilon)_{\varepsilon>0}$ converge uniformément vers une fonction $u_0 \in C(\Omega, \mathbb{R})$. Cela n'implique pas la convergence des dérivées, et ne permet donc pas de passer à la limite $\varepsilon \to 0$ dans (6).

Il serait très pratique de trouver une caractérisation des solutions classiques de (6) qui ne dépende pas des dérivées des solutions, tout du moins en apparence, pour tenter de passer à la limite. C'est l'objet du théorème suivant:

Theorem 0.1.3 (Caractérisation d'être solution de (6))**.**
Considérons une fonction $v \in C^2(\Omega, \mathbb{R})$. La fonction v est une solution classique de (6) si et seulement si pour tout $x_0 \in \Omega$ et toute fonction test $\varphi \in C^2(\Omega, \mathbb{R})$ telle que $v - \varphi$ atteint un minimum local (resp. un maximum local) en x_0, nous avons:

$$-\varepsilon \Delta \varphi(x_0) + F(x_0, v(x_0), D\varphi(x_0)) \geq 0, \quad (\textit{resp. } \leq).$$

Il devient alors possible de passer "formellement" à la limite $\varepsilon \to 0$ afin d'introduire la définition historique de solution de viscosité de $F(x, u, Du) = 0$ pour la limite locale uniforme u_0 de $(u_\varepsilon)_{\varepsilon>0}$. C'est précisément la même formulation que celle de la Définition 0.1.1 à condition que F ne dépende pas de X (avec un argument de densité, on peut étendre la définition pour des fonctions tests seulement $C^1(\Omega, \mathbb{R})$). Cependant, ce raisonnement formel, certes éclairant, n'est en aucun cas une preuve de convergence des solutions de viscosité au sens de la Définition 0.1.1. Celui-ci néglige le fait que les fonctions test pour u_0 et pour la suite $(u_\varepsilon)_{\varepsilon>0}$ sont différentes.

Le théorème précis eu égard à la Définition 0.1.1 est vrai et est donné ci-dessous. Sa preuve repose sur le fait que l'on peut perturber les fonctions test pour u_0 par des constantes additives dépendantes du paramètre $\varepsilon > 0$ pour en faire des fonctions test pour u_ε.

Theorem 0.1.4 (Passage à la limite dans (6))**.** *Chaque point d'accumulation de la suite $(u_\varepsilon)_{\varepsilon>0}$ pour la convergence locale uniforme sur Ω est une solution de viscosité de $F(x, u, Du) = 0$.*

Remark 0.1.5. *Si cette équation a une unique solution de viscosité, alors il n'y a qu'un seul point d'accumulation et toute la suite* $(u_\varepsilon)_{\varepsilon>0}$ *converge vers cette valeur.*

Ce théorème peut être utilisé pour montrer un résultat de convergence dans le cadre de la méthode de la viscosité évanescente utilisée dans (1). Il s'agit d'une convergence locale uniforme vers la fonction définie en (5).

Ce théorème pourra en fait être vu comme un corollaire de la stabilité des solutions de viscosité définies dans le sens le plus général (Définition 0.1.1) vis-à-vis de passages à la limite. Cette compatibilité avec les passages à la limite locale uniforme est présentée dans le Théorème 0.1.6 et est le second objet de ce paragraphe.

Theorem 0.1.6 (Passages à la limite uniforme)**.** *Considérons* $(F_\varepsilon)_{\varepsilon>0}$ *une suite de non-linéarités propres et* $(u_\varepsilon)_{\varepsilon>0}$ *une suite de sous-solutions correspondantes (resp. sur-solutions).*

Si $F_\varepsilon \to F$ *dans* $L^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n))$ *et* $u_\varepsilon \to u$ *dans* $L^\infty(\Omega)$*, alors* u *est une sous-solution (resp. sur-solution) de* (3)*.*

La preuve est détaillée dans le Théorème 3.1 de [9]. Il est important de remarquer que ce théorème permet de passer à la limite dans des équations complètement non-linéaires en les dérivées en supposant seulement la convergence uniforme des fonctions. Cette hypothèse de convergence locale uniforme peut être affaiblie pour des solutions seulement localement bornées. On parle alors de stabilité discontinue. Celle-ci fût étudiée par G. Barles et B. Perthame à l'aide des semi-limites relaxées, voir par exemple [8].

Regardons maintenant deux autres aspects classiques sur les EDP: l'existence puis l'unicité des solutions.

Existence: méthode de Perron

On commence par introduire une notion plus faible de solution de viscosité pour les fonctions discontinues. C'est cette notion qui intervient tout d'abord dans les théorèmes d'existence.

Definition 0.1.7 (Solutions de viscosité discontinues)**.** *Considérons* u*, une fonction localement bornée sur* Ω*. On dit que* u *est une sous-solution (resp. sur-solution) de viscosité discontinue de* (3) *si* u^* *(resp.* u_**) est sous-solution (resp. sur-solution) de* (3)*. La fonction* u *est une solution de viscosité discontinue si elle est sous-solution discontinue et sur-solution discontinue.*

On remarque que si u est continue et est une solution de viscosité discontinue alors elle sera une solution de viscosité au sens de la Définition 0.1.1. On donne maintenant un lemme de stabilité et la méthode de Perron comme corollaire, le lecteur intéressé peut par exemple regarder [19] pour plus de détails.

Lemma 0.1.8 (Supremum de sous-solutions)**.** *Considérons* $(u_\alpha)_{\alpha\in A}$ *une famille de fonctions uniformément localement bornée par au-dessus. Si* $(u_\alpha)_{\alpha\in A}$ *sont sous-solutions de* (3) *alors* $u := \sup_{\alpha\in A} u_\alpha$ *est sous-solution discontinue de* (3)*.*

Un lemme analogue existe pour l'infimum de sur-solutions. Voici maintenant le théorème d'existence de solutions de viscosité:

Theorem 0.1.9 (Méthode de Perron)**.** *Soient* u *et* v *respectivement une sous-solution discontinue et une sur-solution discontinue de* (3) *telles que* $u \le v$ *dans* Ω*. Alors, il existe une solution de viscosité discontinue* w *de* (3) *telle que* $u \le w \le v$*.*

La méthode de Perron pré-suppose l'existence de sous-solutions et sur-solutions discontinues qui sont communément appelées barrières. Ces barrières ne seront pas difficiles à construire en général, bien que parfois techniques. Le lecteur intéressé pourra se référer à l'Exemple 4.6 de [24] ou au Lemme 2.11 de [34] pour une équation non locale.

L'unicité est une conséquence des principes de comparaison que nous allons maintenant étudier. La méthode de Perron est très générale au sens où elle est commune à une grande classe d'équations. Au contraire, l'obtention de principes de comparaison, tant sur leur énoncé que sur les preuves, est un enjeu important pour de nombreuses équations de Hamilton-Jacobi.

Principe de comparaison

On se contente de donner un principe de comparaison classique et très utile pour traiter une large classe d'équations de Hamilton-Jacobi stationnaires du premier ordre. L'énoncé provient de [27] et contient la preuve. Dans le cadre d'équations d'évolution, le principe de comparaison peut s'interpréter comme la conservation de l'ordre initial entre deux solutions au cours du temps.

On considère le cas où $F(x, r, p, X) = r + H(x, p)$ (ce qui revient à étudier $u + H(x, Du) = 0$) avec H un Hamiltonien Lipschitz en la variable x. Plus précisément, on suppose qu'il existe $C > 0$ telle que:

$$|\partial_x H(x, p)| \leq C(1 + |p|), \qquad \forall (x, p) \in \Omega \times \mathbb{R}^n. \tag{7}$$

Le principe de comparaison suivant traduit la conservation de l'ordre des solutions par rapport aux conditions de bord :

Theorem 0.1.10 (Un principe de comparaison pour les équations du premier ordre)**.**
Sous l'hypothèse (7), considérons u et v respectivement une sous-solution et une sur-solution de (3) sur Ω telles que $u \leq v$ sur $\partial\Omega$. Alors on a l'inégalité suivante: $u \leq v$ sur Ω.

L'intérêt premier de ces principes de comparaison est qu'ils induisent l'unicité de la solution au problème considéré. Ces principes sont aussi utiles pour la méthode de Perron car ils impliquent la continuité des solutions construites.

0.1.2 Une vue d'ensemble sur les échelles et modèles de trafic

On rappelle qu'il y a principalement deux échelles que l'on peut distinguer pour modéliser le trafic routier. Tout d'abord, l'échelle microscopique, qui considère les véhicules individuellement et où l'on suit l'évolution de leurs positions au cours du temps. Nous considérerons typiquement une route rectiligne pour les modèles microscopiques dans cette thèse. Les modèles macroscopiques étudient l'évolution temporelle de la densité de trafic à grande échelle. Les figures 2 et 3 donnent une illustration de ces deux échelles, la deuxième image décrivant une situation plus générale que la route rectiligne. L'homogénéisation peut être vue comme le passage rigoureux de l'échelle microscopique à l'échelle macroscopique.

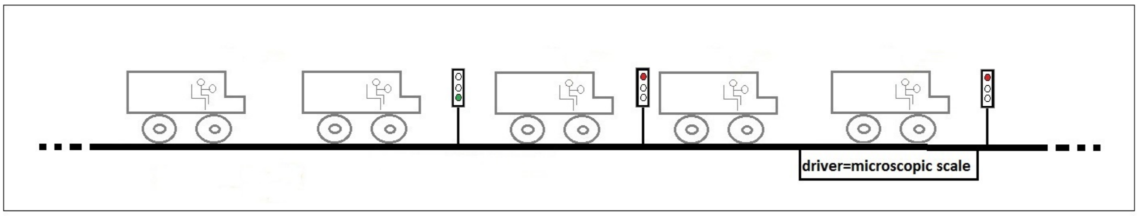

Figure 2: Vue de l'échelle microscopique considérée.

Figure 3: Vue schématique d'une échelle macroscopique générique provenant de sytadin.fr.

Introduction aux modèles microscopiques

Les modèles microscopiques du type "follow the leader" du premier ordre reposent sur l'hypothèse que la vitesse d'un conducteur dépend exclusivement de l'interdistance avec le conducteur qui précède. Ces modèles sont couramment désignés comme des modèles particulaires. Etant donné que la longueur des voitures peut être intégrée directement dans la fonction de vitesse, on peut en effet assimiler les voitures à des points. Il existe aussi des modèles du second ordre; un véhicule dont la vitesse réelle est plus petite que la fonction de vitesse décrite précedemment aura tendance à accélérer, voir par exemple [6]. Les modèles du second ordre ne seront pas traités dans le cadre de cette thèse. On renvoie à [17] ou [50] pour un état de l'art, ou encore aux thèses, [73], [22] et [71] pour un inventaire plus récent.

Il existe une très grande variété dans ces modèles, car ceux-ci peuvent tenir compte de plus ou moins de paramètres.

On peut par exemple:

- tenir compte d'une distance de sécurité en imposant une vitesse nulle en dessous d'une certaine interdistance,
- considérer une fonction de vitesse différente pour chaque type de véhicules,
- ajouter des feux tricolores,
- tenir compte du temps de réaction des conducteurs.

Introduisons maintenant le modèle microscopique le plus simple: tous les véhicules ont la même fonction de vitesse et les conducteurs ont un temps de réaction nul.

On considère une suite de fonctions numériques $(X_i)_{i\in\mathbb{Z}}$ représentant les positions des conducteurs sur une route. On choisit pour convention que les positions sont croissantes en les indices, c'est-à-dire que le conducteur $i+1$ est situé devant le conducteur i.

La vitesse de chaque conducteur est une fonction V continue, croissante et bornée de l'interdistance avec le conducteur de devant. Le modèle microscopique prend alors la forme d'un système infini d'équations différentielles non-linéaires couplées:

$$\frac{dX_i}{dt}(t) = V(X_{i+1}(t) - X_i(t)), \qquad \forall i \in \mathbb{Z}. \tag{8}$$

Introduction aux modèles macroscopiques

Dans ces modèles, le trafic routier est étudié sur de grandes échelles. L'ensemble des véhicules est vu comme un fluide. On y rencontre le vocabulaire typique de la mécanique des fluides en coordonnées euleriennes. La densité instantanée ρ des véhicules est définie comme le nombre de véhicules par unité de longueur. Le débit Q correspond au nombre de véhicules qui passent en un point par unité de temps. La conservation du nombre de véhicules sur une section infinitésimale donne la loi de conservation scalaire suivante:

$$\rho_t(x,t) + Q_x(x,t) = 0, \qquad \forall (x,t) \in \mathbb{R} \times (0,+\infty). \tag{9}$$

On définit par ailleurs la vitesse de flot v comme le quotient du débit par la densité. La quantité $v(t,x)$ donne la vitesse moyenne des véhicules situés en x à l'instant t. On y ajoute par ailleurs la loi d'état suivante: à l'équilibre, la vitesse v est uniquement une fonction (décroissante) de la densité ρ.

Le modèle macroscopique correspondant à cet état d'équilibre permanent est communément appelé modèle LWR. Il fût introduit indépendamment par Lighthill et Whitham d'un côté, et Richards de l'autre dans les années 1950, voir [60] et [69]. Ce modèle, qui est un cas particulier de (9), prend la forme suivante:

$$\rho_t + (\rho v(\rho))_x = 0. \tag{10}$$

Le débit $Q = \rho v(\rho) := f(\rho)$ est ici appelé flux de trafic. La fonction f est appelée diagramme fondamental. Il existe une large littérature autour de ces diagrammes fondamentaux. Le plus classique est le diagramme fondamental parabolique introduit par [45] qui est équivalent à dire que la vitesse de flot est linéairement décroissante en la densité. On pourra se référer à [71] pour une liste de certains diagrammes classiques.

Ces diagrammes fondamentaux partagent des propriétés communes. Pour une très petite densité, les interactions entre véhicules sont quasiment inexistantes, donc ces véhicules roulent à vitesse maximale: c'est le régime libre. À l'inverse, lorsque la densité est plus grande qu'une densité critique, les véhicules ont un effet bloquant les uns sur les autres: on parle de régime congestionné.

Le modèle LWR permet un changement de vitesse instantanée et est donc sujet à l'apparition de discontinuités en temps fini. D'autres modèles macroscopiques (d'ordre supérieur) ont émergé afin de palier ce défaut, voir par exemple [5]. Le lecteur intéressé peut consulter d'autres modèles macroscopiques dans [42] et une étude sur l'instabilité dans des modèles macroscopiques de trafic routier dans [70].

Définissons s comme l'inverse de la densité (s traduit en quelque sorte l'espacement entre les véhicules) et y comme la variable continue généralisant l'indice des véhicules. On sait grâce à [75] que (10) est équivalent à:

$$s_t - (G(s))_y = 0, \tag{11}$$

où $G(s) := v\left(\frac{1}{s}\right)$.

La primitive spatiale de s, qui est l'espacement, désigne la position du véhicule d'indice y à l'instant t. On note $u(y,t)$ cette position.

D'après [59], l'équation précédente (11), qui est écrite en coordonnées lagrangiennes, est équivalente à l'équation de Hamilton-Jacobi suivante:

$$u_t - G(u_y) = 0,$$

et on a la relation suivante concernant le diagramme fondamental: $f(\rho) = \rho v(\rho) = \rho G\left(\frac{1}{\rho}\right)$. On voit donc en quoi la théorie des solutions de viscosité peut être utilisée pour décrire certains modèles de trafic; on considèrera toujours des modèles macroscopiques de ce type dans le cadre de cette thèse. Voir aussi la Section 3 du Chapitre 4 de [22] pour une justification formelle.

0.1.3 Introduction à l'homogénéisation

On va tout d'abord s'intéresser à l'homogénéisation du modèle microscopique (8) et voir de quelle façon la théorie des solutions de viscosité interviendra. On présentera ensuite un résultat classique d'homogénéisation en espace-temps sur des équations de Hamilton-Jacobi.

Transition entre modèles de trafic: exemple via Hamilton-Jacobi

L'homogénéisation consiste à réaliser le passage rigoureux d'un modèle microscopique lagrangien vers le modèle macroscopique eulerien correspondant lorsque cela est possible, voir le schéma générique 4. Formellement, cela correspond à étudier le trafic lorsque la distance d'observation tend vers l'infini, ou de façon équivalente lorsque l'interdistance entre les voitures tend vers zéro.

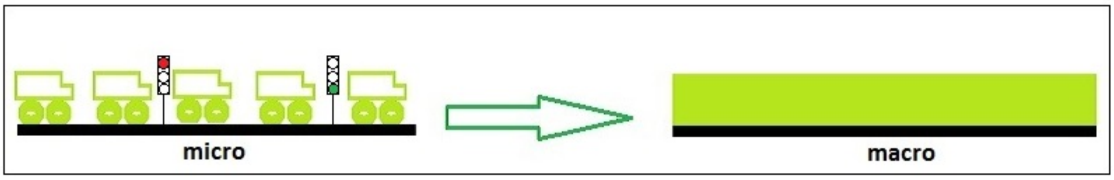

Figure 4: Description schématique du processus d'homogénéisation

Concernant l'homogénéisation de (8), il est pratique d'injecter d'abord le système infini d'EDO dans une seule EDP. Ceci est réalisé par l'interpolation continue de l'indice spatial discret des véhicules. Pour effectuer la transition d'échelle, on effectue tout d'abord une "remise à l'échelle hyperbolique" en introduisant un paramètre $\varepsilon > 0$.

On définit la fonction u^ε de la façon suivante (x dénotera la variable d'espace):

$$u^\varepsilon(x,t) = \varepsilon X_{\lfloor \frac{x}{\varepsilon} \rfloor}(\frac{t}{\varepsilon}). \tag{12}$$

On remplace dans (8) pour obtenir:

$$u_t^\varepsilon(x,t) = V\left(\frac{u^\varepsilon(x+\varepsilon,t) - u^\varepsilon(x,t)}{\varepsilon}\right). \tag{13}$$

Cette injection pourra permettre de montrer le caractère bien posé du modèle microscopique initial à l'aide de la théorie des solutions de viscosité adaptée à des EDP non-locales. Pour cela, il faut considérer le problème de Cauchy associé, avec une condition initiale u_0 Lipschitz:

$$\begin{cases} u_t^\varepsilon(x,t) = V\left(\frac{u^\varepsilon(x+\varepsilon,t)-u^\varepsilon(x,t)}{\varepsilon}\right), & \text{pour } (x,t) \in \mathbb{R} \times (0,T), \\ u^\varepsilon(x,0) = u_0(x), & \text{pour } x \in \mathbb{R}. \end{cases} \tag{14}$$

Sans rentrer dans le détail (ceci sera fait dans le premier chapitre), la fonction u_0 contient les positions initiales des véhicules.

L'homogénéisation consiste à passer à la limite $\varepsilon \to 0$ dans (14). On constate que le taux d'accroissement à l'intérieur de la fonction de vitesse converge vers la dérivée spatiale de u. Plus précisément, on va montrer que l'unique solution de viscosité u^ε de (14) converge localement uniformément vers l'unique solution de viscosité u^0 du problème de Cauchy:

$$\begin{cases} u_t^0(x,t) = V(u_x^0(x,t)), & \text{pour } (x,t) \in \mathbb{R} \times (0,T), \\ u^0(x,0) = u_0(x), & \text{pour } x \in \mathbb{R}. \end{cases} \tag{15}$$

Voici l'énoncé du théorème:

Theorem 0.1.11 (Homogénéisation du cas classique)**.** *Supposons que la fonction V soit Lipschitzienne, croissante et bornée et que la condition initiale u_0 soit aussi Lipschitzienne. Alors l'unique solution de viscosité u^ε de* (14) *converge localement uniformément vers l'unique solution de viscosité u^0 de* (15).

L'existence et l'unicité de la solution u^0 de (15) sont classiques, voir la Sous-Section 0.1.1. L'existence d'une solution u^ε de (14) est aussi classique et on renvoie à [34] pour plus de détails. Le principe de comparaison pour cette équation non-locale en espace est plus spécifique. C'est l'objet du paragraphe suivant.

On considère l'équation suivante:

$$\begin{cases} u_t = V(u(x+1,t) - u(x,t)) & \forall (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R}, \end{cases} \tag{16}$$

où l'on suppose que les fonctions V et u_0 vérifient les hypothèses suivantes:

$$\begin{cases} V \text{ est une fonction Lipschitzienne et croissante sur } \mathbb{R}, \\ u_0 \text{ est une fonction Lipschitzienne sur } \mathbb{R}. \end{cases} \tag{17}$$

La forme de (16) traduit le caractère non-local du modèle "follow-the-leader".

On note $[u]$ la fonction: $(x,t) \mapsto u(x+1,t) - u(x,t)$.

Introduisons maintenant les notions de solution de viscosité et les fonctions tests correspondantes pour ce problème non-local.

Definition 0.1.12 (Solutions de viscosité de (16)). *Supposons* (17) *et considérons* $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ *une fonction localement bornée.*

- *On dit que la fonction* u *est une sous-solution (resp. une sur-solution) de* (16) *dans* $\mathbb{R} \times (0, T)$ *si* u *est semi-continue supérieure (resp. semi-continue inférieure) et pour tout* $(x, t) \in \mathbb{R} \times (0, T)$ *et toute fonction test* $\varphi \in C^1(\mathbb{R} \times (0, T))$ *telle que* $u - \varphi$ *ait un maximum local (resp. un minimum local) en* (x, t)*, alors on a:*

$$\varphi_t \leq V([u]) \quad (\textit{resp. } \geq) \quad \textit{en } (x, t).$$

- *La fonction* u *est dite sous-solution de* (16) *sur* $\mathbb{R} \times [0, T)$ *si c'est une sous-solution (resp. une sur-solution) sur* $\mathbb{R} \times (0, T)$ *et si de plus elle satisfait pour tout* $y \in \mathbb{R}$

$$u(y, 0) \leq u_0(y) \quad (\textit{resp. } \geq).$$

- *La fonction* u *est dite solution de viscosité de* (16) *sur* $\mathbb{R} \times [0, T)$ *si elle est à la fois sous-solution et sur-solution sur* $\mathbb{R} \times [0, T)$.

On énonce maintenant un principe de comparaison pour (16):

Theorem 0.1.13 (Un principe de comparaison).
Sous les hypothèses (17) *et* (18)*, considérons* u *and* v *respectivement une sous-solution et une sur-solution de* (16) *sur* $\mathbb{R} \times [0, T)$ *telles que:*

$$\begin{cases} u(x, t) \leq u_0(x) + C_0 t & \forall (x, t) \in \mathbb{R} \times [0, T) \\ v(y, s) \geq u_0(y) - C_0 s & \forall (y, s) \in \mathbb{R} \times [0, T) \end{cases} \tag{18}$$

où $C_0 > 0$.

Alors on a $u \leq v$ *sur* $\mathbb{R} \times [0, T)$.

Les transitions entre les modèles microscopiques et macroscopiques sont aussi largement étudiées dans la littérature des lois de conservation scalaire. On peut citer [25] ou plus récemment [49] pour la convergence d'un modèle "follow-the-leader" vers un modèle LWR.

Homogénéisation en espace-temps des équations de Hamilton-Jacobi locales: cas périodique

On donne ici les idées générales de l'homogénéisation des équations de Hamilton-Jacobi périodiques. Les idées capitales ont été introduites dans les travaux fondateurs [61], [28] et [29]. Le travail [61] traita un problème d'homogénéisation dans un cadre stationnaire périodique. Dans l'article [28], L. C. Evans a introduit la célèbre méthode des fonctions tests perturbées et l'a appliqué dans [29] pour un autre problème d'homogénéisation stationnaire périodique.

Soit $\varepsilon > 0$, on considère l'unique solution de viscosité u^ε de l'équation de Hamilton-Jacobi oscillante en espace-temps:

$$\begin{cases} u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0, & \text{pour } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u^\varepsilon(0, x) = u_0(x), & \text{pour } x \in \mathbb{R}, \end{cases} \tag{19}$$

où l'on suppose:

(A0) <u>Continuité de H</u>: $H : \mathbb{R}^3 \to \mathbb{R}$ est continue.

(A1) Coercivité de H:

$$\lim_{R\to+\infty} \inf\{H(s,y,p'),(s,y)\in\mathbb{R}^2, |p'|\geq R\} = +\infty.$$

(A2) Périodicité de H: H est $1-$périodique en temps et en espace.

(A3) H uniformément borné pour des gradients bornés:

$$\sup_{(s,y,p')\in\mathbb{R}\times[-R,R]} |H(s,y,p')| < +\infty.$$

(C1) Régularité de la condition initiale: u_0 est une fonction L- Lipschitz sur $\mathbb{R}$.

Comme mentionné dans [61], la périodicité de l'Hamiltonien va engendrer un effet de moyennisation et on espère trouver un problème limite lorsque $\varepsilon\to 0$. En particulier, il s'agit d'identifier une limite éventuelle de la fonction u^ε pour une certaine norme: on parle d'homogénéisation. On aimerait que cette fonction converge localement uniformément vers une fonction moyennée u^0, lorsque $\varepsilon\to 0$. Cela revient à dire que la différence tend vers zéro pour $\varepsilon\to 0$, et il semble raisonnable de supposer que cette différence oscille avec la même fréquence que l'Hamiltonien.

Pour identifier cette limite éventuelle u^0 de la fonction u^ε, on réalise un développement asymptotique formel de la fonction u^ε qu'on injecte dans (19). On obtient alors le problème formel de la cellule pour un correcteur v:

$$\lambda + v_\tau(\tau,y) + H(\tau,y,p+v_y(s,y)) = 0, \qquad \text{pour } (\tau,y)\in\mathbb{R}^2, \tag{20}$$

où $(\lambda,p) := \left(u_t^0(t,x), u_x^0(t,x)\right)$ et $(\tau,y) := \left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)$. Dans ce cadre classique, on peut montrer que ce problème de la cellule est bien posé au sens suivant: pour tout $p\in\mathbb{R}$, il existe un unique $\lambda=\lambda(p) := -\bar{H}(p)$, tel qu'il existe une solution périodique v de (20). Une telle fonction est appelée correcteur. La démonstration repose sur l'introduction d'un problème de la cellule approché et le contrôle des oscillations en temps-espace de ce correcteur approché. Le travail [18] donne la motivation de cette approche.

Avec les définitions formelles de λ et p, on obtient le problème de Cauchy correspondant à u^0:

$$\begin{cases} u_t^0 + \bar{H}\left(u_x^0\right) = 0, & \text{sur } (0,+\infty)\times\mathbb{R}, \\ u^0(0,x) = u_0(x), & \text{pour } x\in\mathbb{R}. \end{cases} \tag{21}$$

La fonction $\bar{H}$ est appelée Hamiltonien effectif, il est également continu et coercif. Le problème de Cauchy associé admet donc une unique solution. Pour donner une preuve rigoureuse de la convergence, on utilise la méthode des fonctions tests perturbées introduite dans [28] et [29]. Le travail [61] qui est antérieur à [28] prouve la convergence à l'aide d'actions de semi-groupes et une comparaison avec une condition initiale affine.

Theorem 0.1.14 (Homogénéisation périodique [61])**.** *Sous les hypothèses (A0)-(A3) et (C1), la solution u^ε de* (19) *converge localement uniformément vers l'unique solution de* (21).

0.1.4 Positionnement et contribution de la thèse

Les principales contributions de cette thèse concernent des problèmes d'homogénéisation sur des équations de Hamilton-Jacobi. Des références sont communes aux trois chapitres et nous démarrons par l'analyse de cette littérature commune avant de se consacrer spécifiquement à chaque chapitre. Comme énoncé plus haut, se distinguent principalement deux travaux fondateurs.

Tout d'abord dans [61], Lions, Papanicolaou et Varadhan s'intéressèrent à un problème d'homogénéisation stationnaire périodique. Ils introduisent de nombreuses idées très largement reprises par la littérature par la suite: introduction de l'ansaltz et obtention du correcteur formel et de la constante ergodique; puis preuve rigoureuse de l'existence de la constance ergodique via le contrôle des oscillations d'un correcteur approché.

La méthode pour démontrer la convergence à l'aide des fonctions test a été introduite par Evans dans le travail pionnier [28]. Elle s'inspire directement de l'ansaltz, à la différence que cet ansaltz est introduit dans la fonction test et non dans la solution oscillante: il s'agit de la méthode de la fonction test perturbée, aussi présente dans [29] pour un problème d'homogénéisation stationnaire périodique.

Homogénéisation d'un modèle de poursuite 1D avec temps de retard et contre-exemple

Les idées et concepts fondamentaux de ce chapitre sont issus de [64]. On étudie l'influence du temps de réaction des conducteurs sur le trafic à grande échelle. On s'attend intuitivement, pour des positions initiales raisonnables, à aboutir à un modèle macroscopique continu pour des temps de réactions en-dessous d'une valeur critique (notion précisée dans les théorèmes du chapitre concerné) et à rencontrer des problèmes pour des temps de réactions supérieurs. L'objectif de ce chapitre est de formaliser ces intuitions dans le cadre d'un modèle du type "follow-the-leader" du premier ordre.

Littérature. Le travail [34] traite de l'homogénéisation d'une équation non-locale du premier ordre. Cette équation traduit la mécanique d'une chaîne infinie de particules couplées soumises à un potentiel périodique. Chaque particule interagit avec ses plus proches voisins. Ce modèle généralise l'équation introduite dans le cadre du modèle de trafic "follow-the-leader", où l'on considérait seulement l'interaction avec le voisin de devant. On y trouve le procédé d'injection d'un système d'équations différentielles couplées dans une seule EDP. La preuve d'homogénéisation repose sur la construction de fonction enveloppe. Le cas d'interaction avec un nombre infini de voisins est traité dans [33]. On renvoie à [22] pour une estimation d'erreur pour le modèle "follow-the-leader" sans temps de réaction.

Le chapitre 9 de [3] est consacré à l'étude de modèles de trafic à retard. On y trouve une discussion autour de la modélisation du temps de réaction lui-même. Il peut lui-même dépendre du temps et provient naturellement du facteur humain mais on peut aussi ajouter d'autres contributions, par exemple venant de la mécanique des véhicules. Sont également présents un inventaire des modèles de trafic tenant compte du temps de réaction, des analyses de stabilités linéaires et des simulations numériques. Le travail [66] considère un modèle macroscopique (et non microscopique) avec temps de retard et étudie la stabilité du modèle de trafic par rapport à ce paramètre.

Contribution. Dans ce chapitre, on montre l'homogénéisation du modèle de trafic pour un temps de réaction inférieur à une valeur critique. Cette valeur critique est donnée explicitement et dépend de la fonction de vitesse. On prouve un principe de comparaison strict car le temps de réaction impose de prendre une marge stricte entre les véhicules aux instants initiaux, et de contrôler l'évolution de leurs positions pendant une période proportionnelle au temps de réaction. On montre notamment la conservation de l'ordre des véhicules en-dessous du temps de réaction critique pour des conditions initiales qui entrent dans ce cadre. À l'inverse, on montre que des collisions peuvent se produire au-delà de ce temps de réaction critique. On finit enfin par donner un contre-exemple explicite à l'homogénéisation pour un temps de réaction situé au-dessus de cette valeur critique. Pour cela, on perturbe légèrement la solution stationnaire instable qui correspond à une interdistance constante entre les véhicules aux instants initiaux.

Le modèle considéré est le même que celui introduit dans le paragraphe 0.1.3 à la différence qu'on tient compte du temps de réaction des conducteurs. Afin de ne pas alourdir les notations, on présente seulement la version des théorèmes où tous les conducteurs possèdent le même temps de réaction mais les résultats restent valables lorsque chaque conducteur a son propre temps de réaction. Cette fois-ci, on désigne la fonction de vitesse par F et le modèle microscopique devient:

$$\frac{dX_i}{dt}(t) = F(X_{i+1}(t-\tau) - X_i(t-\tau)), \qquad \text{pour } (i,t) \in \mathbb{Z} \times (0,T), \tag{22}$$

où $\tau > 0$ désigne le temps de réaction commun à tous les conducteurs.

L'injection du système d'EDOs dans une seule EDP donne cette fois:

$$\begin{cases} \partial_t u^\varepsilon(x,t) = F\left(\frac{u^\varepsilon(x+\varepsilon,t-\varepsilon\tau)-u^\varepsilon(x,t-\varepsilon\tau)}{\varepsilon}\right), & \text{pour } (x,t) \in \mathbb{R}\times(0,T), \\ u^\varepsilon(x,s) = u_0(x,s), & \text{pour } (x,s) \in \mathbb{R}\times[-2\varepsilon\tau,0], \end{cases} \tag{23}$$

avec les hypothèses sur F et u_0:

$$\begin{cases} F \text{ est croissante, bornée et } C_F - \text{Lipschitz sur } \mathbb{R}, \\ u_0 \text{ est } L - \text{Lipschitz sur } \mathbb{R}\times[-2\tau,0]. \end{cases} \tag{24}$$

La seconde différence est que l'on doit se donner la condition initiale sur un intervalle de longueur au moins τ pour pouvoir définir la solution de cette équation à retard. Ici on se donne en fait la condition initiale sur un temps 2τ pour des raisons techniques. On parle de condition initiale épaisse.

Il est à noter que la solution de cette équation à pure retard est solution au sens classique, et elle est construite par intégrations successives. Si l'on se donne les positions initiales sur un temps suffisamment long, la vitesse instantanée dépendant exclusivement du passé, alors on peut reconstruire la position pour tout temps. Le futur est ici totalement déterminé par le passé.

Le modèle macroscopique apparaissant naturellement est le même que le cas sans temps de réaction:

$$\begin{cases} \partial_t u^0(x,t) = F(\partial_x u^0(x,t)), & \text{pour } (x,t) \in \mathbb{R}\times(0,T), \\ u^0(x,0) = u_0(x,0), & \text{pour } x \in \mathbb{R}. \end{cases} \tag{25}$$

Le premier résultat d'homogénéisation de cette thèse est contenu dans le Théorème 0.1.15.

Theorem 0.1.15 (Homogénéisation de l'équation à retard).
Pour $\tau \in \left(0; \frac{1}{eC_F}\right)$ *et sous les hypothèses* (24)*, la solution* u^ε *de* (23) *converge localement uniformément vers la solution de* (25).

L'obtention des principes de comparaison est un enjeu important de la théorie des solutions de viscosité. Ici, à cause du temps de réaction, on va devoir s'assurer d'une marge strict entre les véhicules, et surtout d'une évolution "contrôlée" de leurs positions initiales. On introduit l'équivalent de (16) dans le cas avec temps de réaction:

$$\begin{cases} \partial_t u(x,t) = F(u(x+1,t-\tau) - u(x,t-\tau)), & \text{pour } (x,t) \in \mathbb{R}\times(0,T), \\ u(x,s) = u_0(x,s), & \text{pour } (x,s) \in \mathbb{R}\times[-2\tau;0]. \end{cases} \tag{26}$$

La preuve d'homogénéisation repose donc ici sur un principe de comparaison dit strict, dont la version globale est donnée dans le Théorème 0.1.16. Le second chapitre donne une formulation plus générale contenant également la version sur un compact.

Theorem 0.1.16 (Principe de comparaison strict).
Sous les hypothèses (24)*, soient* v *et* u *respectivement une sur-solution et une sous-solution de* (26)*. Supposons qu'il existe* $\delta > 0$ *et une fonction* ρ *positive et croissante sur* $\mathbb{R}$ *tels que:*

$$\delta \le (v-u)(x,t-\tau') \le \rho(\tau')(v-u)(x,t), \qquad \textit{pour } \tau' \in [0,\tau], (x,t) \in \mathbb{R}\times[-\tau,0], \tag{27}$$

où ρ *est telle que:*

$$1 + C_F\rho(\tau)\int_0^{\tau'} \rho(s)ds < \rho(\tau'), \qquad \textit{pour } \tau' \in [0,\tau]. \tag{28}$$

Alors:

$$0 < \delta e^{-C_F\rho(\tau)t} \le v(x,t) - u(x,t), \qquad \textit{pour } (x,t) \in \mathbb{R}\times[0,+\infty). \tag{29}$$

À travers ce théorème, on voit que les éléments importants sont à la fois la valeur du temps de réaction et la dynamique initiale des véhicules.

Dans le cas particulier où tous les véhicules possèdent une interdistance constante L aux instants initiaux, la solution du modèle macroscopique est de la forme $Lx + Ct$. Les pentes L et C vérifient l'équation macroscopique. Cette solution est dite stationnaire. Les véhicules restent équidistants de L et avancent avec une vitesse uniforme $F(L)$, et ce quelque soit le temps de réaction. La solution stationnaire nous permet d'observer que tout peut très bien se passer pour de très grands temps de réactions avec certaines conditions initiales. Elle est cependant très instable et c'est grâce à cette instabilité que l'on va pouvoir obtenir le contre-exemple à l'homogénéisation.

Theorem 0.1.17 (Un contre-exemple à l'homogénéisation).
Il existe une fonction F, C_F−Lipschitz, une condition initiale u_0, et un temps de réaction $\tau > \frac{1}{eC_F}$ tels que la solution u^ε de (23) ne converge pas localement uniformément vers la solution de (25).

Homogénéisation d'une équation de Hamilton-Jacobi avec feux tricolores: étude de l'effet du déphasage

On considère une route rectiligne munie d'une infinité de feux identiques et équidistants. Ces feux comportent un déphasage constant, c'est-à-dire que la différence de temps entre la mise au vert d'un feu et celui qui le suit est constante. La zone de circulation libre entre les feux est représentée par un modèle macroscopique. Chaque feu est modélisé par un limiteur de flux ponctuel dépendant du temps. On se réfère à [52] qui pose le cadre théorique des solutions de viscosité adapté à ce modèle où l'Hamiltonien est discontinu en espace et dépendant du temps. On montre l'homogénéisation de ce modèle en adaptant la méthode décrite dans 0.1.3. On effectue des simulations numériques pour obtenir des propriétés qualitatives sur l'Hamiltonien effectif et étudier l'effet du déphasage sur les flux de trafic.

Littérature. On rappelle que le cadre théorique est celui introduit dans [52] qui traite des équations de Hamilton-Jacobi sur les réseaux. La notion de limiteur de flux, reprise pour les feux du second chapitre, est introduite pour des Hamiltoniens quasi-convexes (c'est-à-dire décroissants puis croissants). La principale difficulté est de montrer un principe de comparaison sur ces équations à Hamiltonien discontinu en espace, ce qui a été fait via le recours à une "fonction sommet". Ce même article traite de l'homogénéisation dans le cas où l'Hamiltonien est stationnaire. Les travaux [23] et [47] montrent respectivement la convergence et une estimation d'erreur sur des schémas numériques sur ces réseaux. Plus récemment, les travaux [46],[62], et [63] étudient ces équations sur réseaux dans un cadre non convexe. Dans [53], il est montré comment un modèle de trafic routier comportant une jonction peut être vu comme une équation d'Hamilton-Jacobi avec limiteur de flux. L'homogénéisation d'un modèle de trafic comportant un nombre fini de feux tricolores est traité dans [41]. Les feux ne sont pas supposés identiques et équidistants, et peuvent être discontinus en temps. Ces feux sont considérés comme des perturbations locales: on parle d'homogénéisation précisée, terme introduit par P.L. Lions dans ses cours au Collège de France en 2013-2014. Les travaux [35] et [36] portent sur l'homogénéisation précisée de modèles de trafic routier du second ordre avec bifurcations et des conditions de jonction apparaissent au niveau macroscopique. Le travail [40] porte sur l'homogénéisation précisée d'un modèle du premier ordre comportant un feu tricolore au niveau microscopique; là encore il est montré qu'une condition de jonction apparaît au niveau macroscopique.

L'ergodicité pour des correcteurs instationnaires est très liée au comportement de solutions d'équations de Hamilton-Jacobi en temps long, voir [12] et [13] par exemple. Dans les travaux [41], [51] et [10] elle provient du contrôle des oscillations en espace-temps du correcteur, ce dernier n'étant a priori plus de régularité Lipschitz. Dans [30] et [31], l'ergodicité est démontrée à l'aide de la théorie KAM faible, respectivement pour un problème d'homogénéisation périodique stationnaire et pour un problème d'homogénéisation périodique en espace-temps.

Contributions. On montre le résultat d'homogénéisation dans le cas d'un nombre infini de feux tricolores. L'Hamiltonien est périodique en espace seulement lorsque le déphasage est rationnel. Dans tous les cas, on montre qu'on peut se ramener au cadre périodique via un changement de variables en temps. À l'aide des outils introduits dans [52], on montre un principe de comparaison pour l'équation du correcteur approché et on maîtrise ses oscillations globales. Ceci permet d'obtenir la constante ergodique. On commence par montrer quelques propriétés classiques: continuité et coercivité de l'Hamiltonien effectif par rapport à la variable gradient, obtention de la constante ergodique comme une pente en temps long. En utilisant le schéma numérique introduit dans [23] et [47], on effectue des simulations numériques afin d'obtenir d'autres propriétés qualitatives sur cet Hamiltonien dépendant du déphasage. On montre notamment la continuité par rapport au déphasage, la convexité, une symétrie et une estimation. Quelques remarques et interprétations du point de vue du trafic routier sont également présentes.

On introduit maintenant les résultats de façon succincte, les détails techniques étant contenus dans le second chapitre.

Notre équation de Hamilton-Jacobi prend la forme générale suivante:

$$\begin{cases} u_t + H_F(u_x) = 0, & \text{pour } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z}, \\ u_t + G(t-\theta x, u_x) = 0, & \text{pour } (t,x) \in (0,+\infty) \times \mathbb{Z}, \end{cases}$$

où θ représente le déphasage entre les feux. G est un limiteur de flux instationnaire.

Pour $\varepsilon > 0$, on considère le problème de Cauchy oscillant associé et on condense les deux équations précédentes en une seule :

$$\begin{cases} u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0, & \text{pour } (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u^\varepsilon(0,x) = u_0(x), & \text{pour } x \in \mathbb{R}, \end{cases} \tag{30}$$

Hypothèses (A): G est continue et périodique en temps, H_F est continu, coercif et quasi-convexe, u_0 est Lipschitzienne sur $\mathbb{R}$.

Theorem 0.1.18 (Hamiltonien effectif). *Soit $p \in \mathbb{R}$. Sous les hypothèses (A), il existe un unique $\lambda \in \mathbb{R}$ tel que pour tout $\delta > 0$, il existe un correcteur approché continu et borné $v^\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solution de:*

$$\begin{cases} \lambda + v_\tau^\delta(\tau,y) + H(\tau,y,p+v_y^\delta(\tau,y)) \leq \delta, & \textit{pour } (\tau,y) \in \mathbb{R} \times \mathbb{R}, \\ \lambda + v_\tau^\delta(\tau,y) + H(\tau,y,p+v_y^\delta(\tau,y)) \geq -\delta, & \textit{pour } (\tau,y) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{31}$$

On peut alors définir l'Hamiltonien effectif $\bar{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ en posant $\bar{H}(p,\theta) = -\lambda$.

Theorem 0.1.19 (Convergence). *Sous les hypothèses (A), la solution u^ε de (30) converge localement uniformément vers la solution u^0 du problème de Cauchy suivant:*

$$\begin{cases} u_t^0 + \bar{H}\left(u_x^0, \theta\right) = 0, & \textit{sur } (0,+\infty) \times \mathbb{R}, \\ u^0(0,x) = u_0(x), & \textit{pour } x \in \mathbb{R}. \end{cases} \tag{32}$$

Homogénéisation d'équations de Hamilton-Jacobi presque périodiques

Dans ce chapitre, on s'intéresse à deux problèmes d'homogénéisation en milieu presque périodique. On se contente de résoudre les problèmes de la cellule associés. Ce cadre généralise le cadre périodique classique. Il a d'abord été étudié dans [54] qui traita de l'homogénéisation d'une équation de Hamilton-Jacobi stationnaire. Le premier cas que nous étudions est la version évolution de [54] avec le même Hamiltonien stationnaire. Le deuxième problème est celui du second chapitre, que l'on traite avec des arguments presque périodiques. L'Hamiltonien est périodique en espace pour un déphasage rationnel et presque périodique en espace dans le cas général. Il est périodique en temps dans tous les cas. Pour résoudre le problème de la cellule, la clé est de contrôler les oscillations du correcteur approché sur tout l'espace. Pour cela, on se ramène à un compact, comme dans le cas périodique, à une erreur près. Différentes définitions de presque périodicité et méthodes d'obtention de la constante ergodique sont présentées. Des problèmes ouverts sont aussi évoqués à la fin du chapitre: le problème d'homogénéisation pour un Hamiltonien presque périodique en espace et en temps; et l'homogénéisation d'un modèle de trafic microscopique presque périodique. Un enjeu commun à tous ces problèmes est de voir comment la presque périodicité de l'Hamiltonien se transmet à la solution.

Littérature. On réfère à [15] et [14] pour une introduction à la théorie des fonctions presque périodiques. Ishii apporta la première contribution à l'homogénéisation presque périodique pour les équations de Hamilton-Jacobi dans [54], qui fût repris par [1]. Le cadre presque périodique peut être inclus dans le cadre stationnaire ergodique stochastique, voir [74], [58], [72], et le travail plus récent [57] dans le cas d'Hamiltoniens convexes. Pour les équations d'évolution du premier ordre dans le cadre stationnaire ergodique, les travaux [74] et [68] considèrent des Hamiltoniens stationnaires convexes. Le travail [72] traite du cas d'Hamiltoniens convexes instationnaires. Il généralise la version visqueuse [58]. Le travail [56] traite le cas d'un Hamiltonien sous-linéaire avec une forme particulière, supposé périodique en espace et stationnaire ergodique en temps. Ceci peut être vu comme un cas inversé de l'Hamiltonien de notre problème avec feux tricolores, périodique en temps et presque périodique en espace. Le travail [13] montre une convergence vers des états périodiques en temps pour les solutions d'équations de Hamilton-Jacobi sur le cercle avec un Hamiltonien convexe et périodique en temps. Il est mentionné que la période des solutions limites peut être plus grande que la période de l'Hamiltonien.

Contribution. Le premier résultat est la preuve de l'ergodicité pour la version évolution de [54]. La constante ergodique est caractérisée comme la pente en temps long de la solution d'une équation de Hamilton-Jacobi reliée au problème de la cellule. Tout d'abord, on donne des estimations a priori sur cette solution. On contrôle ensuite ses oscillations en espace grâce à la presque périodicité. On montre l'unicité de la constante ergodique en l'identifiant avec la constante obtenue par Ishii dans le cas stationnaire. Pour le second problème, on se contente de montrer l'existence de la constante ergodique. L'unicité a été montrée dans le deuxième chapitre. On montre d'abord que l'Hamiltonien issu du modèle de trafic est presque périodique en espace. On adapte ensuite la méthode de la sous-section 0.1.3 pour maîtriser les oscillations du correcteur approché en se ramenant à un compact à une erreur près. La difficulté est que la taille du compact dépend de cette erreur et tend à exploser plus l'erreur devient petite.

Intuitivement, une fonction presque périodique aura des $\delta-$périodes pour tout $\delta > 0$. Ces presque périodes seront réparties sur $\mathbb{R}$ comme l'étaient les périodes d'une fonction périodique. Cela correspond à la définition donnée par Bohr, voir [15]. C'est celle qui sera utilisée pour le problème avec feux. Une autre définition équivalente, celle de Bochner, sera utilisée pour le premier problème. Elle caractérise la presque périodicité comme une relative compacité des translatés de la fonction.

Les résultats obtenus pour le second problème sont identiques à ceux du second chapitre, seule la méthode change. On se contente donc de donner les résultats pour le premier problème.

Pour $\varepsilon > 0$, on considère la solution ré-échelonnée v^ε de l'équation de Hamilton-Jacobi oscillante sur la droite réelle:

$$\begin{cases} v_t^\varepsilon + h\left(\frac{x}{\varepsilon}, v_x^\varepsilon\right) = 0, & \text{pour } (t,x) \in (0,+\infty) \times \mathbb{R}, \\ v^\varepsilon(0,x) = v_0(x), & \text{pour } x \in \mathbb{R}, \end{cases} \tag{33}$$

sous les hypothèses suivantes (A'0)-(A'4) introduites dans [54] pour le même Hamiltonien:

(A'0) Continuité: $h : \mathbb{R}^2 \to \mathbb{R}$ est continue.

(A'1) Coercivité:

$$\lim_{R \to +\infty} \inf\{h(y,p'), y \in \mathbb{R}, |p'| \geq R\} = +\infty.$$

(A'2) Uniforme continuité en le gradient sur des bornés (uniformément en espace): Pour tout $R > 0$, il existe une fonction $\omega_R \in C([0, +\infty))$, avec $\omega_R(0) = 0$, telle que

$$|h(y, p') - h(y, q)| \leq \omega_R(|p' - q|), \qquad \text{for } y \in \mathbb{R}, (p', q) \in [-R, R] \times [-R, R].$$

(A'3) Uniforme bornitude pour des gradients bornés:

$$\sup_{(y,p') \in \mathbb{R} \times [-R,R]} |h(y, p')| < +\infty.$$

(A'4) Presque périodicité spatiale au sens de Bochner: Pour tout $R > 0$, la famille $\{h(. + z, .), z \in \mathbb{R}\}$ est relativement compacte dans $BUC(\mathbb{R} \times [-R, R])$.

Cela signifie que pour tout $\delta > 0$, il existe $n \in \mathbb{N}$, $(k_1, .., k_n) \in \mathbb{R}^n$ tel que:

$$\forall k \in \mathbb{R}, \exists i \in \{1, n\}, \sup_{(y,q) \in \mathbb{R} \times [-R,R]} |h(y + k, q) - h(y + k_i, q)| < \delta.$$

(B'1) Régularité de la condition initiale: v_0 est une fonction K- Lipschitz sur $\mathbb{R}$.

On énonce maintenant les principaux résultats du chapitre.

Theorem 0.1.20 (Définition de l'Hamiltonien homogénéisé)**.** *Soit $p \in \mathbb{R}$. Sous les hypothèses (A'0)-(A'4), il existe un unique $\lambda \in \mathbb{R}$ tel que:*

$$\sup_{y \in \mathbb{R}} \left| \frac{v(\tau, y)}{\tau} + \lambda \right| \underset{\tau \to +\infty}{\longrightarrow} 0, \tag{34}$$

où v est l'unique solution de viscosité de:

$$\begin{cases} v_\tau(\tau, y) + h(y, p + v_y(\tau, y)) = 0, & \text{pour } (\tau, y) \in (0, +\infty) \times \mathbb{R}, \\ v(0, y) = 0, & \text{pour } y \in \mathbb{R}. \end{cases} \tag{35}$$

Ceci permet de définir $\bar{H} : \mathbb{R} \to \mathbb{R}$ avec $\bar{H}(p) = -\lambda$.

Theorem 0.1.21 (Convergence)**.** *Sous les hypothèses (A'0)-(A'4) et (B'1), la solution v^ε de (33) converge localement uniformément vers la solution v^0 du problème de Cauchy suivant:*

$$\begin{cases} v_t^0 + \bar{H}(v_x^0) = 0, & \text{sur } (0, +\infty) \times \mathbb{R}, \\ v^0(0, x) = v_0(x), & \text{pour } x \in \mathbb{R}. \end{cases} \tag{36}$$

Chapter 1

Homogenization of a 1D pursuit law with delay and a counter-example

Abstract

In this paper, we consider a one dimensional pursuit law with delay which is derived from traffic flow modelling. It takes the form of an infinite system of first order coupled delayed equations. Each equation describes the motion of a driver who interacts with the preceding one, taking into account his reaction time. We derive a macroscopic model, namely a Hamilton-Jacobi equation, by a homogenization process for reaction times that are below an explicit threshold. The key idea is to show, that below this threshold, a strict comparison principle holds for the infinite system. Above this threshold, we show that collisions can occur. In a second time, for well-chosen dynamics and higher reaction times, we show that there exist some microscopic pursuit laws that do not lead to the previous macroscopic model.

Keywords: Hamilton-Jacobi equations, infinite system, delay time, strict comparison principle, homogenization.

1.1 Introduction

In the present paper we consider an infinite system of delay differential equations (DDEs). This form can especially be derived from a one dimensional pursuit law on a straight road with a very simple model. Such models, where we follow the time position of every driver are called microscopic models. The effect of drivers' reaction times on the stability of the corresponding systems of DDEs is studied in [3] and [65]. Presentations of some classical microscopic traffic flow models of first order or second order are made in [20], [43] or [22]. Conversely, macroscopic models describe the time evolution of densities in global traffic flow. We want to study if we can recover a macroscopic model from our microscopic one. Formally, it corresponds to making the interdistance between drivers go to zero or to observe the pursuit law from a height and time that go to infinity. When possible, the passage can be made rigorous by an homogenization process that will be done here in the framework of viscosity solutions of Hamilton-Jacobi equations. The interested reader is referred to the pionnering works [61], [28] and [29] about homogenization of Hamilton-Jacobi equations. The works of [60] or [69] describe the historical approach used to define and analyse macroscopic traffic flow models and links with fluid mechanics. Several examples (and counter-examples) will be exhibited throughout this paper to show that both the initial dynamics and reaction time values will have a significant influence on the homogenization process.

This paper derives from Régis Monneau's work [64] (see also [22]).

1.1.1 Description of the model and main results

The common velocity of each driver is supposed to be a Lipschitz continuous, bounded, nondecreasing function, F say, of the distance that separates each driver from the preceding one (with $F(0) = 0$). Each driver has its own reaction time and we assume that the reaction times are uniformly bounded from above and from below; we denote τ the upper bound and $\xi > 0$ the lower bound. Let us study the evolution of the system during a time $T \in (0, +\infty]$ with $T > \tau$.

Given a sequence $(X_i)_{i\in\mathbb{Z}}$ of drivers' positions on the road, the microscopic model then takes the form of an infinite system of DDEs:

$$\frac{dX_i}{dt}(t) = F(X_{i+1}(t-\tau_i) - X_i(t-\tau_i)), \qquad \text{for } (i,t) \in \mathbb{Z} \times (0,T), \tag{1.1}$$

where $\tau_i \in [\xi, \tau]$ denotes the individual reaction time.

If we suppose that we know the dynamics of the cars during the initial time interval, we can define time functions $(x_i^0)_{i\in\mathbb{Z}}$ such that:

$$X_i(t) = x_i^0(t), \qquad \text{for } t \in [-2\tau, 0]. \tag{1.2}$$

The first step is to embed the microscopic system dynamics into a single PDE. In order to go from the microscopic scale to the macroscopic one, we have to introduce a small parameter, $\varepsilon > 0$ say, so as to rescale properly the function and to take into account the microscopic space and time oscillations with high frequency.

Hence, by hyperbolic change of variables, we define the function u^ε as follows (the space variable will be denoted as x):

$$u^\varepsilon(x,t) = \varepsilon X_{\lfloor \frac{x}{\varepsilon} \rfloor}(\frac{t}{\varepsilon}). \tag{1.3}$$

Let us assume that the vehicles are regularly spaced at initial times. We inspire from [34] and suppose that there exists a Lipschitz continuous function $u_0 : \mathbb{R} \times [-2\tau, 0] \to \mathbb{R}$ such that:

$$x_i^0(t) = \frac{u_0(i\varepsilon, t\varepsilon)}{\varepsilon}, \qquad \text{for } (i,t) \in \mathbb{Z} \times [-2\tau, 0], \tag{1.4}$$

We define the piecewise affine function $\tau_0 : \mathbb{R} \to [\xi, \tau]$ such that:

$$\tau_0(i) = \tau_i, \qquad \text{for } i \in \mathbb{Z}.$$

The rescaled model can thus be embedded into the following equation:

$$\begin{cases} \partial_t u^\varepsilon(x,t) = F\left(\frac{u^\varepsilon(x+\varepsilon, t-\varepsilon\tau_0(\varepsilon^{-1}x)) - u^\varepsilon(x, t-\varepsilon\tau_0(\varepsilon^{-1}x))}{\varepsilon}\right), & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u^\varepsilon(x,s) = u_0(x,s), & \text{for } (x,s) \in \mathbb{R} \times [-2\varepsilon\tau, 0], \end{cases} \tag{1.5}$$

with the following assumptions on F, τ_0 and u_0:

$$\begin{cases} F \text{ is a non-decreasing, bounded and } C_F - \text{Lipschitz continuous function on } \mathbb{R}, \\ \overline{\tau_0(\mathbb{R})} = [\xi, \tau], \\ u_0 \text{ is a globally } L - \text{Lipschitz continuous function on } \mathbb{R} \times [-2\tau, 0]. \end{cases} \tag{1.6}$$

As the argument of F in (1.5) converges towards a space derivative for vanishing ε, it is natural to introduce the following dynamics in the form of a first-order Hamilton-Jacobi equation:

$$\begin{cases} \partial_t u^0(x,t) = F(\partial_x u^0(x,t)), & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u^0(x,0) = u_0(x,0), & \text{for } x \in \mathbb{R}. \end{cases} \tag{1.7}$$

The first goal of the article is to prove that u^ε tends locally uniformly towards u^0.

Theorem 1.1.1 (Homogenization of the delay equation)**.**
For $\tau \in \left(0; \frac{1}{eC_F}\right)$ and under assumptions (1.6), the solution u^ε to (1.5) converges locally uniformly towards the (unique viscosity) solution of (1.7).

The key idea of the convergence proof relies on a strict comparison principle. This one is stated for the microscopic model for $\varepsilon = 1$:

$$\begin{cases} \partial_t u(x,t) = F(u(x+1, t-\tau_0(x)) - u(x, t-\tau_0(x))), & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u(x,s) = u_0(x,s), & \text{for } (x,s) \in \mathbb{R} \times [-2\tau, 0]. \end{cases} \tag{1.8}$$

In contrast with the classical cases where it is sufficient to keep the initial order of the solutions, here, because of the reaction time, those solutions (namely the vehicles in traffic flow modelling) must be sufficiently spaced out at initial times in the sense made precise in the following theorem:

Theorem 1.1.2 (Strict comparison principle)**.**
Under the assumptions (1.6), let v and u be respectively a supersolution and a subsolution of (1.8). Let us assume that there exist $\delta > 0$, $t_0 \in [0,T)$, $R \in [0, +\infty)$, $x_0 \in \mathbb{R}$, and a positive, non decreasing function ρ in $\mathbb{R}$ such that:

$$v(x,t) \geq u(x,t), \qquad \text{for } R \leq |x - x_0| \leq R+1, \quad t \in [t_0 - \tau, T) \tag{1.9}$$

$$\delta \leq (v-u)(x, t-\tau') \leq \rho(\tau')(v-u)(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in [x_0 - R, x_0 + R] \times [t_0 - \tau, t_0], \tag{1.10}$$

where ρ is such that:

$$1 + C_F \rho(\tau) \int_0^{\tau'} \rho(s) ds < \rho(\tau'), \qquad \text{for } \tau' \in [0, \tau]. \tag{1.11}$$

Then we have:

$$0 < \delta e^{-C_F \rho(\tau)(t-t_0)} \leq v(x,t) - u(x,t), \qquad \text{for } (x,t) \in [x_0 - R, x_0 + R] \times [t_0, T). \tag{1.12}$$

Remark 1.1.3. *The theorem is still valid when $R = +\infty$ (substituting $[x_0 - R, x_0 + R]$ by $\mathbb{R}$), (1.9) is defined on $\emptyset$.*

The homogenization proof is different from the one performed in [34] on generalized Frenkel-Kontorova models, concerning an infinite system of particles that interact with a finite number of neighbours and that are subject to a periodic potential (this system takes the form of non-linear ODEs). The convergence proof in [34] relies on the construction of hull functions in a periodic setting. The case where particles interact with an infinite number of other particles is covered in [33]. Other homogenization results in traffic flow modelling without considering the reaction time can be found in [38] when adding a junction condition in the microscopic model and in [39] for a second order microscopic model.

The second goal of the article is to provide a "counter-example to homogenization" in the following sense: we will display an explicit model for which the rescaled solution does not converge locally uniformly towards the solution u^0 of (1.7) in the special case where all the drivers have a common reaction time larger than the threshold.

Theorem 1.1.4 (A counter-example to homogenization)**.**
In the special case $\tau_0 \equiv \tau$, there exist a C_F-Lipschitz continuous F, an initial data u_0, and $\tau > \frac{1}{eC_F}$ such that the rescaled solution does not converge locally uniformly towards the solution of (1.7).

We derive such a counter-example by a small perturbation of a trivial but unstable case where the microscopic and macroscopic models coincide for all reaction times. This trivial case is a stationary one where all the vehicles are initially equally spaced. This also highlights the fact that the initial dynamics is at least as important as the value of the reaction time from the homogenization perspective.

1.1.2 Organisation of the article

We will first show the existence and uniqueness of solutions to (1.5) and (1.7) in Section 1.2. We will then prove the strict comparison principle for (1.5) in Section 1.3 and we will show that the restriction on τ in Theorem 1.1.1 is equivalent to the existence of functions ρ that verify (1.11). We will also state and prove a direct corollary of Theorem 1.1.2 on drivers' positions and give a counter-example to Theorem 1.1.2 when the vehicles are not suitably spaced out at initial times, regardless of the threshold on τ. We will also give an example where collisions occur when the reaction time is higher than the threshold. We will give the explicit convergence proof in Section 1.4 and give the unstable example where homogenization holds for all reaction times, provided the vehicles are perfectly spaced at initial times. In Section 1.5, we will give the explicit model and will prove Theorem 1.1.4.

1.2 Existence and uniqueness

This section is devoted to the study of the existence and uniqueness of solutions to (1.5) and (1.7). Studying existence and uniqueness of solutions to (1.5) is stricly equivalent to studying (1.8).

Proposition 1.2.1. *Under the assumptions* (1.6)*, there exists a unique classical solution to* (1.8)*.*

Proof of Proposition 2.1.
The proof is based on an explicit and incremental construction. Indeed, on the time interval $(0, \xi]$ the equation can be written as follows:

$$\partial_t u(x,t) = F(u_0(x+1, t-\tau_0(x)) - u_0(x, t-\tau_0(x))), \qquad \text{for } (x,t) \in \mathbb{R} \times (0, \xi].$$

Hence, it can be explicitly integrated and we get:

$$u(x,t) = u_0(x,0) + \int_0^t F(u_0(x+1, s-\tau_0(x)) - u_0(x, s-\tau_0(x)))ds, \qquad \text{for } (x,t) \in \mathbb{R} \times (0, \xi].$$

This enables us to define the function u_1 which coincides with u_0 at initial times and that is defined by the previous expression in $[0, \xi]$.

Hence, in the time interval $(\xi, 2\xi]$, the equation can be written as follows:

$$\partial_t u(x,t) = F(u_1(x+1, t-\tau_0(x)) - u_1(x, t-\tau_0(x))), \qquad \text{for } (x,t) \in \mathbb{R} \times (\xi, 2\xi].$$

Here again, it can be explicitly integrated:

$$u(x,t) = u_1(x,\xi) + \int_\xi^t F(u_1(x+1, s-\tau_0(x)) - u_1(x, s-\tau_0(x)))ds, \qquad \text{for } (x,t) \in \mathbb{R} \times [\xi, 2\xi].$$

The process can be iterated to obtain a function u defined piecewise which is, by construction, continuous on $\mathbb{R} \times [0, \infty)$ and which solves equation (1.8) in each $\mathbb{R} \times (k\xi, (k+1)\xi)$, $k = 0, .., \lfloor \frac{T}{\xi} \rfloor - 1$. It remains to prove that it solves the equation globally and so, that u is C^1 in time on $(0, T)$. Indeed, at each $k\xi$, the left and right limits of the derivative coincide and verify the equation.

The uniqueness also comes from the process as we see that on each time interval, the solution is completely determined by its expression on the previous interval which implies global uniqueness for a given initial data u_0. $\square$

Remark 1.2.2 (Initial data)**.** *It is enough for Theorem 1.2.1 to hold, that the initial data is defined in* $[-\tau, 0]$*. The reason why we ask it to be defined in* $[-2\tau, 0]$ *is that we need it in Theorem 1.1.2.*

Equation (1.7) is an Hamilton-Jacobi equation and is therefore studied in the viscosity solutions' framework. The existence and uniqueness are classical and are respectively given by Perron's method and the usual comparison principle, see [9] for instance.

1.3 Strict comparison principle

This section is first devoted to the proof of Theorem 1.1.2. We will then explain the restriction on τ. We will also prove the conservation of initial order of the vehicles as a corollary when those are well spaced out at initial times. Finally, we will present a counter-example to comparison principle for any reaction time $\tau > 0$. Indeed, the existence of ρ functions that verify (1.11) is equivalent to considering τ under the threshold but both (1.11) and (1.10) are necessary for the strict comparison principle to hold and hence, the restriction on τ is not a sufficient condition to ensure the result. The existence of ρ functions is not enough, it has to be connected with the initial dynamics of the vehicles. Namely, we will show that if the vehicles are not suitably spaced out at initial times (in the sense made precise in (1.10)), their initial order can be disturbed even for reaction times that are below the threshold.

1.3.1 Proof of Theorem 1.1.2

Proof of Theorem 1.2.
Let us consider $d := v - u$ and define T^* as:

$$T^* = \sup\{S \in [0, +\infty)/\forall \tau' \in [0, \tau], \forall (x,t) \in [x_0 - R, x_0 + R] \times [t_0 - \tau, S],\ \ d(x, t - \tau') \leq \rho(\tau')d(x,t)\}.$$

The set is not empty as it contains t_0 so T^* is well-defined.

- We first claim that to establish (1.12), it is sufficient to prove that $T^* \geq T$.

Thanks to (1.11) we have $\rho(0) > 1$. Taking $\tau' = 0$ in the definition of T^*, with $T^* \geq T$, implies that:

$$d(x,t) \geq 0, \qquad \text{for } (x,t) \in [x_0 - R, x_0 + R] \times [t_0 - \tau, T). \tag{1.13}$$

By combining with (1.9), we get:

$$d(x,t) \geq 0, \qquad \text{for } (x,t) \in [x_0 - R - 1, x_0 + R + 1] \times [t_0 - \tau, T). \tag{1.14}$$

Let us consider $(x,t) \in [x_0 - R, x_0 + R] \times (t_0, T)$. By definition of d, we have:

$$\partial_t d(x,t) \geq F(v(x+1, t - \tau_0(x)) - v(x, t - \tau_0(x))) - F(u(x+1, t - \tau_0(x)) - u(x, t - \tau_0(x))). \tag{1.15}$$

By combining (1.14) with the fact that F is non-decreasing gives:

$$\partial_t d(x,t) \geq F(u(x+1, t - \tau_0(x)) - v(x, t - \tau_0(x))) - F(u(x+1, t - \tau_0(x)) - u(x, t - \tau_0(x))).$$

Using the fact that F is C_F-Lipschitz, we get:

$$\partial_t d(x,t) \geq -C_F d(x, t - \tau_0(x)).$$

If $T^* \geq T$, then we have:

$$\partial_t d(x,t) \geq -C_F \rho(\tau_0(x)) d(x,t).$$

By using the fact that ρ is non-decreasing, we get:

$$\partial_t d(x,t) \geq -C_F \rho(\tau) d(x,t). \tag{1.16}$$

Hence, d is a supersolution of the problem:

$$\begin{cases} \partial_t w(x,t) = -C_F \rho(\tau) w(x,t), \\ w(x, t_0) = \delta. \end{cases} \tag{1.17}$$

We conclude (1.12) by a standard comparison principle for this ODE.

- Let us now show that $T^* \geq T$.

By contradiction, if we suppose that $T^* < T$, then for all $\beta \in \left(0, \inf\left(\frac{T-T^*}{2}, 1\right)\right)$, there exists $\tau_\beta \in [0, \tau]$, $(x_\beta, t_\beta) \in [x_0 - R, x_0 + R] \times (T^*, T^* + \beta]$ such that:

$$d(x_\beta, t_\beta - \tau_\beta) > \rho(\tau_\beta) d(x_\beta, t_\beta). \tag{1.18}$$

Let us notice that (1.16) holds true for all $t \in [t_0, T^*]$ (and so does (1.12)).

As a preliminary result, let us first show that:

$$d(x, t - \tau') \leq \bar{\rho}(\tau') d(x, t), \qquad \text{for } \tau' \in [0, \tau], (x, t) \in [x_0 - R, x_0 + R] \times [t_0, T^*], \tag{1.19}$$

with $\bar{\rho}$ given by:

$$\bar{\rho}(\tau') = 1 + C_F \rho(\tau) \int_0^{\tau'} \rho(s) ds.$$

By remarking that $\bar{\rho}(0) = 1$, we notice the equality for $\tau' = 0$. For $\tau' \in [0, \tau], (x, t) \in [x_0 - R, x_0 + R] \times [t_0, T^*]$, we have:

$$\partial_{\tau'}(\bar{\rho}(\tau') d(x, t)) = C_F \rho(\tau) \rho(\tau') d(x, t).$$

Let us remark that $\tau' \mapsto d(x, t - \tau')$ is a subsolution of this equation. Indeed:

$$\partial_{\tau'}\left(d(x, t - \tau')\right) = -\partial_t d(x, t - \tau'). \tag{1.20}$$

We can use (1.16) as $t - \tau' \in [t_0 - \tau, T^*]$:

$$\partial_{\tau'}\left(d(x, t - \tau')\right) \leq C_F \rho(\tau) d(x, t - \tau'). \tag{1.21}$$

Finally, by definition of T^*, we get:

$$\partial_{\tau'}\left(d(x, t - \tau')\right) \leq C_F \rho(\tau) \rho(\tau') d(x, t). \tag{1.22}$$

The preliminary result is then obtained by a comparison principle on the equation $\partial_{\tau'} W = C_F \rho(\tau) \rho(\tau') W$.

Let us notice that (1.6) and (1.15) imply that:

$$\partial_t d(x, t) \geq -2||F||_\infty. \tag{1.23}$$

By integrating it, we get:

$$d(x, t) \geq d(x, s) - 2||F||_\infty (t - s), \qquad \text{for } (x, t) \in [x_0 - R, x_0 + R] \times [t_0, T], s \in [t_0, t]. \tag{1.24}$$

Case 1: $t_\beta - \tau_\beta \leq T^*$.

In this case, by defining $\tau'_\beta := T^* - t_\beta + \tau_\beta \in [0, \tau]$ we get:

$$d(x_\beta, t_\beta - \tau_\beta) = d(x_\beta, T^* - \tau'_\beta).$$

Hence, by (1.19) we get:

$$d(x_\beta, t_\beta - \tau_\beta) \leq \bar{\rho}(\tau'_\beta) d(x_\beta, T^*).$$

Thanks to (1.24), we have:

$$d(x_\beta, t_\beta - \tau_\beta) \leq \bar{\rho}(\tau'_\beta)(d(x_\beta, t_\beta) + 2||F||_\infty \beta). \tag{1.25}$$

In order to get a contradiction with (1.18), it is sufficient to consider $\beta < \min(\beta_1, \beta_2)$ with:

$$\begin{cases} \beta_1 := \frac{\delta}{2||F||_\infty + 1} e^{-C_F \rho(\tau)(T^* - t_0)} > 0, \\ \beta_2 := \frac{\beta_1}{2||F||_\infty} \inf_{[0,\tau]} (\frac{\rho}{\bar{\rho}} - 1) > 0. \end{cases} \tag{1.26}$$

Indeed, for $\beta < \beta_1$, thanks to (1.12) (which is still valid in $[x_0 - R, x_0 + R] \times [t_0, T^*]$) and to (1.24), we have:

$$d(x_\beta, t_\beta) \geq d(x_\beta, T^*) - 2||F||_\infty \beta \geq \delta e^{-C_F \rho(\tau)(T^* - t_0)} - 2||F||_\infty \beta_1 = \beta_1,$$

and for $\beta < \min(\beta_1, \beta_2)$ we have:

$$\bar{\rho}(\tau'_\beta)(d(x_\beta, t_\beta) + 2||F||_\infty \beta) \leq \rho(\tau'_\beta) d(x_\beta, t_\beta).$$

Together with (1.25), this implies:

$$d(x_\beta, t_\beta - \tau_\beta) \leq \rho(\tau'_\beta) d(x_\beta, t_\beta).$$

Finally, as ρ is non decreasing and $\tau'_\beta \leq \tau_\beta$, we get a contradiction with (1.18).

Case 2: $t_\beta - \tau_\beta > T^*$.

As we have $t_\beta < T^* + \beta$, this implies that $\tau_\beta \in [0, \beta]$.

Then, thanks to (1.24) and (1.18) we have:

$$d(x_\beta, t_\beta) + 2||F||_\infty \beta \geq d(x_\beta, t_\beta) + 2||F||_\infty \tau_\beta \geq d(x_\beta, t_\beta - \tau_\beta) > \rho(\tau_\beta) d(x_\beta, t_\beta).$$

In particular, we have:

$$(\rho(\tau_\beta) - 1) d(x_\beta, t_\beta) < 2||F||_\infty \beta.$$

Thanks to (1.24), this implies:

$$(\rho(\tau_\beta) - 1) d(x_\beta, T^*) < 2||F||_\infty \rho(\tau_\beta) \beta.$$

Thus,

$$0 < \delta e^{-C_F \rho(\tau)(T^* - t_0)} < \frac{\rho(\tau_\beta)}{\rho(\tau_\beta) - 1} 2||F||_\infty \beta. \tag{1.27}$$

For vanishing β this implies $\delta e^{-C_F \rho(\tau)(T^* - t_0)} = 0$ which is false. □

1.3.2 Restriction on τ and existence of solutions to (1.11)

Condition (1.11) explains why we chose $\tau < \frac{1}{eC_F}$. In fact, we have the following proposition.

Proposition 1.3.1. *Equation (1.11) admits solutions if and only if $\tau < \frac{1}{eC_F}$.*

We now give a proof of Proposition 1.3.1.

Proof of Proposition 3.1. We want to show that condition (1.11) implies $\tau < \frac{1}{eC_F}$.

Let us define:

$$R(\tau') = \int_0^{\tau'} \rho(s) ds.$$

We have $R(0) = 0$. By defining $\alpha = R'(\tau) = \rho(\tau)$, condition (1.11) is equivalent for R to be a strict supersolution of:

$$\begin{cases} y' = C_F \alpha y + 1, \text{ in } [0, \tau], \\ y(0) = 0. \end{cases} \tag{1.28}$$

As $S : \tau' \mapsto \frac{e^{C_F \alpha \tau'} - 1}{C_F \alpha}$ is a solution of this Cauchy problem on $[0, \tau]$, by Gronwall's lemma or a standard comparison principle, we deduce that $R \geq S$ or equivalently that:

$$1 + C_F R(\tau') R'(\tau) \geq e^{C_F R'(\tau) \tau'} \qquad , \text{ for } \tau' \in [0, \tau].$$

Moreover, as R is a strict supersolution of (1.28), we deduce that: $R'(\tau') > e^{C_F R'(\tau)\tau'}$ for all $\tau' \in [0, \tau]$. By considering this inequality for $\tau' = \tau$ and taking the logarithm, we get:

$$C_F \tau < \frac{\ln R'(\tau)}{R'(\tau)} \leq \max_{x>0} \frac{\ln x}{x} = \frac{1}{e}.$$

Let us now show that (1.11) admits solutions as soon as $\tau < \frac{1}{eC_F}$.

It is equivalent to find positive and non-decreasing solutions R of:

$$\begin{cases} R'(\tau') > C_F R'(\tau) R(\tau') + 1, & \text{for } \tau' \in [0, \tau], \\ R(0) = 0. \end{cases} \tag{1.29}$$

Let us consider $\lambda \in \left(1, \sqrt{\frac{1}{e\tau C_F}}\right)$.

The function $h : \gamma \mapsto \ln \gamma - \lambda \gamma \tau C_F$ is smooth on $(0, +\infty)$ and admits a maximum at $\gamma_{max} = \frac{1}{\lambda \tau C_F}$. As $\lambda < \sqrt{\frac{1}{e\tau C_F}}$, it is straightforward to check that $\ln \lambda < h(\gamma_{max})$ and so, there exists $\gamma_0 > 0$ such that $h(\gamma_0) = \ln \lambda$ or equivalently that $\gamma_0 = \lambda e^{\lambda \gamma_0 C_F \tau}$.

Let us define $U_\lambda : \tau' \mapsto \frac{1}{\gamma_0 C_F}(e^{\lambda \gamma_0 C_F \tau'} - 1)$. We have: $U'_\lambda(\tau) = \lambda e^{\lambda \gamma_0 C_F \tau} = \gamma_0$. As $\lambda > 1$, U_λ verifies:

$$\begin{cases} U'_\lambda(\tau') = \lambda \gamma_0 C_F U_\lambda(\tau') + \lambda > \gamma_0 C_F U_\lambda(\tau') + 1 = C_F U'_\lambda(\tau) U_\lambda(\tau') + 1, & \text{for } \tau' \in [0, \tau], \\ U_\lambda(0) = 0. \end{cases}$$

For all $\lambda \in \left(1, \sqrt{\frac{1}{e\tau C_F}}\right)$, we can construct a solution U_λ to (1.29). Therefore, as soon as $\tau < \frac{1}{eC_F}$, there exist (infinitely many) solutions. $\square$

Remark 1.3.2. *Condition* (1.11) *is reduced to:*

$$\rho C_F \tau' < 1 - \frac{1}{\rho},$$

if we consider ρ as a constant. If it is considered for $\tau' = \tau$, we get $\tau < \frac{1}{4C_F}$ because the condition tells that the function $x \mapsto C_F \tau x^2 - x + 1$ has a negative part. Conversely, for all fixed $\tau < \frac{1}{4C_F}$, we can always find constant solutions.

1.3.3 Consequence on drivers' positions and counter-examples

A very practical consequence of this strict comparison principle for the microscopic model derived from traffic flow is the conservation of initial order for vehicles, provided they are suitably spaced out at initial times. In Remark 3.3 of [34], it was highlighted that the solution is not decreasing in space if the initial data satisfies the same assumption. We here give an analogous result.

Corollary 1.3.3 (Conservation of initial order)**.** *Let us consider $(X_i)_{i \in \mathbb{Z}}$ a sequence of drivers' positions that evolve under the dynamics* (1.1) *with initial conditions given by* (1.2) *and for $\tau_0 \equiv \tau$. We consider the solution u to* (1.8) *such that:*

$$u_0(i, t) = x_i^0(t), \qquad \textit{for } (i, t) \in \mathbb{Z} \times [-\tau, 0].$$

We suppose that there exist $\sigma > 0$ and a time function ρ_p which satisfies (1.11) such that:

$$\sigma \leq u_0(x+1, t-\tau') - u_0(x, t-\tau') \leq \rho_p(\tau')(u_0(x+1,t) - u_0(x,t)), \text{ for } \tau' \in [0,\tau], (x,t) \in \mathbb{R} \times [-\tau, 0]. \tag{1.30}$$

Then,

$$X_{i+1}(t) > X_i(t), \qquad \text{for } (i,t) \in \mathbb{Z} \times [0,T).$$

Proof of Corollary 3.3.
By uniqueness, we have:

$$X_i(t) = u(i,t), \qquad \text{for } (i,t) \in \mathbb{Z} \times [0,T). \tag{1.31}$$

We consider $v : (x,t) \mapsto u(x+1,t)$. In particular, we have:

$$X_{i+1}(t) = v(i,t), \qquad \text{for } (i,t) \in \mathbb{Z} \times [0,T). \tag{1.32}$$

For $d := v - u$, thanks to (1.30), the conditions of Theorem 1.1.2 (and its remark) are fulfilled with $\delta = \sigma$, $t_0 = 0$, $R = +\infty$ and $\rho = \rho_p$. Therefore we have $v > u$ on $\mathbb{R} \times [0,T)$ and in particular on $\mathbb{Z} \times [0,T)$. Using (1.31) and (1.32) this is equivalent to having:

$$X_{i+1}(t) > X_i(t), \qquad \text{for } (i,t) \in \mathbb{Z} \times [0,T). \quad \square$$

Remark 1.3.4. *Our microscopic model is valid when the drivers always stay in the same order, otherwise it is not adapted anymore to the concrete situation. In our model, the behaviour of the driver at position X_i does not depend anymore on the position X_{i+1} if at some time the car $i+1$ is not in front of the car i and this corresponds to collisions for real vehicles. In the example below, we show that this can occur for well-chosen dynamics as soon as $\tau > \frac{1}{eC_F}$.*

Example 1.3.5 (Collisions for higher reaction times)**.** *We consider the case: $\tau_0 \equiv \tau \in (\frac{1}{e}, \frac{\pi}{2})$ and F being identically equal to the identity on $[-e\pi, e\pi]$ (with $C_F = 1$).*

We consider that for all $i \in \mathbb{N}$, $x_i^0 = 0$. Then for those vehicles, $X_i = 0$ for all time. Hence, the interdistance $d_{-1} := X_0 - X_{-1}$ is identically equal to $-X_{-1}$ for all time.

We will choose a special initial position x_{-1}^0 for the vehicle of index $i = -1$ so that this interdistance becomes strictly negative for a certain time $t_0 > 0$.

The extended function $f : \theta \in \left[0, \frac{\pi}{2}\right] \mapsto \frac{\theta}{\sin(\theta)} e^{-\frac{\theta}{\tan(\theta)}}$ is continuous and $f(0) = \frac{1}{e}$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.

Hence, there exists at least one $\theta(\tau) \in \left(0, \frac{\pi}{2}\right)$ such that $\tau = f(\theta(\tau))$. We denote by $\Re$ the real part of a complex number and define:

$$\begin{cases} r(\tau) := \frac{\theta(\tau)}{\tau \sin(\theta(\tau))} = e^{\frac{\theta(\tau)}{\tan(\theta(\tau))}}, \\ \lambda(\tau) := r(\tau) e^{i\theta(\tau)}, \\ x_{-1}^0(t) := \Re\left(-e^{-\lambda(\tau)t}\right), \qquad \text{for } t \in [-\tau, 0]. \end{cases}$$

For $t \in [-\tau, 0]$, we have:

$$d_{-1}(t) = e^{-r(\tau)\cos(\theta(\tau))t} \cos\left(\theta(\tau)\frac{t}{\tau}\right) > 0.$$

We claim that there exists $t_0 > 0$ such that $d_{-1}(t_0) < 0$, and thus there is a collision between vehicles. By contradiction, we suppose that $d_{-1} \geq 0$ for all times.

We have:

$$\sup_{t \in [-\tau, 0]} d_{-1}(t) = d_{-1}(\tau) \leq e^{r(\tau)\cos(\theta(\tau))\tau} = e^{\frac{\theta(\tau)}{\tan(\theta(\tau))}} = r(\tau).$$

As $\tau \geq \frac{1}{e}$ *and* $\max_{\alpha\in[0,\frac{\pi}{2}]} \frac{\alpha}{\sin\alpha} = \frac{\pi}{2}$*, then* $\sup_{t\in[-\tau,0]} d_{-1}(t) \leq e\pi$*.*

The equation solved by d_{-1} *in* $(0,\tau]$ *is then:*

$$d'_{-1}(t) = -d_{-1}(t-\tau).$$

For all $t \in (0,\tau]$*, we have:*

$$d'_{-1}(t) = -\Re\left(-e^{-\lambda(\tau)(t-\tau)}\right).$$

Combining with the fact that $e^{\lambda(\tau)\tau} = e^{r(\tau)\cos(\theta(\tau))\tau}e^{ir(\tau)\sin(\theta(\tau))\tau} = r(\tau)e^{i\theta(\tau)} = \lambda(\tau)$*, we get:*

$$d'_{-1}(t) = \Re\left(-\lambda(\tau)e^{-\lambda(\tau)t}\right).$$

We integrate and use the fact that the time derivative and the real part commute to get:

$$d_{-1}(t) = e^{-r(\tau)\cos(\theta(\tau))t}\cos\left(\theta(\tau)\frac{t}{\tau}\right), \qquad \text{for } t \in [0,\tau].$$

By an induction proof, we have:

$$d_{-1}(t) = e^{-r(\tau)\cos(\theta(\tau))t}\cos\left(\theta(\tau)\frac{t}{\tau}\right), \qquad \text{for } t \in [0,+\infty),$$

and for $t_1 = \frac{\pi\tau}{\theta(\tau)}$*, we get the contradiction* $d_{-1}(t_1) < 0$*.*

In contrast with most Hamilton-Jacobi equations, it is not possible to state a classical comparison principle because of the delay time. More accurately, it is compulsory to ensure that the vehicles are suitably spaced out (this space is represented by the function ρ) even for tiny reaction times as it is observed in the example below.

Example 1.3.6 (Non conservation of initial order between solutions for any reaction time)**.** *We consider the case:* $\tau_0 \equiv \tau$*. For any delay time* $\tau > 0$*, it is straightforward to show that the initial order of solutions is not necessarily conserved when the vehicles are not suitably spaced out in the sense made precise in Theorem 1.1.2.*

Let us consider $n_0 \in \mathbb{N}^*$ *such that* $\tau > \frac{2}{n_0}$*. Let us consider the particular case where* F *is identically equal to the identity on* $[0,1]$ *(and that verifies (1.6) in* $\mathbb{R}$*). Let us consider two sets of drivers* $(X_i)_{i\in\mathbb{Z}}$ *and* $(Y_i)_{i\in\mathbb{Z}}$ *that evolve under the same dynamics (1.1) and such that:*

$$y_i^0(t) < x_i^0(t), \qquad \text{for } (i,t) \in \mathbb{Z}\times[-\tau,0],$$

and there exists $j \in \mathbb{Z}$ *such that for* $t \in [-\tau,0]$*:*

$$\begin{cases} y_j^0(t) = j - 1 + \frac{n_0}{n_0+1}e^{n_0 t} < x_j^0(t) = j, \\ y_{j+1}^0(t) = j + \frac{n_0}{n_0+1}e^{n_0 t}, \\ x_{j+1}^0(t) = y_{j+1}^0(t) + \frac{1}{n_0+1} \end{cases} \tag{1.33}$$

Those two sets of drivers do not see each other, that means that they do not evolve on the same physical road but on two identical copies of the real line. We can integrate (1.1) on $[0,\tau]$ *for the two sets and we find that:*

$$(Y_j - X_j)(\tau) = \frac{n_0}{n_0+1}\tau - \frac{1}{n_0+1} - \int_{-\tau}^{0} e^{n_0 u}du = \frac{n_0}{n_0+1}\tau - \frac{1}{n_0+1} - \frac{1-e^{-n_0\tau}}{n_0+1} > 0.$$

Therefore, the initial order is disrupted even if:

$$\frac{1}{n_0+1} \leq x_i^0(t) - y_i^0(t), \qquad \text{for } (i,t) \in \mathbb{Z}\times[-\tau,0].$$

1.4 Convergence

This section is mainly devoted to the proof of Theorem 1.1.1. We will then introduce the unstable stationary case.

1.4.1 Proof of Theorem 1.1.1

Proof of Theorem 1.1.
Let us first show that u^ε is globally Lipschitz continuous in space and time uniformly in ε. This will enable us to define the relaxed semi-limits which will be compared to the macrosopic solution u^0 of (1.7) to prove the convergence result.

In time: By looking at (1.5), we remark that u^ε is $||F||_\infty-$Lipschitz continuous in time.

In space: The equation is translation invariant in space and invariant by addition of constants to the solutions. Let $h > 0$.

The solution corresponding to the initial condition $u_1 : (x,t) \mapsto u_0(x+h,t)$ is the function $w : (x,t) \mapsto u^\varepsilon(x+h,t)$.

The one associated to $u_2 : (x,t) \mapsto u_0(x,t) + 2Lh$ is $v : (x,t) \mapsto u^\varepsilon(x,t) + 2Lh$.

We define $\psi_1 : (x,t) \mapsto \frac{1}{\varepsilon} w(\varepsilon x, \varepsilon t)$ and $\psi_2 : (x,t) \mapsto \frac{1}{\varepsilon} v(\varepsilon x, \varepsilon t)$.

The function ψ_1 solves:

$$\begin{cases} \partial_t u(x,t) = F(u(x+1, t-\tau_0(x)) - u(x, t-\tau_0(x))), & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u(x,t) = \frac{1}{\varepsilon} u_0(\varepsilon x + h, \varepsilon t), & \text{for } (x,t) \in \mathbb{R} \times [-2\tau; 0]. \end{cases} \tag{1.34}$$

The function ψ_2 solves:

$$\begin{cases} \partial_t u(x,t) = F(u(x+1, t-\tau_0(x)) - u(x, t-\tau_0(x))), & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u(x,t) = \frac{1}{\varepsilon} u_0(\varepsilon x, \varepsilon t) + \frac{2Lh}{\varepsilon}, & \text{for } (x,t) \in \mathbb{R} \times [-2\tau; 0]. \end{cases} \tag{1.35}$$

By (1.6), ψ_1 and ψ_2 are $L-$ Lipschitz continuous functions on $\mathbb{R} \times [-2\tau, 0]$.

Hence, $d := \psi_2 - \psi_1$ is a $2L-$ Lipschitz continuous function on $\mathbb{R} \times [-2\tau, 0]$. Moreover, we have:

$$d(x,t) = \frac{1}{\varepsilon}\left(u_0(\varepsilon x, \varepsilon t) - u_0(\varepsilon x + h, \varepsilon t)\right) + \frac{2Lh}{\varepsilon} \geq -\frac{Lh}{\varepsilon} + \frac{2Lh}{\varepsilon} = \frac{Lh}{\varepsilon}, \qquad \text{for } (x,t) \in \mathbb{R} \times [-\tau; 0]. \tag{1.36}$$

Let us now consider any solution ρ_m to (1.11) for $\tau < \frac{1}{eC_F}$ (see Proposition 1.3.1). We recall that (1.11) implies $\rho_m \geq \rho_m(0) > 1$.

We have:

$$d(x, t-\tau') \leq d(x,t) + 2L\tau, \qquad \text{for } \tau' \in [0,\tau], (x,t) \in \mathbb{R} \times [-\tau, 0].$$

Therefore, to get:

$$d(x, t-\tau') \leq \rho_m(\tau') d(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in \mathbb{R} \times [-\tau, 0],$$

it is sufficient to verify:

$$2L\tau \leq (\rho_m(\tau') - 1) d(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in \mathbb{R} \times [-\tau, 0].$$

As ρ_m is a non-decreasing function, and as $d \geq 0$, it is finally sufficient to verify:

$$\frac{2L\tau}{\rho_m(0) - 1} \leq d(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in \mathbb{R} \times [-\tau, 0].$$

Thanks to (1.36), this will be the case for ε small enough. The conditions of the strict comparison principle are fulfilled with $\delta = \frac{Lh}{\varepsilon}$, $t_0 = 0$, $R = +\infty$ and $\rho = \rho_m$. Then, by Theorem 1.1.2 we have $\psi_2 - \psi_1 > 0$ in $\mathbb{R} \times [0,T)$ and this implies for all $(y,s) \in \mathbb{R} \times [0,T)$:

$$u^\varepsilon(y+h,s) \leq u^\varepsilon(y,s) + 2Lh.$$

Analogously, we can show that:

$$u^\varepsilon(y+h,s) \geq u^\varepsilon(y,s) - 2Lh.$$

Hence, u^ε is $2L$-Lipschitz continuous in space. Therefore, we can define the relaxed upper and lower semi-limits $\bar{u}(x,t) = \lim_{\varepsilon\to 0}\sup^* u^\varepsilon(x,t)$ and $\underline{u}(x,t) = \lim_{\varepsilon\to 0}\inf_* u^\varepsilon(x,t)$. By definition, we have $\underline{u} \leq \bar{u}$.

To get the convergence, it is sufficient to show that $\bar{u}$ and $\underline{u}$ are respectively a subsolution and a supersolution of (1.7). Indeed, by using a classical comparison principle we will then get:

$$\underline{u} \leq \bar{u} \leq u^0 \leq \underline{u}.$$

This is performed through two steps: we first start to prove that $\bar{u}$ and $\underline{u}$ both coincide with the macroscopic initial data at $t = 0$ and then we show that they are respectively a subsolution and a supersolution in the open set $\mathbb{R} \times (0,T)$.

Step 1. Let us show that:

$$\bar{u}(x,0) = \underline{u}(x,0) = u_0(x,0), \qquad \text{for } x \in \mathbb{R}.$$

The key idea is to obtain barriers for u^ε. We consider the extended initial data (and we still denote it by u_0) such that for all $t \in [0,T), u_0(x,t) = u_0(x,0)$.

Given $\nu > 0$, for ε small enough, we can apply Theorem 1.1.2 on $v_b : (x,t) \mapsto \frac{1}{\varepsilon}\left(u_0(\varepsilon x, \varepsilon t) + \nu + ||F||_\infty \varepsilon t\right)$ and on $u_b : (x,t) \mapsto \frac{1}{\varepsilon}u^\varepsilon(\varepsilon x, \varepsilon t)$ and we get:

$$u^\varepsilon(x,t) \leq u_0(x,t) + \nu + ||F||_\infty t, \qquad \text{for } (x,t) \in \mathbb{R} \times [-2\varepsilon\tau, T),$$

Similarly, we get:

$$u^\varepsilon(x,t) \geq u_0(x,t) - \nu - ||F||_\infty t, \qquad \text{for } (x,t) \in \mathbb{R} \times [-2\varepsilon\tau, T),$$

and so:

$$|u^\varepsilon(x,t) - u_0(x,t)| \leq \nu + ||F||_\infty t, \qquad \text{for } (x,t) \in \mathbb{R} \times [-2\varepsilon\tau, T).$$

This implies the result for the relaxed semi-limits (for vanishing ν).

Remark 1.4.1. *As F is bounded, the barriers we have obtained here are much simpler than the ones obtained in Lemma 3.1 of [34].*

Step 2. Let us only show by contradiction that $\bar{u}$ is a subsolution of (1.7) in $\mathbb{R} \times (0,T)$ (the other one being very similar). Assume that there exist $(\bar{x},\bar{t})$, $\varphi \in C^1_{x,t}$, $(r,\theta,\eta) \in (0,+\infty)^3$ such that:

$$\begin{cases} \bar{u}(\bar{x},\bar{t}) = \varphi(\bar{x},\bar{t}), \\ \bar{u} < \varphi \text{ in } B_{2r}(\bar{x},\bar{t}) \setminus \{(\bar{x},\bar{t})\}, \\ \bar{u} \leq \varphi - 2\eta \text{ in } B_{2r}(\bar{x},\bar{t}) \setminus B_r(\bar{x},\bar{t}), \\ \partial_t\varphi(\bar{x},\bar{t}) = 2\theta + F(\partial_x\varphi(\bar{x},\bar{t})), \\ \partial_t\varphi(x,t) \geq \theta + F(\partial_x\varphi(x,t)) \text{ in } B_r(\bar{x},\bar{t}), \end{cases} \tag{1.37}$$

where we set $B_s(p,q) := (p-s,p+s) \times (q-s,q+s)$.

As u^ε and φ are continuous, we can define:

$$M_\varepsilon := \max_{B_{2r}(\bar{x},\bar{t})} (u^\varepsilon - \varphi) := (u^\varepsilon - \varphi)(x_\varepsilon, t_\varepsilon). \tag{1.38}$$

We then use the fact that $\bar{u}$ (which is the relaxed upper semi-limit of u^ε) and φ coincide at $(\bar{x}, \bar{t})$ to get:

$$M_\varepsilon \geq -\eta, \tag{1.39}$$

for ε small enough.

Hence $(x_\varepsilon, t_\varepsilon) \in B_r(\bar{x}, \bar{t})$.

Let us define: $\varphi^\varepsilon := \varphi + M_\varepsilon$. This function satisfies:

$$\begin{cases} \varphi^\varepsilon(x_\varepsilon, t_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon), \\ \partial_t \varphi^\varepsilon(x,t) \geq \theta + F(\partial_x \varphi^\varepsilon(x,t)) \text{ in } B_r(\bar{x}, \bar{t}). \end{cases} \tag{1.40}$$

By regularity of φ^ε and F, for ε small enough, we have in $B_r(\bar{x}, \bar{t})$:

$$\partial_t \varphi^\varepsilon(x,t) \geq F\left(\frac{\varphi^\varepsilon(x+\varepsilon, t - \varepsilon\tau_0(\varepsilon^{-1}x)) - \varphi^\varepsilon(x, t - \varepsilon\tau_0(\varepsilon^{-1}x))}{\varepsilon} \right) + \frac{\theta}{2}. \tag{1.41}$$

Let us define: $d^\varepsilon(x,t) = \varphi^\varepsilon(x,t) - u^\varepsilon(x,t)$. We recall that $\bar{u} \leq \varphi - 2\eta$ in $B_{2r}(\bar{x}, \bar{t}) \setminus B_r(\bar{x}, \bar{t})$ and we combine with (1.39) to have:

$$d^\varepsilon(x,t) = \varphi(x,t) - u^\varepsilon(x,t) + M_\varepsilon \geq \frac{3\eta}{2} - \eta = \frac{\eta}{2}, \tag{1.42}$$

where we also used that $\bar{u}$ is the relaxed upper semi-limit of u^ε.

Using the fact that u^ε is $||F||_\infty-$Lipschitz continuous in time and that φ is smooth, we remark that d^ε is Lipschitz continuous in $B_{2r}(\bar{x}, \bar{t})$. Let K denote its Lipschitz constant. In order to apply the strict comparison principle (Theorem 1.1.2) let us define:

$$d_1 : (x,t) \mapsto \frac{1}{\varepsilon} d^\varepsilon(\varepsilon x, \varepsilon t).$$

d_1 is $K-$ Lipschitz continuous in $B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right)$. Thanks to (1.42), we have in $B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right)$:

$$d_1(x,t) \geq \frac{\eta}{2\varepsilon}. \tag{1.43}$$

Let us now consider any solution ρ_1 to (1.11) for $\tau < \frac{1}{eC_F}$ (see Proposition 1.3.1). We recall that (1.11) implies $\rho_1 \geq \rho_1(0) > 1$.

We have:

$$d_1(x, t-\tau') \leq d_1(x,t) + K\tau, \qquad \text{for } \tau' \in [0,\tau], (x,t) \in B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right).$$

Therefore, to get:

$$d_1(x, t-\tau') \leq \rho_1(\tau') d_1(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right),$$

it is sufficient to verify:

$$K\tau \leq (\rho_1(\tau') - 1) d_1(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right).$$

As ρ_1 is a non-decreasing function, and as $d_1 \geq 0$, it is finally sufficient to verify:

$$\frac{K\tau}{\rho_1(0)-1} \leq d_1(x,t), \qquad \text{for } \tau' \in [0,\tau], (x,t) \in B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right).$$

Thanks to (1.43), this will be the case for ε small enough. We choose ε such that $\max(1,\tau) < \frac{r}{8\varepsilon}$. The previous inequalities enable us to apply the strict comparison principle with $\delta = \frac{\eta}{2\varepsilon}$, $R = \frac{3r}{2\varepsilon}$, $x_0 = \frac{\bar{x}}{\varepsilon}$, $T = \frac{\bar{t}}{\varepsilon} + R$ and $t_0 = \frac{\bar{t}}{\varepsilon} - R$. Indeed, we have:

$$\begin{cases} [x_0 - R - 1, x_0 + R + 1] \backslash]x_0 - R, x_0 + R[\times [t_0 - \tau, T) \subset B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \\ [x_0 - R, x_0 + R] \times [t_0 - \tau, t_0] \subset B_{\frac{2r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \setminus B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right). \end{cases} \tag{1.44}$$

Hence, by Theorem 1.1.2, we have $d_1 > 0$ in $[x_0 - R, x_0 + R] \times [t_0, T)$ which is in contradiction with the definition of $(x_\varepsilon, t_\varepsilon)$ as we have:

$$\begin{cases} \left(\frac{x_\varepsilon}{\varepsilon}, \frac{t_\varepsilon}{\varepsilon}\right) \in B_{\frac{r}{\varepsilon}}\left(\frac{\bar{x}}{\varepsilon}, \frac{\bar{t}}{\varepsilon}\right) \subset [x_0 - R, x_0 + R] \times [t_0, T), \\ d_1\left(\frac{x_\varepsilon}{\varepsilon}, \frac{t_\varepsilon}{\varepsilon}\right) = 0. \end{cases}$$

□

Assumption (1.4) is linked with the fact that the vehicles are regularly spaced at initial times. As shown in [37], we can get rid of this assumption under some conditions. Let us consider the following equation:

$$\begin{cases} \partial_t v^\varepsilon(x,t) = F\left(\frac{v^\varepsilon(x+\varepsilon, t-\varepsilon\tau_0(\varepsilon^{-1}x)) - v^\varepsilon(x, t-\varepsilon\tau_0(\varepsilon^{-1}x))}{\varepsilon}\right), & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ v^\varepsilon(x,s) = u_0^\varepsilon(x,s), & \text{for } (x,s) \in \mathbb{R} \times [-2\varepsilon\tau, 0], \end{cases} \tag{1.45}$$

with the following assumptions on u_0^ε:

$$\begin{cases} \forall \varepsilon > 0, u_0^\varepsilon \text{ is a globally } L' - \text{Lipschitz continuous function on } \mathbb{R} \times [-2\tau, 0], \\ \exists D > 0 / \forall \varepsilon > 0, ||u_0^\varepsilon - u_0||_\infty \leq D\varepsilon. \end{cases} \tag{1.46}$$

Then we have:

Corollary 1.4.2 (Convergence under modified initial conditions). *For $\tau \in \left(0; \frac{1}{eC_F}\right)$ and under assumptions (1.6) and (1.46), the solution v^ε to (1.45) converges locally uniformly towards the (unique viscosity) solution of (1.7).*

Proof. Exactly as for u^ε, we show that v^ε is $2L'-$Lipschitz continuous in space and $||F||_\infty-$Lipschitz continuous in time; hence we can define the relaxed upper and lower semi-limits.

As for the barriers for u^ε in the convergence proof, we introduce a constant $\nu > 0$ and use the strict comparison principle for ε small enough to get:

$$||v^\varepsilon - u^\varepsilon||_\infty \leq D\varepsilon + \nu.$$

We then take the relaxed upper and lower semi-limits and combine with Theorem 1.1.1 to get the desired result for vanishing ν. □

1.4.2 The unstable stationary case

A natural question arises. What happens for higher reaction times? Example 1.3.6 highlights the fact that the initial dynamics plays a significant role and that reaction times cannot be considered separately. The answer to this question is not invariable and the following example shows that the expected macrosopic model can be derived for any reaction time for a special initial condition.

Example 1.4.3. *For $T = +\infty$, let us consider the case where all drivers have the same reaction time $\tau \in (0, +\infty)$. Let $L > 0$ be the common interdistance between all the vehicles. We consider that the vehicles do not move at initial times:*

$$x_i^0(t) = Li, \qquad \text{for } t \in [-\tau, 0].$$

This corresponds to:

$$u_0(x, t) = Lx, \qquad \text{for } (x, t) \in \mathbb{R} \times [-\tau, 0].$$

By incremental construction (or directly by uniqueness), we see that the solution u^ε to (1.5) for this initial condition does not depend on ε and is given by:

$$u^\varepsilon(x, t) = Lx + F(L)t, \qquad \text{for } (x, t) \in \mathbb{R} \times [0, +\infty).$$

The unique solution u^0 of (1.7) corresponding to this initial linear data is also given by the same expression. Therefore we have:

$$u^\varepsilon(x, t) = u^0(x, t) = Lx + F(L)t, \qquad \text{for } (x, t) \in \mathbb{R} \times [0, +\infty).$$

1.5 A counter-example to homogenization

The goal of this section is to exhibit a counter-example derived from Example 1.4.3. We will first give the explicit expressions of F, of the vehicles' initial positions and the value of τ and give a list of lemmas that will be useful to finally prove Theorem 1.1.4.

For simplicity, we fix $T = +\infty$. Let us consider the same $L > 0$ and $(k, \beta, \alpha) \in (0, +\infty)^3$ with the condition $\alpha > 4\beta L$. We consider the following F:

$$F(x) = k + \beta(x - L)^2 + \alpha(x - L) \qquad x \in [0, 2L],$$

and we continuously extend F to $\mathbb{R}$ by constants; hence F satisfies (1.6) with $C_F = \alpha + 2\beta L$.

We here choose the common reaction time: $\tau = \frac{\pi}{4\alpha} > \frac{1}{eC_F}$.

Let us consider $A \in (0, \frac{L}{8})$. We now introduce the initial positions of the vehicles:

$$x_i^0(t) = iL + \frac{(-1)^i - 1}{2} A \sin(2\alpha t), \qquad \text{for } (i, t) \in \mathbb{Z} \times [-\tau, 0]. \tag{1.47}$$

Remark 1.5.1. *We have $x_{i+2}^0 - x_i^0 = 2L$ but $x_{i+1}^0 - x_i^0 \neq L$. We recover the initial data of Example 1.4.3 for $A = 0$. This means that in this case, the odd vehicles oscillate around the previous stationary positions at initial times. Those oscillations will be essential in the construction of the counter-example: they will last for all time and will raise the time velocity of the vehicles that will become strictly greater than the value of the velocity function of the corresponding macroscopic space gradient.*

The key relationship between τ and α will allow the periodic oscillations to remain for all times as expressed in the following lemma:

Lemma 1.5.2. *Under the previous initial conditions, we have:*

$$X_{i+1}(t) - X_i(t) = L + A(-1)^{i+1} \sin(2\alpha t), \qquad \text{for } t \geq 0.$$

Remark 1.5.3. *We have $X_{i+2} - X_i = 2L$ for all times.*

Proof of Lemma 1.5.2. The proof is based on the incremental construction of the solutions and is thus an induction proof. We show the result for each $[n\tau, (n+1)\tau], n \in \mathbb{N}$. We only perform the first step as the next ones are identical.

Let us define: $d_i := X_{i+1} - X_i$ and $\bar{d}_i := d_i - L$. Our goal is to prove that:

$$\bar{d}_i(t) = A(-1)^{i+1}\sin(2\alpha t) \qquad \text{for } t \in [0, \tau].$$

For $t \in (0, \tau]$, we have:

$$d_i'(t) = F(X_{i+2}(t-\tau) - X_{i+1}(t-\tau)) - F(d_i(t-\tau)).$$

As $t - \tau \in [-\tau, 0]$:

$$d_i'(t) = F(x_{i+2}^0(t-\tau) - x_{i+1}^0(t-\tau)) - F(d_i(t-\tau)).$$

Thanks to Remark 1.5.1, we get:

$$d_i'(t) = F(2L - d_i(t-\tau)) - F(d_i(t-\tau)).$$

With the expression of F, this equivalently gives for $\bar{d}_i$:

$$\bar{d}_i{}'(t) = -2\alpha \bar{d}_i(t-\tau).$$

Thanks to (1.47), we have:

$$\bar{d}_i{}'(t) = -2\alpha A(-1)^{i+1}\sin(2\alpha t - 2\alpha\tau).$$

We remind that we have chosen $\tau = \frac{\pi}{4\alpha}$ to get:

$$\bar{d}_i{}'(t) = 2\alpha A(-1)^{i+1}\cos(2\alpha t).$$

Given that $\bar{d}_i(0) = X_{i+1}(0) - X_i(0) - L = 0$, we then integrate:

$$\bar{d}_i(t) = A(-1)^{i+1}\sin(2\alpha t),$$

or equivalently:

$$X_{i+1}(t) - X_i(t) = L + A(-1)^{i+1}\sin(2\alpha t).$$

This gives the result for $n = 0$. □

We define the following function v, piecewise affine in space as:

$$v(i,t) := x_i^0(t) = u_0(i,t) + \frac{(-1)^i - 1}{2} A\sin(2\alpha t), \qquad \text{for } (i,t) \in \mathbb{Z} \times [-\tau, 0], \tag{1.48}$$

where $u_0(x,t) = Lx$.

We immediately get that v is Lipschitz continuous and we denote by L' its Lipschitz constant.

For $\varepsilon > 0$, we now consider the solution v^ε of (1.45) associated to the initial data u_0^ε:

$$u_0^\varepsilon : (x,t) \mapsto \varepsilon v\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$$

u_0^ε is also L'-Lipschitz continuous.

From (1.48), we get that:

$$\left| \frac{u_0^\varepsilon(i\varepsilon, t\varepsilon)}{\varepsilon} - \frac{u_0(i\varepsilon, t\varepsilon)}{\varepsilon} \right| \leq \frac{L}{2}, \qquad \text{for } (i,t) \in \mathbb{Z} \times [-\tau, 0],$$

or equivalently:

$$|u_0^\varepsilon(i\varepsilon, s) - u_0(i\varepsilon, s)| \leq \varepsilon\frac{L}{2}, \qquad \text{for } (i,t) \in \mathbb{Z} \times [-\varepsilon\tau, 0].$$

We finally use the fact that both u_0^ε and u_0 are Lipschitz continuous in space to get:

$$||u_0^\varepsilon - u_0||_\infty \leq D_{L,L'}\varepsilon,$$

for some $D_{L,L'} > 0$. This shows that assumptions (1.46) are verified.

Corollary 1.5.4. *The solution of the macroscopic equation* (1.7) *corresponding to this initial data is:*

$$u^0(x,t) = Lx + F(L)t, \qquad \text{for } (x,t) \in \mathbb{R} \times [0, +\infty).$$

Proof. This is a classical solution and the solution is unique. □

Remark 1.5.5. *We see that we have:*

$$\partial_t u^0(0,t) = k = F(L) \qquad \text{for } t > 0.$$

That was also the case for u^ε in Example 1.4.3 but it is not true anymore for $A > 0$.

Indeed, we have for $s > 0$:

$$\partial_t v^\varepsilon(0,s) = F\left(\frac{v^\varepsilon(\varepsilon, s - \varepsilon\tau) - v^\varepsilon(0, s - \varepsilon\tau)}{\varepsilon}\right).$$

We recall the relationship:

$$X_i(t) = \frac{v^\varepsilon(i\varepsilon, t\varepsilon)}{\varepsilon}, \qquad \text{for } (i,t) \in \mathbb{Z} \times [-\tau, +\infty).$$

Hence, we get:

$$\partial_t v^\varepsilon(0,s) = F\left(X_1\left(\frac{s}{\varepsilon} - \tau\right) - X_0\left(\frac{s}{\varepsilon} - \tau\right)\right).$$

From Lemma 1.5.2, we get for the velocity function F considered:

$$\partial_t v^\varepsilon(0,s) = F(L) + \beta A^2 \sin^2\left(2\alpha\left(\frac{s}{\varepsilon} - \tau\right)\right) - \alpha A \sin\left(2\alpha\left(\frac{s}{\varepsilon} - \tau\right)\right),$$

or equivalently, for $\tau = \frac{\pi}{4\alpha}$:

$$\partial_t v^\varepsilon(0,s) = F(L) + \beta A^2 \cos^2\left(2\alpha\frac{s}{\varepsilon}\right) + \alpha A \cos\left(2\alpha\frac{s}{\varepsilon}\right). \tag{1.49}$$

We are now ready to prove Theorem 1.1.4.

Proof of Theorem 1.1.4. By contradiction, we suppose that v^ε converges locally uniformly towards the solution u^0 of (1.7) whose expression is given in corollary 1.5.4.

Let us consider $s > 0$ and $h > 0$. We then have:

$$\lim_{\varepsilon\to 0} \frac{v^\varepsilon(0,s+h) - v^\varepsilon(0,s)}{h} = \frac{u^0(0,s+h) - u^0(0,s)}{h} = F(L). \tag{1.50}$$

Otherwise, we have for $\varepsilon > 0$:

$$\frac{v^\varepsilon(0,s+h) - v^\varepsilon(0,s)}{h} = \frac{1}{h}\int_s^{s+h} \partial_t v^\varepsilon(0,s')ds'.$$

From (1.49), we get:

$$\frac{v^\varepsilon(0,s+h)-v^\varepsilon(0,s)}{h} = F(L)+\beta\frac{A^2}{2}+\frac{\beta\varepsilon A^2}{8\alpha h}\left[\sin\left(\frac{4\alpha s'}{\varepsilon}\right)\right]_s^{s+h}+\frac{\varepsilon A}{2h}\left[\sin\left(\frac{2\alpha s'}{\varepsilon}\right)\right]_s^{s+h}$$

As $|\sin|\leq 1$, we immediately get:

$$\lim_{\varepsilon\to 0}\frac{v^\varepsilon(0,s+h)-u^\varepsilon(0,s)}{h} = F(L)+\beta\frac{A^2}{2},$$

which is in contradiction with (1.50). □

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Chapter 2

Homogenization of a Hamilton-Jacobi equation for traffic lights: study of a phase shift effect

Abstract

In this paper, we study a one dimensional pursuit law with lights which is derived from traffic flow modelling. More precisely, we consider an infinite road where lights are equally spaced and with a constant phase shift between two lights. This model takes the form of a first order Hamilton-Jacobi equation with an Hamiltonian that is discontinuous in the space variable and the notion of viscosity solution is the one introduced in [52]. Each light is modelled as a time-periodic flux limiter and the traffic flow between two lights corresponds to the classical LWR model. The global Hamiltonian will be time-periodic but not periodic in space for a general phase shift. We first show that the rescaled solution converges toward the solution of the expected macroscopic model where the effective Hamiltonian depends on the phase shift. In a second time, numerical simulations are used to analyse the effect of the phase shift on the effective Hamiltonian and to reveal some properties of the effective Hamiltonian from the numerical observations.

Keywords: Hamilton-Jacobi equations, flux limiter, phase shift, homogenization.

2.1 Introduction

In the present paper we consider a Hamilton-Jacobi equation where the Hamiltonian is continuous and periodic in time and discontinuous in space, which here appears through a one dimensional pursuit law on a straight road that contains equidistant lights. We suppose a macroscopic model between two lights, namely a Hamilton Jacobi equation for the density of vehicles. We consider a Hamiltonian which is quasiconvex and coercive in the gradient variable between two lights. It was shown in [40] that a light can be seen as a flux limiter at the macroscopic level. In the present work, each light is viewed as a time dependent flux limiter, like in [41], and can be considered as a junction in the sense introduced in [52] where a specific notion of viscosity solution is defined. Our Hamilton-Jacobi equation takes the general form:

$$\begin{cases} u_t + H_F(u_x) = 0, & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z}, \\ u_t + G(t - \theta x, u_x) = 0, & \text{for } (t,x) \in (0,+\infty) \times \mathbb{Z}, \end{cases}$$

where θ denotes the constant phase shift between the lights.

Existence and uniqueness of the viscosity solution is contained in the network case of [52]. We use a formal asymptotic expansion of the rescaled solution (see [61] for instance) to figure out what the macroscopic model should be. The homogenization process is then performed using the perturbed test function method [28]. The corresponding cell problem is periodic if and only if the phase shift is rational. Thanks to a change of variables in time, we will be able to get back to a periodic setting for any value of the phase shift and hence show the existence of correctors. The effective Hamiltonian can be classicaly approximated by the large time slope of the corrector. We use this ergodicity property to get its numerical approximation and to study the phase shift effect on this effective Hamiltonian for the particular case where we consider a LWR traffic flow model (see [60] or [69]) between two lights. The numerical simulations are performed using the scheme introduced in [23] and [47]. We then retrieve some classical properties (continuity, convexity, coercivity) of the effective Hamiltonian but also less usual properties (symmetry, dependance with the distance between lights) and give some traffic flow interpretations and some conjectures.

2.1.1 Description of the model, notations and main results

We consider an infinite straight road where lights are located at each $i \in \mathbb{Z}$. Each light is modelled as a continuous 1-time periodic flux limiter, denoted by A_i, and $\theta \geq 0$ is the constant phase shift between two lights. We then have the following relationship:

$$A_{i+1}(t) = A_i(t-\theta), \qquad \text{for } t > 0.$$

We then consider a unique flux limiter A and define all the A_i in the following way:

$$A_i(t) := A(t - i\theta), \qquad \text{for } t > 0.$$

For a smooth real-valued function u defined on $(0,+\infty) \times \mathbb{R}$, $\partial_x u(t,x)$ denotes the spatial derivative of u at $x \in \mathbb{R} \setminus \mathbb{Z}$ and we define its space gradient on the whole real line as follows:

$$u_x(t,x) := \begin{cases} \partial_x u(t,x), & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z}, \\ \left(\partial_x u(t,i^-), \partial_x u(t,i^+)\right), & \text{for } (t, x=i) \in (0,+\infty) \times \mathbb{Z}. \end{cases}$$

In the free zone between two lights, we suppose that the traffic flow corresponds to the following macroscopic model:

$$u_t + H_F(u_x) = 0, \qquad \text{for } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z},$$

where H_F is supposed to be continuous, coercive and quasiconvex in the gradient variable:

(A0) Continuity of H_F: $H_F : \mathbb{R} \to \mathbb{R}$ is continuous.

(A1) Coercivity of H_F:

$$\lim_{|p| \to +\infty} H_F(p) = +\infty.$$

(A2) Quasiconvexity of H_F: There exists some $p_0 \in \mathbb{R}$ such that H_F is nonincreasing in $(-\infty, p_0]$ and is nondecreasing on $(p_0, +\infty)$. We denote by a_0 the minimum value of H_F.

The last assumption (A2) allows us to use the notion of flux-limited solution introduced in [52]. The non-increasing part H_F^- and the non-decreasing part H_F^+ of the Hamiltonian H_F are used in the definition of flux-limited junction conditions:

$$H_F^-(p) := \begin{cases} H_F(p), & \text{for } p \leq p_0, \\ H_F(p_0) = a_0, & \text{for } p > p_0. \end{cases}$$

$$H_F^+(p) := \begin{cases} H_F(p_0) = a_0, & \text{for } p < p_0, \\ H_F(p), & \text{for } p \geq p_0. \end{cases}$$

We now recall the definition of flux-limited junction function F_A (see also [52]):

$$F_A(t, x = i, p_L, p_R) = \max\left(A(t - i\theta), H_F^+(p_L), H_F^-(p_R)\right), \text{ for } (t, i) \in \mathbb{R} \times \mathbb{Z}, \ (p_L, p_R) \in \mathbb{R}^2,$$

where we recall the following assumptions on A:

(B1) Time periodicity of A:

$$A(t+1) = A(t), \qquad \text{for } t \in \mathbb{R}.$$

(B2) Uniform continuity of A: A is (uniformly) continuous with a modulus of continuity ω.

We now consider the following junction condition at each light:

$$u_t + F_A(t, i, u_x) = 0, \qquad \text{for } (t, i) \in (0, +\infty) \times \mathbb{Z}.$$

We now introduce a shorthand notation:

$$H(t, x, p) := \begin{cases} H_F(p), & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R} \setminus \mathbb{Z}, \ p \in \mathbb{R}, \\ F_A(t, i, p), & \text{for } (t, i) \in \mathbb{R} \times \mathbb{Z}, \ p \in \mathbb{R}^2. \end{cases}$$

A schematic picture is shown in Figure 2.1.

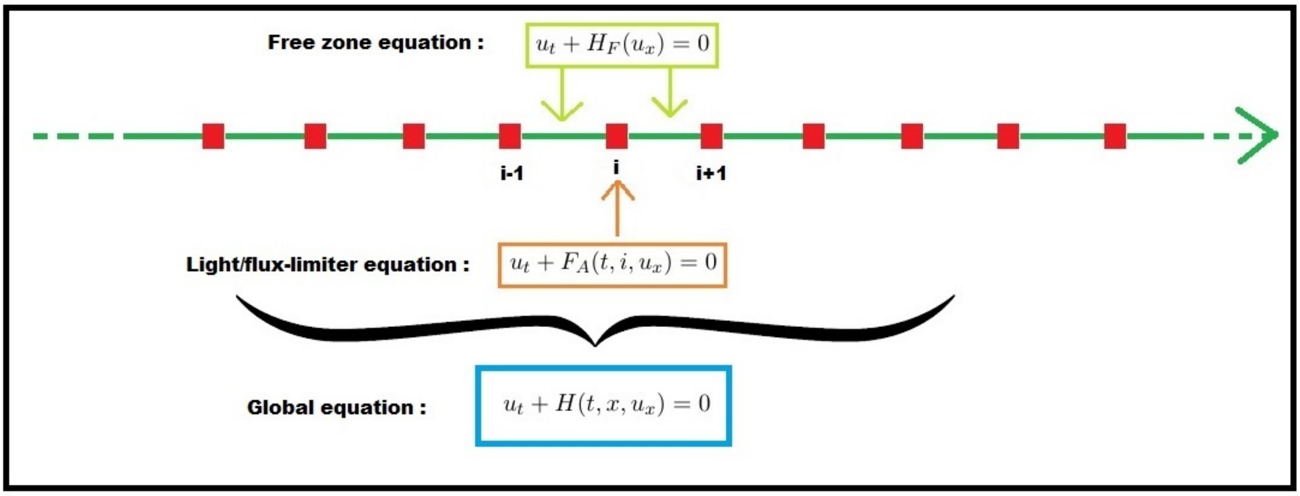

Figure 2.1: Sketch of the road.

Let us consider $\varepsilon > 0$. We consider the rescaled solution u^ε of the corresponding Hamilton-Jacobi equation on the whole straight road which takes the form of an oscillating Hamilton-Jacobi equation on the real line:

$$\begin{cases} u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0, & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u^\varepsilon(0, x) = u_0(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where we suppose that:

(C1) Regularity of the initial condition: u_0 is a L- Lipschitz continuous function in $\mathbb{R}$.

In the setting described in Section 2.2, there exists a unique viscosity solution of (2.1). The existence is given by Perron's method and the uniqueness by the comparison principle proved in [52] using the so-called vertex test function.

Our main theoretical result is the identification of the limit of u^ε for vanishing ε. This is part of homogenization of Hamilton-Jacobi equations. The stationary case was completely solved in [52].

To foresee the possible limit of u^ε and the equation solved by the limit, we perform a formal asymptotic expansion of u^ε as it was originally introduced in [61] and get the formal cell problem for a corrector v:

$$\lambda + v_\tau(\tau, y) + H(\tau, y, p + v_y(s, y)) = 0, \qquad \text{for } (\tau, y) \in \mathbb{R}^2, \tag{2.2}$$

where $(\lambda, p) := \left(u_t^0(t, x), u_x^0(t, x)\right)$ and $(\tau, y) := \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$.

Remark 2.1.1 (General method in the usual space-time periodic setting)**.** *In the classical space-time periodic setting, we can prove that this cell problem is well posed in the following sense: for each $p \in \mathbb{R}$, there exists a unique $\lambda = \lambda(p) := -\bar{H}(p)$, such that there exists a periodic solution v of (2.2). Such a function will be referred to as a corrector.*

With the formal definitions of λ and p, we get the Cauchy problem solved by u^0:

$$\begin{cases} u_t^0 + \bar{H}\left(u_x^0\right) = 0, & \text{in } (0, +\infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.3}$$

The function $\bar{H}$ is called the effective (or homogenized) Hamiltonian. This Cauchy problem has a unique viscosity solution (because $\bar{H}$ inherits the good properties of H). We finally use the perturbed test function method [28] (or [29]) to perform a rigorous proof of convergence, that is to say: u^ε converges locally uniformly to the unique solution of (2.3).

Changes and adaptations in our case : Let us recall that in our case, H is time dependent and discontinuous in space and that for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we are not in a space-time periodic setting. Here, we will perform a change of variables in time to get back to the periodic frame. Hence, we will be able to show that there exists a corrector, i.e. a solution to the cell problem (2.2) for any value of the phase shift. This is not necessary to define λ as the existence of approximate correctors is sufficient. We can show the existence of such approximate correctors and hence define λ for any value of the phase shift without performing this change of variables. This is what we do in Chapter 3, where we study the same problem in the almost periodic setting.

Main theoretical results.

Theorem 2.1.2 (Definition of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}$. There exists a unique $\lambda \in \mathbb{R}$ such that for all $\delta > 0$, there exists a bounded continuous approximate corrector $v^\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solution of:*

$$\begin{cases} \lambda + v_\tau^\delta(\tau, y) + H(\tau, y, p + v_y^\delta(\tau, y)) \leq \delta, & \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}, \\ \lambda + v_\tau^\delta(\tau, y) + H(\tau, y, p + v_y^\delta(\tau, y)) \geq -\delta, & \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{2.4}$$

According to this theorem, we can define the effective Hamiltonian $\bar{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting $\bar{H}(p, \theta) = -\lambda$.

We finally state the main theoretical result of this paper.

Theorem 2.1.3 (Convergence)**.** *Under the assumptions (A0)-(A2), (B1)-(B2) and (C1), the solution u^ε of (2.1) converges locally uniformly to the unique solution u^0 of the following Cauchy problem:*

$$\begin{cases} u_t^0 + \bar{H}\left(u_x^0, \theta\right) = 0, & \text{in } (0, +\infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.5}$$

Brief review of the litterature. The homogenization result has already been covered in [52] for a stationary periodic Hamiltonian. Homogenization results in a stationary almost periodic setting can be found in [54] and [1] and our problem has been solved in Chapter 3 using almost periodic arguments (in space). Some homogenization results in traffic flow modelling can be found in [35] when adding a junction condition in the microscopic model, in [36] for a second order microscopic model, in [40] when considering a light at the microscopic level, in [41] where a macroscopic junction condition is derived from a continuous microscopic model, and in [32] when considering the reaction time of drivers. See also [55] for other connections between Hamilton–Jacobi equations and traffic light problems and [2] for green waves modelling. In [30] and [31], the ergodic cell problem is studied with the use of the weak KAM theory, respectively for stationary periodic homogenization and space-time periodic homogenization.

The existence of λ in Theorem 2.1.2 is the essential step (its uniqueness being easier to obtain), as the convergence proof will classically follow the well-known perturbed test function method. To prove this theorem, we follow the idea developed in [61] and consider the following approximate cell problem for $\alpha > 0$:

$$\alpha v^\alpha(\tau,y) + v^\alpha_\tau(\tau,y) + H(\tau,y,p+v^\alpha_y(\tau,y)) = 0, \qquad \text{for } (\tau,y)\in\mathbb{R}\times\mathbb{R}. \tag{2.6}$$

With the help of the vertex test function introduced in [52] for the network setting, we can show a comparison principle for equation (2.6) (see Subsection 2.2.2). Thus, it is well-posed, and we classically get that αv^α is bounded in $\mathbb{R}^2$ and for a bound which does not depend on α (see Subsection 2.4.1).

In the usual space-time periodic setting, the original proof of the existence of λ relies on the estimate of the oscillation $\max_{\mathbb{R}^2} v^\alpha - \min_{\mathbb{R}^2} v^\alpha$ independently from α, see [51] for instance. This estimate replaces the classical Lipschitz estimate (see [61]). This was combined with the fact that αv^α is bounded independently from α to get λ as the uniform limit of αv^α for vanishing α: this property is called ergodicity (see also [10]).

In our framework, we have to perform a change of variables in time to get such an estimate.

Another goal of this paper is to analyse the phase shift effect on the effective flux. As there is no obvious way to get an explicit expression of the effective Hamiltonian, we will use another characterization of ergodicity (see [10] for instance). We consider the following equation:

$$\begin{cases} v_\tau + H(\tau,y,p+v_y) = 0, & \text{for } (\tau,y)\in(0,+\infty)\times\mathbb{R},\\ v(0,y) = 0, & \text{for } y\in\mathbb{R}.\end{cases} \tag{2.7}$$

Theorem 2.1.4 (Asymptotic time slope)**.** *Let us consider the unique solution v of (2.7). Under the assumptions (A0)-(A2) and (B1)-(B2), we have:*

$$\lim_{\tau\to+\infty} \frac{-v(\tau,y)}{\tau} = \bar{H}(p,\theta) \text{ uniformly in } y. \tag{2.8}$$

The effective Hamiltonian can then be approximated by the asymptotic time slope of the solution of (2.7). We will numerically compute the solution v of this equation for long time and for various values of p and θ. To do so, we will use the numerical schemes introduced in [23] and [47]. A key feature for the numerical simulations is that, for a rational phase shift, the Hamiltonian and hence the solution of (2.7) become periodic in space.

The homogenization also holds for Hamiltonians that are discontinuous in time (with the tools introduced in [41]). On the one hand, we will modelize the lights as time periodic crenels. On the other hand, between lights, we will consider the classical Hamiltonian $H : p \mapsto -\zeta p(1-p)$, where ζ will encode information about the distance between lights, the time periodicity and the maximum velocity of drivers.

We observe that the effective Hamiltonian is a continuous, coercive and convex function and also has other properties that are stated, illustrated, proved and interpreted in Sections 2.5 and 2.6. For example, we retrieve the fact that the blocking effect of several lights can be much higher than the effect of a single one, and that the distance between lights and their phase shift have a deep interaction.

2.1.2 Organisation of the article

In Section 2.2, we recall the notion of viscosity solutions on networks and prove a comparison principle for the approximate cell problem. Section 2.3 is devoted to the convergence proof (Theorem 2.1.3). In Section 2.4, we show the well-posedness of the approximate cell problem and the control of the oscillation via the change of time scaling. In Section 2.5, we show the existence of λ (Theorem 2.1.2) and give classical properties of the effective Hamiltonian (including Theorem 2.1.4 about the asymptotic time slope). Finally, in Section 2.6 we prove some other properties that are observed on the numerical simulations, state some conjectures and give some interpretations in terms of traffic flow.

2.2 Viscosity solutions on networks and comparison principle

In this section, we give all the technical tools and settings that we will need in this paper. We will first recall how to define viscosity solutions on networks (see [52]) and then will prove the comparison principle for the approximate cell problem.

2.2.1 Viscosity solutions on networks

We recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined in $[0,+\infty)\times\mathbb{R}$:

$$u^*(t,x)=\limsup_{(s,y)\to(t,x)} u(s,y) \quad \text{and} \quad u_*(t,x)=\liminf_{(s,y)\to(t,x)} u(s,y).$$

We recall the Cauchy problem on the whole straight road which takes the form of a Hamilton-Jacobi equation on the real line:

$$\begin{cases} u_t+H(t,x,u_x)=0, & \text{for } (t,x)\in(0,+\infty)\times\mathbb{R},\\ u(0,x)=u_0(x), & \text{for } x\in\mathbb{R}, \end{cases} \tag{2.9}$$

We remind the class of test functions for this type of equation introduced in [52]:

Class of test functions. We define the class of test functions in $N_T:=(0,+\infty)\times\mathbb{R}$ by

$$C_1(N_T)=\{\varphi\in C(N_T),\ \text{the restriction of } \varphi \text{ to } (0,+\infty)\times[i,i+1] \text{ is } C^1,\ \text{for all } i\in\mathbb{Z}\}.$$

Definition 2.2.1 (Viscosity solutions). *Assume (A0)-(A2), (B1)-(B2) and (C1), and let* $u:[0,+\infty)\to\mathbb{R}$.

i) We say that u is a sub-solution (resp. super-solution) of $u_t+H(t,x,u_x)=0$ in N_T if for all $\varphi\in C_1(N_T)$ such that

$$u^*\le\varphi \ \textit{(resp. } u_*\ge\varphi\textit{) in a neighbourhood of } (t_0,x_0)\in N_T,$$

with equality at (t_0,x_0), we have

$$\varphi_t+H(t,x,\varphi_x)\le 0 \ \textit{(resp. } \ge 0\textit{) at } (t_0,x_0).$$

ii) We say that u is a sub-solution (resp. super-solution) of (2.9) *in* $[0,+\infty)\times\mathbb{R}$ *if additionally*

$$u^*(0,x)\le u_0(x) \ \textit{(resp. } u_*(0,x)\ge u_0(x)\textit{) for all } x\in\mathbb{R}.$$

iii) We say that u is a (viscosity) solution if it is both a sub-solution and a super-solution.

2.2.2 Comparison principle for the approximate cell problem

We now give a comparison principle for (2.6). This one is not directly contained in [52] as we consider a time dependent equation in $\mathbb{R}\times\mathbb{R}$.

Theorem 2.2.2 (Comparison principle for the approximate cell problem)**.** *Under the assumptions (A0)-(A2) and (B1)-(B2), let u and v be respectively a bounded subsolution and a bounded supersolution of* (2.6)*. If u is upper semi-continuous and v is lower semi-continuous, we have: $u\le v$.*

The arguments for Theorem 2.2.2 are mainly contained in the newtork case of [52] (Section 5.5). To face with the spatial discontinuity of the Hamiltonian, the key idea is to use the vertex test function in the doubling variable approach.

Proof. We only give a sketch of proof. For that reason, we make two simplifying assumptions: we will suppose that the following supremum is reached at some point and that the penalized maximisers in the doubling variable approach converge towards this point (see the Remark 2.2.4 below for the general case). We fix $\alpha\in(0,1)$. By contradiction, we hence suppose that there exists $(t_0,y_0)\in\mathbb{R}^2$ such that:

$$M:=\sup_{\mathbb{R}^2}(u-v)=(u-v)(t_0,y_0)>0.$$

Step 1: the penalization procedure: localization around (t_0,y_0).

We will redefine the Hamiltonian and the vertex test function locally. For that purpose, we define $B:=\overline{B_r(t_0,y_0)}$ to be the closed ball of radius $r\in(0,1)$ centered at (t_0,y_0).

If $y_0\notin\mathbb{Z}$, we impose $r<dist(y_0,\mathbb{Z}):=\inf_{z\in\mathbb{Z}}|y_0-z|$.

For $(\eta,\nu,\beta,\gamma)\in(0,1)^4$, we consider :

$$M_{\eta,\nu,\beta}=\sup_{(t,x),(s,y)\in B}\left(u(t,x)-v(s,y)-\eta G^\gamma\left(\frac{x}{\eta},\frac{y}{\eta}\right)-\frac{(t-s)^2}{2\nu}-\frac{\beta(t-t_0)^2}{2}-\frac{\beta(s-t_0)^2}{2}-\frac{\beta(x-y_0)^2}{2}\right),$$

where we choose:

$$G^\gamma(x,y)=\begin{cases}\frac{(x-y)^2}{2}, & \text{if } y_0\notin\mathbb{Z},\\ G^{y_0,\gamma}(x,y), & \text{if } y_0\in\mathbb{Z},\end{cases}$$

where $G^{y_0,\gamma}$ is the vertex test function of parameter $\gamma>0$ built on the junction problem associated to the vertex y_0 at fixed time t_0 (see [52]), *i.e.* associated to junction problem for the Hamiltonian H^{t_0,y_0} given by:

$$H^{t_0,y_0}(x,q):=\begin{cases}H_F(p+q), & \text{if } x\neq y_0,\\ F_A(t_0,y_0,p+q), & \text{if } x=y_0.\end{cases}$$

We here recall some useful properties of $G^{y_0,\gamma}$:

i) (Bound from below) $G^{y_0,\gamma}\ge 0=G^{y_0,\gamma}(0,0)$.

ii) (Regularity) $G^{y_0,\gamma} \in C(\mathbb{R}^2)$, $G^{y_0,\gamma}(x,.) \in C^1(\mathbb{R})$ for all $x \in \mathbb{R}$ (resp. for $G^{y_0,\gamma}(.,y)$).

iii)(Compatibility condition on the diagonal) For all $x \in \mathbb{R}$,

$$0 \leq G^{y_0,\gamma}(x,x) - G^{y_0,\gamma}(0,0) \leq \gamma.$$

iv) (Compatibility condition on the gradients) For all $(x,y) \in \mathbb{R}^2$,

$$H^{t_0,y_0}\left(y, -G_y^{y_0,\gamma}(x,y)\right) - H^{t_0,y_0}\left(x, G_x^{y_0,\gamma}(x,y)\right) \leq \gamma. \tag{2.10}$$

v) (Superlinearity) For all $K > 0$, there exists a constant $C_K > 0$ such that for all $(x,y) \in \mathbb{R}^2$,

$$K|x-y| - C_K \leq G^{y_0,\gamma}(x,y).$$

vi) (Gradient bounds) For all $K > 0$, there exists a constant $C'_K > 0$ such that for all $(x,y) \in \mathbb{R}^2$,

$$|x-y| \leq K \Rightarrow |G_x^{y_0,\gamma}(x,y)| + |G_y^{y_0,\gamma}(x,y)| \leq C'_K.$$

The supremum in the definition of $M_{\eta,\nu,\beta}$ is reached at some point $(\bar{t},\bar{x})$, $(\bar{s},\bar{y}) \in B$. These maximizers satisfy the following penalization estimates for $\gamma \in \left(0, \min\left(1, \frac{M}{4}\right)\right)$.

Lemma 2.2.3 (Penalization)**.**

$$u(\bar{t},\bar{x}) - v(\bar{s},\bar{y}) \geq M_{\eta,\nu,\beta} \geq M - \eta\gamma \geq \frac{M}{2} > 0, \tag{2.11}$$

$$\frac{(\bar{t}-\bar{s})^2}{2\nu} + \frac{\beta(\bar{t}-t_0)^2}{2} + \frac{\beta(\bar{s}-t_0)^2}{2} + \frac{\beta(\bar{x}-y_0)^2}{2} \leq C_0 := ||u||_\infty + ||v||_\infty, \tag{2.12}$$

$$|\bar{x}-\bar{y}| \leq \omega(\eta), \tag{2.13}$$

for some modulus of continuity ω (depending on γ).

The proof of this lemma is given in [52], see Section 5.5. Moreover, we stick to the simplified case where:

$$(\bar{t},\bar{s},\bar{x},\bar{y}) \to (t_0,t_0,y_0,y_0) \text{ as } (\eta,\gamma) \to (0,0).$$

In particular, $(\bar{t},\bar{s},\bar{x},\bar{y}) \in \text{int}B$ for η,γ small enough.

Step 2: Viscosity inequalities. Hence, we can write the viscosity inequalities at $(\bar{t},\bar{x})$ and $(\bar{s},\bar{y})$,

$$\begin{cases} \alpha u(\bar{t},\bar{x}) + \frac{\bar{t}-\bar{s}}{\nu} + \beta(\bar{t}-t_0) + H\left(\bar{t},\bar{x},p + G_x^\gamma\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right) + \beta(\bar{x}-y_0)\right) \leq 0, \\ \alpha v(\bar{s},\bar{y}) + \frac{\bar{t}-\bar{s}}{\nu} + \beta(t_0-\bar{s}) + H\left(\bar{s},\bar{y},p - G_y^\gamma\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right)\right) \geq 0. \end{cases} \tag{2.14}$$

From (2.12), we get $\beta|\bar{t}-t_0| \leq C_0\sqrt{\beta}$ and $\beta|\bar{s}-t_0| \leq C_0\sqrt{\beta}$ and thus we can choose β small enough such that we have:

$$\beta|\bar{s}-t_0| + \beta|\bar{t}-t_0| \leq \alpha\frac{M}{4}.$$

Substracting the two viscosity inequalities, we get:

$$\alpha\frac{M}{4} \leq H\left(\bar{s},\bar{y},p - G_y^\gamma\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right)\right) - H\left(\bar{t},\bar{x},p + G_x^\gamma\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right) + \beta(\bar{x}-y_0)\right). \tag{2.15}$$

Step 3: Conclusion using gradient estimates. The case $y_0 \notin \mathbb{Z}$ is classical and we do not treat this one. We hence consider that $y_0 \in \mathbb{Z}$.

The key feature is to get estimates on the gradients $G_x^{y_0,\gamma}, G_y^{y_0,\gamma}$ written in Equation (2.15) uniformly in η and γ.

If we have such estimates, we can consider converging subsequences and denote the limits by respectively $p_x^{\nu,\beta}$ for the gradient in x and $p_y^{\nu,\beta}$ for the gradient in y. Given the assumption about the convergence of maximizers, we can pass to the limit $(\eta,\gamma)\to(0,0)$ in (2.15):

$$\alpha\frac{M}{4}\leq H(t_0,y_0,p-p_y^{\nu,\beta})-H(t_0,y_0,p+p_x^{\nu,\beta})=H^{t_0,y_0}\left(y_0,-p_y^{\nu,\beta}\right)-H^{t_0,y_0}\left(y_0,p_x^{\nu,\beta}\right)\leq 0,$$

where we get the last inequality by passing to the limit $(\eta,\gamma)\to(0,0)$ in (2.10).

We then get the desired contradiction $M\leq 0$.

<u>Gradient estimates:</u> We now focus on the bounds and start from the subsolution inequality in (2.14) and we combine with the penalization estimates to get:

$$H\left(\bar{t},\bar{x},p+G_x^{y_0,\gamma}\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right)+\beta(\bar{x}-y_0)\right)\leq 2C_0+\frac{C_0}{\sqrt{\nu}}. \tag{2.16}$$

$*$ When $\bar{x}\neq y_0$, we first get a bound C_ν (independent of β, η and γ) on $|G_x^{y_0,\gamma}\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right)|$ by using (2.12) and the coercivity of H_F in (2.16):

$$\left|G_x^{y_0,\gamma}\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right)\right|\leq C_\nu. \tag{2.17}$$

- If $\bar{y}\neq y_0$, we then get the same bound on $|-G_y^{y_0,\gamma}\left(\frac{\bar{x}}{\eta},\frac{\bar{y}}{\eta}\right)|$ by combining (2.17) with the compatibility condition on the gradients (2.10) written for H^{t_0,y_0} and the coercivity of H_F.

- If $\bar{y}=y_0$, using (2.10), (2.17) and the non-increasing property of the flux limiter, we get a bound from below for $-G_y^{y_0,\gamma}\left(\frac{\bar{x}}{\eta},\frac{y_0}{\eta}\right)$, denoted by $-C_\nu$ (independent of β, η and γ).

$*$ When $\bar{x}=y_0$, using the non-increasing property of the flux limiter, we get a bound by below on $G_x^{y_0,\gamma}\left(\frac{y_0}{\eta},\frac{\bar{y}}{\eta}\right)$, denoted $-C_\nu$ (independent of β, η and γ) such that in (2.16) we get:

$$F_A\left(\bar{t},y_0,p+G_x^{y_0,\gamma}\left(\frac{y_0}{\eta},\frac{\bar{y}}{\eta}\right)\right)\leq 2C_0+\frac{C_0}{\sqrt{\nu}}\quad\Rightarrow\quad G_x^{y_0,\gamma}\left(\frac{y_0}{\eta},\frac{\bar{y}}{\eta}\right)\geq -C_\nu. \tag{2.18}$$

- If $\bar{y}=y_0$, still using (2.10) and the non-increasing property of the flux limiter, we get the same lower bound on $-G_y^{y_0,\gamma}\left(\frac{y_0}{\eta},\frac{y_0}{\eta}\right)$.

- If $\bar{y}\neq y_0$, we use the lower bound in (2.18) and the non-increasing property of the flux limiter to get an upper bound on $F_A\left(t_0,y_0,G_x^{y_0,\gamma}\left(\frac{y_0}{\eta},\frac{\bar{y}}{\eta}\right)\right)$. We then use (2.10) and the coercivity of H_F to get a bound on $\left|-G_y^{y_0,\gamma}\left(\frac{y_0}{\eta},\frac{\bar{y}}{\eta}\right)\right|$.

To summarize, in the best cases we have bounds by above and by below on the gradients (when we can use coercivity), but, in every case we have at least a lower bound for the gradients' components (when we can only use the non-increasing property of the flux limiter).

We get the bound by above and by below in every case by using a remark from [52]. In the worst case $\bar{x}=\bar{y}=y_0$, the gradients are vectors and their components are only bounded from below. But when writing viscosity inequalities, they appear as variables of the nonincreasing part of Hamiltonians. Hence, if they are too large, they can be replaced with the point minimizing the Hamiltonian, without changing the viscosity inequalities. Upon using a truncation, we can hence consider that the constant C_ν is a bound by above and by below. $\square$

Remark 2.2.4. *To overcome the two simplifying assumptions we made at the beginning (the supremum is reached and the convergence of maximizers), it is enough to introduce a penalized version of M (in space and time), reached at some point $(t_0, s_0, y_0) \in \mathbb{R}^3$ as it was made in Lemma 5.12 (see Step 1) in [52]. At the end of the proof (see Step 6 of the Proof of Theorem 5.8 in [52]), we will also pass to the limit $(\eta, \gamma) \to (0, 0)$ in (2.16) and use the uniform continuity of H in time and for bounded gradients.*

2.3 Convergence

In this section, we will use the approximate correctors introduced in Theorem 2.1.2 to prove the convergence (Theorem 2.1.3) using the celebrated perturbed test function method of Evans (see [28]). The proof will be a proof by contradiction and we will use the relaxed semi-limits thanks to a-priori estimates. The proof is inspired from the one performed in Section 8 of [52] for a stationary Hamiltonian.

2.3.1 A-priori estimates and relaxed semi-limits

We first give classical barriers on u^ε uniformly in ε so as to introduce the relaxed semi-limits.

Lemma 2.3.1 (Barriers). *There exists $C > 0$ such that for all $\varepsilon > 0$,*

$$|u^\varepsilon(t,x) - u_0(x)| \leq Ct, \qquad \text{for } (t,x) \in [0, +\infty) \times \mathbb{R}.$$

Sketch of proof. It is sufficient to take $C := \max(||A||_\infty, \sup_{q\in[-L,L]} |H_F(q)|)$ where L is the Lipschitz constant of u_0 (see (C1)).

To get the barriers, we combine the comparison principle for (2.1) with the fact that $(x,t) \mapsto u_0(x) - Ct$ and $(x,t) \mapsto u_0(x) + Ct$ are respectively sub-solution and super-solution. □

We can hence define the following relaxed semi-limits :

$$\bar{u}(t,x) = \limsup_{\varepsilon\to 0,(s,y)\to(t,x)} u^\varepsilon(s,y) \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\varepsilon\to 0,(s,y)\to(t,x)} u^\varepsilon(s,y).$$

We now turn to the proof of convergence.

2.3.2 Convergence proof: perturbed test function method

Proof of Theorem 2.1.3. To get the convergence of u^ε towards u^0, it is sufficient to show that $\bar{u}$ and $\underline{u}$ are respectively a subsolution and a supersolution of (2.5). Indeed, by using a classical comparison principle we will then get:

$$\underline{u} \leq \bar{u} \leq u^0 \leq \underline{u}.$$

Those inequalities are true at $t = 0$ thanks to Lemma 2.3.1. Let us only show that $\bar{u}$ is a subsolution of (2.5) (the other one being very similar). By contradiction, we assume that there exist $(t_0, x_0) \in N_T$, $\varphi \in C^1((0, +\infty) \times \mathbb{R})$, $(r_0, \sigma, \eta) \in (0, +\infty)^3$ such that:

$$\begin{cases} \bar{u}(t_0,x_0) = \varphi(t_0,x_0), \\ \bar{u} < \varphi \text{ in } B_{2r_0}(t_0,x_0) \setminus \{(t_0,x_0)\}, \\ \bar{u} \leq \varphi - 2\eta \text{ in } B_{2r_0}(t_0,x_0) \setminus B_{r_0}(t_0,x_0), \\ \varphi_t(t_0,x_0) - \lambda = \varphi_t(t_0,x_0) + \bar{H}(\varphi_x(t_0,x_0),\theta) = \sigma > 0. \end{cases} \tag{2.19}$$

For fixed $\delta \in \left(0, \frac{\sigma}{2}\right)$, we consider the following perturbed test function $\varphi^\varepsilon : N_T \to \mathbb{R}$ [28],

$$\varphi^\varepsilon(t,x) = \varphi(t,x) + \varepsilon v^\delta \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

where (λ, v^δ) solves (2.4) for $p = \varphi_x(t_0, x_0)$.

Lemma 2.3.2. *For $r \leq r_0$ small enough, we have $\varphi_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varphi_x^\varepsilon\right) \geq 0$ in $B_r(t_0, x_0) \subset N_T$.*

Moreover, there exists $\eta_r > 0$ such that: $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ on $\partial B_r(t_0, x_0)$.

Remark 2.3.3. *Both the proof of Lemma 2.3.2 and the localized comparison principle we will invoke hereafter are derived from [52] (see Section 8.3).*

Proof of Lemma 2.3.2. Let us consider a test function ψ touching φ^ε from below at $(t,x) \in N_T$. Then the function

$$\psi_\varepsilon(\tau, y) = \varepsilon^{-1}\left(\psi(\varepsilon\tau, \varepsilon y) - \varphi(\varepsilon\tau, \varepsilon y)\right)$$

touches v^δ from below at $(\tau, y) = \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$. Hence by (2.4),

$$\lambda + \psi_t(t,x) - \varphi_t(t,x) + H(\tau, y, \varphi_x(t_0,x_0) + \psi_x(t,x) - \varphi_x(t,x)) \geq -\delta \geq -\frac{\sigma}{2}.$$

We combine with (2.19) to get:

$$\psi_t(t,x) + H(\tau, y, \psi_x(t,x)) \geq \frac{\sigma}{2} + E,$$

where

$$E = \varphi_t(t,x) - \varphi_t(t_0,x_0) + H(\tau, y, \psi_x(t,x)) - H(\tau, y, \psi_x(t,x) + \varphi_x(t_0,x_0) - \varphi_x(t,x)).$$

The fact that φ is C^1 implies that we can choose $r > 0$ small enough so that for all $(t,x) \in B_r(t_0,x_0)$,

$$E \geq -\frac{\sigma}{2}.$$

Moreover, as φ is strictly above $\bar{u}$ in (2.19), we conclude that $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ on $\partial B_r(t_0,x_0)$ for some $\eta_r > 0$. This achieves the proof of the lemma. □

From the lemma, we deduce thanks to the (localized) comparison principle (see Section 8.3 of [52]) that $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ in $B_r(t_0, x_0)$. In particular, this implies

$$\bar{u}(t_0,x_0) = \varphi(t_0,x_0) \geq \bar{u}(t_0,x_0) + \eta_r > \bar{u}(t_0,x_0),$$

which is the desired contradiction. □

2.4 Study of the approximate cell problem

We here show that the approximate cell problem is well posed and give the result about the control of the oscillation.

2.4.1 Well-posedness and regularity

Theorem 2.4.1 (Well-posedness of the approximate cell problem)**.** *Let us consider* $(p, \theta) \in \mathbb{R}^2$ *and* $\alpha > 0$*. Under the assumptions (A0)-(A2) and (B1)-(B2), for any* α*, there exists a unique bounded continuous solution* v^α *of* (2.6)*. Moreover,* v^α *is periodic in time and there exists* $C > 0$*, only depending on* H *and* p *such that:*

$$||\alpha v^\alpha||_{L^\infty(\mathbb{R}^2)} \leq C.$$

Proof. The existence of viscosity solutions to (2.6) is given by Perron's method as proved in [52] (the only difference is that the barriers are constant in our case). The constant C in Theorem 2.4.1 is actually the constant barrier chosen in Perron's method:

$$C = \max(||A||_\infty, |H_F(p)|).$$

The comparison principle of Theorem 2.2.2 classically leads to the uniqueness and the continuity of the solution v^α as $(v^\alpha)^* \leq (v^\alpha)_*$. Indeed, for u^α another solution, we get:

$$(v^\alpha)_* \leq (v^\alpha)^* \leq (u^\alpha)_* \leq (u^\alpha)^* \leq (v^\alpha)_*.$$

By translation invariance of (2.6), the solution for the time translated Hamiltonian $(\tau, y, q) \mapsto H(\tau + 1, y, q)$ is $v_1 : (\tau, y) \mapsto v^\alpha(\tau + 1, y)$.

By time periodicity of A, we have $H(\tau + 1, y, p + q) = H(\tau, y, p + q)$ and hence v_1 is also a solution of (2.6). By uniqueness, we deduce that $v_1 = v^\alpha$ and hence, v^α is time periodic. □

2.4.2 Control of the oscillation

We here state and prove the result about the control of the oscillation.

Theorem 2.4.2 (Control of the oscillation)**.** *Let us consider* $(p, \theta) \in \mathbb{R}^2$*. Under the assumptions (A0)-(A2) and (B1)-(B2), there exists a constant* $C_0 > 0$*, only depending on* H*, such that:*

$$\max_{\mathbb{R}^2} v^\alpha - \min_{\mathbb{R}^2} v^\alpha \leq C_0. \tag{2.20}$$

Proof of Theorem 2.4.2. **Step 1: Change of variables.**

To reduce to a classical periodic frame, we introduce the following linear change of variables. For $\alpha > 0$, we define: $u^\alpha(s, y) := v^\alpha(s + y\theta, y)$ (this corresponds to $s := \tau - y\theta$). We get that u^α is the unique bounded solution of:

$$\alpha u^\alpha(s, y) + u^\alpha_s(s, y) + h(s, y, p + u^\alpha_y(s, y) - \theta u^\alpha_s(s, y)) = 0, \qquad \text{for } (s, y) \in \mathbb{R} \times \mathbb{R}, \tag{2.21}$$

where the new Hamiltonian h does not depend on θ:

$$h(s, y, q) := \begin{cases} H_F(q), & \text{for } (s, y) \in \mathbb{R} \times \mathbb{R} \setminus \mathbb{Z},\ q \in \mathbb{R}, \\ \max\left(A(s), H_F^+(q_L), H_F^-(q_R)\right), & \text{for } (s, y) \in \mathbb{R} \times \mathbb{Z},\ q = (q_L, q_R) \in \mathbb{R}^2. \end{cases} \tag{2.22}$$

From the fact that u^α is a solution of (2.21) if and only if v^α is a solution of (2.6), we deduce that the same comparison principle (Theorem 2.2.2)holds for (2.21).

The Hamiltonian h is 1-periodic in s and y. Using this comparison principle, we get that u^α is one periodic in (the new) time and space.

We then get:

$$\max_{\mathbb{R}^2} v^\alpha - \min_{\mathbb{R}^2} v^\alpha = \max_{\mathbb{R}^2} u^\alpha - \min_{\mathbb{R}^2} u^\alpha = \max_{[0,1]^2} u^\alpha - \min_{[0,1]^2} u^\alpha. \tag{2.23}$$

Step 2: Time oscillation of u^α.

By coercivity and continuity of H_F and the continuity of A (see (A0)-(A2), (B1)-(B2)), H is bounded from below. We deduce from Theorem 2.4.1 that there exists $C > 0$ independent of α such that:

$$u^\alpha_s \leq C.$$

From a classical comparison principle, we get:

$$u^\alpha(s+h, y) \leq u^\alpha(s, y) + Ch, \qquad \text{for } (s, y) \in [0,1] \times \mathbb{R}, h \geq 0.$$

We combine with the time periodicity of u^α to get:

$$|u^\alpha(s, y) - u^\alpha(S, y)| \leq C, \qquad \text{for } (s, S, y) \in [0,1]^2 \times \mathbb{R}.$$

Step 3: Space oscillation. In particular, we define $M^\alpha(y) = \sup_{s \in \mathbb{R}} u^\alpha(s, y) := u^\alpha(s_\alpha(y), y)$ where $s_\alpha(y) \in [0,1]$ and get:

$$|u^\alpha(s, y) - M^\alpha(y)| \leq C, \qquad \text{for } y \in \mathbb{R}. \tag{2.24}$$

Lemma 2.4.3. *M^α is a continuous function and is a viscosity subsolution of $H(s_\alpha(y), y, p + M^\alpha_y(y)) \leq C'$ for some $C' > 0$ independent of α.*

Moreover there exists a constant independent of α, still denoted by $C > 0$, such that we have: $|M^\alpha_y(y)| \leq C$, for all $y \notin \mathbb{Z}$.

Proof. Consider a sequence $(y_n)_{n \in \mathbb{N}}$ that converges towards $y \in \mathbb{R}$. We want to show that $(M^\alpha(y_n))_{n \in \mathbb{N}}$ converges towards $M^\alpha(y)$.

As $s_\alpha(y_n) \in [0,1]$, we can consider a converging subsequence (denoted by ψ) towards some $s' \in [0,1]$. By definition, we have:

$$M^\alpha(y_n) = u^\alpha(s_\alpha(y_n), y_n) \geq u^\alpha(s, y_n), \qquad \text{for } s \in \mathbb{R}.$$

Taking the limit for the subsequence, we get by continuity of u^α:

$$\lim_{n \to +\infty} M^\alpha(y_{\psi(n)}) = u^\alpha(s', y) \geq u^\alpha(s, y), \qquad \text{for } s \in \mathbb{R}.$$

Thus, by definition of M^α we get: $M^\alpha(y) = u^\alpha(s', y)$ (but s' is not necessarily equal to $s_\alpha(y)$) and $(M^\alpha(y_{\psi(n)}))$ converges towards $M^\alpha(y)$ which does not depend on ψ and thus all the sequence $(M^\alpha(y_n))$ converges to the same value.

Consider $y_0 \in \mathbb{R}$ and φ a space test-function such that $M^\alpha - \varphi$ has a strict local maximum at y_0:

$$\begin{cases} M^\alpha(y_0) = \varphi(y_0), \\ M^\alpha < \varphi \text{ in } (y_0 - r; y_0 + r) \setminus \{y_0\}. \end{cases}$$

This implies:

$$\begin{cases} u^\alpha(s_\alpha(y_0), y_0) = \varphi(y_0), \\ u^\alpha(s, y) < \varphi(y) \text{ in } \mathbb{R} \times (y_0 - r; y_0 + r) \setminus \{y_0\}. \end{cases}$$

$u^\alpha - \varphi$ has a strict local maximum at $(s_\alpha(y_0), y_0)$ and thus:

$$\alpha u^\alpha(s_\alpha(y_0), y_0) + H(s_\alpha(y_0), y_0, p + \varphi_y(y_0)) \leq 0.$$

Combining with the estimate of Theorem 2.4.1, we get the first part of the result.

Finally, we get the Lipschitz estimate for $y \notin \mathbb{Z}$ by using the coercivity of H_F. □

From Lemma 2.4.3, we get:

$$|M^\alpha(x) - M^\alpha(y)| \le C, \qquad \forall k \in \mathbb{Z}, \forall (x,y) \in (k,k+1)^2.$$

Using the continuity of M^α, we have:

$$|M^\alpha(x) - M^\alpha(y)| \le C, \qquad \forall k \in \mathbb{Z}, \forall (x,y) \in [k,k+1]^2. \tag{2.25}$$

Step 4: Conclusion. For $(s,y,s',y') \in [0,1]^4$, we have:

$$|u^\alpha(s,y) - u^\alpha(s',y')| \le |u^\alpha(s,y) - M^\alpha(y)| + |M^\alpha(y) - M^\alpha(y')| + |u^\alpha(s',y') - M^\alpha(y')|.$$

We combine with (2.24), (2.25) and get the desired result:

$$|u^\alpha(s,y) - u^\alpha(s',y')| \le 3C := C_0, \qquad \forall (s,y,s',y') \in [0,1]^4.$$

The proof is now complete. □

2.5 Effective Hamiltonian

In this section we give classical properties of the effective Hamiltonian (see [41] for instance). Thanks to section 2.4, we will first show Theorem 2.1.2.

2.5.1 Existence and uniqueness of λ: Proof of Theorem 2.1.2

The existence of λ is a direct consequence of the previous section, as it was shown in [54]. The uniqueness of λ is also classical (see [10] for instance) and is a consequence of the following Lemma.

Lemma 2.5.1. *Let us consider $(\lambda, \bar\lambda) \in \mathbb{R}^2$. If for any $\delta > 0$, there exist bounded functions v^δ and w^δ such that:*

$$\begin{cases} \lambda + v^\delta_\tau(\tau,y) + H(\tau,y,p+v^\delta_y(\tau,y)) \ge -\delta, & \text{for } (\tau,y) \in \mathbb{R}\times\mathbb{R}, \\ \bar\lambda + w^\delta_\tau(\tau,y) + H(\tau,y,p+w^\delta_y(\tau,y)) \le \delta, & \text{for } (\tau,y) \in \mathbb{R}\times\mathbb{R}, \end{cases}$$

then, $\bar\lambda \le \lambda$.

Proof of Lemma 2.5.1. For fixed $\delta > 0$, we define:

$$\begin{cases} v(\tau,y) := v^\delta(\tau,y) + (\lambda+\delta)\tau + ||v^\delta - w^\delta||_{L^\infty(\mathbb{R}^2)}, \\ w(\tau,y) := w^\delta(\tau,y) + (\bar\lambda - \delta)\tau. \end{cases}$$

The functions v and w are respectively super-solution and sub-solution of $u_\tau + H(\tau,y,p+u_y) = 0$ and we have $v(0,.) \ge w(0,.)$. Thus, by comparison principle, we have $v \ge w$ in $[0,+\infty)\times\mathbb{R}$.

Since v^δ and w^δ are bounded, by dividing this inequality by $\tau > 0$ and letting τ tend to $+\infty$, we obtain: $2\delta \ge \bar\lambda - \lambda$ for all $\delta > 0$. Thus, we get $\bar\lambda \le \lambda$. □

Proof of Theorem 2.1.2.

Existence: From Theorem 2.4.1 we see that the set $\{\alpha v^\alpha(0,0), \alpha > 0\} \subset \mathbb{R}$ is bounded. We choose a subsequence α_j such that $\alpha_j v^{\alpha_j}(0,0) \to \lambda$ for some $\lambda \in \mathbb{R}$ as $j \to +\infty$.

For any fixed $\delta > 0$, according to Theorem 2.4.2, there is a $j \in \mathbb{N}$ such that $|\alpha_j v^{\alpha_j}(\tau, y) - \lambda| \leq \delta$ for all $(\tau, y) \in \mathbb{R}^2$. Then, $v^\delta := v^{\alpha_j}$ is bounded (the bound depends on α_j) and satisfies the viscosity inequalities (2.4).

Uniqueness: We suppose that there exist λ and $\bar{\lambda}$ that verify Theorem 2.1.2. For fixed $\delta > 0$, there exist bounded approximate correctors v^δ and w^δ respectively associated to λ and $\bar{\lambda}$ in (2.4). By using Lemma 2.5.1 in the case where v^δ is a supersolution and w^δ is a subsolution, we get $\bar{\lambda} \leq \lambda$, and the reverse inequality is obtained by the same argument in the symmetrical case.

Remark 2.5.2 (Exact correctors)**.** *Theorem 2.4.2 enables us to define the half-relaxed limits*

$$\bar{v}(\tau, y) = \limsup_{\alpha \to 0, (t,x) \to (\tau, y)} v^\alpha(t, x) - v^\alpha(0, 0) \quad \text{and} \quad \underline{v}(\tau, y) = \liminf_{\alpha \to 0, (t,x) \to (\tau, y)} v^\alpha(t, x) - v^\alpha(0, 0).$$

As stated in Lemma 3.1 of [41], the discontinuous stability of viscosity solutions implies that $\bar{v}$ and $\underline{v}$ are respectively subsolution and supersolution of the following equation:

$$\lambda + v_\tau(\tau, y) + H(\tau, y, p + v_y(\tau, y)) = 0, \qquad \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}.$$

Perron's method then allows us to construct an exact corrector between $\bar{v}$ and $\underline{v} + ||\bar{v} - \underline{v}||_{L^\infty(\mathbb{R}^2)}$.

□

2.5.2 Regularity, estimate and coercivity

We first start with the continuity of the effective Hamiltonian in the gradient variable as it is a direct consequence of the uniqueness of λ. We introduce an elementary but useful lemma and its corollary:

Lemma 2.5.3. *F_A is continuous with respect to $q \in \mathbb{R}^2$ uniformly in θ, space and time.*

Proof. The continuity of H_F^+ and H_F^- is naturally inherited from the one of H_F.

For $p' := (p'_L, p'_R) \in \mathbb{R}^2$ and $q' := (q'_L, q'_R) \in \mathbb{R}^2$, $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, we have:

$$|F_A(t, i, p') - F_A(t, i, q')| = |\max\left(A(t - i\theta), H_F^+(p'_L), H_F^-(p'_R)\right) - \max\left(A(t - i\theta), H_F^+(q'_L), H_F^-(q'_R)\right)|.$$

We use the fact that max is 1−Lipschitz continuous to get rid out of time, space and θ:

$$|F_A(t, i, p') - F_A(t, i, q')| \leq |\max\left(H_F^+(p'_L), H_F^-(p'_R)\right) - \max\left(H_F^+(q'_L), H_F^-(q'_R)\right)|.$$

Using the triangular inequality, we get:

$$\begin{aligned} |F_A(t, i, p') - F_A(t, i, q')| \leq {} & |\max\left(H_F^+(p'_L), H_F^-(p'_R)\right) - \max\left(H_F^+(p'_L), H_F^-(q'_R)\right)| + \\ & |\max\left(H_F^+(p'_L), H_F^-(q'_R)\right) - \max\left(H_F^+(q'_L), H_F^-(q'_R)\right)|. \end{aligned} \tag{2.26}$$

By using the fact that max is 1−Lipschitz continuous again, we get:

$$|F_A(t, i, p') - F_A(t, i, q')| \leq |H_F^-(p'_R) - H_F^-(q'_R)| + |H_F^+(p'_L) - H_F^+(q'_L)|,$$

which gives the desired result by combining with the continuity of H_F^+ and H_F^-. □

Lemma 2.5.3 implies that H is continuous in the gradient variable, as $H = H_F$ far from the lights.

Theorem 2.5.4 (Continuity in the gradient variable)**.** *Let us consider $\theta \in [0, 1]$. Under the assumptions (A0)-(A2), $\bar{H}$ is continuous in the gradient variable.*

Remark 2.5.5. *We have observed from the numerical simulations that the effective Hamiltonian is also continuous in the phase shift and this is proved in Theorem 2.6.3.*

Proof of Theorem 2.5.4. Let us consider $\theta \in [0,1]$ and $p \in \mathbb{R}$. If the sequence $(p_k)_k \subset [p-1, p+1]$ converges to p, then, up to a subsequence, $(\bar{H}(p_k,\theta))_k$ converges to some $\mu \in \mathbb{R}$. Indeed, even if the constant C in Theorem 2.4.1 depends on p_k, this constant can be replaced by $C_1 = \max(||A||_\infty, \sup_{q\in[p-1,p+1]} |H_F(q)|)$ which does not depend on k.

For each $k \in \mathbb{N}$, we denote by v^k a possibly discontinuous solution of (see Remark 2.5.2):

$$-\bar{H}(p_k,\theta) + v^k_\tau(\tau,y) + H(\tau,y,p_k + v^k_y(\tau,y)) = 0, \qquad \text{for } (\tau,y) \in \mathbb{R}\times\mathbb{R}.$$

The correctors v^k are uniformly bounded and hence the half-relaxed limits

$$\bar{w}(\tau,y) = \limsup_{k\to+\infty,(t,x)\to(\tau,y)} v^k(t,x) \quad \text{and} \quad \underline{w}(\tau,y) = \liminf_{k\to+\infty,(t,x)\to(\tau,y)} v^k(t,x)$$

are finite.

The discontinuous stability of viscosity solutions implies that $\bar{w}$ and $\underline{w}$ are respectively a bounded periodic subsolution and supersolution of:

$$-\mu + w_\tau(\tau,y) + H(\tau,y,p + w_y(\tau,y)) = 0, \qquad \text{for } (\tau,y) \in \mathbb{R}\times\mathbb{R}.$$

By uniqueness of the ergodic constant (see Theorem 2.1.2 or Lemma 2.5.1), this implies that $\mu = \bar{H}(p,\theta)$ and since this is true for any converging subsequence $(\bar{H}(p_k,\theta))_k$, the continuity is proved. □

Theorem 2.5.6 (Coercivity of the effective Hamiltonian and estimate)**.** *For $a := \inf_{\mathbb{R}} A$ and $b := \sup_{\mathbb{R}} A$, we have:*

$$\bar{H}_a(p) \le \bar{H}(p,\theta) \le \bar{H}_b(p), \qquad \forall (p,\theta) \in \mathbb{R}^2,$$

where H_a and H_b are respectively the stationary Hamiltonians associated to the stationary flux a and b (i.e where A is replaced by respectively a and b).

In particular, $\bar{H}$ is coercive in the gradient variable uniformly in θ.

Proof. We consider $v^{a,\alpha}$ and $v^{b,\alpha}$ the unique solutions of the stationary approximate cell problems for H_a and H_b:

$$\alpha v^{j,\alpha}(y) + H_j(y, p + v^{j,\alpha}_y(y)) = 0, \qquad \text{for } j = a,b \text{ and } y \in \mathbb{R}.$$

The functions $v^{a,\alpha}$ and $v^{b,\alpha}$ are respectively supersolution and subsolution of (2.6) and hence by comparison principle:

$$\alpha v^{a,\alpha}(y) \ge \alpha v^{\alpha}(\tau,y) \ge \alpha v^{b,\alpha}(y), \qquad \text{for } (\tau,y) \in \mathbb{R}^2.$$

By definition of $\bar{H} = -\lambda$, we get the desired inequalities by letting α tend to 0.

The uniform coercivity of $\bar{H}$ is inherited from the coercivity of $\bar{H}_a$ whose explicit form was given in Subsection 8.4 of [52]. □

Theorem 2.5.7. *$\bar{H}$ is $1-$periodic in θ.*

Proof. We explicitly write the dependence on θ for H by writing $H = H_\theta$. Thanks to the $1-$periodicity of A, we have $H_\theta = H_{\theta+1}$. We then use the comparison principle for the approximate cell problem and take the limit $\alpha \to 0$ to conclude. □

2.5.3 Proof of Theorem 2.1.4 (asymptotic time slope)

The proof is very classical and can be found in [10] for instance. We give it here for the reader's convenience.

Proof. For fixed $\delta > 0$, we consider an approximate corrector v^δ described in Theorem 2.1.2 for $\lambda = -\bar{H}(p, \theta)$.

The functions defined by $(\tau, y) \to (\lambda - \delta)\tau + v^\delta(\tau, y) - ||v^\delta||_\infty$ and $(\tau, y) \to (\lambda + \delta)\tau + v^\delta(\tau, y) + ||v^\delta||_\infty$ are respectively subsolution and supersolution of (2.7).

Hence, by comparison we get:

$$(\lambda - \delta)\tau + v^\delta(\tau, y) - ||v^\delta||_\infty \leq v(\tau, y) \leq (\lambda + \delta)\tau + v^\delta(\tau, y) + ||v^\delta||_\infty.$$

We use the fact that v^δ is bounded to get:

$$-2||v^\delta||_\infty + (\lambda - \delta)\tau \leq v(\tau, y) \leq (\lambda + \delta)\tau + 2||v^\delta||_\infty, \qquad \text{for } (\tau, y) \in [0, \infty) \times \mathbb{R}.$$

We finally divide by $\tau > 0$ and let τ tend to $+\infty$ to get the desired result as it is true for all $\delta > 0$. □

2.6 Numerical estimation of the flux: study of the phase shift effect

In this section, we analyse the effect of a rational phase shift $\theta \in \mathbb{Q}$ on the effective Hamiltonian. The microscopic Hamiltonian is periodic in time and space and the effective Hamiltonian is approximated by the asymptotic time slope of the solution of (2.7).

In the previous parts, we supposed that the lights were equally spaced and that the spacing was rescaled to 1, like their common time period. We here suppose that the spacing is $d > 0$, the time period is $T \in (0, +\infty)$, the phase shift is $\theta \in [0, T]$; we denote v_{max} the constant maximum velocity of the vehicles and ρ_{max} the constant maximum density. We first give the explicit expression of the free zone Hamiltonian, H_f say, that corresponds to the classical Greenshields fundamental diagram [45] for LWR model in our setting (where v is the average velocity):

$$H_f : \rho \mapsto -\rho v_{max} \left(1 - \frac{\rho}{\rho_{max}}\right) = -\rho v(\rho).$$

Hence, the equation solved by u in the free zone is:

$$u_t - u_x v_{max}(1 - \frac{u_x}{\rho_{max}}) = 0, \tag{2.27}$$

where u_x represents the density of vehicles.

We will then compute the corresponding Hamiltonian for the rescaled parameters, by a nondimensionalization procedure. A parameter ζ will then naturally appear and encode all the information about the road and the vehicles.

Remark 2.6.1 (Definition of the free zone Hamiltonian H_f). *The reason why H_f is defined as the opposite of the fundamental diagram is given in Section 2.2 of [53].*

2.6.1 Nondimensionalization

For $\sigma > 0$, we define $\underline{u}(\underline{t},\underline{x}) = \frac{1}{\sigma}u(t,x)$ where $\underline{t} := \frac{t}{T}$ and $\underline{x} := \frac{x}{d}$. Equivalently, we have: $u(t,x) = \sigma\underline{u}(\frac{t}{T},\frac{x}{d})$.

From (2.27), we get the equation solved by $\underline{u}$ in the free zone:

$$\frac{\sigma}{T}\underline{u}_{\underline{t}} - v_{max}\frac{\sigma}{d}\underline{u}_{\underline{x}}\left(1 - \frac{\sigma}{\rho_{max}d}\underline{u}_{\underline{x}}\right) = 0.$$

We choose $\sigma = \rho_{max}d$ and we get:

$$\underline{u}_{\underline{t}} - \zeta\underline{u}_{\underline{x}}(1 - \underline{u}_{\underline{x}}) = 0,$$

where $\zeta := \frac{v_{max}T}{d} > 0$.

We here fit into the theoretical framework of the previous sections by defining $H_F : p \mapsto -\zeta p(1-p)$.

For instance, in town we have the following order of magnitude for ζ: $v_{\max} \sim 10\ m.s^{-1}$, $T \sim 60\ s$, $d \sim 200\ m$ and $\zeta \sim 3$.

The rescaled flux limiter is defined as:

$$A(\underline{t}) = \begin{cases} 0 \text{ when } \underline{t} - \lfloor \underline{t} \rfloor < \frac{1}{2} \\ \zeta a_0 \text{ else,} \end{cases}$$

where $a_0 = -\frac{1}{4}$ is the minimum value of $p \mapsto -p(1-p)$.

It is important to state that such a flux limiter is discontinuous and the corresponding Hamilton-Jacobi equation does not fit into the theoretical frame of the previous sections. All the results of the present work still hold for a discontinuous flux limiter but are much more difficult to prove. The technical arguments are given in Section 5 of [41]. We only use this discontinuous flux limiter for the numerical simulations. For sake of simplicity, we consider a regularized version of this flux limiter for all the following theorems. For example, we stick to the case where A is Lipschitz continuous.

The rescaled phase shift is: $\underline{\theta} := \frac{\theta}{T} \in [0,1]$ and we now state (and sometimes prove) different observations on the effective Hamiltonian defined for $(p,\underline{\theta}) \in \mathbb{R} \times [0,1]$. The rescaled phase shift will still be denoted as θ to avoid heavy notation.

We now use Theorem 2.10 and Remark 2.11 of [40]. Using a comparison principle on (2.1), we show that if the initial density of vehicles is bounded, then it stays bounded with the same bound for all time. This is why the Hamiltonian H_F only needs to be defined for $p \in [0,1]$. However, for the construction of the correctors it is necessary to work with a coercive Hamiltonian, that is why we actually define the function H_F in $\mathbb{R}$.

Remark 2.6.2 (Additional hypotheses on H_f and A)**.** *All the additional hypotheses given in the beginning of Section 2.6 hold throughout the whole section.*

Before giving the different properties and conjectures, let us mention that when we set the flux limiter to a constant value smaller than $\min H_F$ (which modelizes a road without any light), we numerically recover that $\bar{H} = H_F$ as it is shown in Figure 2.2.

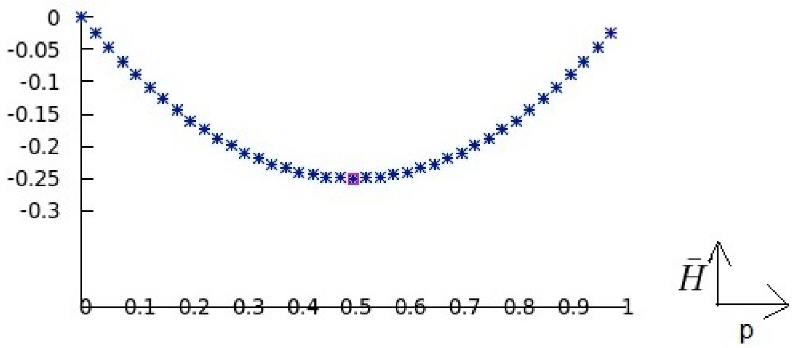

Figure 2.2: Hamiltonian without lights (for $\zeta = 1$).

2.6.2 Continuity of the effective Hamiltonian in the phase shift

As shown in Figure 2.3, we observe that the effective Hamiltonian seems to be continuous in θ and this is the object of Theorem 2.6.3.

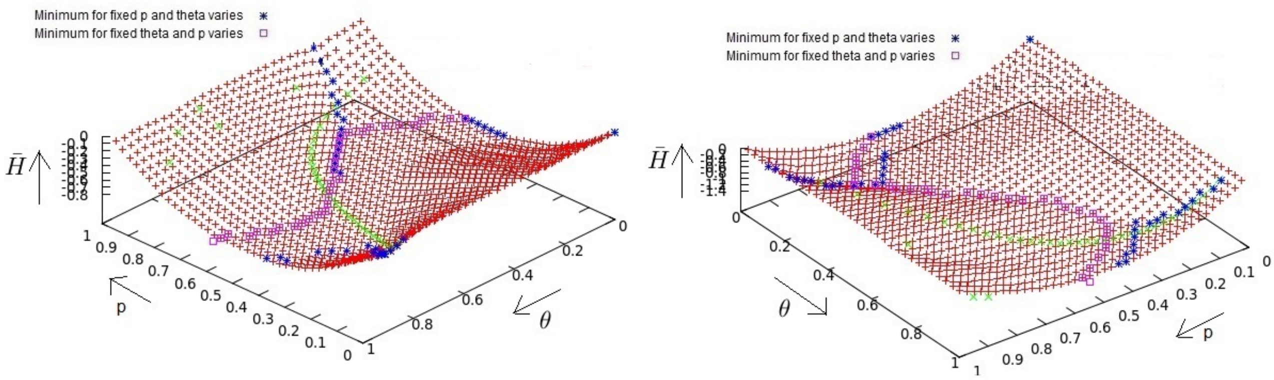

Figure 2.3: Continuity of the effective Hamiltonian for $\zeta = 6$ and $\zeta = 10$.

Theorem 2.6.3 (Continuity in the phase shift)**.** *Under the assumptions (A0)-(A2), $\bar{H}$ is continuous in the phase shift variable.*

Before starting the proof, we highlight the fact that the correctors in (2.6) are not Lipschitz continuous in space uniformly in α, unlike the stationary case. Therefore, we cannot use a proof that involves Ascoli Theorem.

Here we will use the change of time scaling to get the desired result and will consider the solution u^α of (2.21) where we explicitly see the influence of the phase shift in the problem. As $u^\alpha(0,0) = v^\alpha(0,0)$, the corresponding ergodic constants are equal and the effective Hamiltonian $\bar{H}$ is the same for the two problems.

We can now turn to the proof of Theorem 2.6.3 (see [10] for instance):

Proof of Theorem 2.6.3. Let us consider a fixed p. If the sequence $(\theta_k)_k$ converges to $\theta \in [0,1]$, then, up to a subsequence, $(\bar{H}(p,\theta_k))_k$ converges to some $\mu \in \mathbb{R}$ thanks to the estimate of Theorem 2.4.1.

For each $k \in \mathbb{N}$, we denote by u^k a periodic, possibly discontinuous solution of:

$$-\bar{H}(p,\theta_k) + u^k_s(s,y) + h(s,y,p+u^k_y(s,y) - \theta_k u^k_s(s,y)) = 0, \qquad \text{for } (s,y) \in \mathbb{R}\times\mathbb{R}.$$

The correctors u^k are uniformly bounded and hence the half-relaxed limits

$$\bar{w}(s,y) = \limsup_{k\to+\infty,(t,x)\to(s,y)} u^k(t,x) \quad \text{and} \quad \underline{w}(s,y) = \liminf_{k\to+\infty,(t,x)\to(s,y)} u^k(t,x)$$

are finite.

The discontinuous stability of viscosity solutions implies that $\bar{w}$ and $\underline{w}$ are respectively a bounded periodic subsolution and supersolution of:

$$-\mu + w_s(s,y) + h(s,y,p+w_y(s,y) - \theta w_s(s,y)) = 0, \qquad \text{for } (s,y) \in \mathbb{R}\times\mathbb{R}.$$

By uniqueness of the ergodic constant (see Theorem 2.1.2), this implies that $\mu = \bar{H}(p,\theta)$ and since this is true for any converging subsequence $(\bar{H}(p,\theta_k))_k$, the continuity is proved. □

Remark 2.6.4 (Continuity in the phase shift for general quasiconvex Hamiltonians)**.** *We observed the continuity in the phase shift variable only for one Hamiltonian but the proof does not rely on the form of H_f. The result is hence true for any Hamiltonian that fits into the theoretical frame of the previous sections.*

2.6.3 Estimates on the effective Hamiltonian

We also observe in Figure 2.4 a common lower bound for all phase shifts (shown in blue) with contact zones in some cases.

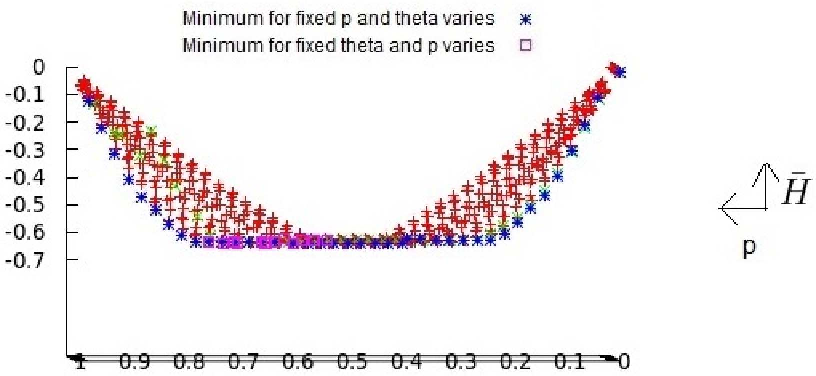

Figure 2.4: Common lower bound for all phase shifts, example for $\zeta = 5$.

This is summarized in the following theorem:

Theorem 2.6.5 (Common lower bound)**.** *The following estimate holds for all values of $\zeta > 0$ and $\theta \in [0,1]$:*

$$\bar{H}(p,\theta) \geq \max(\langle A\rangle, H_F(p)),$$

where $\langle A\rangle = \frac{\zeta a_0}{2}$ denotes $\int_0^1 A(s)ds$.

Proof. We recall that we sticked to the case where A is Lipschitz continuous and we denote by D its Lipschitz constant. Then, h given by (2.22), is also $D-$Lipschitz continuous in time.

By comparison principle for (2.21), we immediately get that u^α is $\frac{D}{\alpha}$-Lipschitz continuous in time. By coercivity and the estimate of Theorem 2.4.1, we get that u^α is also Lipschitz continuous in space (as it was performed in Lemma 5.1 of [41]).

Hence, by Rademacher Theorem, we get that u^α is differentiable almost everywhere and that for $\delta > 0$ and α small enough:

$$u_s^\alpha(s,y) + h(s,y,p + u_y^\alpha(s,y) - \theta u_s^\alpha(s,y)) \leq \bar{H}(p,\theta) + \delta, \qquad \text{for a.e } (s,y) \in (0,1)^2,$$

which simply gives:

$$u_s^\alpha(s,y) + H_F(p + u_y^\alpha(s,y) - \theta u_s^\alpha(s,y)) \leq \bar{H}(p,\theta) + \delta, \qquad \text{for a.e } (s,y) \in (0,1)^2. \tag{2.28}$$

We integrate (2.28) over $(0,1)^2$ and use the fact that u^α is 1-periodic in time and space, and use Jensen inequality as H_F is convex to get:

$$H_F(p) \leq \bar{H}(p,\theta) + \delta. \tag{2.29}$$

Let us now consider what happens at the junction point $y = 0$. As stated in Step 2.1 of proof of Lemma 5.2 in [41], from Theorem 2.11 of [52], we know that $u^\alpha(s,0)$ as a function of time only, satisfies, in the viscosity sense:

$$u_s^\alpha(s,0) + A(s) \leq \bar{H}(p,\theta) + \delta, \qquad \text{for } s \in (0,1).$$

Using the 1-periodicity in time of u^α, we see that the integration in time on one period implies:

$$\langle A \rangle \leq \bar{H}(p,\theta) + \delta. \tag{2.30}$$

As (2.29) and (2.30) are true for any $\delta > 0$, this gives the result. □

In order to get some traffic flow analysis, we first recall that the effective Hamiltonian is the opposite of the flux of vehicles at large scales. Therefore, the more the Hamiltonian increases, the more the global traffic flow gets congested (or equivalently the more the average limitation increases).

As pointed out in [41], Theorem 2.6.5 has a natural traffic interpretation, saying that the average limitation on the traffic flow, created by several traffic lights on a single road is greater than or equal to the one created by a single traffic light. The authors also mentionned that this inequality can be strict for large ζ: if we have two traffic lights very close to each other (let us say that the distance in between is at most the space for only one car) and if the traffic lights are exactly in opposite phases (with, for instance, one minute for the green phase and one minute for the red phase), then the effect of the two traffic lights together gives a very low flux which is much lower than the effect of a single traffic light alone (i.e., here at most one car every two minutes will go through the two traffic lights). This is also visible in Figure 2.5 as the surface is strictly above the time average of the flux limiter, $\langle A \rangle = -\frac{\zeta}{8}$, for some phase shifts around $\theta = \frac{1}{2}$.

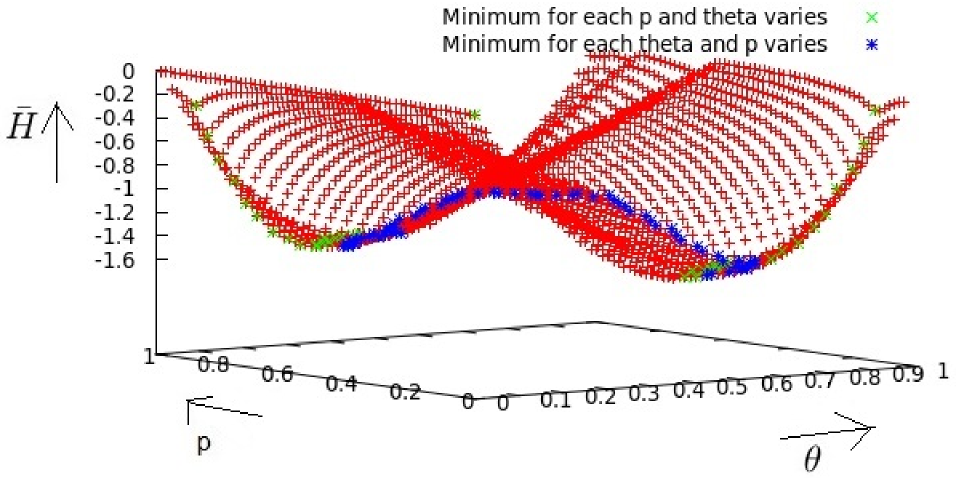

Figure 2.5: Strict inequality, example for $\zeta = 12$ and $\langle A \rangle = -1.5$.

2.6.4 Influence of the distance between lights and a weak monotony result

We here obtain a difference from [41] where this average limitation is smaller if the distances between traffic lights are bigger which was leading to a monotony of the effective Hamiltonian in the distance between lights (or equivalently in ζ). Here, $\bar{H}(p, \theta)$ can have different behaviours because of the interaction between the phase shift θ and the distance between lights (or ζ). Let us consider two practical situations:

-If $\theta = \frac{1}{2}$, as we said above, if two consecutive lights are very close from one to each other (i.e ζ is very large) then the average limitation is very high. If we put the second light to a larger distance (i.e ζ decreases), then the average limitation may decrease. This example suggests that $\bar{H}(p, \theta)$ cannot be a nonincreasing function of ζ in that case.

-If $\theta = 0$ and if two successive lights are separated from 2 meters say (i.e a very large ζ), then the average limitation is very low. If we put the second light to larger a distance d (i.e ζ decreases) and consider vehicles which velocity is exactly $\frac{2d}{T}$, then if the first light becomes green, those vehicles will meet the second light when this latter becomes red. Therefore, the average limitation has increased. This example suggests that $\bar{H}(p, \theta)$ cannot be a nondecreasing function of ζ in that case.

We can observe some behaviours numerically. For example, when $\zeta \leq 1$, we observe in Figure 2.6 that $\bar{H}(p, \theta)$ is a nonincreasing function of ζ.

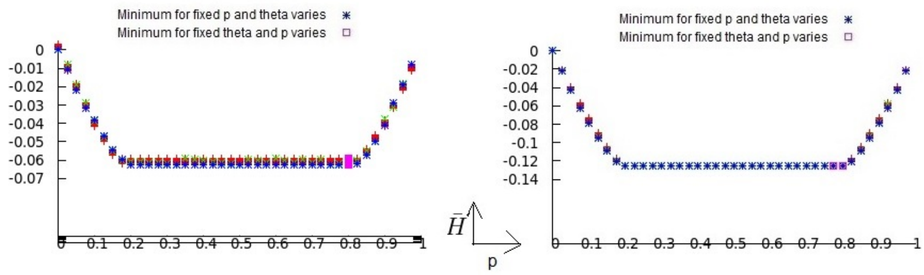

Figure 2.6: Effective Hamiltonian for $\zeta = \frac{1}{2}$ and $\zeta = 1$.

However, we have a weaker result of monotony.

Theorem 2.6.6 (Weak monotony in ζ). *For any $(p, \theta) \in [0, 1]$, $\zeta \mapsto \bar{H}(p, \theta) - \zeta a_0$ is nondecreasing, where $a_0 = -\frac{1}{4}$ is the minimum value of $p \mapsto -p(1-p)$.*

Proof. We respectively denote by H_{F_1}, A_1, F_{A_1}, H_1 the functions H_F, A, F_A, H for $\zeta = 1$ (we have $a_0 = \min_{\mathbb{R}} H_{F_1} = -\frac{1}{4}$).

Let us remark that $= \min_{\mathbb{R}} A_1 = a_0$. Hence, $F_{A_1} \geq a_0$ and so $H_1 \geq a_0$.

For $\delta > 0$, we consider v^δ, the bounded approximate corrector of (2.4) and define $w^\delta = v^\delta + \zeta a_0 \tau$. The viscosity inequalities becomes:

$$\begin{cases} -\bar{H}(p,\theta) + w^\delta_\tau(\tau, y) + \zeta \left(H_1(\tau, y, p + w^\delta_y(\tau, y)) - a_0 \right) \leq \delta, & \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}, \\ -\bar{H}(p,\theta) + w^\delta_\tau(\tau, y) + \zeta \left(H_1(\tau, y, p + w^\delta_y(\tau, y)) - a_0 \right) \geq -\delta, & \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{2.31}$$

Let us consider $0 < \zeta_1 \leq \zeta_2$. We explicitly write the dependence on ζ for $\bar{H}$, v^δ and w^δ and aim at showing that $\bar{H}^{\zeta_1}(p, \theta) - \zeta_1 a_0 \leq \bar{H}^{\zeta_2}(p, \theta) - \zeta_2 a_0$.

As $\zeta_1 \leq \zeta_2$ and $H_1 - a_0 \geq 0$, we immediately get that:

$$-\bar{H}^{\zeta_1}(p,\theta) + w^{\delta,\zeta_1}_\tau(\tau, y) + \zeta_2 \left(H_1(\tau, y, p + w^{\delta,\zeta_1}_y(\tau, y)) - a_0 \right) \geq -\delta, \qquad \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}.$$

We define:

$$\begin{cases} w_1(\tau, y) := w^{\delta,\zeta_1}(\tau, y) + (\delta - \bar{H}^{\zeta_1}(p,\theta))\tau + ||v^{\delta,\zeta_1} - v^{\delta,\zeta_2}||_{\infty,\mathbb{R}^2}, \\ w_2(\tau, y) := w^{\delta,\zeta_2}(\tau, y) + (\delta - \bar{H}^{\zeta_2}(p,\theta))\tau. \end{cases}$$

The functions v and w are respectively super-solution and sub-solution of $u_\tau + \zeta_2 H_1(\tau, y, p + u_y) = 0$ and we have $w_1(0, .) \geq w_2(0, .)$. Thus, by comparison principle, we have $w_1 \geq w_2$ in $[0, +\infty) \times \mathbb{R}$ or equivalently:

$$v^{\delta,\zeta_1}(\tau, y) + a_0\zeta_1\tau + (\delta - \bar{H}^{\zeta_1}(p,\theta))\tau + ||v^{\delta,\zeta_1} - v^{\delta,\zeta_2}||_{\infty,\mathbb{R}^2} \geq v^{\delta,\zeta_2}(\tau, y) + a_0\zeta_2\tau + (\delta - \bar{H}^{\zeta_2}(p,\theta))\tau.$$

We finally divide by $\tau > 0$ and let τ tend to $+\infty$ to get the desired result (the correctors are bounded functions). $\square$

We also observe that for $\zeta \leq 1$ (i.e $v_{max}T \leq d$), the effective Hamiltonian is the same for all phase shifts and coincides with the common lower bound (which still depends on ζ). This is consistent with the fact that in this case, the vehicles do not have the time to reach one light from the preceding one within one period and therefore the lights do not really interact one with each other as it was pointed out in [41] when the distance between lights is large enough. Figures 2.7 and 2.8 show the effective Hamiltonian in that case.

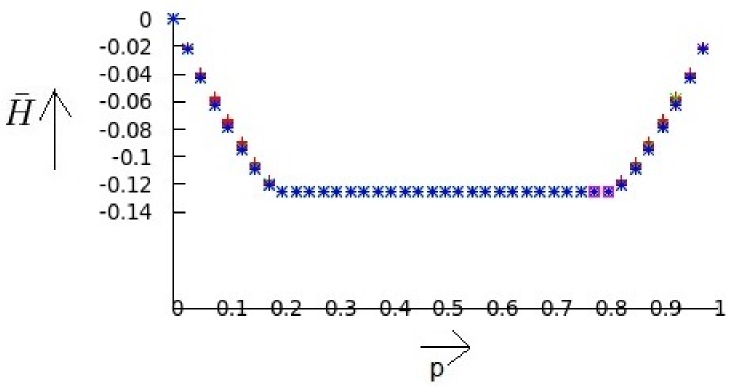

Figure 2.7: Same effective Hamiltonian for all phase shifts, example for $\zeta = 1$.

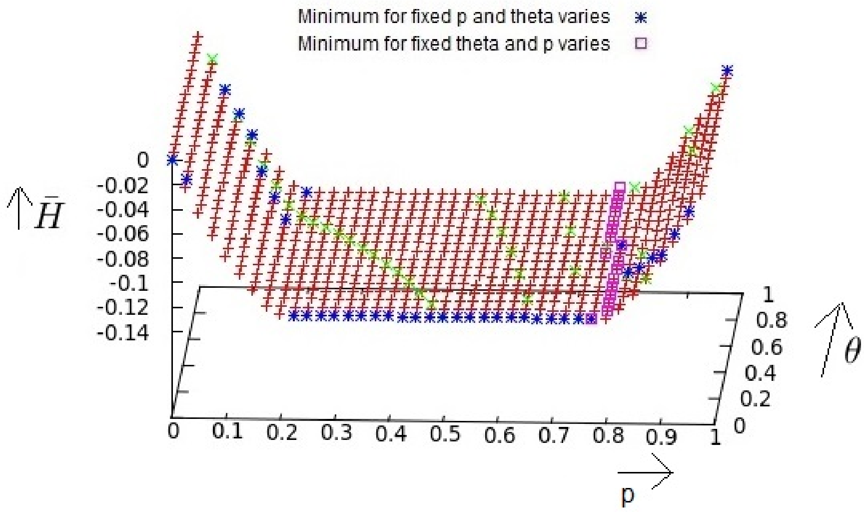

Figure 2.8: Same effective Hamiltonian for all phase shifts, example for $\zeta = 1$.

This is stated in the following conjecture.

Conjecture 2.6.7 (Conjecture on the effective Hamiltonian)**.** *For $\zeta \in (0, 1)$, the effective Hamiltonian seems not to depend on the phase shift. More precisely, for all $\theta \in [0, 1]$ we expect that:*

$$\bar{H}(p, \theta) = \max(\langle A \rangle, H_F(p)),$$

where $\langle A \rangle$ denotes $\int_0^1 A(s)ds$.

2.6.5 Convexity in the gradient variable

We observe in Figures 2.9 and 2.10 that the effective Hamiltonian is convex in the gradient variable, for all fixed values of ζ and θ.

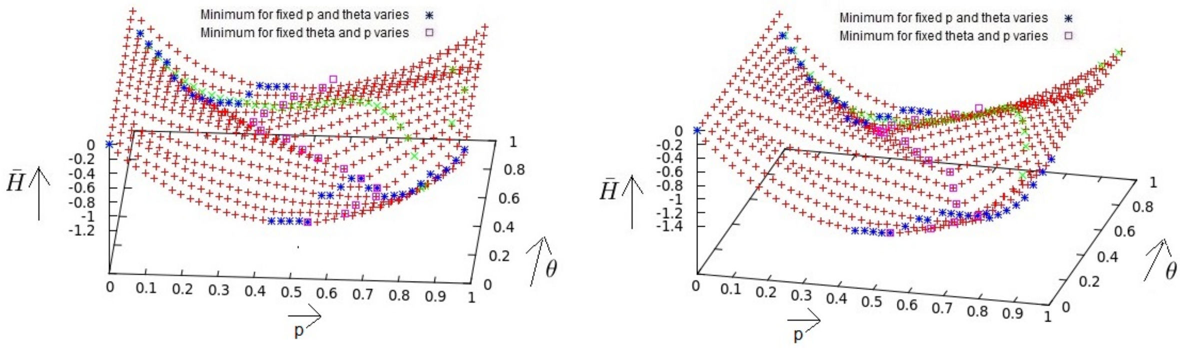

Figure 2.9: Convexity of the effective Hamiltonian for $\zeta = 8$ and $\zeta = 10$.

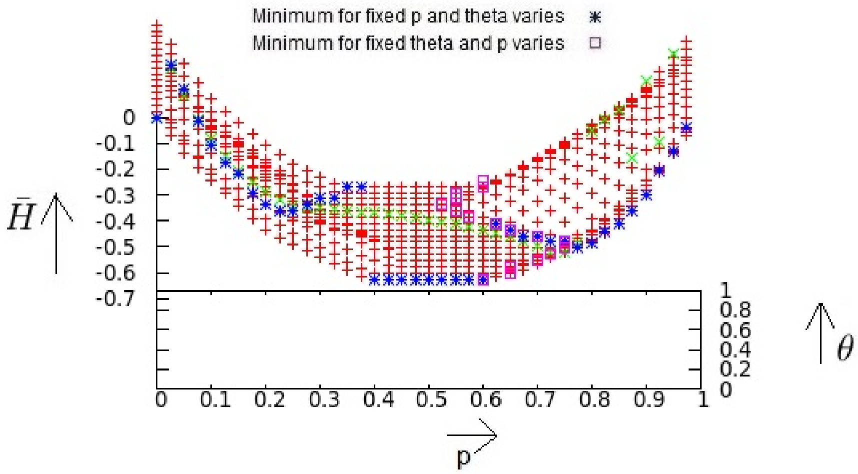

Figure 2.10: Convexity of the effective Hamiltonian for $\zeta = 5$.

We give the proof of this classical result for the reader's convenience (see [41] for instance).

Theorem 2.6.8 (Convexity in the gradient variable)**.** *For any $\theta \in [0,1]$, $\bar{H}(.,\theta)$ is a convex function.*

Proof. Let us notice that H_F is a C^2 convex function and that H_F^+ and H_F^- are hence C^1 convex functions. Therefore, F_A is a convex function of the gradient variable $(p_L, p_R) \in \mathbb{R}^2$ as a supremum of convex functions. Finally, H is convex in the gradient variable.

Let us consider $\mu \in [0,1]$ and $(p,q) \in \mathbb{R}^2$ and $\theta \in [0,1]$. We want to show that:

$$\bar{H}(\mu p + (1-\mu)q, \theta) \leq \mu \bar{H}(p,\theta) + (1-\mu)\bar{H}(q,\theta).$$

For $\delta > 0$, we consider v^δ, w^δ and z^δ the approximate correctors respectively associated to $\bar{H}(p,\theta)$, $\bar{H}(q,\theta)$ and $\bar{H}(\mu p + (1-\mu)q, \theta)$ in (2.4). As mentionned in the proof of Theorem 2.6.5, v^δ, w^δ and z^δ are Lipschitz continuous and hence differentiable almost everywhere.

We define $r^\delta := \mu v^\delta + (1-\mu)w^\delta$. By convexity, we get, almost everywhere:

$$r^\delta_\tau + H(\tau, y, \mu p + (1-\mu)q + r^\delta_y) \leq \mu \bar{H}(p,\theta) + (1-\mu)\bar{H}(q,\theta) + \delta.$$

As mentioned in the proof of Lemma 3.5 in [41], the convexity of H ensures that r^δ is in fact a viscosity subsolution (using a mollifier and an argument from [7] in Proposition 5.1).

Then the comparison principle implies the desired inequality (using the supersolution inequality for z^δ):

$$\bar{H}(\mu p + (1-\mu)q, \theta) \leq \mu \bar{H}(p, \theta) + (1-\mu)\bar{H}(q, \theta).$$

□

Remark 2.6.9 (Convexity for general convex Hamiltonians)**.** *We observed the convexity only for one Hamiltonian but the proof does not rely on the form of H_f. The result is hence true for any convex Hamiltonian that fits into the theoretical frame of the previous sections.*

We observe some simple properties (see Figure 2.11): the effective Hamiltonian is negative (it is the opposite of the flux of vehicles), it vanishes at $p = 0$ (the flux is zero if there are no cars) and $p = 1$ (the flux is zero if there are cars everywhere).

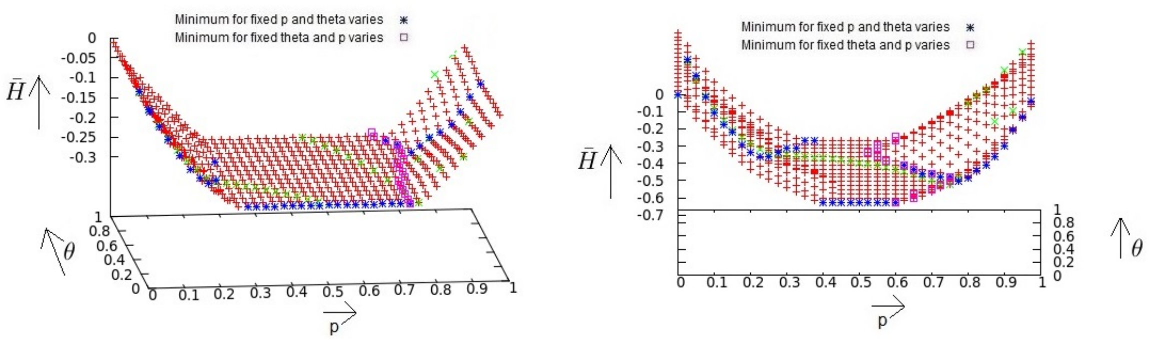

Figure 2.11: Sign of the effective Hamiltonian and vanishment, example for $\zeta = 2$ and $\zeta = 5$.

Corollary 2.6.10 (Sign of the effective Hamiltonian)**.** *For all $(p, \theta) \in [0,1]^2$, $\bar{H}(p, \theta) \leq 0 = \bar{H}(0, \theta) = \bar{H}(1, \theta)$.*

Proof. Let us first prove that $\bar{H}(0, \theta) = 0$ (the proof for $p = 1$ is the same).

We remark that $H_F(0) = F_A(\tau, y, 0) = 0$ for all $(\tau, y) \in \mathbb{R} \times \mathbb{Z}$. Hence, the unique solution of (2.6) in the case $p = 0$ is $v^\alpha \equiv 0$. Hence, we get that: $\bar{H}(0, \theta) = 0$. The sign of $\bar{H}$ comes from its convexity. □

The convexity and Theorem 2.6.5 lead to the following remark.

Remark 2.6.11 (Intersections with the lower bound)**.** *For all $\theta \in [0,1]$, we either have:*

$$\begin{cases} \forall p \in [0,1], \bar{H}(p, \theta) > \langle A \rangle, \\ \exists! p_\theta \in [0,1] / \bar{H}(p_\theta, \theta) = \langle A \rangle, \\ \exists (p_\theta^1, p_\theta^2) \in [0,1]^2 / \forall p \in [p_\theta^1, p_\theta^2], \bar{H}(p, \theta) = \langle A \rangle. \end{cases}$$

The last case can have a useful meaning in traffic flow modelling: if for a given phase shift, we observe that there exist two different densities such that the corresponding flux under the effect of several lights are the same as the flux under a single light, then it is also the case for all intermediate densities.

For a single value of ζ, we can even observe the three cases, see Figure 2.12.

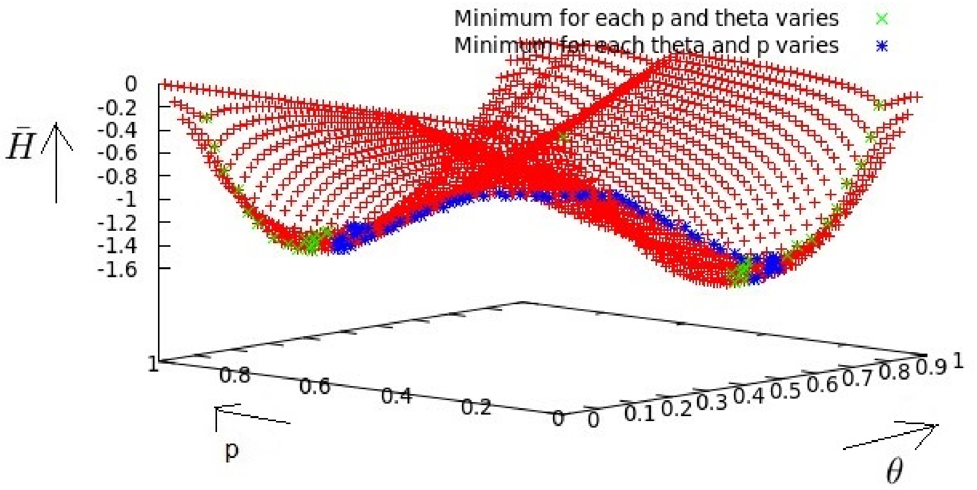

Figure 2.12: Intersections with the lower bound, example for $\zeta = 12$ and $\langle A \rangle = -1.5$.

2.6.6 Symmetry of the effective Hamiltonian

We also observe a symmetry on the graph of the effective Hamiltonian, see Figure 2.13.

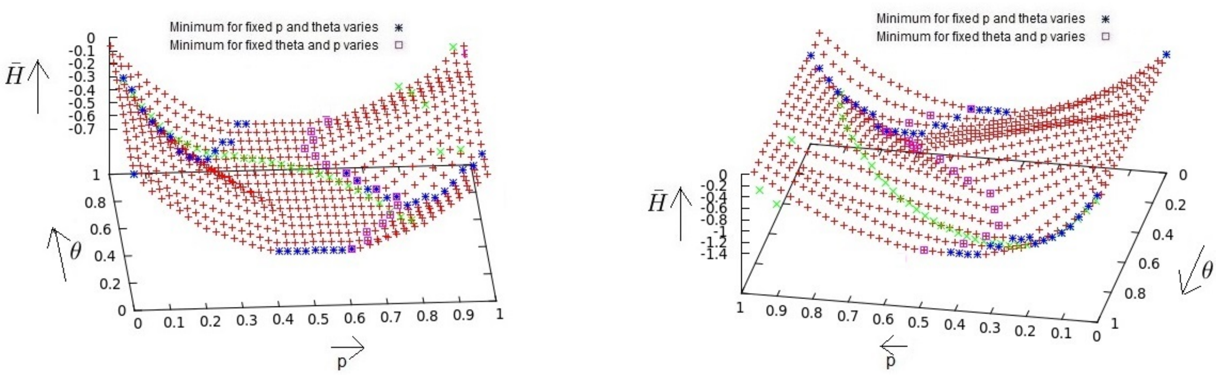

Figure 2.13: Symmetry of the effective Hamiltonian for $\zeta = 5$ and $\zeta = 8$.

Theorem 2.6.12 (Symmetry of the effective Hamiltonian)**.** *For all* $(p, \theta) \in [0,1]^2$, $\bar{H}(1-p, 1-\theta) = \bar{H}(p, \theta)$.

Proof. We explicitly write the dependence on θ for H by writing $H = H_\theta$. Let us consider the approximate cell problem for the couple $(1-p, 1-\theta)$:

$$\alpha w^\alpha(\tau, y) + w^\alpha_\tau(\tau, y) + H_{1-\theta}(\tau, y, 1-p+w^\alpha_y(\tau, y)) = 0, \qquad \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}. \tag{2.32}$$

We define $r^\alpha(\tau, y) = w^\alpha(\tau, -y)$ and get the following equalities for the derivatives:

$$\begin{cases} r^\alpha_\tau(\tau, y) = w^\alpha_\tau(\tau, -y), \\ r^\alpha_y(\tau, y) = -w^\alpha_y(\tau, -y), \text{ in particular,} \\ r^\alpha_y(\tau, i^+) = -w^\alpha_y(\tau, -i^-), \text{ for } i \in \mathbb{Z}. \end{cases} \tag{2.33}$$

Hence, for $y \notin \mathbb{Z}$, we have:

$$H_{1-\theta}(\tau, -y, 1-p+w_y^\alpha(\tau,-y)) = H_F(1-p+w_y^\alpha(\tau,-y)) = H_F(1-(p+r_y^\alpha(\tau,y))) = H_F(p+r_y^\alpha(\tau,y))),$$

by using the symmetry of H_F in the last equality.

For $y = i \in \mathbb{Z}$, we get:

$$H_{1-\theta}(\tau,-i,1-p+w_y^\alpha(\tau,-i)) = \max\left(A(\tau+i(1-\theta)), H_F^+(1-p+w_y^\alpha(\tau,-i^-)), H_F^-(1-p+w_y^\alpha(\tau,-i^+))\right).$$

By using the periodicity of A and (2.33), we get:

$$H_{1-\theta}(\tau,-i,1-p+w_y^\alpha(\tau,-i)) = \max\left(A(\tau-i\theta), H_F^+(1-(p+r_y^\alpha(\tau,i^+))), H_F^-(1-(p+r_y^\alpha(\tau,i^-)))\right).$$

We finally use the symmetry properties of H_F^- and H_F^+ to get:

$$H_{1-\theta}(\tau,-i,1-p+w_y^\alpha(\tau,-i)) = \max\left(A(\tau-i\theta), H_F^-(p+r_y^\alpha(\tau,i^+)), H_F^+(p+r_y^\alpha(\tau,i^-))\right).$$

By writing (2.32) at $(\tau,-y) \in \mathbb{R}^2$, we hence have:

$$\alpha r^\alpha(\tau,y) + r_\tau^\alpha(\tau,y) + H_\theta(\tau,y,p+r_y^\alpha(\tau,y)) = 0.$$

By uniqueness, we have $r^\alpha = v^\alpha$ and as $r^\alpha(0,0) = w^\alpha(0,0)$ we get the equality between the ergodic constants and thus the desired symmetry on the effective Hamiltonian. □

The convexity of the effective Hamiltonian and its symmetry lead to the following remark.

Remark 2.6.13. *For all $p \in [0,1]$, $\bar{H}\left(p,\frac{1}{2}\right) \geq \bar{H}\left(\frac{1}{2},\frac{1}{2}\right)$.*

This means that ,when the lights are in opposite phases, the average limitation is the smallest for $p = \frac{1}{2}$ which was also the maximum of the flux without lights.

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Chapter 3

Homogenization of some almost periodic Hamilton-Jacobi equations

Abstract

In this chapter, we are interested in some homogenization problems of Hamilton-Jacobi equations within the almost periodic setting which generalizes the usual periodic one. The first problem studied in this paper is the evolutionary version of the work [54], with the same stationary Hamiltonian. The second problem has already been solved in Chapter 2 but we use here almost periodic arguments for the time periodic and space almost periodic Hamiltonian. We will characterize the ergodic constant as an asymptotic time slope for the first problem, and through the equivalent estimate of an approximate corrector's oscillation for the second one. Bochner's definition of almost periodic functions will be employed in the first problem while Bohr's definition will be used in the second one. Those two definitions are known to be equivalent, see [14] for instance. We finally discuss open problems, the first one concerning a space and time almost periodic Hamiltonian and the second one being a microscopic model for traffic flow modelling where the Hamiltonian is almost periodic in space.

Keywords: Hamilton-Jacobi equations, almost periodic homogenization.

3.1 Introduction

The homogenization of Hamilton-Jacobi equations started with [61]. It holds in a periodic setting. This periodic assumption can be relaxed to an almost periodic one, like in the works [54] and [1] which deal with stationary homogenization. This latter setting is itself a particular case of the stationary ergodic framework; see [74], [58] and [72] for the case of convex Hamiltonians. We stick here to the almost periodic setting. We will only study the ergodicity of the cell problems associated to the following Hamilton-Jacobi equations.

Problem 1 (evolutionary version of [54]):

$$v_t(t,x) + h(x, v_x(t,x)) = 0, \qquad \text{for } (t,x) \in (0,+\infty) \times \mathbb{R},$$

where $h(.,p)$ is supposed to be almost periodic in space for al $p \in \mathbb{R}$.

Problem 2 (shifted lights):

$$\begin{cases} u_t + H_F(u_x) = 0, & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z}, \\ u_t + G(t - \theta x, u_x) = 0, & \text{for } (t,x) \in (0,+\infty) \times \mathbb{Z}, \end{cases}$$

where the first equation models the free zone between two consecutive lights separated from $d := 1$ and the second equation represents what happens at a light. The parameter $\theta > 0$ denotes the constant phase shift between the lights and G is $T-$periodic in time.

The almost periodic structure in space arises when $\frac{\theta}{T} \notin \mathbb{Q}$.

We first give our intuitive point of view about almost periodic functions, inspired from [26] and [67], before introducing precise definitions in the next subsections. Roughly speaking, the usual notion of period is replaced by the notion of (approximate) $\delta-$period for any $\delta > 0$. By definition, a translation along a $\delta-$period will imply an error of order δ. It is not enough for a bounded function to possess some $\delta-$periods for all δ because this is the case for all uniformly continuous functions (it is sufficient to take η small enough to have $|f(x+\eta) - f(x)| \leq \delta$ for all x in that case).

For a $T-$periodic function, every kT, for $k \in \mathbb{Z}$ is a period, which means that all the periods are uniformly located on the real line. For an almost periodic function f, the $\delta-$periods will also be distributed on the real line in a precise sense which is referred to as relative density. This is part of Bohr's definition, see [15]. This implies that every real number is not "too far" from a $\delta-$period and leads to Bochner's definition (see Theorem 2.1 in [1]). This definition is based on the compactness of the translates of f (see [14] Chapter 1, Paragraph 2 for the proof of the equivalence between the two definitions). It is useful to highlight that the set of the almost periodic functions is not the L^∞ closure of the periodic ones (the author has been thanked in [48] for this remark). The most usual introductory example of almost periodic function is: $x \mapsto \sin(x) + \sin(\sqrt{2}x)$ (see [21] for some details).

From the homogenization perspective, the periodic assumption enables one to reduce to compact sets, especially for the so-called cell problem. It enables one to get a global estimate on the (exact) correctors oscillation and in turn to define the effective Hamiltonian. In the almost periodic case, the domain of the correctors' equation is now the whole space and only approximate correctors are constructed.

We now introduce the two problems and give the main results.

3.1.1 Homogenization in space of a Cauchy problem with stationary almost periodic Hamiltonian

For $\varepsilon > 0$, we consider the rescaled solution v^ε of the oscillating Hamilton-Jacobi equation on the real line:

$$\begin{cases} v^\varepsilon_t + h\left(\frac{x}{\varepsilon}, v^\varepsilon_x\right) = 0, & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R}, \\ v^\varepsilon(0,x) = v_0(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{3.1}$$

under the following assumptions (A'0)-(A'4) firstly introduced in [54] for the same Hamiltonian:

(A'0) Continuity: $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(A'1) Coercivity:

$$\lim_{R \to +\infty} \inf\{h(y,p'), y \in \mathbb{R}, |p'| \geq R\} = +\infty.$$

(A'2) Uniform continuity in the gradient variable on bounded sets (uniformly in space): For each $R > 0$, there is a function $\omega_R \in C([0,+\infty))$, with $\omega_R(0) = 0$, such that

$$|h(y,p') - h(y,q)| \leq \omega_R(|p'-q|), \qquad \text{for } y \in \mathbb{R}, (p',q) \in [-R,R] \times [-R,R].$$

(A'3) Uniform boundedness for bounded gradients:

$$\sup_{(y,p') \in \mathbb{R} \times [-R,R]} |h(y,p')| < +\infty.$$

(A'4) Bochner's almost periodicity in space: For each $R > 0$, the family $\{h(.+z,.), z \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R} \times [-R,R])$.

This means that for each $\delta > 0$, there exist $n \in \mathbb{N}$, $(k_1,..,k_n) \in \mathbb{R}^n$ such that:

$$\forall k \in \mathbb{R}, \exists i \in \{1,n\}, \sup_{(y,q)\in\mathbb{R}\times[-R,R]} |h(y+k,q) - h(y+k_i,q)| < \delta.$$

(B'1) Regularity of the initial condition: v_0 is a K- Lipschitz continuous function in $\mathbb{R}$.

In this setting, there exists a unique viscosity solution of (3.1). The existence is given by Perron's method and the uniqueness by a classical comparison principle (see [11] for instance). We now give the main results of this section.

Theorem 3.1.1 (Definition of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}$. There exists a unique $\lambda \in \mathbb{R}$ such that:*

$$\sup_{y\in\mathbb{R}} \left| \frac{v(\tau,y)}{\tau} + \lambda \right| \underset{\tau\to+\infty}{\longrightarrow} 0, \tag{3.2}$$

where v is the unique viscosity solution of:

$$\begin{cases} v_\tau(\tau,y) + h(y, p + v_y(\tau,y)) = 0, & \textit{for } (\tau,y) \in (0,+\infty)\times\mathbb{R}, \\ v(0,y) = 0, & \textit{for } y \in \mathbb{R}. \end{cases} \tag{3.3}$$

We first prove that the unique solution of (3.3) is Lipschitz in time and space and give a space almost periodicity result on this function. This almost periodicity property is precisely described in Theorem 3.2.2. This enables us to get the existence of λ by combining with a control of its space oscillation (see Lemma 3.2.3). The uniqueness comes from the identification with the ergodic constant obtained in [54].

Thanks to this theorem, we can define the effective Hamiltonian $\bar{H} : \mathbb{R} \to \mathbb{R}$ by setting $\bar{H}(p) = -\lambda$.

We finally get to the desired convergence result.

Theorem 3.1.2 (Convergence)**.** *Under the assumptions (A'0)-(A'4) and (B'1), the solution v^ε of* (3.1) *converges locally uniformly to the unique solution v^0 of the following Cauchy problem:*

$$\begin{cases} v^0_t + \bar{H}(v^0_x) = 0, & \textit{in } (0,+\infty)\times\mathbb{R}, \\ v^0(0,x) = v_0(x), & \textit{for } x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Brief review of the litterature. The pioneering works [61], which solved the stationary periodic case, [28] and [29], which present the celebrated perturbed test function method, have introduced the main techniques and concepts used in the periodic homogenization of Hamilton-Jacobi equations. The work [51] deals with space and time homogenization for a Hamiltonian that depends on $\frac{u}{\varepsilon}$ and [10] for a non coercive Hamiltonian. Homogenization results in a stationary almost periodic setting can be found in [54] and [1]. For first order evolution equations within the stationary ergodic framework, the works [74] and [68] deal with stationary convex Hamiltonians and the work [72] with time and space dependent convex Hamiltonians. The work [72] generalized the viscous version [58]. It was mentioned in [13], while dealing with the long time behaviour of solutions to time-periodic equations on the circle with convex Hamiltonian, that the period of limiting solutions may be greater than the period of the Hamiltonian.

3.1.2 Homogenization in time and space with a time periodic and space almost periodic Hamiltonian

The model and the notations are exactly the same as the ones used in Chapter 2. We write them here again for the reader's convenience.

In the present subsection, we consider a Hamilton-Jacobi equation where the Hamiltonian is continuous and periodic in time and discontinuous in space. It models a one dimensional pursuit law on a straight road that contains equidistant lights. We consider a macroscopic model between two lights, namely a Hamilton-Jacobi equation for the density of vehicles. We consider a Hamiltonian which is quasiconvex (decreasing and then increasing) and coercive in the gradient variable between two lights. Each light is viewed as a time dependent flux limiter and can be considered as a junction in the sense introduced in [52]. A specific notion of viscosity solution is defined in that paper. Existence and uniqueness of the viscosity solution is contained in the network case of the same paper.

We will show the existence of approximate solutions to the cell problem on the real line (the so-called approximate correctors). They will turn to be almost periodic functions.

We consider an infinite straight road where lights are located at each $i \in \mathbb{Z}$. Each light is modelled as a continuous 1-time periodic flux limiter, denoted by A_i; $\theta \geq 0$ is the constant phase shift between two lights.

We then have the following relationship:

$$A_{i+1}(t) = A_i(t - \theta), \qquad \text{for } t > 0.$$

We then consider a unique flux limiter A and define all the A_i in the following way:

$$A_i(t) := A(t - i\theta), \qquad \text{for } t > 0.$$

The assumptions on A are:

(B1) Time periodicity of A:

$$A(t+1) = A(t), \qquad \text{for } t \in \mathbb{R}.$$

(B2) Uniform continuity of A: A is (uniformly) continuous with a modulus of continuity ω.

For a smooth real-valued function u defined on $(0, +\infty) \times \mathbb{R}$, $\partial_x u(t, x)$ denotes the spatial derivative of u at $x \in \mathbb{R} \setminus \mathbb{Z}$ and we define its space gradient on the whole real line as follows:

$$u_x(t,x) := \begin{cases} \partial_x u(t,x), & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z}, \\ \left(\partial_x u(t,i^-), \partial_x u(t,i^+)\right), & \text{for } (t, x = i) \in (0,+\infty) \times \mathbb{Z}. \end{cases}$$

In the free zone between two lights, we suppose that the traffic flow corresponds to the following macroscopic model:

$$u_t + H_F(u_x) = 0, \qquad \text{for } (t,x) \in (0,+\infty) \times \mathbb{R} \setminus \mathbb{Z},$$

where H_F is supposed to be continuous, coercive and quasiconvex in the gradient variable:

(A0) Uniform continuity of H_F: $H_F : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

(A1) Coercivity of H_F:

$$\lim_{|p| \to +\infty} H_F(p) = +\infty.$$

(A2) Quasiconvexity of H_F: There exists some $p_0 \in \mathbb{R}$ such that H_F is nonincreasing in $(-\infty, p_0]$ and is nondecreasing on $(p_0, +\infty)$.

The last assumption allows us to use the notion of flux-limited solution introduced in [52]. We denote by a_0 the minimum value of H_F. The non-increasing part H_F^- and the non-decreasing part H_F^+ of the Hamiltonian H_F are used in the definition of flux-limited junction conditions:

$$H_F^-(p) := \begin{cases} H_F(p), & \text{for } p \leq p_0, \\ H_F(p_0) = a_0, & \text{for } p > p_0. \end{cases}$$

$$H_F^+(p) := \begin{cases} H_F(p_0) = a_0, & \text{for } p < p_0, \\ H_F(p), & \text{for } p \geq p_0. \end{cases}$$

We now recall the definition of the flux-limited junction function F_A (see also [52]):

$$F_A(t, x = i, p_L, p_R) = \max\left(A(t - i\theta), H_F^+(p_L), H_F^-(p_R)\right), \qquad \text{for } (t,i) \in \mathbb{R} \times \mathbb{Z},\ (p_L, p_R) \in \mathbb{R}^2.$$

We now consider the following junction condition at each light:

$$u_t + F_A(t, i, u_x) = 0, \qquad \text{for } (t,i) \in (0, +\infty) \times \mathbb{Z}.$$

We now introduce a shorthand notation:

$$H(t, x, p) := \begin{cases} H_F(p), & \text{for } (t,x) \in \mathbb{R} \times \mathbb{R} \setminus \mathbb{Z},\ p \in \mathbb{R}, \\ F_A(t, i, p), & \text{for } (t,i) \in \mathbb{R} \times \mathbb{Z},\ p \in \mathbb{R}^2. \end{cases}$$

A schematic picture is shown in Figure 3.1.

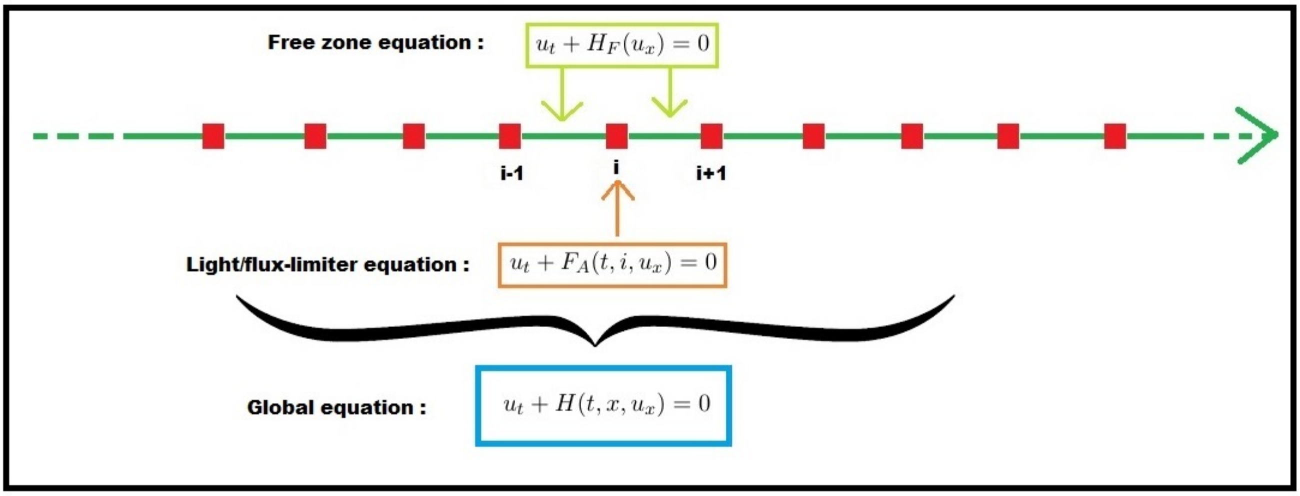

Figure 3.1: Sketch of the road.

Let us consider $\varepsilon > 0$. We consider the rescaled solution u^ε of the corresponding Hamilton-Jacobi equation on the whole straight road which takes the form of an oscillating Hamilton-Jacobi equation on the real line:

$$\begin{cases} u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0, & \text{for } (t,x) \in (0, +\infty) \times \mathbb{R}, \\ u^\varepsilon(0, x) = u_0(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{3.5}$$

where we suppose that:

(C1) Regularity of the initial condition: u_0 is a L- Lipschitz continuous function in $\mathbb{R}$.

There exists a unique viscosity solution of (3.5) (see Chapter 2 for more details and references).

Our main theoretical result is the identification of the limit of u^ε for vanishing ε.

For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we are not in a space-time periodic setting. Though the cell problem may only have approximate solutions, that is still sufficient to define λ. We will use the tools of the almost periodic setting described in Subsection 3.3.1. Let us recall that in our case, H is time periodic and discontinuous in space.

Main results of this section.

Theorem 3.1.3 (Characterization of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}$. There exists a unique $\lambda \in \mathbb{R}$ such that for all $\delta > 0$, there exists a bounded continuous approximate corrector $v^\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solution of:*

$$\begin{cases} \lambda + v^\delta_\tau(\tau, y) + H(\tau, y, p + v^\delta_y(\tau, y)) \leq \delta, & \text{for all } (\tau, y) \in \mathbb{R} \times \mathbb{R}, \\ \lambda + v^\delta_\tau(\tau, y) + H(\tau, y, p + v^\delta_y(\tau, y)) \geq -\delta, & \text{for all } (\tau, y) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{3.6}$$

According to this theorem, we can define the effective Hamiltonian $\bar{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting $\bar{H}(p, \theta) = -\lambda$.

We finally get to the main theoretical result of this section.

Theorem 3.1.4 (Convergence)**.** *Under the assumptions (A0)-(A2), (B1)-(B2) and (C1), the solution u^ε of (3.5) converges locally uniformly to the unique solution u^0 of the following Cauchy problem:*

$$\begin{cases} u^0_t + \bar{H}\left(u^0_x, \theta\right) = 0, & \text{in } (0, +\infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{3.7}$$

Brief review of the litterature. The homogenization result has already been proved in [52] for a stationary periodic Hamiltonian. Our problem has already been solved in Chapter 2 by using a change of variables. Some homogenization results in traffic flow modelling can be found in [35] when adding a junction condition in the microscopic model, in [36] for a second order microscopic model, in [41] where a macroscopic junction condition is derived from a continuous microscopic model, and in [32] when considering the reaction time of drivers. The work [56] considers a sublinear Hamiltonian with a particular form: it is supposed periodic in space and stationary ergodic in time. This could be seen as a reverse case of our problem, where the Hamiltonian is periodic in time and almost periodic in space. The work [72] deals with first order equations with time and space dependent convex continuous Hamiltonians in the stationary ergodic setting. It is not possible to construct any corrector for this problem, contrary to the viscous second order work [57] where correctors are constructed their oscillations are controlled in big cylinders. To study the ergodicity, the author of [72] rather uses the representation formula of the oscillating solution and shows a convergence result for the associated Lagrangian. An inf-sup formula for the effective Hamiltonian is obtained.

To get the existence of λ in Theorem 3.1.3, we consider the following approximate cell problem for $\alpha > 0$:

$$\alpha v^\alpha(\tau, y) + v^\alpha_\tau(\tau, y) + H(\tau, y, p + v^\alpha_y(\tau, y)) = 0, \qquad \text{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}. \tag{3.8}$$

We can show a comparison principle for equation (3.8) (see Chapter 2). Thus, it is well-posed and we classically get that αv^α is bounded in $\mathbb{R}^2$, for a bound which does not depend on α.

In the usual space-time periodic setting, the proof of the existence of λ relies on the estimate of the oscillation $\max_{\mathbb{R}^2} v^\alpha - \min_{\mathbb{R}^2} v^\alpha$ independently from α, see [51] for instance. This estimate replaces the classical Lipschitz estimate (see [61]). This was combined with the fact that αv^α is bounded independently from α to get λ as the uniform limit of αv^α for vanishing α: this property is called ergodicity (see also [10]).

Following [54], we obtain a weaker control of the oscillation but still sufficient to get ergodicity (see Theorem 3.3.10).

Remark 3.1.5 (Focus of the chapter)**.** *We will only study existence and uniqueness of the ergodic constants as the convergence proofs are classical. The convergence proof for the first problem (Theorem 3.1.2) is contained in [52] (the only difference is that we use approximate correctors in the almost periodic framework) and is a particular case of the one for the second problem (Theorem 3.1.4) which is contained in Chapter 2.*

3.1.3 Organisation of the chapter

In Section 3.2, we give some regularity results on the solution of (3.3) which will turn out to be almost periodic in space (see Theorem 3.2.2) and prove Theorem 3.1.1. Section 3.3 is devoted to the control of the oscillation after showing the almost periodicity in space for the second problem. We also show that the effective Hamiltonian $\bar{H}(p, \theta)$ is uniformly continuous in p. In Section 3.4, we discuss some open related problems: the homogenization in time and space with almost periodicity in all variables and the case of a non local traffic flow microscopic model with almost periodicity in space.

3.2 Ergodicity for the Cauchy problem with stationary Hamiltonian

We now give the proof of the existence and uniqueness of the ergodic constant. To do so, we first give some results about the regularity of the solution v to (3.3). This will enable us to use compactness results and get the existence of the ergodic constant. The uniqueness will come from the identification with the ergodic constant of [54].

3.2.1 Regularity of the solution of (3.3) and weak almost periodicity in space

By construction (following Perron's method), we have the following estimate on the unique solution v of (3.3):

$$|v(\tau, y)| \leq C_p \tau, \qquad \text{for } (\tau, y) \in [0, +\infty) \times \mathbb{R}, \tag{3.9}$$

where $C_p := \sup_{y \in \mathbb{R}} |h(y, p)|$.

Let us now show that v is Lipschitz continuous in time and space.

Theorem 3.2.1 (Lipschitz regularity in time and space)**.** *Under (A'0)-(A'3), the solution v of (3.3) is Lipschitz continuous in time and space. Moreover, the Lipschitz constants only depend on h and p.*

Proof. Let us show that v is Lipschitz continuous in time uniformly in space. More precisely, we have the following result:

$$|v(\tau + s, y) - v(\tau, y)| \leq C_p s, \qquad \text{for } (\tau, y) \in [0, +\infty) \times \mathbb{R}, s > 0, \tag{3.10}$$

where C_p is the constant defined in (3.9).

The equation is invariant by addition of constants to the solutions and is translation invariant in time (as h is stationary). Let $s > 0$.

The solution corresponding to the initial condition $y \mapsto v(s, y)$ is the function $(\tau, y) \mapsto v(\tau + s, y)$.

The one associated to $y \mapsto v(0,y) + C_p s = C_p s$ is $(\tau, y) \mapsto v(\tau, y) + C_p s$.

From equation (3.9), we have $v(s,y) \le v(0,y) + C_p s$ and hence by comparison principle:

$$v(\tau + s, y) - v(\tau, y) \le C_p s, \qquad \text{for } (\tau, y) \in [0, +\infty) \times \mathbb{R}, s > 0.$$

We get the reverse inequality in the same way.

We deduce that $||v_\tau||_{L^\infty((0,+\infty)\times\mathbb{R})} \le C_p$ and by combining with the coercivity assumption (A'1) in (3.3) we get the existence of a constant (still denoted by C_p) which only depends on h and p such that:

$$||v_y||_{L^\infty((0,+\infty)\times\mathbb{R})} \le C_p. \quad \square$$

We now give the precise statement about what we called almost periodicity in space for the solution v of (3.3).

Theorem 3.2.2 (Almost periodicity in space)**.** *Let (A'0)-(A'4) hold true and v be the solution of* (3.3)*.*

For any family of space translates $\{v^z : (\tau, y) \mapsto \frac{v(\tau, y+z)}{\tau},\ z \in \mathbb{R}\}$, there exists a subsequence $(z_j) \in \mathbb{R}^{\mathbb{N}}$ and a function $G \in C(\mathbb{R})$ such that:

$$\sup_{y \in \mathbb{R}} \left| \frac{v(\tau, y + z_j)}{\tau} - G(y) \right| \underset{j \to +\infty}{\longrightarrow} 0.$$

Proof. It is equivalent to prove that for each $\delta > 0$, there exist $n \in \mathbb{N}$, $(k_1, .., k_n) \in \mathbb{R}^n$ such that:

$$\forall k \in \mathbb{R}, \exists i \in \{1, n\}, \ \sup_{y \in \mathbb{R}} |v(\tau, y + k) - v(\tau, y + k_i)| \le \delta\tau, \qquad \text{for } \tau \in [0, +\infty).$$

Let us consider $\delta > 0$. Using assumption (A'4) for $R = C_p + |p|$, there exist $n \in \mathbb{N}$, $(k_1, .., k_n) \in \mathbb{R}^n$ such that:

$$\forall k \in \mathbb{R}, \exists i \in \{1, n\}, \quad \sup_{(y,q) \in \mathbb{R}\times[-R,R]} |h(y + k, q) - h(y + k_i, q)| < \delta.$$

We now follow the same argument as [1] (see Section 7) and define for $k \in \mathbb{R}$:

$$\begin{cases} w^+(\tau, y) = v(\tau, y + k) + \delta\tau, & \text{for } (\tau, y) \in [0, +\infty) \times \mathbb{R}, \\ w^-(\tau, y) = v(\tau, y + k) - \delta\tau, & \text{for } (\tau, y) \in [0, +\infty) \times \mathbb{R}. \end{cases}$$

By translation invariance, w^+ is a solution of:

$$\begin{cases} w_\tau(\tau, y) - \delta + h(y + k, p + w_y(\tau, y)) = 0, & \text{for } (\tau, y) \in (0, +\infty) \times \mathbb{R}, \\ w(0, y) = 0, & \text{for } y \in \mathbb{R}. \end{cases}$$

For this $k \in \mathbb{R}$, there exists k_i, for $i \in \{1, n\}$, such that:

$$\sup_{(y,q) \in \mathbb{R}\times[-R,R]} |h(y + k, q) - h(y + k_i, q)| < \delta. \tag{3.11}$$

By translation invariance, the function $v_i : (\tau, y) \mapsto v(\tau, y + k_i)$ is the solution of:

$$\begin{cases} g_\tau(\tau, y) + h(y + k_i, p + g_y(\tau, y)) = 0, & \text{for } (\tau, y) \in (0, +\infty) \times \mathbb{R}, \\ g(0, y) = 0, & \text{for } y \in \mathbb{R}. \end{cases} \tag{3.12}$$

From (3.11), we get that w^+ is a supersolution of (3.12). Using the comparison principle, we get that $w^+ \ge v_i$.

We perform similar computations for w^- and combine both inequalities to get the desired result. $\square$

3.2.2 Proof of Theorem 3.1.1

This Subsection is devoted to the proof of Theorem 3.1.1. We first start with a lemma about the control of the oscillation in space for large times thanks to the almost periodicity in space.

Lemma 3.2.3 (Control of the oscillation in space (for large times)). *Under (A'0)-(A'4), we have:*

$$\sup_{y\in\mathbb{R}}\left|\frac{v(\tau,y)}{\tau}-\frac{v(\tau,0)}{\tau}\right| \underset{\tau\to+\infty}{\longrightarrow} 0.$$

Remark 3.2.4. *The proof of this lemma is similar to the proof of Theorem 8 in [54].*

Proof of Lemma 3.2.3. We argue by contradiction.

We thus suppose that there exist $\delta>0$ and sequences $(y_j)\in\mathbb{R}^{\mathbb{N}}$,$(\tau_j)\in(0,+\infty)^{\mathbb{N}}$ with $\lim_{j\to+\infty}\tau_j=+\infty$ such that:

$$|v(\tau_j,y_j)-v(\tau_j,0)|\geq\delta\tau_j,\qquad \text{for } j\in\mathbb{N}. \tag{3.13}$$

We then use Theorem 3.2.2 to get for $(j,k)\in\mathbb{N}^2$ large enough:

$$\sup_{y\in\mathbb{R}}|v(\tau,y+y_j)-v(\tau,y+y_k)|\leq\frac{\delta}{2}\tau, \tag{3.14}$$

along a subsequence.

Relabeling the sequence (y_j) if necessary, we may assume that (3.14) holds for all $(j,k)\in\mathbb{N}^2$.

We deduce for $y=0$ and $k=1$ that:

$$|v(\tau,y_j)-v(\tau,y_1)|\leq\frac{\delta}{2}\tau,\qquad \text{for } \tau\in[0,+\infty),$$

which is in particular true for $\tau=\tau_j$.

We then deduce:

$$|v(\tau_j,y_j)-v(\tau_j,0)|\leq\frac{\delta}{2}\tau_j+|v(\tau_j,y_1)-v(\tau_j,0)|.$$

We then use Theorem 3.2.1, i.e the fact that v is Lipschitz continuous in space to get:

$$|v(\tau_j,y_j)-v(\tau_j,0)|\leq\frac{\delta}{2}\tau_j+C_p|y_1|.$$

As $\lim_{j\to+\infty}\tau_j=+\infty$, we deduce the following contradiction with (3.13) for j large enough:

$$|v(\tau_j,y_j)-v(\tau_j,0)|<\delta\tau_j.$$

□

We now recall Theorem 2 of [54] as we will identify our ergodic constant with the one of the stationary case.

Theorem 3.2.5 (Existence of approximate corrector, [54]). *Let $p\in\mathbb{R}$. There exists a unique $\mu\in\mathbb{R}$ such that for all $\delta>0$, there exists a bounded continuous function $v^\delta:\mathbb{R}\to\mathbb{R}$ solution of:*

$$\begin{cases} h(y,p+v^\delta_y(y))\leq\mu+\delta, & \textit{for all } y\in\mathbb{R},\\ h(y,p+v^\delta_y(y))\geq\mu-\delta, & \textit{for all } y\in\mathbb{R}.\end{cases} \tag{3.15}$$

We now come to the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. From (3.9), there exist $\lambda \in \mathbb{R}$ and $(\tau'_n) \in (0, +\infty)^{\mathbb{N}}$ with $\lim_{n\to+\infty} \tau'_n = +\infty$, such that:

$$\lim_{n\to+\infty} \frac{v(\tau'_n, 0)}{\tau'_n} = -\lambda.$$

Let us show that λ coincides with μ given by Theorem 3.2.5.

By combining with Lemma 3.2.3, we get that:

$$\sup_{y\in\mathbb{R}} \left| \frac{v(\tau'_n, y)}{\tau'_n} + \lambda \right| \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.16}$$

For fixed $\delta > 0$, we consider a bounded corrector v^δ of Theorem 3.2.5 and define:

$$v^\delta_+(\tau, y) := v^\delta(y) + (-\mu + \delta)\tau + ||v^\delta||_{L^\infty(\mathbb{R})}.$$

The function v^δ_+ is a supersolution of (3.3) and by comparison principle we get:

$$-\mu + \delta + \frac{v^\delta(y) + ||v^\delta||_{L^\infty(\mathbb{R})}}{\tau} \geq \frac{v(\tau, y)}{\tau}, \qquad \text{for } (\tau, y) \in (0, +\infty) \times \mathbb{R}.$$

We apply at $\tau = \tau'_n$ and let $n \to +\infty$. Thanks to (3.16), we get:

$$-\mu + \delta \geq -\lambda.$$

As this is true for any $\delta > 0$, we have: $\mu \leq \lambda$. We similarly get the reverse inequality and hence the equality.

This identification also gives us the uniqueness of $\lambda = \mu$ thanks to Theorem 3.2.5 and as this value does not depend on the converging subsequence, (3.16) turns into the desired result (3.2).

□

Remark 3.2.6 (Stationary approximate corrector)**.** *As the homogenization is only performed in space, and as it was suggested in the previous proof, the approximate cell problem in our case is actually stationary (see Remark 1.1 in [10]). It is the same as the one of [54] and we could also have obtained the existence of λ by this way. By identification with the ergodic constant obtained in [54], we automically get the continuity and the coercivity of the effective Hamiltonian. As mentioned earlier, the convergence proof also uses the stationary approximate corrector and is a straightforward adaptation of the one contained in [52], where the exact stationary correctors are replaced by approximate ones. We preferred presenting this first method and use the approximate cell problem in the second part.*

3.3 Ergodicity for the traffic flow model with shifted traffic lights

We now study the approximate cell problem (3.8) and introduce the precise statements concerning almost periodic functions. Contrary to the first problem, the almost periodic structure (in space) is not an assumption but a consequence of the assumptions and has to be shown.

3.3.1 Bohr's almost periodic functions and application to the traffic lights problem

For a rational phase shift $\theta = \frac{r}{q}$, the Hamiltonian H is q-periodic in space. For a general phase shift, we enter into the almost periodic framework. This framework is also described in [1].

Roughly speaking, the space periods will be replaced by $\delta-$almost periods for any $\delta > 0$, meaning that small errors will appear. Here is the precise definition:

Definition 3.3.1 (Almost period)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Let $\delta > 0$. We say that $X \in \mathbb{R}$ is a $\delta-$almost period of f if:*

$$\sup_{x \in \mathbb{R}} |f(x+X) - f(x)| \leq \delta.$$

Those almost periods will also be regularly distributed over the real line. To be more precise, the set of those almost periods will be "relatively dense" in $\mathbb{R}$. Here is the exact meaning of relative density:

Definition 3.3.2 (Relative density)**.** *A set $E \subset \mathbb{R}$ is called relatively dense (in $\mathbb{R}$) if there exists $l > 0$, such that any interval of length l contains at least one element of E.*

We now come to the historical definition of almost periodicity:

Definition 3.3.3 (Bohr's almost periodicity)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. We say that f is almost periodic if for each $\delta > 0$, the set of its $\delta-$almost periods is relatively dense.*

This means that there exists $l(\delta) > 0$ such that:

$$\forall x_0 \in \mathbb{R}, \exists X_\delta \in [x_0, x_0 + l(\delta)], \quad \sup_{x \in \mathbb{R}} |f(x+X_\delta) - f(x)| \leq \delta.$$

Remark 3.3.4 (Distribution of the δ-periods)**.** *For periodic functions, the previous length of interval does not depend on the error and the length of the interval coincides with the period. Moreover, the length $l(\delta)$ blows up for vanishing δ, except for periodic functions. Indeed, if the sequence $(l(\delta))_{\delta>0}$ were bounded, then it would be the case for $X_\delta^0 \in [0, l(\delta)]$ and we could extract an exact period.*

This definition extends to almost periodicity in space uniformly in time, for a function that depends on time and space:

Definition 3.3.5 (Almost periodicity in space uniformly in time)**.** *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. We say that g is almost periodic in space uniformly in time, if for each $\delta > 0$, there exists $l(\delta) > 0$ such that:*

$$\forall x_0 \in \mathbb{R}, \exists X_\delta \in [x_0, x_0 + l(\delta)], \qquad \sup_{(t,x) \in \mathbb{R}^2} |g(t, x+X_\delta) - g(t,x)| \leq \delta.$$

The elements X_δ are still called $\delta-$almost periods and the set of those elements is relatively dense.

The Hamiltonian H also depends on the gradient variable. Moreover, as H is not continuous in space, it cannot be considered as almost periodic in the usual sense. We have the following property:

Theorem 3.3.6 (Extended almost periodicity for H)**.** *Under the assumptions (A0)-(A2) and (B1)-(B2), for each $\delta > 0$, there exists $l(\delta) > 0$ such that:*

$$\forall x_0 \in \mathbb{R}, \exists X_\delta \in [x_0, x_0 + l(\delta)], \qquad \sup_{(t,x) \in \mathbb{R}^2, q \in \mathbb{R}} |H(t, x+X_\delta, q) - H(t,x,q)| \leq \delta.$$

The elements X_δ are still called $\delta-$almost periods and the set of those elements is relatively dense. Moreover, the $\delta-$almost periods are integers.

This property is linked with the time periodicity of A (the time period is set to 1). Hence, it is very related to what happens at lights' positions, as H does not depend on space in the free zone (outside lights). The desired result is a direct corollary of the following lemma (see [44]):

Lemma 3.3.7. *Let us consider a nonzero phase shift θ. For any $\mu > 0$, there exists a number $n_\mu \in \mathbb{Z}$ such that:*

$$dist(n_\mu \theta, \mathbb{Z}) < \mu.$$

Moreover, the set of n_μ is relatively dense.

Proof of Theorem 3.3.6. For $\delta > 0$, we consider $\mu(\delta) > 0$ such that $\omega(\mu(\delta)) < \delta$ where ω is the modulus of continuity of A, see (B2). The set of $n_{\mu(\delta)} \in \mathbb{Z}$ defined in Lemma 3.3.7 for $\mu = \mu(\delta)$ is relatively dense.

Let us now show that any $n_{\mu(\delta)}$ is a $\delta-$almost period for H to conclude. For $x \notin \mathbb{Z}$, the result is direct as $n_{\mu(\delta)}$ is an integer:

$$|H(t, x + n_{\mu(\delta)}, q) - H(t, x, q)| = |H_F(q) - H_F(q)| = 0, \qquad \text{for all } t \in \mathbb{R}, \text{ for all } q \in \mathbb{R}.$$

When $x = i \in \mathbb{Z}$, we first use the fact that max is $1-$Lipschitz continuous:

$$|H(t, i + n_{\mu(\delta)}, q) - H(t, i, q)| \leq |A(t - (i + n_{\mu(\delta)})\theta) - A(t - i\theta)|, \qquad \text{for all } t \in \mathbb{R}, \text{ for all } q \in \mathbb{R}. \tag{3.17}$$

We consider $m \in \mathbb{Z}$ such that $dist(n_{\mu(\delta)}\theta, \mathbb{Z}) = |m - n_{\mu(\delta)}\theta| < \mu(\delta)$. By periodicity of A, we get:

$$|A(t - (i + n_{\mu(\delta)})\theta) - A(t - i\theta)| = |A(t - i\theta + m - n_{\mu(\delta)}\theta) - A(t - i\theta)|.$$

By uniform continuity we therefore get in (3.17):

$$|H(t, i + n_{\mu(\delta)}, q) - H(t, i, q)| \leq \omega(\mu(\delta)) < \delta, \qquad \text{for all } t \in \mathbb{R}, \text{ for all } q \in \mathbb{R}.$$

Hence, we get the desired result for all $x \in \mathbb{R}$. □

3.3.2 Well-posedness and regularity

We here show that the approximate corrector is almost periodic in space and give the result about the "weak control" of the oscillation.

Theorem 3.3.8 (Well-posedness of the approximate cell problem)**.** *Let us consider $(p, \theta) \in \mathbb{R}^2$ and $\alpha > 0$. Under the assumptions (A0)-(A2) and (B1)-(B2), for any α, there exists a unique bounded continuous solution v^α of (3.8). Moreover, v^α is periodic in time and almost periodic in space (uniformly in time); and there exists $C > 0$, only depending on H such that:*

$$||\alpha v^\alpha||_{L^\infty(\mathbb{R}^2)} \leq C.$$

Partial proof of Theorem 3.3.8. Let us show that αv^α is almost periodic in space uniformly in time (see Definition 3.3.5). The almost periods will actually coincide with the ones of H. Let $\delta > 0$, we consider an almost period X_δ of Theorem 3.3.6.

We now follow the same argument as [1] (see Section 7) and define:

$$\begin{cases} v^+(\tau, y) = v^\alpha(\tau, y + X_\delta) + \frac{\delta}{\alpha}, & \text{for } (\tau, y) \in \mathbb{R}^2, \\ v^-(\tau, y) = v^\alpha(\tau, y + X_\delta) - \frac{\delta}{\alpha}, & \text{for } (\tau, y) \in \mathbb{R}^2. \end{cases}$$

By translation invariance, v^+ is a solution of:

$$\alpha v^+(\tau, y) + v^+_\tau(\tau, y) + H(\tau, y + X_\delta, p + v^+_y(\tau, y)) - \delta = 0.$$

By definition of the $\delta-$almost periods, we get:

$$\alpha v^+(\tau, y) + v^+_\tau(\tau, y) + H(\tau, y, p + v^+_y(\tau, y)) \geq 0.$$

Using the comparison principle for the approximate cell problem, we get $v^+ \geq v^\alpha$ and thus:

$$\alpha v^\alpha(\tau, y) - \alpha v^\alpha(\tau, y + X_\delta) \leq \delta, \qquad \text{for } (\tau, y) \in \mathbb{R}^2.$$

We perform similar computations for v^- and combine both results to get the desired result:

$$|\alpha v^\alpha(\tau, y + X_\delta) - \alpha v^\alpha(\tau, y)| \leq \delta, \qquad \text{for } (\tau, y) \in \mathbb{R}^2.$$

□

Remark 3.3.9. *The whole proof of Theorem 3.3.8 is given in Chapter 2.*

3.3.3 Weak control of the oscillation

We here state and prove the result about the weak control of the oscillation and adapt the proof made in [54] and [1]:

Theorem 3.3.10 (Weak control of the oscillation)**.** *Let us consider $(p,\theta) \in \mathbb{R}^2$. Under the assumptions (A0)-(A2) and (B1)-(B2), we have:*

$$\lim_{\alpha\to 0} ||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(\mathbb{R}^2)} = 0.$$

Proof. **Step 1: Reduction to a compact set.**

Thanks to the time periodicity and almost periodicity in space, it will be enough to control the oscillation on a compact set as we have the following inequality:

Lemma 3.3.11 (Reduction to a compact set)**.** *For any $\delta > 0$, we have:*

$$||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(\mathbb{R}^2)} \leq ||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(K_\delta)} + \delta, \tag{3.18}$$

where $K_\delta := [0,1] \times [0, l(\delta)]$ with $l(\delta)$ the length associated to the relative density of the set of the $\delta-$almost periods for αv^α (see Definition 3.3.5).

Proof. By time periodicity of αv^α, we have:

$$||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(\mathbb{R}^2)} = ||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty([0,1]\times\mathbb{R})}.$$

Let us consider $\delta > 0$ and $(\tau, y) \in [0,1] \times \mathbb{R}$. By uniform almost periodicity in space of αv^α, there exists a $\delta-$almost period $X_\delta \in [-y, -y + l(\delta)]$. Thus we have:

$$|\alpha v^\alpha(\tau,y) - \alpha v^\alpha(0,0)| \leq |\alpha v^\alpha(\tau, y + X_\delta) - \alpha v^\alpha(0,0)| + \delta \leq ||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(K_\delta)} + \delta, \tag{3.19}$$

for any $(\tau, y) \in [0,1] \times \mathbb{R}$ and this gives the result. □

Step 2: Time oscillation

By coercivity and uniform continuity of H_F and the continuity of A (see (A0)-(A2), (B1)-(B2)), H is bounded from below. We deduce from Theorem 3.3.8 that there exists $C > 0$ independent of α such that:

$$v^\alpha_\tau \leq C.$$

From a classical comparison principle, we get:

$$v^\alpha(\tau + h, y) \leq v^\alpha(\tau, y) + Ch, \qquad \text{for } (\tau, y) \in [0,1] \times \mathbb{R}, h \geq 0.$$

We combine with the time periodicity of v^α to get:

$$|v^\alpha(s,y) - v^\alpha(\tau,y)| \leq C, \qquad \text{for } (s,\tau,y) \in [0,1]^2 \times \mathbb{R}.$$

Step 3: Space oscillation

In particular, we define $M^\alpha(y) = \sup_{\tau\in\mathbb{R}} v^\alpha(\tau,y) := v^\alpha(\tau_\alpha(y), y)$ where $\tau_\alpha(y) \in [0,1]$ and get:

$$|v^\alpha(\tau,y) - M^\alpha(y)| \leq C, \qquad \text{for all } y \in \mathbb{R}. \tag{3.20}$$

Lemma 3.3.12. *M^α is a continuous function and is a viscosity subsolution of $H(\tau_\alpha(y), y, p + M^\alpha_y(y)) \leq C'$ for some $C' > 0$ independent of α.*

Moreover there exists a constant independent of α, still denoted by $C > 0$, such that we have: $|M^\alpha_y(y)| \leq C$, for all $y \notin \mathbb{Z}$. In particular:

$$|M^\alpha(x) - M^\alpha(y)| \leq C, \qquad \textit{for all } k \in \mathbb{Z}, \textit{ for all } (x,y) \in [k, k+1]^2. \tag{3.21}$$

Remark 3.3.13 (Proof of Lemma 3.3.12)**.** *The proof of Lemma 3.3.12 is given in Chapter 2.*

Step 4: Conclusion. For $\delta > 0$, $(\tau, y) \in K_\delta$, we have:

$$|\alpha v^\alpha(\tau, y) - \alpha v^\alpha(0,0)| \leq |\alpha v^\alpha(\tau, y) - \alpha M^\alpha(y)| + |\alpha M^\alpha(y) - \alpha M^\alpha(0)| + |\alpha v^\alpha(\tau, 0) - \alpha M^\alpha(0)|.$$

We combine with (3.20), (3.21):

$$|\alpha v^\alpha(\tau, y) - \alpha v^\alpha(0,0)| \leq 2\alpha C + \alpha C l(\delta), \qquad \text{for all } (\tau, y) \in [0,1] \times K_\delta.$$

Using Lemma 3.3.11, we get:

$$||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(\mathbb{R}^2)} \leq 2\alpha C + \alpha C l(\delta) + \delta.$$

Thanks to the estimate of Theorem 3.3.8, we can take the upper limit for vanishing α on both sides for fixed δ to get:

$$\limsup_{\alpha \to 0} ||\alpha v^\alpha - \alpha v^\alpha(0,0)||_{L^\infty(\mathbb{R}^2)} \leq \delta,$$

for all $\delta > 0$ which ends the proof. □

Remark 3.3.14 (Comparison with the estimate in Chapter 2)**.** *The result is weaker than the one obtained in Chapter 2 where we used a change of variables to prove that* $\max_{\mathbb{R}^2} v^\alpha - \min_{\mathbb{R}^2} v^\alpha$ *was controlled independently from* α*. Such a result is not possible to get anymore, because the ratio* $\frac{\delta}{\alpha}$ *can blow up for vanishing* α*, while we had* $\delta = 0$ *in the space periodic frame.*

Remark 3.3.15 (Proof of Theorem 3.1.3)**.** *The proof of Theorem 3.1.3 is given in Chapter 2 (existence of approximate correctors, uniqueness of* λ*).*

We now analyse the regularity of the effective Hamiltonian.

3.3.4 Uniform continuity of the effective Hamiltonian

The uniform continuity of the effective Hamiltonian in the gradient variable is a direct consequence of the uniqueness of λ. We introduce an elementary but useful lemma and its corollary:

Lemma 3.3.16. *F_A is uniformly continuous with respect to $q \in \mathbb{R}^2$ (uniformly in θ, space and time).*

Remark 3.3.17 (Proof of Lemma 3.3.16)**.** *The proof is exactly the same as the one performed in Chapter 2, by replacing "continuity" by "uniform continuity".*

Lemma 3.3.16 implies that H is uniformly continuous in the gradient variable, as $H = H_F$ far from the lights. We denote by ω_H the modulus of continuity of H in the gradient variable. We also give a second lemma which was proved in Chapter 2.

Lemma 3.3.18. *Let us consider $(\lambda, \bar{\lambda}) \in \mathbb{R}^2$. If for any $\delta > 0$, there exist bounded functions v^δ and w^δ such that:*

$$\begin{cases} \lambda + v^\delta_\tau(\tau, y) + H(\tau, y, p + v^\delta_y(\tau, y)) \geq -\delta, & \textit{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}, \\ \bar{\lambda} + w^\delta_\tau(\tau, y) + H(\tau, y, p + w^\delta_y(\tau, y)) \leq \delta, & \textit{for } (\tau, y) \in \mathbb{R} \times \mathbb{R}, \end{cases}$$

then, $\bar{\lambda} \leq \lambda$.

Theorem 3.3.19 (Uniform continuity in the gradient variable)**.** *Under the assumptions (A0)-(A2), $\bar{H}$ is uniformly continuous in the gradient variable uniformly in θ.*

Proof. For $\delta > 0$, $(p,q) \in \mathbb{R}^2$, and $\theta \in [0,1]$, we consider v^δ and w^δ the approximate correctors respectively associated to $\bar{H}(p,\theta)$ and $\bar{H}(q,\theta)$ in (3.6). Combining with the uniform continuity of H, we get for v^δ:

$$v^\delta_\tau(\tau,y) + H(\tau,y,q+v^\delta_y(\tau,y)) \geq \bar{H}(p,\theta) - \omega_H(|p-q|) - \delta, \qquad \text{for } (\tau,y) \in \mathbb{R}\times\mathbb{R}.$$

Hence, by applying Lemma 3.3.18, we obtain (we replace λ by $\omega_H(|p-q|) - \bar{H}(p,\theta)$ and $\bar{\lambda}$ by $-\bar{H}(q,\theta)$):

$$\bar{H}(p,\theta) \leq \bar{H}(q,\theta) + \omega_H(|p-q|).$$

Using a similar argument, we get the desired inequality:

$$|\bar{H}(q,\theta) - \bar{H}(p,\theta)| \leq \omega_H(|p-q|).$$

□

Remark 3.3.20 (Difference with Chapter 2)**.** *As we do not have any control on the Lipschitz constant (contrary to [54]), we see that the assumption (A0) is very important and is stronger than the one made in Chapter 2 where we only needed continuity instead of uniform continuity of the Hamiltonian in the free zone.*

3.4 Open questions

We gather here some open questions for related problems. We only give intuitive ideas and do not enter into technical aspects.

3.4.1 Homogenization in time and space with almost periodicity in all variables

For $\varepsilon > 0$, we consider the rescaled solution v^ε of the oscillating Hamilton-Jacobi equation on the real line:

$$\begin{cases} v^\varepsilon_t + h_1\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v^\varepsilon_x\right) = 0, & \text{for } (t,x) \in (0,+\infty)\times\mathbb{R}, \\ v^\varepsilon(0,x) = v_0(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

where we suppose similar assumptions as (A'0)-(A'3) but where the almost periodic assumption (A'4) is replaced by:

For each $R > 0$, the family $\{h(.+s,.+z,.), (s,z) \in \mathbb{R}^2\}$ is relatively compact in $BUC(\mathbb{R}\times\mathbb{R}\times[-R,R])$.

This means that for each $\delta > 0$, there exist $n \in \mathbb{N}$, $(r_1,..,r_n) \in \mathbb{R}^n$, $(k_1,..,k_n) \in \mathbb{R}^n$, such that:

$$\forall (r,k) \in \mathbb{R}^2, \exists (i,j) \in \{1,n\}^2, \quad \sup_{(t,y,q)\in\mathbb{R}\times\mathbb{R}\times[-R,R]} |h(t+r,y+k,q) - h(t+r_j,y+k_i,q)| < \delta.$$

We have not been able to control the oscillation of the approximate corrector by using almost periodic arguments (this kind of problem can be handled in the stationary ergodic setting for convex Hamiltonians, see [72]).

Exactly as it has been performed in Section 3.3, we can reduce to a compact set. The time oscillation is still controled by the same arguments.

The main difference with Section 3.3 occurs when dealing with the space oscillation (see Steps 2 and 3 of the proof of Theorem 3.3.10). Indeed, as the approximate corrector is not time periodic anymore (but only almost periodic in time), it does not necessarily attain its time supremum in $\mathbb{R}$. When reducing to the compact set, the supremum can be attained on the boundaries and Lemma 3.3.12 is not true anymore. We could penalize the right boundary of the compact domain to avoid this case but we cannot deal with the case where the maximum is attained on the left boundary.

3.4.2 Homogenization of a non local microscopic traffic flow model with almost periodicity in space

We study here a microscopic version of the second problem, which will turn out to be a non local problem. More precisely, we suppose that the lights are not very far one from each other and thus consider a microscopic model between two lights.

We consider an infinite 1D road with equally spaced lights. To transpose mathematically their presence and their influence on drivers' velocities, we can choose a periodic function in time and space, ψ say, which take values in $[0, 1]$. $\psi \equiv 1$ corresponds to the case where there is no light.

Let us consider that the distance d between lights and their common time period are respectively rescaled to one meter and one minute, so that ψ is 1-periodic in space and time.

The effect of each light situated at $k \in \mathbb{Z}$ is mainly local. We will suppose that it is contained in an open interval around each light, $i_r(k) =]k - r; k + r[$, where $r > 0$ is chosen to be constant for all lights, and r has to be stricly less than d. Between two lights, there is then a non-empty free zone where ψ identically equals 1, that is where there is no limiting effect on the traffic. The whole free zone on the real line, that is the complementary of the union of all the $i_r(k)$ for $k \in \mathbb{Z}$, is denoted by L. This setting is drawn in Figure 3.2.

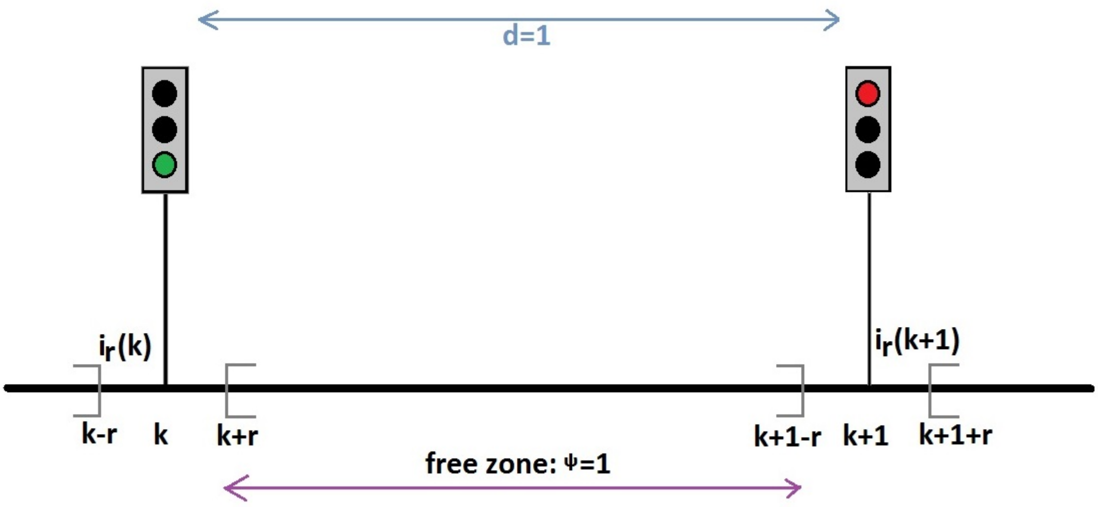

Figure 3.2: Sketch of the different zones.

We can remark that transposing the presence of a light on the road into a mathematical function is different from transposing the light itself into such a function, that is without considering the effect on its local environment. A fixed and isolated light can be seen as a function of the time variable, one-periodic in our example, and which varies between 0 and 1, that is, a periodic continuous function, q say. To be consistent, we then expect to have $\psi(k, t) = q(t)$ for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, that means that a car precisely placed in front of a light, will be blocked if the light is red and will move forward normally if the light is green, as if there were no light.

The local effect of the light is explained by the fact that if a driver sees a red light at a given $x = k$, then he will brake and will spend some time to retrieve his initial velocity after the light turned green. This is the case when the light is not too far away from him (the driver is in $i_r(k)$), otherwise he does not have any reason to stop, the latter case corresponding to the free zone. In each $i_r(k)$, ψ will take values in the whole interval $[0, 1]$, so as to take into account the blocking effect of the light placed at $x = k$. To ensure a continuous transition between the two zones, it is reasonable to state that $\psi(k - r, t) = \psi(k + r, t) = 1$ for all $(k, t) \in \mathbb{Z} \times \mathbb{R}_+$, that is ψ is continuous in the space variable, exactly as it was made in [37] when dealing with a bifurcation.

Now, let us try to give a simple traffic model on this road. Basically, the velocity of each driver is a nondecreasing and positive function, V say, of the distance that separates him from the preceding driver. Such a model is refered to as a "follow-the-leader" model, see [17] for instance. It is reasonable to state V as bounded, like common car velocities. When there are lights on the road, this function V has to be weighted by ψ.

Let us suppose that the phase difference between two successive lights, is constant. This corresponds to c being constant in (3.22), where $c > 0$ can represent the propagation velocity of a green wave throughout lights when all the lights are initially red and one (for example at $k = 0$) suddenly becomes green. The phase difference between two consecutive lights is d/c, where $d = 1$. If the phase of the light at $k = 0$ is described by $\psi(0, t)$ then the phase of the k-th light will be given by $\psi(k, t - k/c)$. In first approximation, we choose a similar form for the phase for each driver between two lights, by replacing k by the driver's position in (3.22).

Given a sequence $(X_i)_{i\in\mathbb{Z}}$ of drivers' positions on the road, which are functions of time, the microscopic model then takes the form of an infinite coupled system of non-linear ODEs:

$$\frac{dX_i}{dt}(t) = \psi(X_i(t), t - X_i(t)/c)V(X_{i+1}(t) - X_i(t)), \qquad \forall i \in \mathbb{Z}. \tag{3.22}$$

For $\varepsilon > 0$, we define the function u^ε as follows (the space variable will be denoted as x):

$$u^\varepsilon(x,t) = \varepsilon X_{\lfloor \frac{x}{\varepsilon} \rfloor}(\frac{t}{\varepsilon}). \tag{3.23}$$

Substituting in (3.22) leads to the following equation:

$$\partial_t u^\varepsilon(x,t) = \psi\left(\frac{u^\varepsilon(x,t)}{\varepsilon}, \frac{t}{\varepsilon} - \frac{u^\varepsilon(x,t)}{c\varepsilon}\right) V\left(\frac{u^\varepsilon(x+\varepsilon,t) - u^\varepsilon(x,t)}{\varepsilon}\right). \tag{3.24}$$

Let us introduce a slight change of functions. We define φ as: $\varphi(x,t) = \psi(x, t - \frac{x}{c})$. Then we get a non local equation:

$$\partial_t u^\varepsilon(x,t) = \varphi\left(\frac{u^\varepsilon(x,t)}{\varepsilon}, \frac{t}{\varepsilon}\right) V\left(\frac{u^\varepsilon(x+\varepsilon,t) - u^\varepsilon(x,t)}{\varepsilon}\right) := F\left(\frac{t}{\varepsilon}, \frac{u^\varepsilon(x,t)}{\varepsilon}, \frac{u^\varepsilon(x+\varepsilon,t)}{\varepsilon}\right). \tag{3.25}$$

As ψ is one-periodic in space and time, we show that φ is space and time periodic if and only if $c \in \mathbb{Q}$. In that case, the homogenization result for the Cauchy problem corresponding to (3.25) is contained in [34]. For $c \notin \mathbb{Q}$, we fall into the almost periodic setting in space.

We accordingly adapt the method of Section 3.2 and consider the following equation for $p > 0$:

$$\begin{cases} s_\tau(\tau,y) = F(\tau, s(\tau,y) + py, s(\tau,y+1) + p(y+1)), & \text{for } (\tau,y) \in (0,+\infty) \times \mathbb{R}, \\ s(0,y) = 0, & \text{for } y \in \mathbb{R}. \end{cases} \tag{3.26}$$

The main idea to get the ergodic constant is to prove a control on the space oscillation of the solution to (3.26), which will imply a control on the time oscillation (see Proposition 4.1 of [34]). The periodic assumption plays a significant role as it enables one to get invariance by addition of integers to the solutions. For $c \notin \mathbb{Q}$, the solution s of (3.26) is proved to be almost periodic in space but not uniformly in time (the difference between the space translates of s blows up for large time as it is proportional to the time variable like in Theorem 3.2.2).

Let us mention some difficulties arising in this almost periodic non local case:

- The Hamiltonian is not coercive and we cannot get any Lipschitz estimate in space.

- The time dependance implies some difficulties for the uniqueness of λ because we do not have invariance by time translation.

What does precisely happen on the control of the space oscillation without the invariance by addition of integers?

As $p > 0$, as in Step 1 of Proposition 4.2 in [34], we get that s is non-decreasing in y. For $(\tau, y) \in (0, +\infty) \times \mathbb{R}$, we try to control $|s(\tau, y) - s(\tau, 0)|$ uniformly in time and space.

In [34], the invariance by addition of integers and the fact that there exists $k \in \mathbb{Z}$ such that $|py - k| \leq 1$ were combined with the monotony of s to get a direct control. The approximate periods will replace the periods but the problem is that the more we want to decrease the error by translation along those periods, the more those approximate periods will be far one from each other (which is not the case in the periodic framework). More precisely, the length of the interval introduced in Definition 3.3.2 will blow up for a vanishing error (see Remark 3.3.4).

We try to follow the same method as [34]. In our setting, for any $\delta > 0$, there exist a positive length of interval $l(\delta) > 0$ and a $\delta\tau-$period $X \in [py - l(\delta), py]$.

We define $a := py - X \in [0, l(\delta)]$. By using the monotony of s, we get:

$$-\delta\tau - a \leq s(\tau, y) - s(\tau, 0) \leq \delta\tau + s\left(\tau, \frac{l(\delta)}{p}\right) - s(0) + l(\delta) - a, \tag{3.27}$$

which does not give a meaningful estimate.

Remark 3.4.1. *In the periodic case, we have* $\delta = 0$*,* $l(0) = 1$*,* $a \in [0, 1]$ *and* $s\left(\tau, \frac{1}{p}\right) = s(\tau, 0)$ *and hence we recover the same result as [34]:* $|s(\tau, y) - s(\tau, 0)| \leq 1$.

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