

Pricing of XVA adjustments : from expected exposures to wrong-way risks

Marouan Iben Taarit

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par

Marouan IBEN TAARIT

École Doctorale : MATHEMATIQUES ET SCIENCES ET TECHNOLOGIES DE L'INFORMATION ET DE LA COMMUNICATION

Valorisation des ajustements Xva : De l'exposition espérée aux risques adverses de corrélation

Thèse dirigée par Professeur Bernard Lapeyre

Soutenue le 08 janvier 2018 devant le jury composé de

Fréderic Abergel	CentralSupélec
Stéphane Crépey	Université d'Évry
Romuald Elie	Université Paris-Est
Monique Jeanblanc	Université d'Évry
Bernard Lapeyre	École des Ponts ParisTech
Étienne Varloot	Natixis



Thèse réalisée au Laboratoire Cermics
École des Ponts ParisTech
Bâtiment Coriolis
6 et 8 avenue Blaise Pascal Cité Descartes – Champs sur Marne
77455 Marne la Vallée Cedex 2
Tél : +33 1 64 15 30 00
Web : https://cermics-lab.enpc.fr/

Ainsi qu'à Banque Natixis
30 Avenue Pierre Mendès-France
75013 Paris
Tél: +33 1 58 32 30 00
Web : https://www.natixis.com/

Sous la direction de Professeur Bernard Lapeyre
Courriel : Bernard.Lapeyre@cermics.enpc.fr

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Résumé

Nous entamons ce rapport de thèse par l'évaluation de l'espérance espérée qui représente une des composantes majeures des ajustements XVA. Sous l'hypothèse d'indépendance entre l'exposition et les coûts de financement et de crédit, nous dérivons dans le chapitre 3 une représentation nouvelle de l'exposition espérée comme la solution d'une équation différentielle ordinaire par rapport au temps d'observation du défaut. Nous nous basons, pour le cas unidimensionnel, sur des arguments similaires à ceux de la volatilité locale de Dupire. Et pour le cas multidimensionnel, nous nous référons à la formule de la Co-aire. Cette représentation permet d'expliciter l'impact de la volatilité sur l'exposition espérée : Cette valeur temps fait intervenir la volatilité des sous-jacents ainsi que la sensibilité au premier ordre du prix, évalués sur un ensemble fini de points. Malgré des limitations numériques, cette méthode est une approche précise et rapide pour la valorisation de la XVA unitaire en dimension 1 et 2.

Les chapitres suivants sont dédiés aux aspects du risque de corrélations entre les enveloppes d'expositions et les coûts XVA. Nous présentons une modélisation du risque général de corrélation à travers une diffusion stochastique multivariée, comprenant à la fois les sous-jacents des dérivés et les intensités de défaut. Dans ce cadre, nous exposons une nouvelle approche de valorisation par développements asymptotiques, telle que le prix d'un ajustement XVA correspond au prix de l'ajustement à corrélation nulle, auquel s'ajoute une somme explicite de termes correctifs. Le chapitre 4 est consacré à la dérivation technique et à l'étude de l'erreur numérique dans le cadre de la valorisation de dérivés contingents au défaut. La qualité des approximations numériques dépend uniquement de la régularité du processus de diffusion de l'intensité de crédit, et elle est indépendante de la régularité de la fonction payoff. Les formules de valorisation pour CVA et FVA sont présentées dans le chapitre 5. Une généralisation des développements asymptotiques pour le cadre bilatéral de défaut est adressée dans le chapitre 6.

Le chapitre 7 termine ce mémoire en abordant un cas du risque spécifique de corrélation lié aux contrats de migration de rating. Au delà des formules de valorisation, notre contribution consiste à présenter une approche robuste pour la construction et la calibration d'un modèle de transition de ratings consistant avec les probabilités de défaut implicites de marché.

Mots clés : Ajustements XVA, Exposition espérée, Formules de la Co-aire, Dérivés contingent au crédit, Développements asymptotiques, XVA bilateral, *Wrong-Way risk* général, *Wrong-Way risk* spécifique, Collatéral avec migration de *Ratings*

Abstract

The point of departure of this thesis is the valuation of the expected exposure which represents one of the major components of XVA adjustments. Under independence assumptions with credit and funding costs, we derive in Chapter 3 a new representation of the expected exposure as the solution of an ordinary differential equation w.r.t the default time variable. We rely on PDE arguments in the spirit of Dupire's local volatility equation for the one dimensional problem. The multidimensional extension is addressed using the co-area formula. This forward representation gives an explicit expression of the exposure's time value, involving the local volatility of the underlying diffusion process and the first order Greek *delta*, both evaluated only on finite set of points. From a numerical perspective, dimensionality is the main limitation of this approach. Though, we highlight high accuracy and time efficiency for standalone calculations in dimensions 1 and 2.

The remaining chapters are dedicated to aspects of the correlation risk between the exposure and XVA costs. We start with the general correlation risk which is classically modeled in a joint diffusion process for market variables and the credit/funding spreads. We present a novel approach based on asymptotic expansions in a way that the price of an XVA adjustment with correlation risk is given by the classical correlation-free adjustment to which is added a sum of explicit correction terms depending on the exposure Greeks. Chapter 4 is consecrated to the technical derivation and error analysis of the expansion formulas in the context of pricing credit-contingent derivatives. The accuracy of the valuation approach is independent of the smoothness of the payoff function, but it is related to the regularity of the credit intensity model. This finding is of special interest for pricing in a real financial context. Pricing formulas for CVA and FVA adjustments are derived in Chapter 5, along with numerical experiments. A generalization of the asymptotic expansions to a bilateral default risk setting is addressed in Chapter 6.

Our thesis ends by tackling, in Chapter 7, the problem of modeling the specific *Right-Way Risk* induced by rating trigger events within the collateral agreements. Our major contribution is the calibration of a rating transition model to market implied default probabilities.

Keywords: XVA adjustments, Expected exposure, Co-area formula, Credit-contingent derivatives, Asymptotic expansions, General Wrong-Way Risk, Specific Wrong-Way Risk, Collateral with Rating migrations

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Introduction

En 2007, la chute du marché des prêts hypothécaires « *Subprime* » aux États-Unis a déclenché une crise bancaire, financière et économique mondiale. Les évènements de faillite de nombreuses sociétés de crédit spécialisé et de la banque Lehman Brothers ont provoqué un changement significatif dans le fonctionnement des marchés financiers et du cadre règlementaire. La crise financière a montré l'importance du suivi des risques de marché, mais aussi celui du risque de crédit sous-jacent aux activités de dérivés de gré-à-gré (ou OTC[1]). Même s'ils n'ont pas été à l'origine de la crise, les dérivés OTC ont largement contribué à amplifier les problèmes de solvabilité et de contagion.

Une autre évolution aussi inévitable a été que les spreads entre le taux au jour-le-jour[2] et le Libor d'emprunts longs non garantis des banques sur le marché interbancaire ont considérablement augmenté, passant de quelques points de base [3] historiquement à plusieurs centaines de *bps*. Le risque perçu sur les banques a été amplifié pendant la crise de la dette européenne de 2011-2012 pour le zone euro. Il devient désormais clair que le coût de financement représente une composante significative du prix d'un dérivé, s'additionnant à celle du risque bilatéral de contrepartie.

A travers cette introduction, nous souhaitons situer le sujet de thèse dans un contexte opérationnel, lié à l'industrie bancaire et marqué par une évolution constante des pratiques et de la règlementation. Mais d'abord, il nous parait nécessaire de revenir sur quelques éléments clés du fonctionnement du marché des dérivés de gré à gré.

Marché des dérivés OTC et conséquences de la crise financière

Un produit dérivé définit une relation contractuelle sur une période allant de plusieurs jours (exemple des contrats futures) à quelques décennies (exemple d'un swap de taux long terme). Pendant la durée de vie du contrat, les deux contreparties échangent des flux dont le montant est fixe ou variant en fonction d'actifs sous-jacents ou de conditions de marché. Pour les produits simples, la maturité du contrat correspond à une échéance fixe. Pour des produits plus complexes, cette échéance peut varier en fonction de la valeur d'une variable de référence (sortie

[1] Over the counter

[2] Sur le marché monétaire, le taux au jour-le-jour (TJJ) désigne un taux d'intérêt in fine applicable d'un jour ouvrable au suivant. Il existe des marchés de swaps contre des fixings de TJJ connus sous l'appellation d'OIS. En particulier, l'indice de référence est *Eonia* pour la zone Euro et *Fed Funds* pour les marchés US.

[3] *bps* 0.01%

obligatoire), ou implicite à une décision de rappel (sortie optionnelle) au gré de l'une des deux parties prenantes.

Les produits dérivés représentent des instruments de couverture contre l'exposition à des risques de marché (crédit, taux, change, actions, matières premières). Ils permettent aussi de structurer des solutions d'investissement très variées à travers des notes complexes (voir [BO10]). Parmi les utilisateurs des produits dérivés, on compte à la fois des hedge funds, des asset managers, des compagnies d'assurances, mais aussi des agences d'états, des banques supranationales, des banques centrales ou régionales ainsi que des entreprises industrielles.

Le marché des dérivés OTC

Le marché des dérivés, au sens large, est structuré autour d'un marché organisé à travers des plateformes d'échange pour les produits les plus simples, et d'un marché de gré à gré. Les instruments échangés sur le marché organisé sont des instruments standardisés comme les contrats à termes (e.g. contrats futures) et les options vanilles. Le marché organisé favorise la liquidité et l'efficience d'information via une découverte plus efficace du prix. Il fournit également un moyen standard pour mitiger le risque de contrepartie à travers l'organisation des échanges des flux via une chambre de compensation.

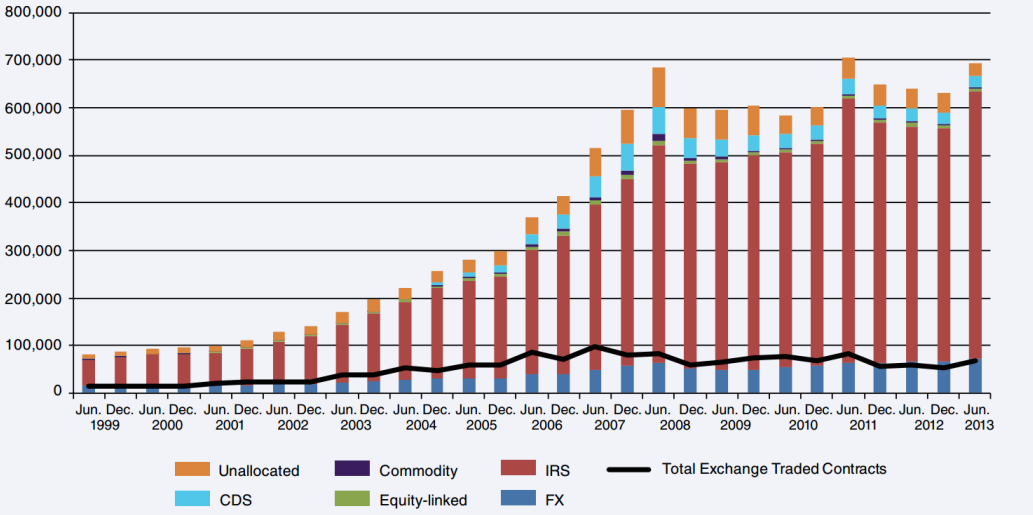


FIGURE 0.0.1 : Marché des dérivés OTC - Nominaux en milliard de dollars US (source BIS `https://www.bis.org/`)

Les produits négociés de gré à gré sont moins standards et traités dans le cadre d'une transaction bilatérale privée entre deux contreparties. Ces contrats précisent les règles d'appel de marge et de compensation, en outre les clauses définissants les échanges de flux.

La figure 0.0.1 montre l'évolution des volumes de nominaux échangés sur les deux marchés. Le marché des dérivés OTC est beaucoup plus important ($> 80\%$) de part la longue maturité des produits échangés notamment les swaps de taux et les *cross-currency swap* sur le marché

de change[4]. La part des produits de taux et de change représente à elle seule plus de 90% du marché OTC. Le volume restant représente majoritairement le périmètre Crédit via les *Credit Default Swaps* ou *CDS* qui ne cesse de diminuer depuis la crise de 2007-2008.

	CSA	No CSA
Commodity derivatives	59.1%	40.9%
Credit derivatives	97.0%	3.0%
Equity derivatives	91.3%	8.7%
Fixed-income derivatives	88.9%	11.1%
Foreign Exchange derivatives (excluding spot transactions)	73.0%	27.0%

TABLE 0.0.1 : Appel de marge bilatéral par classes d'actifs au 31 Decembre 2014 (Source ISDA [ISD15])

Les dérivés OTC sont de plus en plus collatéralisés, via des chambres de compensation ou des appels de marges bilatéraux. Le recours aux chambres de compensation est de plus en plus important. Cependant, il requiert toujours un degré de standardisation afin d'assurer une transparence au niveau de la valorisation et des appels de marges. Concernant la collatéralisation bilatérale, le tableau 0.0.1 présente le pourcentage des dérivés OTC parfaitement collatéralisés par des accords bilatéraux par classe d'actifs. Les dérivés sous-collatéralisés représentent en moyenne 20% du périmètre. Malgré cette faible proportion, les enjeux restent extrêmement importants à l'échelle du portefeuille d'une contrepartie, ainsi que du point de vue du coût de financement des positions non-collatéralisées et des charges en capital règlementaire.

Le processus de *trading* varie en fonction de la complexité du dérivé sous-jacent. Il implique des commerciaux, juristes, traders, risk-manageurs, quants, comptables, support informatiques, ect. 2 étapes principales définissent ce processus :

Pre-trade

La première étape définit les besoins client en terme de produits dérivés ou en solutions d'investissement. Pour les produits non standards, la définition de la structure est exprimée dans une notice[5] qui précise la fonction payoff, les évènements pouvant affecter le paiement de flux, ainsi que le cadre juridique de la transaction. Les fonctions de trading et d'ingénierie quantitative ont la responsabilité d'analyser le profil de risque du produit et de préciser leur capacité (fonctionnelle et d'infrastructure informatique) à gérer sa complexité. En cas de retour favorable du trading, les premières cotations sont transmises au client, déclenchant la négociation.

[4] également appelé *Forex* ou *FX*
[5] *Termsheet*

Les transactions avec des nouvelles contreparties nécessitent l'approbation du département juridique qui étudie les termes spécifiques du contrat, les clauses de règlement[6] en cas d'évènement de terminaison anticipée, ainsi que les implications légales du point de vue du risque de crédit.

L'approbation du Département des Risques est la dernière étape, et certainement la plus complexe. Celui-ci mesure les impacts de la nouvelle transaction sur l'exposition agrégée du point de vue du risque de marché et de celui du risque de contrepartie. Ces expositions sont encadrées à travers deux types de limites suivies quotidiennement : Des limites pour le risque de marché, et des limites pour le risque de contrepartie. Les limites pour le risque de marché se basent sur des mesures comme la Value-At-Risk, les sensibilités (Delta, Vega,...) ou des stress-tests spécifiques[7] ou globaux[8]. La calibration des limites dépend de la nature des sous-jacents, leurs corrélations et de leurs niveaux de liquidité. Un dépassement de limite implique l'annulation de la transaction, une demande de revue de limites, ou la mise en place de couvertures en vue de réduire l'exposition. Le monitoring du risque de marché se fait au niveau d'une activité délimitée dans une section ou portefeuille. Le risque de contrepartie est encadré par une ligne de crédit affectée à chaque contrepartie selon une notation interne. Cette limite tient compte des contrats de collatéral et autres mécanismes de réduction de l'exposition crédit. Elle est l'objet de revues périodiques pour prendre en considération l'évolution de notation de chaque contrepartie.

Le département des risques analyse l'adéquation des outils de valorisation et de gestion aux spécificités du payoff, ainsi que les schémas de saisie[9] dans les systèmes. Il mesure le risque de modèle induit par le choix de dynamiques pour les axes de risques, l'observabilité des paramètres de modèle, la dérivation de formules de valorisation ou l'utilisation de méthodes numériques (Monte Carlo, EDP).

Post-trade

Après l'acceptation et l'exécution, la transaction est désormais saisie dans un système de gestion jusqu'à sa maturité ou l'activation d'une clause de sortie. Le système de gestion doit permettre de :

- Valoriser à la demande, calculer les flux espérés, les probabilités d'entrées/sorties, les maturités effectives.
- Calculer les sensibilités du produit et saisie des positions de couverture.
- Effectuer des calculs règlementaires (VAR, ES[10]), des stress-tests ou des provisions.

Modélisation et valorisation

Le département des risques veille à l'application des règlementations et recommandations émises par les autorités de régulations. Ceci concerne en premier lieu l'intégrité des données de valori-

[6] *Settlement*
[7] définis sur les facteurs spécifiques à l'activité
[8] comme les stress tests historiques ou hypothétiques
[9] *Booking*
[10] Expected Shortfall

sations, la validité des modèles de valorisation [11], et le calcul de sensibilités nécessaires au suivi des limites et à l'attribution du P&L.

La valorisation d'un produit complexe peut être assimilée à un processus d'extrapolation à partir d'une base de prix d'instruments de référence. L'approche la plus répondue s'appuie sur la théorie de valorisation risque-neutre et d'un modèle de diffusion stochastique (i.e. équations différentielles stochastiques) pour les facteurs latents. Les paramètres du modèle de diffusion sont déterminés de manière à reproduire les prix des instruments de référence. La conception d'un modèle de valorisation est sujet à de multiples hypothèses dues à des contraintes d'observabilité, opérationnelles ou de puissances de calcul. Ces hypothèses tendent à représenter des approximations réalistes mais peuvent engendrer une incertitude matérielle sur le prix d'un dérivé. Quelques exemples issus de la crise financière des *Subprimes* sont discutés par Morini dans [Mor11].

Pour un produit simple, la validité du modèle de valorisation peut être vérifiée à travers une stratégie de réplication dynamique appliquée sur des scénarios économiques. La pertinence du modèle est ainsi donnée par la distribution de l'erreur de réplication, dite aussi distribution des fuites de P&L. Dans le cas de produits complexes, il est plus difficile d'appliquer ce *backtest* de réplication, particulièrement en présence de plusieurs facteurs de risque corrélés. Cette complexité explique une prime de risque significative qui sert de réserve contre des risques non ou difficilement couvrables comme les risques de volatilité et de corrélation. Ceci se traduit par un marquage défensif des paramètres de modèle ou par l'utilisation d'un modèle alternatif, jugé plus pertinent, pour le calcul des réserves.

Le processus de conception, développement et validation des modèles quantitatifs fait l'objet aujourd'hui d'un suivi règlementaire stricte dont le but est de vérifier l'adéquation des modèles avec leurs utilisations. La règlementation tend vers une homogénéisations des pratiques entre les banques à travers des textes comme le *Supervisory Guidance on Model Risk Management* publié par le *Office of the Comptroller of the Currency* en 2011 aux États-Unis. Cette surveillance a été significativement renforcée suite aux pratiques défaillantes aux MBS[12] et autres produits structurés de crédit. Nous nous référons à nouveau à [Mor11] pour son analyse de ces défaillances.

Crise financière de 2007 - Un changement majeur

Entre 2004 et 2006, les taux d'intérêt américain ont significativement remonté, déclenchant un ralentissement du marché de l'immobilier. La hausse des taux a conduit à un record d'insolvabilité sur les emprunts immobiliers, particulièrement les emprunts hypothécaires de type *Subprime* contractés par des ménages de faible pouvoir d'achat. Ces emprunts rapidement qualifiés de toxiques, étaient initialement détenus par des banques de détail, ainsi que les organismes d'état *Fannie Mae* et *Freddie Mac*. Le marché des subprimes s'est élargi car les emprunts sous-jacents ont été redistribués dans des structures complexes comme les MBS. Ces structures sont ensuite émises sur les marchés financiers à travers le monde comme des produits d'investissement jouissants d'une signature avantageuse de la part des agences de notation. Dès la moitié de 2007, le *Credit crunch* a commencé par des défaillances sévères de liquidité et une volatilité excessive sur le marché de la dette. L'effet de contagion était immédiat.

[11] mark-to-model

[12] Mortgage-backed securities

A la fin de 2009, les craintes d'une crise de la dette souveraine se sont accentuées en Europe à cause d'un niveau d'endettement record et la détérioration de la signature de quelques états de l'Europe du sud due à la crise de 2007. En mai 2010, la Grèce a reçu un plan de sauvetage de plus de 100 milliards d'euro de la Zone Euro et du FMI. D'autres pays ont suivi comme le Portugal, l'Irlande et l'Espagne. Les banques étaient à nouveau fortement exposées aux faiblesses potentielles des pays souverains européens. Encore une fois, le risque de contrepartie de ces entités avait été considéré comme faible, mais devenu problématique et aggravé par le fait que les états n'apportent plus à leurs banques l'assurance d'une garantie totale.

L'importance accordée au risque de contrepartie remonte à la fin des années 1990, lorsque la crise économique asiatique (1997) et le défaut de la Russie (1998) ont révélé la gravité du risque de contrepartie dans les opérations de marché. En particulier, les conséquences de la faillite du fond LTCM [13] montre l'étendue du risque systémique sur le marché des dérivés à travers des mécanismes comme l'effet de levier[14].

Les exemples de LTCM (1998), d'Enron (2001), WorldCom (2002) et Parmalat (2003) ont conduit à l'encadrement du risque de contrepartie par les autorités de régulation, à travers les textes du *Counterparty Risk Management Policy Group* (i.e. CRMPG [CRM08]) et Bâle II (1999) dans le cadre du Pilier 1 qui définit les exigences minimales de fonds propres.

CVA/DVA

Avec le renforcement de la règlementation et la croissance du marché des dérivés à partir de l'année 2000, les banques ont commencé à valoriser le risque de contrepartie sur les opérations OTC, avec une attention renforcée pour les opérations les plus risquées. C'est le début des ajustements CVA/DVA[15] considérés comme une assurance au risque de contrepartie. Le risque de marché inhérent à ces ajustements reste cependant non géré. Cette approche a commencé à changer en 2005 avec le concept de *Fair Value* et de *Exit Price*, énoncé à partir des textes comptables IAS 39 [16] et FAS 157[17], impliquant une prise en compte systématique de CVA et DVA. En particulier, FAS 157 applicable aux banques américaines, requiert l'utilisation des probabilités de défaut de marché à la place des probabilités historiques, et la comptabilisation de la DVA comme un bénéfice.

Les mesures règlementaires et les règles comptables décrites précédemment ainsi que les provisions CVA calculées par les banques n'ont pas permis de résorber les pertes enregistrées à partir de 2007. Les charges dues au risque de contrepartie ont été sous-évaluées afin de limiter les besoins en fonds propres. L'absence de gestion du risque de contrepartie a amplifié les conséquences de la crise.

La définition des ajustements CVA et DVA a été re-précisée par la norme IFRS13[18] entrée

[13] Long-Term Capital Management

[14] Les US$4.7 milliards de capitaux investis par les membres du fonds avaient été multipliés par un levier de 27, et avaient atteint US$129 milliards. Et pour ne rien arranger, une partie de l'argent avait été utilisée pour acheter des produits dérivés, portant les actifs fictifs du fonds à près de US$1 mille milliards

[15] Credit/Debt Valuation Adjustment

[16] Financial Instruments : Recognition and Measurement

[17] Financial Accounting Standard 157 : Fair Value Measurement

[18] International Financial Reporting Standards : C'est le référentiel comptable applicable aux sociétés cotées sur le marché européen.

en vigueur au 31 mars 2013 en remplacement des règles IAS39[19] et FAS157[20]. C'est un complément de valorisation des opérations de marché comptabilisées à la juste valeur. La norme IFRS13 insiste sur l'utilisation maximale de données de marché observables pour la quantification de ces ajustements qui jusqu'alors étaient estimés selon des paramètres de crédit historiques peu volatiles. La notion de prix de sortie (*exit price*) devient une notion centrale pour le calcul des ajustements de valeurs.

À l'heure actuelle, il est devenu clair qu'aucune contrepartie (entités triple-A, banques d'investissement mondiales, banques de détail et banques souveraines) ne peut être considérée comme sans risque. Le risque de contrepartie, précédemment négligé par la présence de garanties ou des hypothèses juridiques, est désormais significatif sur tous les marchés financiers. L'ajustement de la valeur de crédit (CVA) est passé d'un terme règlementaire rarement utilisé à un concept constamment associé à des dérivés. Les banques ont également pris conscience de leurs coûts croissants de financement et de capitalisation de leurs bilans.

FVA

L'hypothèse fondamentale de valorisation avant la crise de 2007 était d'approcher le taux sans risque par le taux LIBOR, un indice de taux d'intérêt publié pour chaque devise reflétant le taux moyen d'emprunt « en blanc »[21] sur le marché interbancaire obtenu à partir d'un échantillon de seize grandes banques établies à Londres. Les spreads ou bases de liquidité nécessaires pour la valorisation au pair des *cross-currency swaps* et des *tenor swaps* sont communément negligés. La Figure 0.0.2 montre que l'écart Euribor-Eonia passe de 10 bps à 200 bps au moment de défaut de Lehman Brothers (10bps à 350 bps pour le spread USD Libor 3M - OIS), les *tenors spread* (Euribor 6x12 dans Figure 0.0.2) ainsi que les *cross-currency spreads* ont réagi de la même manière. Cet écartement explique un accroissement des primes de liquidité et de crédit (voir [Mor11]) et s'est traduit par une segmentation du marché de taux d'intérêt par ténor et par devise.

La première conséquence majeure a été le passage vers une base *multi-courbe* pour la projection des risques de taux en replacement de l'approche classique *mono-courbe*. Cela a montré que même les types de dérivés les plus simples (comme les FRA, voir Morini [Mor13]), qui avaient été évalués de la même manière pendant des décennies, devaient être évalués différemment, de manière plus sophistiquée. La deuxième conséquence est le recours à l'indice OIS pour le taux de référence sans risque pour les transactions adossées à des contrats CSA, c'est à dire des transactions avec perte marginale en cas de défaut. En effet, les contrats CSA rémunèrent le collatéral posté au taux au jour le jour dans la devise de paiement. Piterbarg ([Pit12]) montre que, pour ces contrats, l'indice de paiement des intérêts du collatéral correspond aussi à l'indice d'actualisation. L'actualisation basée sur l'indice OIS est devenue rapidement une référence de marché.

[19] International Accounting Standard 39 : L'IAS 39 est une norme internationale d'information financière faisant partie du corpus des normes IFRS.

[20] Financial Accounting Standards 157 : Règles américaines pour la mesure de la valeur juste (ou *fair value*)

[21] sans que le prêt soit gagé par des titres

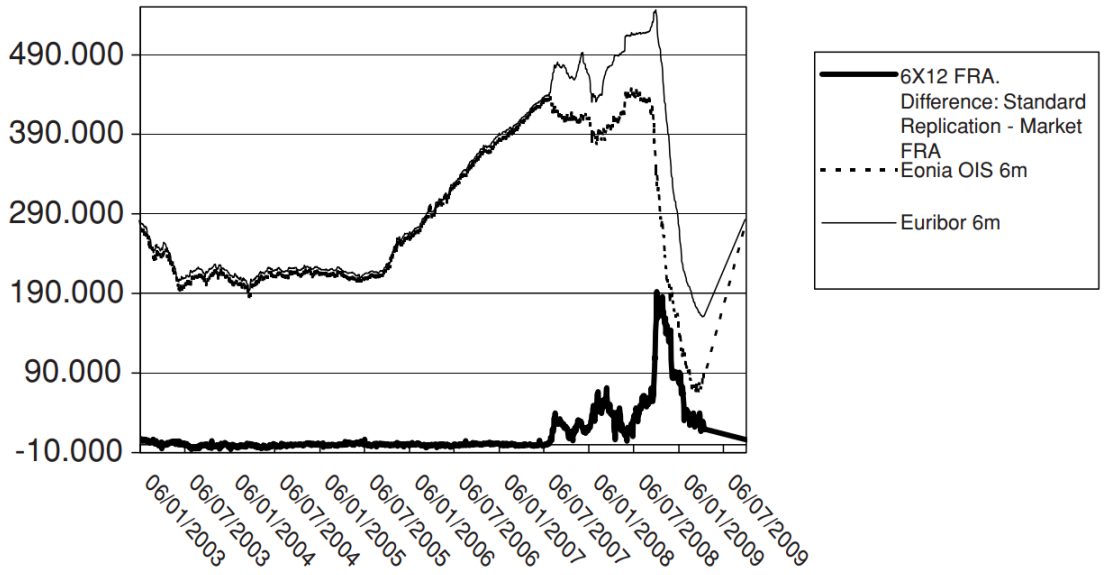


FIGURE 0.0.2 : Spread (bp) Eonia 6M vs. Euribor 6M - Inconsistence de la replication mono-courbe FRA vs. marché (source [Mor11])

ColVA

L'augmentation des coûts de financement a conduit les banques à élever leurs exigences en matière de collatéral. Ceci a affecté les contreparties mal disposées vis à vis de ces exigences pour des raisons opérationnelles ou de liquidité. Le recours à la collatéralisation a fini par être accepté par la plupart des intervenants non seulement pour mitiger les coûts en risque de crédit et de financement auxquels ils font face, mais aussi pour éviter les effets d'un éclatement de leurs spreads provoqués par les couvertures engagés par les banques via des achats massifs de CDS[22]. Pour minimiser le coût de financement, certaines contreparties postent leurs propres obligations en collatéral. Le recours au collatéral partiel ou optionnel nécessite la prise en compte d'un facteur d'actualisation non-conventionnel. Ceci se traduit par un ajustement de valeur du collatéral, ou ColVA pour *Collateral Valuation Adjustment*.

MVA et KVA

Enfin, l'augmentation spectaculaire des besoins en capital conduit à la prise en compte de l'ajustement de valeur en capital (*Capital Valuation Adjustment* - KVA), ainsi que les exigences en financement supplémentaires (*Margin Valuation Adjutment* - MVA) induites par le paiement de marge initiale règlementaire (*Initial Margin*).

Le tableau 0.0.2 donne un comparatif des composantes du prix d'un produit OTC avant et après la crise de 2007, indépendamment de la complexité du produit sous-jacent. La valorisation d'un swap vanilla de taux fait intervenir désormais un modèle probabiliste, ainsi que plusieurs courbes de taux et des courbes de spreads de crédit. Nous soulignons également que la valorisation prend en considération les accords de netting (e.g. ISDA) conclus avec la contrepartie pour agréger l'exposition. Les différents ajustements XVA correspondent donc à des charges incrémentales.

[22]Credit Default Swap : C'est une assurance contre le defaut adossé à un nominal fixe engagé.

Pre-Crisis	Post-Crisis
Risk-neutral price (LIBOR discounting)	Risk-neutral price (OIS discounting)
Hedging costs	Hedging costs
CVA	CVA (DVA?)
Profit	Profit
	FVA (including cost of liquidity buffers)
	KVA (lifetime cost of capital)
	MVA/CCP costs
	TVA (tax on profits/losses)

TABLE 0.0.2 : Composante du prix d'un dérivé avant et après la crise de 2007-2009 (source [Gre15])

La présence des ajustements XVA dépend de la nature de la garantie sous-jacente à la transition. 3 types sont possibles, les composantes de prix inhérents à chaque type sont résumés dans Tableau 0.0.3 :

- **Valorisation non-sécurisée** : En l'absence de garantie, les ajustements XVA sont calculés sur la base de l'espérance de la valeur future du contrat jusqu'à la maturité. Le spread de financement appliqué à la FVA dépend de la courbe de discount choisie pour le prix risque neutre. La KVA intègre le coût de maintien du capital règlementaire dû au risque de contrepartie, pendant la durée de vie de la transaction.
- **Valorisation CSA** : En présence d'un contrat bilatéral CSA, la courbe d'actualisation pour la valorisation risque neutre est la courbe OIS. L'exposition espérée correspond à la valeur future du contrat diminuée de la valeur de collatéral posté. Ceci donne lieu à une exposition marginale due notamment à la fréquence discrète d'appel de marge. Certains contrats de collatéral comportent des clauses non standards pour le calcul de l'exposition comme l'application d'un seuil de collatéral, ou la présence d'un option de choix du collatéral à livrer. Ne pouvant pas être capturé à travers le discount, l'impact sur la valorisation donne lieu à l'ajustement *ColVA*. Un coût de financement supplémentaire est généré par la présence d'*Initial Margin*. Ce coût donne lieu à l'ajustement MVA. Finalement, l'ajustement KVA tient compte du coût du capital règlementaire, même si celui-ci est réduit par la présence de collatéral.
- **Valorisation CCP :** Pour les contrats traités en chambre de compensation, les composantes de prix sont similaires au cas des transactions appuyées par des accords CSA : Présence de CVA/DVA et FVA/ColVA marginales. 3 paiements supplémentaires sont effectués aux CCPs :
  - *Initial margin :* La *marge intiale* est conçue pour couvrir le risque de dérive sur la période d'appel de marge et de constatation du défaut (*cure period*)
  - *Liquidity Buffer* : représente une exigence supplémentaire en liquidité en cas d'une exposition large d'un membre

– *Default Fund* : représente une contribution postée par tous les membres du CCP utilisable en cas de défaut d'un membre.

Unsecured Pricing	CSA Pricing	CCP Pricing
OIS or LIBOR discount	CSA-based (OIS) discount	CCP curves discount
Hedging/management costs	Hedging/Management costs	Hedging/Management costs
CVA	Residual CVA	Residual CVA
DVA	Residual FVA	Residual FVA
FVA	COLVA/Collateral effects	COLVA/Collateral effects
KVA	KVA	KVA
Profit (Tax/TVA)	Bilateral Initial Margin MVA	Initial margin
	Profit	Liquidity buffers
		Default fund
		Profit

TABLE 0.0.3 : Composante du prix d'un dérivé en fonction du mécanisme de garantie

Le tableau 0.0.4 résume la contribution des ajustements XVA en fonction des situations discutées précédemment.

	ColVA	CVA	DVA	FVA	KVA	MVA
Uncollateralised		✓	✓	✓	✓	
One-way collateralised	(✓)	✓	(✓)	✓	✓	
Traditional two-way collateralised	✓	(✓)	(✓)		✓	
Single-currency two-way collateralised	(✓)	(✓)	(✓)		✓	
Collateralised with bilateral initial margin	(✓)				(✓)	✓
Centrally cleared (direct)		(✓)			(✓)	✓
Centrally cleared (indirect)		(✓)			(✓)	✓
Centrally cleared (CCP view)						

TABLE 0.0.4 : Contribution des ajustements XVA (source [Gre15])

L'introduction des ajustements XVA représente un véritable changement de paradigme. La notion de *private valuation* devient dominante en contradiction avec les hypothèses standards de réplication parfaite de Black et Scholes. Le prix d'un dérivé est désormais spécifique aux contreparties qui l'échangent, et reflète au delà du coût de sa construction, le risque de défaut et les coûts de liquidité et de capital.

La régulation visant à rendre les dérivés OTC plus simples et sécurisés a conduit à l'accroissement du nombre des ajustements de prix XVA. Cet accroissement s'accompagne d'une complexification des méthodes de valorisation, une augmentation des risques de modèles et des facteurs inobservables.

Dans ce contexte, ce travail de thèse porte sur la proposition de méthodes numériques pour la valorisation des enveloppes d'exposition d'une part et d'autre part pour la prise en compte du risque de corrélation entre ces enveloppements et les intensités de défaut des contreparties ou les taux de financement. Ces objectifs s'inscrivent dans une logique de construction « bottom to up » de méthodologies de valorisation où nous cherchons à garder les systèmes *pricing* existants et les enrichir par des termes correctifs au titre des nouveaux facteurs propres aux ajustements XVA.

Objectifs et contributions

Mes travaux de recherche ont été organisés autour de deux thématiques :

Dans l'esprit de l'équation de Dupire pour la volatilité locale ([Dup94]), le premier résultat consiste à déterminer une équation qui décrit l'évolution de l'exposition positive (ou négative) espérée en fonction de la date d'observation de l'évènement de défaut. Cette exposition espérée à l'horizon t est ainsi vue comme une option d'achat ou de vente à exercice européen de maturité t et de prix d'exercice 0 pour une position non-collatéralisée. Le sous-jacent de l'option est la valeur agrégée à la date t d'un portefeuille face à une contrepartie. La dérivation analytique dans le cas d'un seul facteur de risque se base sur l'équation de Fokker-Planck couplée à l'équation de Black-Scholes que vérifie la valeur du portefeuille en dehors des dates de discontinuité de tombée de coupons. L'équation obtenue est une équation différentielle ordinaire en t. Elle permet une écriture explicite de l'exposition comme la somme d'une exposition intrinsèque, déduite de la valeur présente du portefeuille, et d'une valeur temps faisant intervenir la volatilité des facteurs de risques sous-jacents et la sensibilité au premier ordre du portefeuille à ces facteurs. La dérivation dans le cas de dimension supérieure repose sur une application de la formule de la Co-aire.

Le deuxième objectif de la thèse est le calcul de l'exposition espérée ajustée du risque de corrélation entre la valeur future du portefeuille et l'intensité de défaut de la contrepartie. Ce risque est communément appelé *Wrong-Way Risk* (WWR). Nous formalisons dans un premier chapitre notre approche dans le cas de la valorisation d'un contrat dont le paiement de flux est conditionnel à la survie d'une entité de référence. Nous nous basons principalement sur l'approche de développements asymptotiques utilisée par Benhamou *et al.* dans [BGM09, BGM10a, BGM10b]. En particulier, nous appliquons cette approche au cas de la diffusion de l'intensité de défaut, sans changer la diffusion des facteurs de risque sous-jacents à la valeur du payoff non risqué. Ceci représente une différence majeure avec les travaux de Benhamou *et al.* que nous illustrons notamment dans l'étude de l'erreur numérique. Le prix du dérivé risqué est alors obtenu par le prix du dérivé non risqué, auquel s'additionne une somme de termes correctifs faisant intervenir les « Grecs » du prix non risqué. En supposant que le cas du dérivé non risqué est un cas standard pour lequel le prix et les grecs sont données par des algorithmes existants, notre approche permet de traiter le cas du dérivé risqué, ou hybride, et d'appréhender, via des ajustements explicites, la valorisation en présence d'un facteur de défaut supplémentaire corrélé aux facteurs existants.

L'approche développée dans le cas d'un dérivé contingent au crédit est ensuite appliquée au cas des ajustements CVA et DVA. L'exposition espérée en présence du WWR correspond à l'exposition sans WWR à laquelle s'ajoute une combinaison explicite de sensibilités. Par des

arguments de symétrie, nous étendons ce principe au cas des ajustements de financement FCA et FBA afin de valoriser le risque de corrélation entre les assiettes de financement et les enveloppes d'expositions. Le résultat final est formulé comme une exposition espérée ajustée par le risque WWR (WWR-EPE). Le cas du risque bilatéral, qui introduit la contingence aux évènements de défaut de deux contreparties, est également traité. L'ajustement WWR obtenu est la somme des deux ajustements WWR pour chaque contrepartie, auquel s'ajoute un dernier terme correctif reflétant la corrélation entre les deux intensités de défaut.

Un autre aspect du risque WWR est lié aux contrats de *Rating Triggers* qui introduisent une dépendance entre les expositions et les évènements de migrations de rating. Les ajustements CVA/DVA expriment alors la perte en cas de défaut à partir d'un rating où l'exposition est sous-collatéralisée (i.e. risque de *gap*). La valorisation de ces contrats nécessite l'introduction d'un modèle risque neutre de migration de rating, c'est à dire, une chaine de Markov pour les transitions de rating, tel que la probabilité de transition terminale vers le défaut est égale à la probabilité de défaut implicite de marché[23]. Partant d'une matrice de transition historique, nous proposons une approche opérationnelle pour la calibration d'un modèle de transition risque neutre dans les cas d'un risque unilatéral ou bilatéral. Nous montons que le calage obtenu aux probabilités de marché est très satisfaisant, et nous illustrons l'utilisation de ce modèle par le calcul des expositions espérées ajustées au Wrong-Way Risk.

Organisation de la thèse

Le mémoire est constitué de deux parties :

La première partie est introductive. Nous présentons dans chapitre 1 les définitions mathématiques des ajustements XVA et soulignons l'importance de leurs impacts en fonction des types de garanties adossées aux contrats. Nous dédions la dernière section du chapitre 1 à une description des pratiques de gestion du risque de contrepartie ainsi qu'au rôle opérationnel d'un desk de trading centralisateur, dit desk XVA. Dans le chapitre 2, nous exposons une revue succincte des principales méthodes numériques adressées dans la littérature pour la valorisation des enveloppes d'expositions.

La deuxième partie est consacrée aux contributions développées au cours de cette thèse. Le chapitre 3 présente notre résultat sur l'équation différentielle ordinaire en temps que vérifie l'exposition espérée, donnant lieu à un calcul *forward* de l'exposition par un schéma incrémental. Les 3 chapitres suivants sont consacrés aux développements d'approximations stochastiques pour la prise en compte du risque de corrélation WWR. Le chapitre 4 restitue les hypothèses de dérivation et les résultats de cette approche dans le cadre de la valorisation d'un produit contingent au défaut d'une entité de référence. Les principaux théorèmes (4.3.9 pour l'approximation à l'ordre 2 et 4.3.14 pour l'ordre 3) sont ensuite transposés dans le chapitre 5 au cadre des ajustements XVA unilatéraux. Nous donnons dans le chapitre 6 une généralisation au cadre bilatéral, où l'exposition au risque de contrepartie et de financement est conditionnée par l'évolution des intensités de défauts des deux contreparties à la fois. Ces intensités sont elles-mêmes corrélées.

Nous clôturons cette thèse par le chapitre 7 où nous abordons le risque WWR spécifique sous l'angle des contrats de *Rating Triggers*. Nous présentons notre approche pour la calibration d'un

[23] Calculé à partir d'instrument de marché, i.e. CDS ou Bond

modèle de transition de rating aux probabilités de marché.

Information Auteur

Marouan IBEN TAARIT
Recherche quantitative XVA, Global Markets
Natixis, Banque de Grande Clientièle
47 quai Austerlitz, 75013, Paris France
marouan.ibentaarit@natixis.com

Une introduction aux ajustements de valeur XVA et à la collatéralisation

« Credit is a system whereby a person who can not pay gets another person who can not pay to guarantee that he can pay. »

(Charles Dickens, 1812-1870)

Chapitre 1

Introduction aux ajustements XVA

Depuis la crise de 2007-2008, la valorisation des opérations de dérivés a évolué, notamment pour prendre en compte la notion de risque de contrepartie qui était largement sous-estimée.

La première évolution a été la modification des courbes d'actualisation pour tenir compte des caractéristiques des conventions cadre (*netting agreements*). En effet, le développement simultané des chambres de compensation et la systématisation de la mise en place d'annexe de remise en garantie (*Credit Support Annex* – CSA »), a eu pour conséquence une adaptation du marché, se traduisant par un changement de courbe d'actualisation afin de refléter la rémunération du collatéral déposé en liquide (généralement basés sur EONIA pour l'EUR, OIS pour les autres devises).

La faillite de la banque Lehman Brothers a rappelé au marché la nécessité d'inclure le risque de contrepartie dans la valorisation des dérivés. Ce risque est bilatéral, il existe un risque lié au risque de défaut de la contrepartie (*Credit Valuation Adjustment* - CVA) et un risque lié au défaut de la banque (*Debit Valuation Adjustment* - DVA). L'impact global appelé BVA (pour « bilateral CVA ») est donc égal à DVA – CVA et permet toujours de trouver un point d'équilibre dans le marché, la contrepartie calculant l'ajustement symétrique au calcul de BVA, les prix s'équilibrent des deux côtés.

La dernière étape de cette évolution est l'introduction d'une notion de liquidité (*Funding Valuation Adjustment* - FVA) pour différencier les transactions collatéralisées des opérations sans collatéral en termes de coût de financement futur.

Les changements de fonctionnement pour le marché OTC ont été initiés par les banques elles-mêmes afin de gérer correctement leurs risques de crédit et coûts de financement. Des mesures règlementaires ont ensuite renforcé les pratiques et les dispositifs de gestion, émanant de Bâle III, des banques centrales ou des autorités de régulation. Deux changements sont majeurs : D'abords, le recours de plus en plus fréquent au collatéral et aux chambres de compensation dans un souci de plus de transparence dans les opérations. Ensuite, pour les échanges partiellement ou totalement non-collatéralisés, l'intégration des ajustements de prix dans la juste valeur et les mesures économiques du risque de contrepartie. Ces ajustements répondent à une pléthore d'acronymes CVA, DVA, FVA, LVA, RVA, MVA, KVA ... ayant pour terminaison commune les lettres VA pour *Valuation Adjustment.*

1.1 Les ajustements XVA

Les ajustements XVA permettent d'ajuster le prix standard non risqué afin de tenir compte de facteurs de risque additionnels comme le crédit et le financement

$$\text{Prix risqué} = \text{Prix non risqué} + \sum_i XVA_i$$

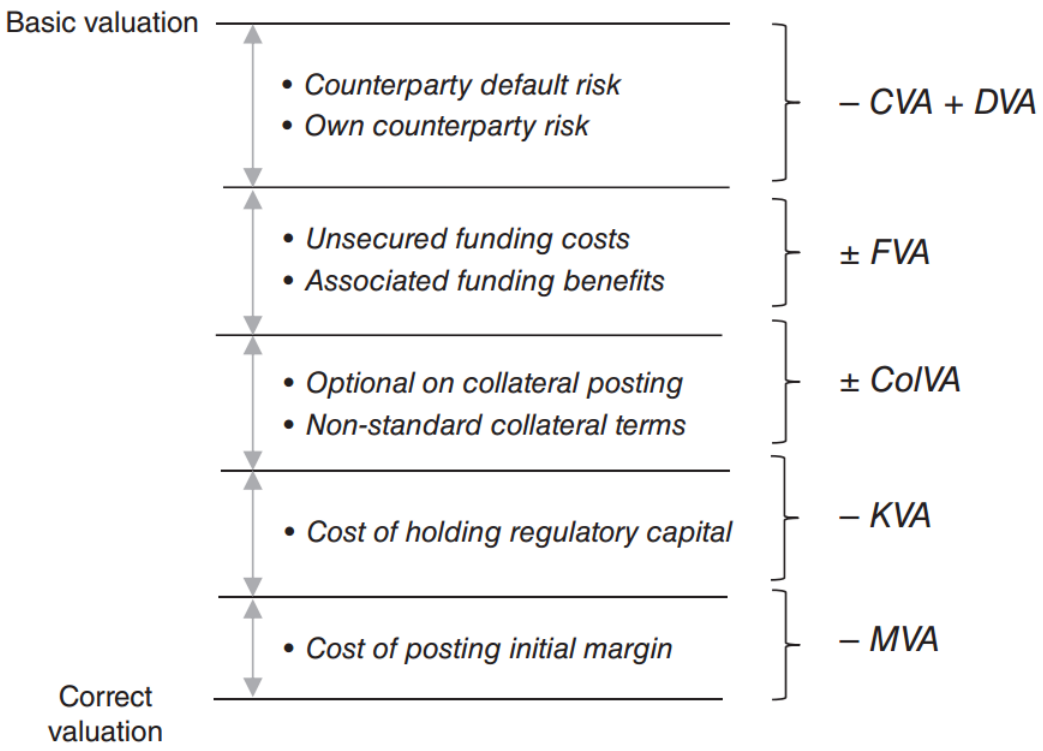


FIGURE 1.1.1 : Rôle des ajustements XVA

1.1.1 Les ajustements aux risques de contrepartie

Les ajustements CVA et DVA mesurent le risque bilatéral de défaut de l'une des deux contreparties avant la maturité du contrat. Le risque de perte correspond à la mesure algébrique de l'exposition au moment du défaut. En particulier l'exposition est nulle si la contrepartie en défaut est préteuse.

Nous nous plaçons du point de vue de l'investisseur I. L'exposition positive espérée (EPE) mesure la perte en t en cas de défaut d'une contrepartie C à un horizon futur T :

$$EPE(t,T) \triangleq (1-R_C)\,\mathbb{E}\left[\max\left(V(T),0\right)|\mathscr{F}_t\right] \tag{1.1.1}$$

L'espérance négative espérée (ENE) mesure la perte pour C en cas de défaut de I

$$ENE(t,T) \triangleq (1-R_I)\,\mathbb{E}\left[\max\left(-V(T),0\right)|\mathscr{F}_t\right] \tag{1.1.2}$$

On dénote par $(\mathscr{G}_t)_t$ la filtration représentant l'ensemble de l'information de marché et par $\mathbb{Q}$ l'unique mesure risque neutre. L'espace $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbb{Q})$ est doté d'une sous-filtration complète et continue à droite $(\mathscr{F}_t)_t$ représentant l'ensemble de l'information de marché à horizon t excluant les évènements de défaut : $\mathscr{F}_t \subseteq \mathscr{G}_t \triangleq \mathscr{F}_t \vee \mathscr{H}_t^I \vee \mathscr{H}_t^C$ avec $\mathscr{H}_t^k = \sigma(\{\tau_k \leq u\} : u \leq t)$, τ_I et τ_C étant respectivement les temps de défaut de I et C.

V est la valeur signée de l'exposition. R_I et R_C sont les pourcentages de recouvrement en cas de défaut resp. de I et C. On notera

$$LGD_I = 1 - R_I \tag{1.1.3}$$
$$LGD_C = 1 - R_C \tag{1.1.4}$$

L'ajustement CVA est le prix de marché du risque de crédit sous-jacent à une transaction non garantie. Il peut être défini comme la différence entre

- le prix de l'instrument sans risque de contrepartie
- le prix de l'instrument traité avec possibilité de défaut de C et paiement du recouvrement proportionnel au moment du défaut

$$CVA = \text{Prix non risqué - Prix risqué} \geq 0$$

L'ajustement DVA est symétrique à l'ajustement CVA du point de vue de la contrepartie C. Cet ajustement est comptabilisé en tant que gain pour I. Le gain est d'autant plus grand que la détérioration du rating de I et la hausse de son spread de crédit. Ce comportement controversé a soulevé un débat sur la démonétisation de la DVA et sur sa couverture.

Il convient de noter la nature bilatérale du risque de contrepartie. Les enveloppes d'exposition positive et négative sont conditionnés par les évènements joints de défaut et de survie des deux contreparties.

$$CVA + DVA = \text{Prix non risqué - Prix risqué}$$

Le terme CVA augmente le taux d'actualisation par le risque de défaut de C alors que le terme DVA le diminue.

Les conventions de calcul de valeur en cas défaut, valeur de clôture ou *Close-out value*, jouent un rôle significatif pour déterminer l'exposition et donc l'ajustement $CVA + DVA$. Plusieurs conventions sont possibles :

- **Clôture à la valeur non risquée (Risk-free close-out) :** La valeur de clôture de transaction est déterminée en cas de défaut par la valeur du portefeuille sans prise en compte du risque de contrepartie. C'est la convention la plus simple à implémenter car la valeur du portefeuille correspond à la somme des valeurs des transactions éligibles aux accords de compensations mis en place entre les deux contreparties.
- **Clôture à la valeur risquée (Risky close-out) :** La valeur de clôture du portefeuille est sa valeur immédiatement avant l'avènement du défaut, y compris tous les ajustements de valeur. Ces ajustements, déterminés par la contrepartie non défaillante, pourraient inclure la CVA, DVA et FVA, immédiatement avant le défaut. Une valeur totale négative des ajustements XVA réduit le montant final de recouvrement.

- **Clôture à la valeur de remplacement (Replacement close-out) - DVA+FVA :** Dans le modèle de CVA bilatérale, il est possible d'ignorer la CVA et de comptabiliser la DVA. Ceci se justifie par la prise en compte de l'ajustement CVA par la nouvelle contrepartie reprenant la transaction, correspondant à la DVA de la contrepartie non défaillante. La prise en compte de la DVA uniquement augmente le montant de recouvrement. Cette convention de clôture correspond au protocole ISDA : "*quotations (either firm or indicative) for replacement transactions supplied by one or more third parties that may take into account the creditworthiness of the Determining Party at the time of the quotation*" (ISDA, [ISD09]). Le protocole de l'ISDA permet également d'inclure des coûts de financement FVA en plus de la DVA.

Nous donnons dans la suite une dérivation par « espérance » des ajustements CVA et DVA. Cette approche se base sur la comptabilisation des flux futurs, elle est communément utilisée dans un cadre Monte Carlo qu'on décrit dans le chapitre suivant. Nous nous référons principalement aux dérivations de [BMP13] (aussi pour les ajustements FVA, ColVA). Une autre approche se basant sur des arguments de réplication à l'instar du modèle de Black-Scholes est proposée par Burgard et Kjaer [BK11] pour CVA/DVA et [BK12] pour FVA. Nous renvoyons le lecteur à ces deux références ainsi qu'au livre d'Andrew Green [Gre15] pour une revue plus détaillée.

On considère donc un portefeuille de transactions entre I et C non collatéralisé, générant des flux $\Pi(t,T)$ entre t et la maturité finale T en l'absence d'évènements de défaut. Le paiement de Π peut être optionnel ou contient des clauses contractuelles. La valeur risquée contingente au défaut de C s'écrit

$$\hat{V}(t) = V(t)\,1_{t<\tau_C} + V_{Recovery,C}(t)\,1_{t\geq\tau_C}$$

avec

$$\begin{cases} V(t) & = \mathbb{E}\left[\Pi(t,\tau_C)\,1_{\tau_C<T} + \Pi(t,T)\,1_{\tau_C\geq T}|\mathscr{G}_t\right] \\ V_{Recovery,C}(t) & = \mathbb{E}\left[D(t,\tau_C)\left(R_C\left(V(\tau_C)\right)^+ + \left(V(\tau_C)\right)^-\right)|\mathscr{G}_t\right] \end{cases}$$

D étant le facteur d'actualisation. En particulier, la valeur de clôture en cas de défaut est $V(\tau_C)$ correspondant au cas *Risk-free close-out.* La valeur du portefeuille s'écrit alors

$$\begin{aligned} \hat{V}(t) &= \mathbb{E}\left[\Pi(t,T)\,1_{\tau_C\geq T} + \Pi(t,\tau_C)\,1_{\tau_C<T} + D(t,\tau_C)\left\{R_C\left(V(\tau_C)\right)^+ + \left(V(\tau_C)\right)^-\right\}1_{\tau_C<T}|\mathscr{G}_t\right] \\ &= \mathbb{E}\left[\Pi(t,T)\,1_{\tau_C\geq T} + \left(\Pi(t,\tau_C) + D(t,\tau_C)\,\Pi(\tau_C,T)\right)1_{\tau_C<T} + LGD_C\left(V(\tau_C)\right)^+ 1_{\tau_C<T}|\mathscr{G}_t\right] \\ &= \mathbb{E}\left[\Pi(t,T)\,1_{\tau_C\geq T} + \Pi(t,T)\,1_{\tau_C<T} - LGD_C\left(V(\tau_C)\right)^+ 1_{\tau_C<T}|\mathscr{G}_t\right] \\ &= V(t) - \mathbb{E}\left[D(t,\tau_C)\,LGD_C\left(V(\tau_C)\right)^+ 1_{\tau_C<T}|\mathscr{G}_t\right] \end{aligned}$$

Par symétrie, nous déduisons les définitions suivantes des ajustements CVA et DVA unilatéraux :

$$UCVA(t) \triangleq \mathbb{E}\left[D(t,\tau_C)\,LGD_C\left(V(\tau_C)\right)^+ 1_{\tau_C<T}|\mathscr{G}_t\right] \tag{1.1.5}$$

$$UDVA(t) \triangleq \mathbb{E}\left[D(t,\tau_I)\,LGD_I\left(V(\tau_I)\right)^- 1_{\tau_I<T}|\mathscr{G}_t\right] \tag{1.1.6}$$

Si on se place sous l'hypothèse standard d'indépendance entre le risque de crédit et l'évolution de V ($\tau_k \perp V$ pour $k \in \{I,C\}$), alors on a

$$UCVA(t) = \int_t^T \lambda_C(s)\, e^{-\int_0^s \lambda_C(u)du} \mathbb{E}\left[D(t,s)\, LGD_C\, (V(s))^+ |\mathscr{F}_t\right] ds \tag{1.1.7}$$

$$UDVA(t) = \int_t^T \lambda_I(s)\, e^{-\int_0^s \lambda_I(u)du} \mathbb{E}\left[D(t,s)\, LGD_I\, (V(s))^- |\mathscr{F}_t\right] ds \tag{1.1.8}$$

avec λ_I et λ_C les intensités de défaut déterministes respectivement pour I et C. L'approche standard consiste à approximer les deux intégrales par

$$UCVA(t) = \sum_{i=1}^N \mathbb{Q}(\tau_C \in]T_{i-1}, T_i]) \mathbb{E}\left[D(t,T_i)\, LGD_C\, (V(T_i))^+ |\mathscr{F}_t\right] \tag{1.1.9}$$

$$UDVA(t) = \sum_{i=1}^N \mathbb{Q}(\tau_C \in]T_{i-1}, T_i]) \mathbb{E}\left[D(t,T_i)\, LGD_I\, (V(T_i))^- |\mathscr{F}_t\right] \tag{1.1.10}$$

avec $t = T_0 < T_1 < \cdots < T_N = T$. En présence d'appel de marge, les expositions positives et négatives prennent compte le collatéral posté dont le montant à une date T_i est représenté par $X_{k\in\{I,C\}}$:

$$UCVA(t) = \sum_{i=1}^N \mathbb{Q}(\tau_C \in]T_{i-1}, T_i]) \mathbb{E}\left[D(t,T_i)\, LGD_C\, (V(T_i) - X_C(T_i))^+ |\mathscr{F}_t\right] \tag{1.1.11}$$

$$UDVA(t) = \sum_{i=1}^N \mathbb{Q}(\tau_C \in]T_{i-1}, T_i]) \mathbb{E}\left[D(t,T_i)\, LGD_I\, (V(T_i) - X_I(T_i))^- |\mathscr{F}_t\right] \tag{1.1.12}$$

Dans les cas usuels, $X_{k\in\{I,C\}}(t) = g_k\left(t, (V(u))_{u\in[0,t]}\right)$ où g_I et g_C sont déterminés par les accords de collatéral.

Dans le cas d'un contrat d'échange, il convient d'inclure les flux de recouvrement au défaut des deux contreparties. On parle dans ce cas d'ajustements CVA et DVA bilatéraux. La valeur risquée $\hat{V}$ du portefeuille en t est représentée par

$$\begin{aligned}\hat{V}(t) &= \mathbb{E}[\Pi(t, \tau_I \wedge \tau_C)\, 1_{\tau_I \wedge \tau_C < T} + \Pi(t,T)\, 1_{\tau_I \wedge \tau_C \geq T} |\mathscr{G}_t] \\ &+ \mathbb{E}\left[\left\{D(t, \tau_I \wedge \tau_C) R_C (V(\tau_I \wedge \tau_C))^+ + (V(\tau_I \wedge \tau_C))^-\right\} 1_{\tau_C < \tau_I < T} |\mathscr{G}_t\right] \\ &+ \mathbb{E}\left[\left\{D(t, \tau_I \wedge \tau_C) R_I (V(\tau_I \wedge \tau_C))^+ + (V(\tau_I \wedge \tau_C))^-\right\} 1_{\tau_I < \tau_C < T} |\mathscr{G}_t\right]\end{aligned}$$

Les deux derniers termes correspondent aux recouvrements en cas de défaut de C (avant I) ou de I (avant C). Comme dans le cas unilatéral, on a

$$\begin{aligned}&\mathbb{E}\left[\left\{D(t, \tau_I \wedge \tau_C) R_C (V(\tau_I \wedge \tau_C))^+ + (V(\tau_I \wedge \tau_C))^-\right\} 1_{\tau_C < \tau_I < T} |\mathscr{G}_t\right] \\ &= \mathbb{E}[D(t, \tau_I \wedge \tau_C)\, \Pi(\tau_I \wedge \tau_C, T)\, 1_{\tau_C < \tau_I < T} |\mathscr{G}_t] \\ &+ \mathbb{E}\left[D(t, \tau_I \wedge \tau_C)\, LGD_C\, (V(\tau_I \wedge \tau_C))^+ 1_{\tau_C < \tau_I < T} |\mathscr{G}_t\right]\end{aligned}$$

et

$$\begin{aligned}&\mathbb{E}\left[\left\{D(t, \tau_I \wedge \tau_C) R_I (V(\tau_I \wedge \tau_C))^+ + (V(\tau_I \wedge \tau_C))^-\right\} 1_{\tau_I < \tau_C < T} |\mathscr{G}_t\right] \\ &= \mathbb{E}\left[D(t, \tau_I \wedge \tau_C)\, LGD_I\, (V(\tau_I \wedge \tau_C))^+ 1_{\tau_I < \tau_C < T} |\mathscr{G}_t\right] \\ &+ \mathbb{E}[D(t, \tau_I \wedge \tau_C)\, \Pi(\tau_I \wedge \tau_C, T)\, 1_{\tau_I < \tau_C < T} |\mathscr{G}_t]\end{aligned}$$

Au final, nous obtenons

$$\hat{V}(t) = V(t) + \mathbb{E}\left[D(t,\tau_I \wedge \tau_C)\, LGD_C\left(V(\tau_I \wedge \tau_C)\right)^+ 1_{\tau_C<\tau_I<T} | \mathscr{G}_t\right] + \mathbb{E}\left[D(t,\tau_I \wedge \tau_C)\, LGD_I\left(V(\tau_I \wedge \tau_C)\right)^+ 1_{\tau_I<\tau_C<T} | \mathscr{G}_t\right]$$

L'ajustement CVA bilatéral est défini par

$$BVA(t) \triangleq \underbrace{\mathbb{E}\left[D(t,\tau_C)\, LGD_C\left(V(\tau_C)\right)^+ 1_{\tau_C<\tau_I<T} | \mathscr{G}_t\right]}_{BCVA} + \underbrace{\mathbb{E}\left[D(t,\tau_I)\, LGD_I\left(V(\tau_I)\right)^- 1_{\tau_I<\tau_C<T} | \mathscr{G}_t\right]}_{BDVA} \quad (1.1.13)$$

faisant intervenir la loi jointe des temps de défaut (τ_I, τ_C). Un modèle de copule de type gaussienne ou de Marsall-Olkin représente l'approche usuelle de marché pour introduire une dépendance entre ces temps de défaut. Pour finir, l'hypothèse d'indépendance entre les temps de défaut et V permet d'écrire BVA sous forme intégrale

$$BVA(t) = \int_t^T e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \lambda_C(s)\, \mathbb{E}\left[D(t,s)\, LGD_C\left(V(s)\right)^+ | \mathscr{F}_t\right] ds + \int_t^T e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \lambda_I(s)\, \mathbb{E}\left[D(t,s)\, LGD_I\left(V(s)\right)^- | \mathscr{F}_t\right] ds \quad (1.1.14)$$

pouvant être approché par

$$BVA(t) = \sum_{i=1}^{N} \mathbb{Q}\left(\tau_C \in]T_{i-1}, T_i], \tau_I > T_i\right) \mathbb{E}\left[D(t,T_i)\, LGD_C\left(V(T_i)\right)^+ | \mathscr{F}_t\right] + \sum_{i=1}^{N} \mathbb{Q}\left(\tau_I \in]T_{i-1}, T_i], \tau_C > T_i\right) \mathbb{E}\left[D(t,T_i)\, LGD_I\left(V(T_i)\right)^- | \mathscr{F}_t\right] \quad (1.1.15)$$

Cette quantité peut être positive ou négative en fonction des niveaux d'intensité de I et C et des expositions positives et négatives.

1.1.2 Les ajustements aux coûts de financement

L'ajustement FVA est lié aux coûts de financement générés par les transactions OTC :

- Non collatéralisées : Les transactions non collatéralisées génèrent des coûts et des bénéfices de financement. C'est le cas aussi du collatéral partiel (e.g. avec seuil d'appel de marge) ou du collatéral asymétrique (posté par une seule contrepartie)

- Avec collatéral ségrégué ou non ré-hypothecable : Le coût de financement est aussi lié à la capacité d'une contrepartie à utiliser le collatéral reçu comme une source de financement. Le collatéral ségrégué est une source de coût de financement, car non compensé par le collatéral reçu.

Dans une approche standard, l'ajustement FVA peut être expliqué par la présence d'un portefeuille de couverture parfaitement collatéralisé (CSA) face à la transaction OTC. Quand la valeur de celle-ci est positive pour I, la valeur du portefeuille de couverture est négative engendrant des appels de marges, et donc un coût de financement de ce collatéral. Quand la valeur de la transaction est négative, celle du portefeuille de couverture est positive, générant un bénéfice de financement. L'ensemble de ces flux est résumé dans Figure 1.1.2b. En particulier, il en résulte que le coût/bénéfice de financement pour une transaction parfaitement collatéralisée est nulle (Figure 1.1.2a). Le collatéral posté est rémunéré à un taux de collatéral, alors qu'il est financé par des emprunts à blanc sur le marché. Si le collatéral est re-hypothécable, le collatéral reçu face à une contrepartie peut être utilisé pour poster du collatéral face à une autre.

En ce basant sur ces arguments, une définition simple de l'ajustement FVA est similaire à celle des ajustements CVA et DVA :

$$FVA(t) = FCA(t) + FBA(t)$$

avec

$$FCA(t) = \mathbb{E}\left[\int_t^{\tau_I \wedge \tau_C} D(t,s)\, s_B(s)\, (V(s))^+ ds | \mathscr{G}_t\right] \tag{1.1.16}$$

$$FBA(t) = \mathbb{E}\left[\int_t^{\tau_I \wedge \tau_C} D(t,s)\, s_L(s)\, (V(s))^- ds | \mathscr{G}_t\right] \tag{1.1.17}$$

où s_B et s_L représentent le spread instantané d'emprunt et de prêt par rapport à taux d'actualisation. Dans le cas d'indépendance entre $s_{B,L}$, V , τ_I et τ_C, on a

$$FCA(t) = \int_t^T s_B(s)\, e^{-\int_0^s (\lambda_I + \lambda_C)(u)} \mathbb{E}\left[D(t,s)\, (V(s))^+ | \mathscr{F}_t\right] ds \tag{1.1.18}$$

$$FBA(t) = \int_t^T s_L(s)\, e^{-\int_0^s (\lambda_I + \lambda_C)(u)} \left[D(t,s)\, (V(s))^- | \mathscr{F}_t\right] ds \tag{1.1.19}$$

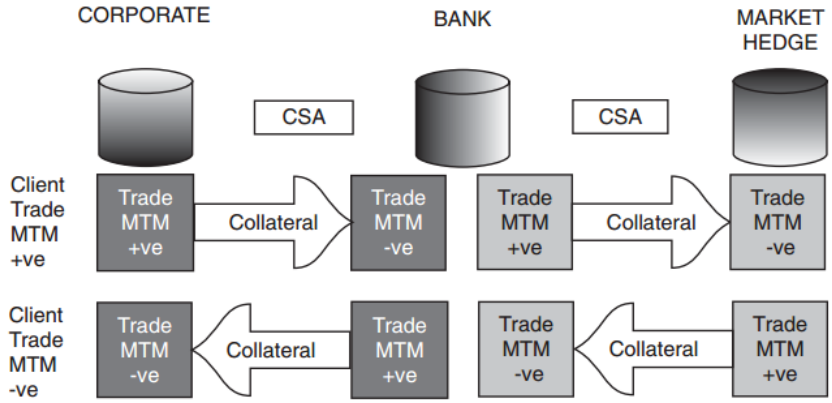
ou encore

$$FCA(t) = \sum_{i=1}^N \mathbb{Q}(\tau_I \wedge \tau_C > T_i)\, f_{s_B}(t, T_{i-1}, T_i)\, \mathbb{E}\left[D(t,T_i)\, (V(T_i))^+ | \mathscr{F}_t\right] \tag{1.1.20}$$

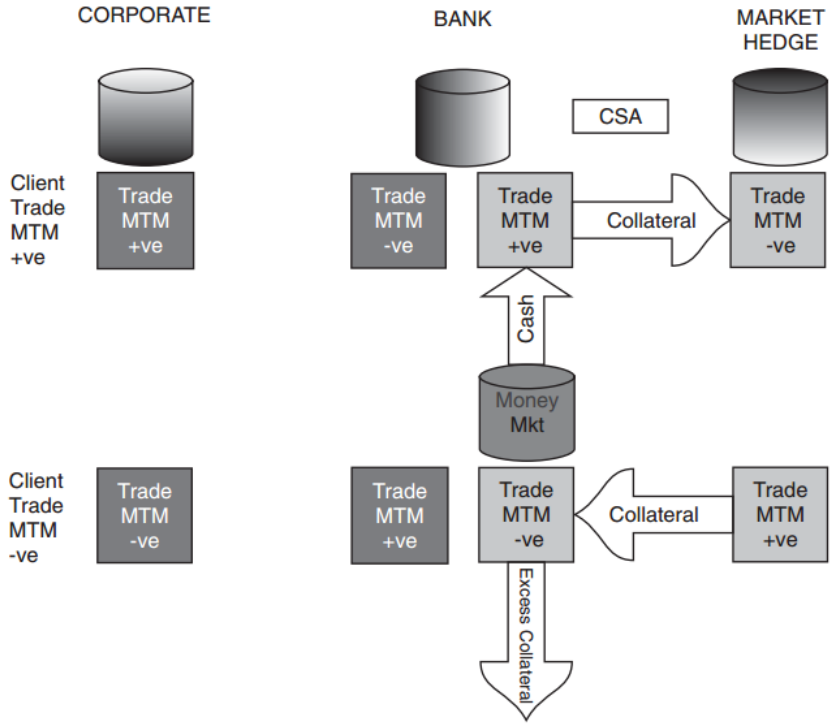
$$FBA(t) = \sum_{i=1}^N \mathbb{Q}(\tau_I \wedge \tau_C > T)\, f_{s_L}(t, T_{i-1}, T_i)\, \mathbb{E}\left[D(t,T_i)\, (V(T_i))^- | \mathscr{F}_t\right] \tag{1.1.21}$$

avec f_{s_B} et f_{s_L} les taux forward instantanés associés à s_B et s_L.

Avec l'avènement du discounting OIS, les premières tentatives de prise en compte de l'ajustement FVA consistaient à actualiser les flux avec une courbe OIS pour les transactions collatéralisées et une courbe Libor ou Libor + spread pour les opérations non garanties. Cette approche est désormais jugée insuffisante car elle ne tient pas compte du risque de crédit, et elle suppose une symétrie entre les coûts de prêt d'emprunt, i.e. $f_{s_B} = f_{s_L}$. D'autre part, elle ne permet pas de traiter les cas de collatéral partiel ou asymétrique.



(a) Transaction collatéralisée : Financement des appels de marge du portefeuille de couverture par le collatéral reçu



(b) Transaction non collatéralisée : Financement d'appels de marge du portefeuille de couverture sur le marché monétaire

FIGURE 1.1.2 : Financement et placement d'appels de marge vs. couverture sur le marché standard (source [Gre15])

Dans le cas d'une intensité de défaut constante λ_I, nous avons la relation suivante λ_I et le spread CDS de I, $s_{CDS,I}$

$$\lambda_I = \frac{s_{CDS,I}}{LGD_I}$$

On peut écrire alors le deuxième terme de DVA dans (1.1.14) comme

$$\int_t^T e^{-\int_0^s \lambda_C(u)+\lambda_I(u)du} \lambda_I \mathbb{E}\left[D(t,s)\, LGD_I\, (V(s))^- \,|\mathscr{F}_t\right] ds$$
$$= \int_t^T e^{-\int_0^s \lambda_C(u)+\lambda_I(u)du} s_{CDS,I} \mathbb{E}\left[D(t,s)\, (V(s))^- \,|\mathscr{F}_t\right] ds$$

Sous cette approximation, la seule différence entre *FBA* (1.1.19) et *BDVA* est la différence entre le spread de financement $s_{B,L}$, i.e. spread d'émission de bonds non garantis, et le spread de crédit, i.e. spread du CDS sur I $s_{CDS,I}$. $s_{CDS,I}$ représente une composante crédit de $s_{B,L}$, à laquelle s'ajoute une composante de financement qui correspond à la base Bond-CDS (cf. Figure 1.1.3) : comptabiliser les deux ajustements *FBA* et *BDVA* à la fois conduit à un double comptage de $s_{CDS,I}$.

Afin d'éviter le double comptage *FBA* et *BDVA*, une solution possible consiste à calculer *FBA* en prenant pour s_L la base Bond-CDS alors que *FCA* est calculé avec le spread de financement s_B entier. Il convient tout de même noter que cette base ne donne pas exactement le coût de financement :

- Les spread crédit du bond, ou Z-spread, sont composés d'autres éléments de liquidité, et dépendent de la taille de l'émission, la maturité, le segment de marché, ...
- Les spread de CDS ne sont pas suffisamment liquides et peuvent contenir une composante de liquidité due à la couverture de CVA notamment.

Le bénéfice de financement *FBA* est un gain réalisé en P&L contrairement à *BDVA* qui doit cependant être comptabilisé pour répondre aux prescriptions des normes comptables IFRS. L'approche de marché consiste à calculer l'ajustement *FBA* avec un spread de financement totale au delà de l'OIS, et de ne comptabiliser que l'écart $FBA - DVA$ en plus de *DVA*.

L'ajustement ColVA regroupe les coûts et bénéfices dus aux optionalités et clauses de collatéral qui différent de ceux d'un contrat de collatéral parfait (CSA) :

1. Différence en taux de rémunération du collatéral posté ou reçu : OIS $\pm$ spread
2. Possibilité un cap ou d'un floor à 0 sur le taux de collatéral : e.g. Eonia flooré à 0
3. Option de choix sur le collatéral à poster/recevoir : devise du collatéral, collatéral en cash vs. en titre, et application d'un *haircut.*

Le premier cas est le plus simple car il se réduit à appliquer un spread sur la courbe de rémunération standard. Les deux cas suivants sont plus complexes. La détermination d'une courbe de collatéral synthétique nécessite le recours à des méthodes numériques. Étant donnée cette

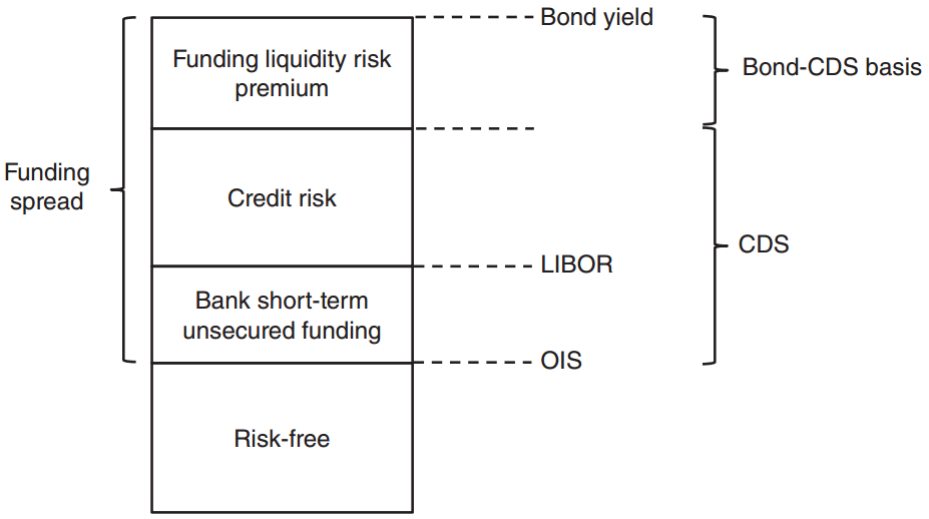


FIGURE 1.1.3 : Composante du spread de financement vs. le spread de CDS

courbe de spread par rapport à l'OIS de rémunération s_{ColCA} pour le collatéral reçu et s_{ColBA} pour le collatéral posté, l'ajustement *ColVA* s'écrit

$$ColVA(t) = ColCA + ColBA(t)$$

avec

$$ColCA(t) = \int_t^T s_{ColCA}(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\,(X_C(s))^+ \,|\mathscr{F}_t\right] ds \tag{1.1.22}$$

$$ColBA(t) = \int_t^T s_{ColBA}(t,s,s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\,(X_I(s))^- \,|\mathscr{F}_t\right] ds \tag{1.1.23}$$

où X_k représente la valeur du collatéral posté par $k \in \{I,C\}$.

$$ColCA(t) = \sum_{i=1}^{N} f_{s_{ColCA}}(t,T_{i-1},T_i)\,\mathbb{Q}(\tau_I \wedge \tau_C > T)\,\mathbb{E}\left[D(t,u)\,(X_C(u))^+ \,|\mathscr{F}_t\right] \tag{1.1.24}$$

$$ColBA(t) = \sum_{i=1}^{N} f_{s_{ColBA}}(t,T_{i-1},T_i)\,\mathbb{Q}(\tau_I \wedge \tau_C > T)\,\mathbb{E}\left[D(t,u)\,(X_I(u))^- \,|\mathscr{F}_t\right] \tag{1.1.25}$$

après discrétisation de l'intégrale.

La principale source de FVA provient des positions non ou partiellement collatéralisées. Cependant, il existe d'autres sources de coûts :

- Coûts de financement face aux chambres de compensations : En plus des marges de variation standards (*Variation Margins*)
  - Marge initiale *(Initial Margin) :* Il s'agit d'un coussin de collatéral posté initialement, indépendamment du sens de l'exposition, et dont le rôle est de couvrir toute variation de la valeur du portefeuille sous-jacent sur la période de clôture au moment de défaut d'un membre. La méthodologie de calcul de l'IM varie entre les opérateurs, mais l'approche usuelle consiste à calculer une Value-At-Risk de la variation de l'exposition sur la période de *close-out* (e.g. 10 jours)

  - Un *Buffer* de volatilité : Des appels de marges supplémentaires sont requis quand des mesures de liquidité du marché indiquent des délais de remplacement de transactions supérieurs à la période de clôture standard. Ceci se traduit par un risque de variation de l'exposition plus important.

  - Un fond de défaut : Représente une soulte de frais supplémentaires afin de couvrir des pertes résiduelles en cas de défaut d'un membre. Il s'agit d'un mécanisme de prévention contre le défaut de la chambre de compensation elle même.

- Marges initiales bilateral ou *Bilateral Initial Margins* introduits par BCBS 226 et 261 : Le comité de Bâle a proposé que toutes les banques et institutions financières systémiques versent un *Initial Margin* bilatétral en complément du collatéral posté pour couvrir les opérations OTC. Cette orientation permet de réduire la volatilité des positions due au risque de contrepartie, mais elle représente aussi une mesure incitative en faveur des chambres de compensations. Une proposition de méthodologie de calcul est celle du modèle SIMM de l'ISDA [ISD16].

- Autres mécanismes de sur-collatéralsation bilatérale comme les *Buffers* de liquidité et/ou de volatilité basés sur des évènements de rating.

Les coûts de financement supplémentaires dus à l'introduction des *Initial Margins* ségrégués constituent l'ajustement de valeur MVA. L'introduction des IM modifient également les valeurs de CVA/DVA et de ColVA, en réduisant le risque bilatéral de contrepartie, et en augmentant le coût de rémunération du collatéral posté ou reçu.

On note IM_k l'Initial Margin posté par $k \in \{I,C\}$ et mis dans un compte ségrégué et $r_{IM,k} = r_{OIS} + s_{IM,k}$ son taux de rémunération. Du point de vue de l'investisseur I, on a :

$$CVA(t) = \int_t^T \lambda_C(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\, LGD_C \left(V(s) - X_C(s) - IM_C\right)^+ |\mathscr{F}_t\right] ds \tag{1.1.26}$$

$$DVA(t) = \int_t^T \lambda_I(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\, LGD_I \left(V(s) - X_I(s) - IM_I\right)^- |\mathscr{F}_t\right] ds \tag{1.1.27}$$

$$FCA(t) = \int_t^T s_B(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s) \left(V(s) - X_C(s)\right)^+ |\mathscr{F}_t\right] ds \tag{1.1.28}$$

$$FBA(t) = \int_t^T s_L(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s) \left(V(s) - X_I(s)\right)^- |\mathscr{F}_t\right] ds \tag{1.1.29}$$

$$ColCA(t) = \int_t^T s_{ColCA}(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s) \left(X_C(s)\right)^+ |\mathscr{F}_t\right] ds \tag{1.1.30}$$

$$+ \int_t^T s_{IM,C}(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\, IM_C(s) \,|\mathscr{F}_t\right] ds \tag{1.1.31}$$

et

$$ColBA(t) = \int_t^T s_{ColBA}(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\,(X_I(s))^- \,|\mathscr{F}_t\right] ds \tag{1.1.32}$$

$$+ \int_t^T s_{IM,I}(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\, IM_I(s)\,|\mathscr{F}_t\right] ds \tag{1.1.33}$$

$$MVA(t) = \int_t^T s_B(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\, IM_I(s)\,|\mathscr{F}_t\right] ds \tag{1.1.34}$$

L'IM posté par I ne peut être utilisé pour financer du collatéral réclamé à I. Par conséquent, MVA est une charge supplémentaire de FCA.

1.1.3 Autres ajustements XVA

Dans la suite de l'ajustement MVA règlementaire (BCBS 226 et 261), la charge en capital due au risque de contrepartie est une composante significative de la règlementation Bâle 3. Historiquement, le capital, plutôt économique et non règlementaire, est implicitement pris en compte à travers les taux d'actualisation appliqués à la valorisation. Cependant, l'importance du capital règlementaire dans Bâle 3 a donné lieu à un ajustement de prix dit *KVA* pour une prise en compte plus précise. Cet ajustement valorise le coût de détention du capital règlementaire sur la durée de vie d'une transaction :

- Risque de contrepartie (CCR) : Calcule le capital requis au défaut des contreparties via le modèle standard SA-CCR (ex. CEM), ou en approche interne.
- Risque *CVA* : Calcule la charge en capital due à la volatilité du *mark-to-market* suite à l'impact sur l'ajustement *CVA* des variations des *spreads* de crédit sans évènement de défaut. Le capital requis est calculé via une approche standard, ou interne se basant sur une VaR sur *CVA* règlementaire, ou utilisant les sensibilités aux spreads de crédit.
- Risque de marché (e.g. VaR) : Les banques couvrent le risque de marché. Le risque résiduel donne lieu à une charge règlementaire
- Charge en risque incrémental (*Incremental Risk Charge*, IRC) : Charge spécifique au titre du risque de gap[1] suite à un évènement de défaut ou de migration de rating.
- Ratio de levier minimal : Le ratio de levier Bâle III est égal à la mesure de fonds propres divisée par la mesure de l'exposition. La règlementation impose un ratio minimal de 3%.

Le calcul de l'ajustement *KVA* fait intervenir la charge future en capital sur la durée de vie d'un portefeuille de dérivés. Le coût future est projeté sur les types de risques mentionnés. Les charges en capital CVA et CCR sont calculés par accords de compensation alors que la charge en risque de marché dépend du portefeuille global de la banque. L'ajustement KVA s'écrit :

$$KVA(t) = \int_t^T \gamma_K(s)\, e^{-\int_0^s (\lambda_I+\lambda_C)(u)} \mathbb{E}\left[D(t,s)\, K(s)\,|\mathscr{F}_t\right] ds$$

avec

[1] Saut de valorisation

- K : le profile du capital requis
- γ_K : le coût du capital, incluant le taux de dividendes des actifs immobilisés

Il est important de noter que cet ajustement ne représente pas le prix d'une risque de perte, mais plutôt le coût supporté par une institution pour porter une ligne d'activité.

Similaire au *FVA*, l'ajustement *KVA* n'est pas un ajustement de prix pour atteindre le prix risque neutre théorique, mais un coût réel non considéré dans la théorie classique de valorisation de Black-Scholes. Il s'agit du coût réel de construction du portefeuille.

Pour finir, il existe d'autres ajustements de valeurs :

- *RVA* : C'est l'ajustement de valeurs au rating ou *Rating Valuation Adjustment*. Cet ajustement prend en compte l'impact de la migration des notations de contrepartie sur la valeur du collatéral dans le cas des contrat dits de Rating Triggers. La chapitre 7 est dédié à la valorisation des enveloppes des expositions des ajustements XVA dans le cadre de ce type de contrat.
- *TVA* : C'est l'ajustement de valeurs de taxation ou *Tax Valuation Adjustment*. La prise en compte des taxes dans la valorisation est un élément nouveau en finance quantitative. La valorisation risque-neutre se base sur un argument de réplication parfaite ne donnant pas lieu à des profits ou des pertes. Dans une approche plus réaliste, l'existence des profits et pertes engendre des coûts ou des bénéfices de taxes suivant la juridiction appliquée. Le rôle de l'ajustement TVA est d'inclure ces coûts et bénéfices potentiels au modèle de valorisation.

1.2 Rôle du desk XVA et gestion du risque de contrepartie

La complexité inhérente au calcul et à la gestion des ajustements XVA par les desks de trading traditionnels a conduit à l'émergence d'un desk centralisateur. Les objectifs de ce desk sont :

- Prendre en charge la valorisation et la gestion des risques XVA pour les lignes métiers individuelles. Suivant la phase pré-trade, le desk a la responsabilité de fournir la valorisation officielle de la charge XVA inclue dans le prix de la transaction.
- Gestion des risques inhérents au stock XVA, comprenant le risque de marché et de crédit, ainsi que les évènements de dégradation de la note et de défaut. Cette gestion se traduit par la couverture du risque de marché et de crédit, en prenant en compte la structure du collatéral et les coûts de financement.
- Gestion des opérations liées aux ajustements XVA, notamment les opérations de collatéralisation, la renégociation des contrats de collatéral ou de compensation, le monitoring des évènements de seuil ou de terminaison de transaction. Ces opérations sont menées conjointement avec les lignes métiers concernés ainsi que le département juridique.
- Optimisation des charges CVA et FVA à travers la recherche d'opportunité d'intermédiation, de novation, la décision de transfert d'un portefeuille vers une chambre de compensation, ou d'un exercice optimal d'une clause de sortie de type *Break-Clauses*.

Un aspect clé de la gestion XVA est la capacité de séparer la valorisation classique d'un dérivé de celle des ajustements de valeur. Il en va de même pour la couverture. Alors que le risque de marché est géré au niveau portefeuille par le desk métier, le desk XVA cherche à couvrir le risque de marché lié aux enveloppes d'exposition associées à une contrepartie et un contrat de collatéral ; ces enveloppes faisant souvent intervenir plusieurs classes d'actifs. En plus du risque de marché, le risque de crédit à travers la variation des spreads CDS, et le risque lié à l'assiette de financement représentent des sources supplémentaires de risques à couvrir. Il convient de noter que :

- La valorisation des ajustements XVA peut être assimilée à la valorisation d'une option complexe (multi sous-jacents) à horizon long terme. Les incertitudes de modèle (diffusion, marquage des paramètres) ainsi que les contraintes de calcul sont significatives.
- Les coûts XVA nécessitent une modélisation des clauses et options de collatéralisation. Cette modélisation accroit la complexité du traitement, et la couverture de ces coûts peut être difficile à mettre en œuvre.
- La dépendance entre les facteurs de marché peut introduire une exposition aux mouvements croisées, ce qui augmente le risque de fuite de P&L et sa volatilité.

1.2.1 Facturation et allocation des charges

Quand une nouvelle transaction est ajoutée, le stock des ajustements XVA augmente du montant de la facture incrémentale qui est proportionnel à la variation de l'exposition avant et après la transaction

$$EPE^{incremental}_{Trade_i}(t) \quad = \quad (1-R_C)\,\mathbb{E}\left[(V_{Stock}(T)+V_{Trade_i}(T))^+ - (V_{Stock}(T))^+ \,|\mathscr{F}_t\right]$$

La charge incrémentale peut être positive ou négative en fonction des effets de compensations avec le portefeuille existant. Cette charge est facturée ou remise à la contrepartie par le desk métier.

Le schéma de facturation incrémental est simple à calculer dans un environnement de valorisation XVA. Cependant, ce schéma n'est pas additif, i.e. la somme des expositions incrémentales n'est pas égale à l'exposition totale. En outre, les facturations incrémentales dépendent de l'ordre dans lequel les opérations sont traités. Ces deux inconvénients majeurs sont abordés dans [PR10] où un schéma additif de contributions marginales est proposé en lien avec le théorème de fonction homogène d'Euler[2]

$$EPE_{Stock}(t) = \sum_{Trade_i \in Stock} EPE_{Trade_i}(t).$$

1.2.2 Gestion du risque de marché

Le risque de marché est déterminé par la sensibilité des ajustements XVA aux paramètres de marché qui entrent dans le calcul de l'enveloppe d'exposition. La répartition de cette sensibilité dépend de la structure des lignes de métiers. Les activités long terme de taux génèrent une

[2]Une fonction homogène g d'ordre m satisfait $m.g(x) = x.\nabla g(x)$.

sensibilité la plus significative sur les facteurs de taux d'intérêt, et il en est de même pour les facteurs de taux d'inflation. Pour les banques internationales, le calcul des expositions fait souvent intervenir un panier de devises de risque. La sensibilité de change correspond à la sensibilité à ces devises, mais aussi à la conversion vers la devise d'expression des ajustements XVA. Le risque de change représente l'un des axes majeur de gestion pour un desk XVA. Le calcul étant fait par contrepartie, il représente le plus souvent aussi une sensibilité aux axes actions, matières premières ou crédit.

La sensibilité au premier ordre ou *delta* se couvre à partir d'instruments linéaires tels que les swap de taux d'intérêt ou les swap de change. La couverture est exécutée face au marché organisé ou en interne face un desk spécialisé.

Comme décrit précédemment, l'exposition positive ou négative est une position optionnelle sur le portefeuille d'une contrepartie, dont la maturité est l'horizon d'exposition. Le sensibilité à la volatilité des axes de risque, ou *vega*, est due à la convexité de cette position. Elle est exprimée par rapport à la volatilité de marché de ces axes à l'horizon de l'exposition, et en particulier pour les axes de taux, aux maturités résiduelles (*tenor*) des swaps sous-jacents aux options swaptions. La couverture du risque de volatilité est effectuée par axe de risque à travers des options vanilles.

La valorisation de l'exposition nécessite la prise en compte d'une structure de corrélation entre tous les axes contributeurs. La non-linéarité du calcul engendre un risque de sensibilités de second ordre *gamma* et de sensibilité croisées *vanna* ,i.e. la sensibilité à un mouvement joint de chaque pair d'axes de risques. Outre le nombre important de sensibilités à calculer, l'exploitation de ces sensibilités à travers une stratégie de *hedge* n'est que limitée par l'absence d'instruments de marché adéquats à la couverture des risques croisés entre des axes hybrides (e.g. taux/change ou taux/action). Ces sensibilités d'ordre supérieur peuvent tout de même servir à quantifier les erreurs de couverture (fuites de P&L), conséquence d'une stratégie de couverture à l'ordre 1 uniquement.

Le desk XVA est amené aussi à calculer la sensibilité de ses portefeuilles aux différentes corrélations utilisées pour le pricing. Ces risques dits *cega* sont utiles pour étudier l'incertitude liée au marquage des corrélations notamment entre les axes hybrides, souvent inobservables ou déduits à partir d'estimations statistiques.

1.2.3 Gestion du risque de contrepartie

Le risque de contrepartie a été historiquement encadré par les banques à travers des limites en ligne de crédit affectées à chaque contrepartie en fonction de son rating externe ou interne, ainsi que l'appétence au risque. La consommation de la ligne de crédit est déterminée par la métrique PFE, ou *Potential Future Exposure*, qui correspond au quantile 95% du niveau de l'exposition, par analogie à la mesure VaR (cf. Figure 1.2.1). Le dépassement de la limite en PFE est suivie dynamiquement en fonction des mouvements de marché et des nouvelles transactions effectuées. Toute nouvelle transaction provoquant un dépassement de la limite peut ne pas être autorisée.

Cependant, l'encadrement par des limites statiques ne représente pas une stratégie de gestion en-soi et ne permet pas de le quantifier et de bénéficier des effets de compensation provenant de

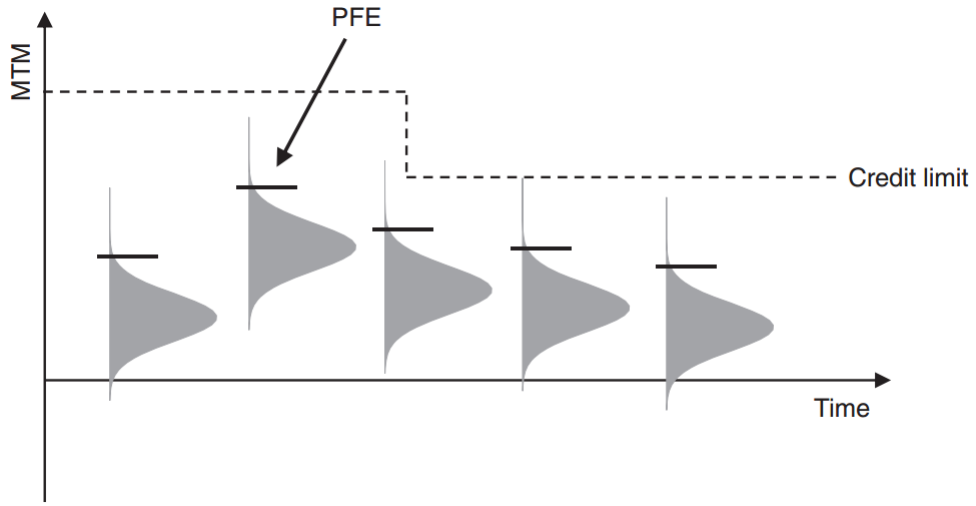


FIGURE 1.2.1 : Ligne de crédit et contrôle de l'exposition par PFE (source [Gre15])

plusieurs transactions. La quantification de ce risque est désormais le rôle des ajustements CVA et DVA calculés au niveau des accords de compensation par contrepartie.

Avant de parler d'une couverture marché de CVA, ou même d'échange de collatéral, plusieurs mécanismes contractuels permettent de réduire le risque de contrepartie.

Accords de compensation

Les accords de compensation bilatérale (*Netting Agreements*) sont historiquement développés sur le marché OTC afin de permettre à deux contreparties d'échanger des montants nets de flux après compensation des positions. Ces accords concernent les échanges quotidiens de flux (*Payment netting*), et également au moment d'un évènement de défaut (*Close-out netting*).

Les accords de type ISDA[3] représentent le cadre légal le plus adopté sur le marché OTC pour l'échange de collatéral et pour définir les règles de compensations.

L'impact de la compensation sur les ajustements CVA et DVA provient de la convexité inhérente au calcul des expositions positives et négatives. Si on définit la valeur d'un portefeuille V comme la somme de la valeur de ses composantes

$$V(t) = \sum_{i=1}^{n} V_i(t)$$

alors la valeur de l'exposition positive (1.1.1) est inférieure à la somme des expositions positives individuelles

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} V_i(T)\right)^+ |\mathscr{F}_t\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[(V_i(T))^+ |\mathscr{F}_t\right]$$

En outre la réduction du risque de contrepartie, les accords de compensation diminuent la complexité des échanges en particulier en cas de défaut.

Les accords ISDA opèrent d'une façon bilatérale. Ils existent cependant des accords de compensations multilatérales dont le but est diminuer plus significativement les échanges de flux et

[3]International Swaps and Derivatives Association

par conséquent de réduire le risque de contrepartie. Ces accords de compensation multilatérales sont souvent appelés « Accords de compression ». Le processus de compression est le plus souvent géré par des fournisseurs tiers comme TriReduce, ou par des chambres de compensation comme LCH. La compression permet de réduire significativement l'ensemble des nominaux échangés et par conséquent de diminuer globalement le risque de contrepartie, le risque opérationnels, les coût en capital des banques et aussi le risque systémique. Malgré le recours de plus en plus important à la compression (cf. Figure (1.2.2)), ce mécanisme requiert un niveau de standardisation des produits, ce qui limite son usage.

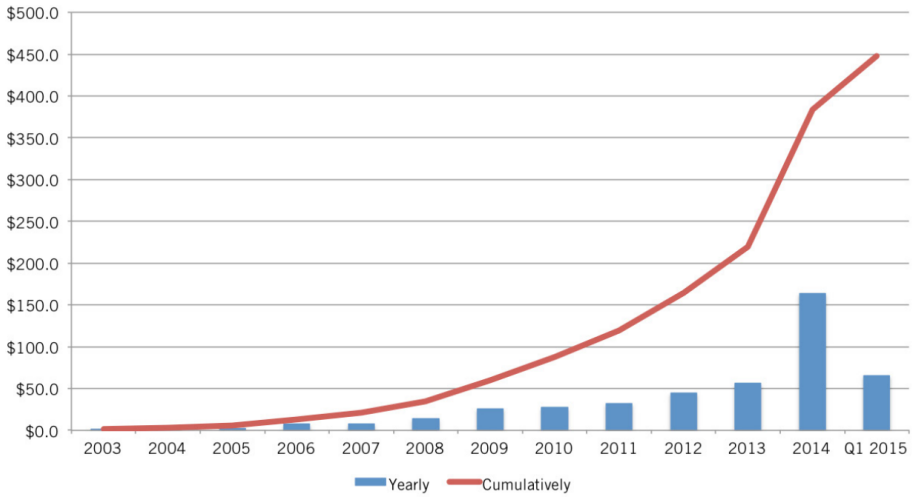


FIGURE 1.2.2 : Compression dans le marché de taux d'intérêt : Total des nominaux des produits soumis (en millier de milliards de dollar US). (Source ISDA 2015)

Évènements de terminaison

Plusieurs contrat OTC contiennent des options de terminaison/règlement anticipé(e) dont le but est prévenir le défaut d'une contrepartie. Deux types d'évènements de terminaison existent :

- Les *Break-Clauses* : Ces termes permettent de terminer une transaction à sa valeur de marché à une ou plusieurs dates prédéfinies. Les *Break-Clauses* obligatoires sont associées à une seule date et exercés automatiquement. Les *Break-Clauses* optionnelles donnent la possibilité à une ou aux deux contreparties de terminer le dérivé à une ou plusieurs dates. Faisant abstraction des coûts XVA, la présence de *Break-Clauses* ne change pas la valeur de la transaction car le règlement est fait à la valeur de marché. Cependant, la présence de *Break-Clauses* obligatoires rend l'exposition nulle au delà de la date de terminaison, ce qui réduit les charges XVA. Le traitement des *Break-Clauses* optionnels est différent car dans ce cas, l'exercice de l'option de terminaison n'est pas déterminé par un critère d'optimalité mais par des considérations commerciales (voir [Cam12]).

- Les *Downgrade Triggers* : Ces évènements déclenchent la terminaison d'une transaction à sa valeur de marché conditionnellement au passage du rating d'une contrepartie en dessous d'un seuil contractuel. La date de sortie est donc aléatoire. Étant donné que le défaut

est souvent précédé par une baisse de la note crédit, ces clauses peuvent réduire considérablement le risque de contrepartie en diminuant de manière préventive l'exposition (en augmentant le collatéral requis). Cependant, il existe encore le risque qu'une contrepartie fasse défaut sans passer par un déclassement de notation. Ce risque donne lieu à un ajustement CVA résiduel (voir [Cam13])

Mécanisme de remise au pair

L'accord de remise au pair ou *Reset* introduit une modification de la structure de la transaction afin ré-equilibrer les échanges de flux entre les deux contreparties. La remise au pair est effectuée à chaque date de paiement ou déclenchée par un évènement d'activation. Ce mécanisme est très utilisé dans des produits comme les FX Cross-Currency Swaps et les Equity Total Return Swaps dit à *Reset* ou *à formule*. Dans ces deux cas, les nominaux sont échangés à chaque date de paiement et non seulement à la fin (cf. Figure 1.2.3).

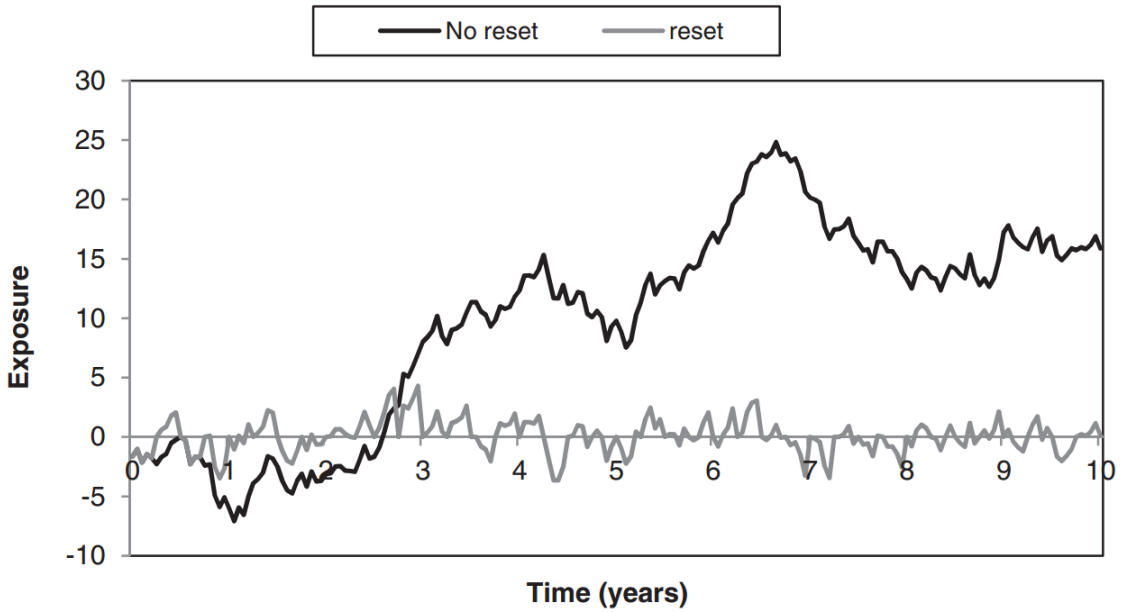


FIGURE 1.2.3 : Impact de la remise au pair sur l'exposition

Le collatéral ou Credit Support Annex

Le risque de contrepartie peut être réduit ou entièrement éliminé si chaque contrepartie fournit un actif au titre d'une garantie qui couvre la perte en cas de son défaut. Cette garantie ou collatéral est règlementée par un contrat cadre appelé CSA ou *Credit Support Annex.* Ce contrat est annexe aux accords ISDA et détermine les modalités d'échange bilatéral du collatéral, dans la grande majorité des cas où la perte en cas de défaut est calculée sur un portefeuille de compensation ou *netting set*. Un support CSA spécifie les paramètres qui précisent le mode de calcul du collatéral et la fréquence d'échange :

Devises de base

Devise de référence dans la quelle les valeurs du portefeuille et du collatéral sont calculées. C'est également la devise des paramètres du contrat.

Devises éligibles

Liste des devises éligibles dans lesquelles le collatéral en cash peut être posté.

Actifs éligibles

Liste des actifs acceptables en collatéral. Cette liste contient habituellement le cash dans les devises éligibles ainsi que d'autres actifs comme les obligations. Le contrat spécifie le pourcentage ou *haircut* à appliquer à la valeur de l'actif pour trouver la valeur du collatéral associé. En fonction du type de l'actif posté, ce pourcentage permet d'avoir une sur-collatéralisation afin de couvrir le risque de dépréciation du collatéral.

Seuil d'appel de marge

A chaque contrepartie s'applique un seuil d'exposition au delà duquel des appels de marges sont effectués. Typiquement, les seuils nuls sont souvent appliqués entre des contreparties bancaires afin d'assurer une collatéralisation complète. Un seuil non nul réduit le nombre d'appel de marge mais augmente l'exposition en cas de défaut. Les seuils peuvent être fixés en fonction de la nature des contreparties. On retrouve souvent des contrats de collatéral asymétrique contre des contreparties supra-nationales où celles-ci bénéficient d'un seuil infini. Dans les contrats de Rating Triggers, les seuils d'appels de marge sont conditionnés par les évènements de migrations de ratings.

Seuil minimum de transfert

Ou *Minimum Transfer Amount*. Ce seuil définit le montant minimum de variation de collatéral pouvant être échangée entre deux contreparties. Ce seuil permet d'éviter les appels de marges associés à un élargissement marginal de l'exposition.

Collatéral indépendant

Ou *Independent Amount.* C'est un montant de collatéral échangés entre les deux contreparties indépendamment de la valeur du portefeuille des transactions couvert. Les montant d'IA peuvent être également conditionnés par les évènements de ratings.

Fréquence d'appel de marge

La fréquence des appels de marge détermine la périodicité à la quelle la variation de l'exposition est ré-évaluée et comparée aux seuils d'appels de marge et de transfert minimum. Cette fréquence est quotidienne pour les contreparties bancaires, mais elle peut être plus large pour les autres types de contreparties.

Seuil de baisse de rating

Certains contrats CSA imposent des provisions supplémentaires en fonction de seuils pour le ratings des contreparties. Il s'agit d'un mécanisme qui augment les appels de marge pour la contrepartie dont la note de crédit baisse.

La présence d'un contrat de collatéral réduit les expositions crédit mais elle ne les annule pas, comme le montre la Figure 1.2.4. En pratique, la période entre deux appels de marges successifs représente une période de risque où la valeur de la transaction peut dériver de la valeur du collatéral reçu. Ce risque existe aussi entre la date du dernier appel de marge et la date de terminaison (close-out) du trade en cas de défaut. Cette période s'appelle Période de marge en risque ou *Margin Period of Risk*. Le contrat CSA ouvre le droit à plusieurs actifs éligibles en tant que collatéral : cash en plusieurs devises, bond, equity... ou bien à l'option de choisir le collatéral le moins cher à livrer, i.e. l'option *Cheapest To Deliver*. Le collatéral posté donne droit à des intérêts rémunérés souvent au taux OIS + spread. Avec la présence de taux négatif sur l'indice Eonia, les intérêts libellés en EUR sont souvent rémunérés à max $(Eonia, 0)$. Finalement, le CSA encadre l'utilisation du collatéral reçu pour financer des appels de marge contre d'autres contreparties. On parle de Re-hypothécation.

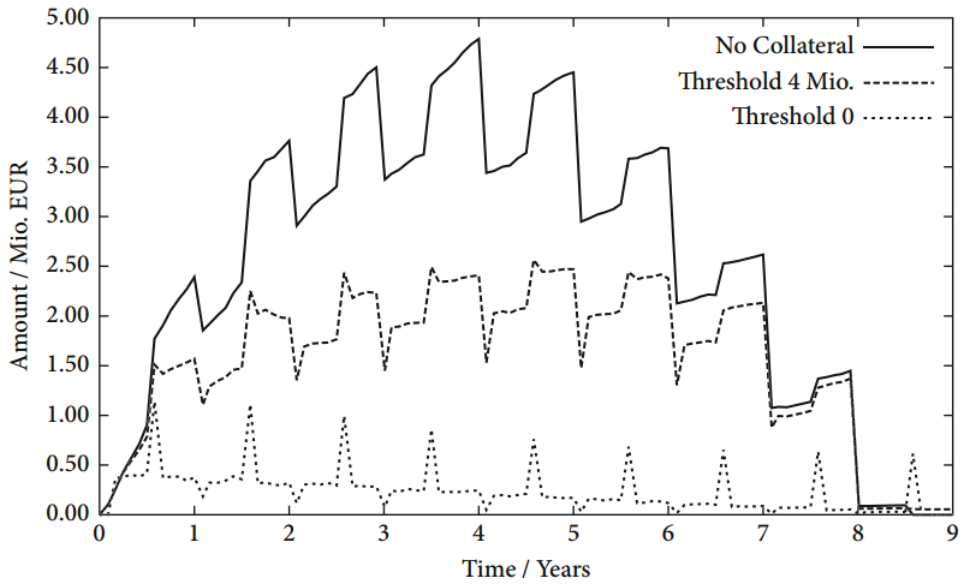


FIGURE 1.2.4 : Exposition d'un swap non-collatéralisé vs. présence d'un collatéral avec seuils d'appels de marge

Couverture dynamique du risque de crédit

Au delà des mécanismes qu'on vient de voir pour réduire le risque de contrepartie, une exposition résiduelle est quantifiée à travers les ajustements CVA et DVA. La gestion du risque de marché sous-jacent se fait à travers des instruments de couverture standards (CDS et obligations risquées) comme nous l'avons décrit précédemment.

La couverture du risque de crédit provenant des mouvements des spreads CDS est moins évidente. La raison principale étant le manque de liquidité sur le marché des CDS sur les noms des contreparties. Ces CDS *Single-name* représentent un instrument de couverture idéal. Cependant, il convient d'examiner les liens de dépendances entre la contrepartie sous-jacente des CDS et la contrepartie vendeuse de protection afin d'éviter tout *Wrong-Way Risk* spécifique. L'achat de CDS traités en chambres de compensation permet de réduire ce risque.

En cas d'absence de CDS *Single-name*, des stratégies de couverture approximatives peuvent être mise en place en utilisant des CDS sur indice ou un proxy du CDS *Single-name.* Les CDS sur indices sont des instruments liquides qui offrent la possibilité d'une couverture macro pour un élargissement général des spreads. Le bénéfice d'une telle approche dépend donc du degré de corrélation de l'indice et de la contrepartie. L'approche CDS *Single-name* proxy repose, elle, sur un rapprochement entre la contrepartie et le nom sous-jacent au CDS (secteur, zone géographique, ...). Ces deux approches peuvent s'avérer efficaces pour couvrir les variations de spreads, mais restent pauvres face à un évènement de défaut (voir [Car13]).

La sensibilité des ajustements XVA au spread de crédit de la banque qui les calcule ne peut être couverte directement car les banques ne sont pas autorisées à acheter une protection sur leurs propres risques de défaut. Néanmoins, la couverture de cette sensibilité reste possible à travers des approximations de type CDS Index ou CDS *Single-name* proxy. Comme pour la couverture de CVA, cette couverture approximative de DVA conduit à un risque résiduel.

1.2.4 Gestion du risque de financement

L'ajustement FVA est un nouvel élément reporté dans la comptabilité dans banques. La comptabilisation de cet ajustement pose le besoin de couvrir les risques associés. L'exposition aux risque de marché est traitée en amont de tous les ajustements XVA facturés.

Si la courbe de spread de financement utilisée correspond à un coût de financement interne face à la trésorerie, alors la sensibilité au coût FVA est couverte en interne avec la trésorerie qui gère tous les besoins de financement de la banque.

Si les coûts FVA sont marqués par une courbe de financement externe face au marché, alors la sensibilité de l'ajustement FVA à ce spread ne peut être couverte en l'absence d'instrument de marché. L'utilisation d'instruments sur une courbe proxy reste une stratégie de couverture plausible.

Chapter 2

Numerical methods for credit and funding exposure valuation

In the previous chapter, we have introduced the concept of counterparty and funding exposure. We focus in this chapter on a more formal presentation of numerical methods that have been traditionally involved or recently introduced for the valuation of XVA adjustments.

Given a portfolio of positions traded with a counterparty, which value is denoted by V and last maturity date by T, all XVA metrics can be recast to integrals of the form

$$\begin{aligned} XVA(t) &= \int_t^T a(s)\, e^{-\int_t^s b(u)du} \mathbb{E}\left[(V(s) - \mathscr{C}(s))^{\pm} | \mathscr{F}_t\right] ds \\ &\approx \sum_{i=1}^{N} \omega(t, T_{i-1}, T_i)\, \mathbb{E}\left[(V(T_i) - \mathscr{C}(T_i))^{\pm} | \mathscr{F}_t\right] \end{aligned} \tag{2.0.1}$$

where $\mathscr{C}$ refers to the value of the collateral held, ω to the $t-$forward discounted cost of XVA. The conditional expectation $\mathbb{E}[.|\mathscr{F}_t]$ represents the risk-neutral measure associated to the cash *numeraire* of the XVA expression currency.

The key element of the calculation of (2.0.1) is the $t-$conditional distributions of V and C at times $(T_i)_{i\in[\![1,N]\!]}$ (see Figure 2.0.1)

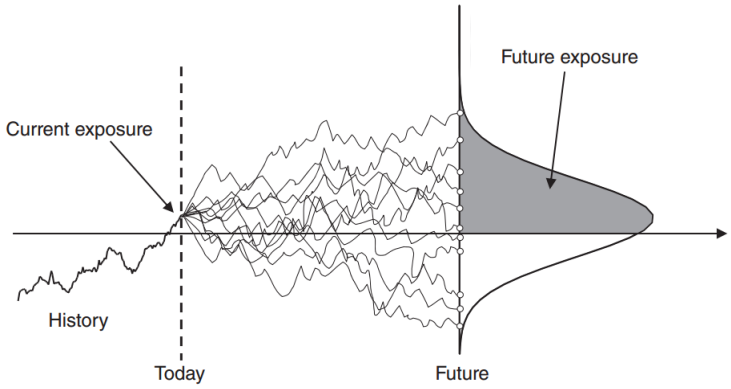


Figure 2.0.1: Current *versus* future exposure distribution

2.1 Monte Carlo valuation framework

To evaluate the expected exposure at a given time T_i, the most standard approach is based on Monte Carlo simulations up to the exposure date and then to calculate the portfolio and collateral values on each path. The simulation model is usually a highly dimensional cross asset diffusion process $(X_t)_t$ that should be jointly calibrated to market standard instruments of all involved asset classes. Given a simulation path of (X_t), the portfolio's value is given by:

- The T_i-conditional value $V_j(T_i)$ of each underlying trade j is computed analytically in case of linear products
- One can also use known closed formulas to price vanilla and a family of complex derivatives such a Barrier, Lookback and American options. This is usually subject of modeling inconsistencies between diffusion dynamics of X and the model assumptions behind the pricing formulas.
- In case of complex derivatives such as Bermuda Swaptions or Autocallable Equity swaps, pricing is usually performed using PDE and in most cases by Monte Carlo. This leads to heavy computation constraints due to the nested Monte Carlo valuation.

Cesari *et al.* define in [CAC+09] a general framework for the computation of counterparty exposure built on regression-based Monte Carlo in order to avoid the nested simulations problem. The American Monte Carlo (AMC) or Least Square (LS) Monte Carlo was introduced by Longstaff and Schwartz for handling early exercise features within Monte Carlo pricing engines.

Let $\mathscr{T} = \{\tau_1, \tau_2, \ldots, \tau_{n_E}\}$ be the set of n_E times at which the contract may be terminated. The holder of the contract is entitled to exchange cash flows CF_t at time t up to a final maturity T or any early termination time $\tau_E \in \mathscr{T}$. In particular, exercise features usually lead to the payment of exit coupons in case of termination events, or to an alternative structure of cash flows, in case of physical settlements for instance. In case of optional termination events, optimal exercise policies are usually calibrated by comparing continuation and exercises values within Monte Carlo simulation. CF_t denotes then the maximum of these values if t is an exercise time.

If the early termination occurs at time $\tau_E \in \mathscr{T}$, then the $t-$conditional value of the contingent claim is given by

$$V(t) = 1_{\tau_E \geq t} \mathbb{E}\left[\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du | \mathscr{F}_t\right] \tag{2.1.1}$$

The value of the exposure is defined as the value of the derivative when the option is still alive, i.e. $t \leq \tau_E$. When the contract is knocked out, the exposure value become 0.

The determination of the right-hand expectation is based on techniques used in optimal decision algorithms within the American Monte Carlo. The key is that at any time t of the exposure observation schedule, the conditional expectation is driven by the values of payoff observables or *predictors*

$$\Theta(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_{n_{Obs}}(t)) \tag{2.1.2}$$

depending on $(X_u)_{u\in[0,t]}$ such that the strong Markov property holds:

$$V(t) \equiv V(t, \Theta(t)) = 1_{\tau_E \geq t} \mathbb{E}\left[\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du | \Theta(t)\right] \tag{2.1.3}$$

We give in the sequel a brief description of numerical approximations of the conditional expectation. Tilley's algorithm operates by first sorting the values of the observables $\Theta(t)$ into equally sized bundles $(\mathfrak{B}_k(t))_k$. The conditional expectation given $\Theta(t) \in \mathfrak{B}_k(t)$ is then estimated using the Monte Carlo associated samples of $\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du$

$$\mathbb{E}\left[\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du | \Theta(t)\right] \approx \sum_k \mathbb{E}\left[\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du | \Theta(t) \in \mathfrak{B}_k(t)\right] \times \mathbb{Q}(\Theta(t) \in \mathfrak{B}_k(t)) \quad (2.1.4)$$

The continuation value is then approximated by a piece-wise constant function. The bundles $(\mathfrak{B}_k(t))_k$ can be chosen either equally-distributed (equally sized), equally distant, or given by nearest neighbor algorithm such as K-Means algorithm, Recursive Bifurcation, ect (see [DGFKO14, FO14]).

The Longstaff-Schwartz approach is a parametric regression where a non-linear continuous function is fitted to the realizations of $\left(\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du\right)_t$. The regression algorithm is based on least-square projection on a finite function space of order H such that

$$\mathbb{E}\left[\int_t^T D(t,u) CF_u 1_{\tau_E \geq u} du | \Theta(t)\right] \approx \sum_{k=0}^{H} \beta_k(t) \psi_k(\Theta(t)) \quad (2.1.5)$$

There are many possibilities for choosing the set of basis functions $(\psi_k)_{k \in [\![1,H]\!]}$. The standard choice correspond to a monomial basis with an arbitrary order H. If the dimension of Θ is d, it is easy to see that the total number of the basis functions is equal to $\frac{1}{(d-1)!} \sum_{n=0}^{H} \frac{(n+d-1)!}{n!}$.

[DGFKO14, FO14] addressed a mixture approach called the *Stochastic-Grid-Bundling* method that was initially introduced for pricing Bermuda options. SGBM employs first bundling to approximate the conditional distribution using MC simulation. The aim of bundling in SGBM is to cluster grid points based on proximity. The idea is to perform regression within each bundle $\mathfrak{B}_k(t)$. For the k^{th} bundle at time t, a set of basis functions $\left(\psi_{k,n}\right)_{n \in [\![1,H]\!]}$ is defined w.r.t a set of coefficients $\left(\beta_{k,n}(t)\right)_{n \in [\![1,H]\!]}$.

The SGBM method has some advantages w.r.t. the Longstaff-Schwartz method. The latter represents a regression on one bundle using at the same time in-the-money and out-the-money paths. In the SGBM method, clustering allows to have an automated decomposition of the regressed variable w.r.t. the payoff patterns. In the case of polynomial regression, we argue that using SGBM also allows to fix a small order H.

2.2 PDE representation of XVA adjustments

With Markovian models, the valuation of XVA adjustments can be represented as the solution of second order partial differential equations (see [BK11]). The representation is derived using a self-financing replication portfolio that comprises bonds issued by both counterparties, and a cash account which growth rate can be decomposed into position dividend income as well as the unsecured funding spread s_F. Usually s_F is given by the yield of the unsecured bond selling with recovery R_I, i.e. $s_F = \lambda_I(1 - R_I)$. The resultant PDE depends on the close-out convention in case of default:

- The *Risk-Free* close-out states that the payments in case of default are based on the risk-free value V.
- The *Substitution* close-out suggests that payments in case of default are calculated w.r.t. the risky value $\hat{V} = V + XVA$.

We recall that the risk-free value V satisfies the regular Black-Scholes PDE

$$\begin{cases} \partial_t V + \mathscr{L}_t V - rV & = 0 \\ V(T,X) & = P(X) \end{cases} \tag{2.2.1}$$

where P is the payoff function, and $\mathscr{L}$ is the Dynkin operator. Then, referring to [BK11], the risky value $\hat{V} = V + XVA$ satisfies

$$\begin{cases} \partial_t \hat{V} + \mathscr{L}_t \hat{V} - r\hat{V} & = (\lambda_I + \lambda_C)\hat{V} + s_F M^+ - \lambda_I (R_I M^- + M^+) - \lambda_C (R_C M^+ - M^-) \\ \hat{V}(T,X) & = P(X) \end{cases} \tag{2.2.2}$$

where

- the first term on the right hand side, i.e. $(\lambda_I + \lambda_C)\hat{V}$, is the additional growth rate required on the risky derivative $\hat{V}$ for the default risk of I or C.
- the second term is the additional funding costs for positive values of the mark-to-market, i.e. the negative values of the cash account of the CSA hedging portfolio.
- the third term is the adjustment in growth rate due cash flow occurrence at the default of I
- the last term is the adjustment in growth rate due the cash flow occurrence at the default of C

Risk-Free close-out

The payments in case of default are based on V, and hence $M(t,X) = V(t,X)$. Equation (2.2.2) becomes

$$\begin{cases} \partial_t \hat{V} + \mathscr{L}_t \hat{V} - (r + \lambda_I + \lambda_C)\hat{V} & = s_F V^+ - (R_I \lambda_I + \lambda_C) V^- - (\lambda_I + R_C \lambda_C) V^+ \\ \hat{V}(T,X) & = P(X) \end{cases} \tag{2.2.3}$$

with V solving (2.2.1). Writing $\hat{V} = V + XVA$, we obtain the following linear PDE for XVA

$$\begin{cases} \partial_t XVA + \mathscr{L}_t XVA - (r + \lambda_I + \lambda_C) XVA & = (1 - R_I)\lambda_I V^- + (1 - R_C)\lambda_C V^+ + s_F V^+ \\ XVA(T,X) & = 0 \end{cases} \tag{2.2.4}$$

Applying the Feynman-Kac theorem yields

$$\begin{aligned} XVA(t,X_t) = &-(1-R_I)\int_t^T \lambda_I(u)\, e^{-\int_t^u (r+\lambda_I+\lambda_C)(\omega)d\omega} \mathbb{E}\left[V^-(u,X_u)\,|\mathscr{F}_t\right] du \\ &-(1-R_C)\int_t^T \lambda_C(u)\, e^{-\int_t^u (r+\lambda_I+\lambda_C)(\omega)d\omega} \mathbb{E}\left[V^+(u,X_u)\,|\mathscr{F}_t\right] du \\ &-\int_t^T s_F(u)\, e^{-\int_t^u (r+\lambda_I+\lambda_C)(\omega)d\omega} \mathbb{E}\left[V^+(u,X_u)\,|\mathscr{F}_t\right] du \end{aligned} \tag{2.2.5}$$

Substitution close-out

Let us consider the case where the payments in case of default are based on $\hat{V}$ so that $M(t,X_t) = \hat{V}(t,X_t)$. The PDE (2.2.2) reads

$$\begin{cases} \partial_t \hat{V} + \mathscr{L}_t \hat{V} - r\hat{V} & = (1-R_I)\lambda_I \hat{V}^- + (1-R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \\ \hat{V}(T,X) & = P(X) \end{cases} \tag{2.2.6}$$

Then, XVA satisfies

$$\begin{cases} \partial_t XVA + \mathscr{L}_t XVA - rXVA & = (1-R_I)\lambda_I (V+XVA)^- + (1-R_C)\lambda_C (V+XVA)^+ \\ & \quad + s_F (V+XVA)^+ \\ XVA(T,X) & = 0 \end{cases} \tag{2.2.7}$$

as well as the following non-linear integral equation, obtained by the means of the Feynman-Kac theorem

$$\begin{aligned} XVA(t,X_t) = &-(1-R_I)\int_t^T \lambda_I(u)\, e^{-\int_t^u (r+\lambda_I+\lambda_C)(\omega)d\omega} \mathbb{E}\left[(V+XVA)^-(u,X_u)\,|\mathscr{F}_t\right] du \\ &-(1-R_C)\int_t^T \lambda_C(u)\, e^{-\int_t^u (r+\lambda_I+\lambda_C)(\omega)d\omega} \mathbb{E}\left[(V+XVA)^+(u,X_u)\,|\mathscr{F}_t\right] du \\ &-\int_t^T s_F(u)\, e^{-\int_t^u (r+\lambda_I+\lambda_C)(\omega)d\omega} \mathbb{E}\left[(V+XVA)^+(u,X_u)\,|\mathscr{F}_t\right] du \end{aligned} \tag{2.2.8}$$

Non-linear Monte Carlo methods are usually involved in solving Equation (2.2.6). A first approach is to simulate a backward stochastic differential equation (BSDE). This topic will be addressed in the following section. An innovative approach based on marked branching diffusion algorithm is proposed by [HL12]. Branching diffusions were first introduced by McKean (1975), in order to give a probabilistic solution for the Kolmogorov-Petrovskii-Piskunov PDE, and more generally for semi-linear PDEs of the form:

$$\begin{aligned} \partial_t u + \mathscr{L}_t u + \beta(t)\left(\sum_{k=0}^{\infty} p_k u^k - u\right) = 0 \\ u(T,x) = P(x) \end{aligned} \tag{2.2.9}$$

with $\beta(t) \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$, $0 \leq p_k \leq 1$. The interpretation of such an equation is closely related to Galton-Watson trees where a single particle starting at the origin, performs an Itô diffusion with generator $\mathscr{L}$, and then, after an exponential time with mean β (independent of X), vanishes and produces k descendants with probability p_k. The descendants restart the process with an independent Itô diffusion from their birth locations. If we denote by $Z_t \equiv \left(z_t^1, \ldots, z_t^{N_t}\right)$ the position of particles alive at time t and N_t their number, then the function

$$\hat{u}(t,x) = \mathbb{E}\left[\prod_{i=1}^{N_T} P\left(z_T^i\right) | N_t = 1, z_t^1 = x\right] \tag{2.2.10}$$

solves the semi-linear PDE (2.2.9). Henry-Labordère ([HL12]) extends this result to an arbitrary polynomial $F(u) = \sum_{k=0}^{M} a_k u^k$ for which the PDE (2.2.9) becomes

$$\partial_t u + \mathscr{L}_t u + \beta(F(u) - u) = 0 \tag{2.2.11}$$

The solution is given by

$$\hat{u}(t,x) = \mathbb{E}\left[\prod_{i=1}^{N_T} P\left(z_T^i\right) \prod_{k=0}^{M} \left(\frac{a_k}{p_k}\right)^{\omega_k} | N_t = 1, z_t^1 = x\right]$$

where ω_k is the number of particles that branch into $k \in \{1,\dots,M\}$ descendants. In the case of unilateral CVA adjustment, Equation (2.2.6) reads

$$\begin{cases} \partial_t \hat{V} + \mathscr{L}_t \hat{V} - (r + (1-R_C)\lambda_C)\hat{V} & = (1-R_C)\lambda_C\left(\hat{V}^+ - \hat{V}\right) \\ \hat{V}(T,X) & = P(X) \end{cases} \tag{2.2.12}$$

As proper discounting allows to write the above equation

$$\partial_t \hat{V} + \mathscr{L}_t \hat{V} + \beta\left(\hat{V}^+ - \hat{V}\right) = 0\,,\ \hat{V}(T,X) = P(X)$$

where $\beta = (1-R_C)\lambda_C$. The last step is to assume that function $x \mapsto x^+$ can be well approximation by a polynomial $F(x)$ of an arbitrary order M. The descendants are drawn with an arbitrary distribution $(p_k)_k$, for instance the uniform distribution such that $p_k = \frac{1}{(M+1)}$. The optimal probability p_k is obtained by minimizing the variance of the algorithm, it reads

$$p_k = \frac{|a_k| \, \|P\|_\infty^k}{\sum_{i=0}^{M} |a_i| \, \|P\|_\infty^i} \tag{2.2.13}$$

2.3 Using Backward Stochastic Differential Equations

While XVA semi-linear PDEs are proper to Markov setups, the problem of pricing and hedging valuation adjustments can be generalized in a BSDE representation. We recall that a BSDE is generally written as

$$\begin{cases} -dY_t & = f(t,Y_t,Z_t)\,dt - Z_t dW_t \\ Y_T & = \xi \end{cases} \tag{2.3.1}$$

or equivalently

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,dt - \int_t^T Z_s dW_s \tag{2.3.2}$$

where:

- The terminal value ξ is an $\mathscr{F}_T$-measurable random variable $\xi : \Omega \mapsto \mathbb{R}$.
- The driver f maps $\Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$ onto $\mathbb{R}$ is $\mathbb{Q} \times dt \times dt \times (dt)^d$-measurable.

A solution of the equation (2.3.2) is a pair (Y,Z) such that $(Y_t)_{t\in[0,T]}$ is a continuous $\mathbb{R}$-valued adapted process and $(Z_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued predictable process such that $\int_0^T |Z_s|^2 ds < +\infty$ almost surely. Conditions for existence and uniqueness of the BSDE solution are discussed by Pardoux and Peng [PP90]. In addition, Pardoux and Peng established the relationship between BSDE and PDE through a generalized Feynman-Kac theorem:

Theorem 2.3.1 (Generalized Feynman-Kac theorem). *Let υ be a $\mathscr{C}^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}\right)$ function (or smooth enough that Itô's formula holds) and suppose that there exists a constant C such that for each $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$*

$$|\partial_t \upsilon(s,x)| + |\nabla_x \upsilon(s,x)\,\sigma(s,x)| \leq C(1+|x|) \tag{2.3.3}$$

Suppose that υ is the solution of the following quasi-linear parabolic equation

$$\begin{cases} \partial_t \upsilon(t,x) + \mathscr{L}_t \upsilon(t,x) + f(t,x,\upsilon(t,x),\nabla_x \upsilon(t,x)\,\sigma(t,x)) = 0 \\ \upsilon(T,x) = P(x) \end{cases} \tag{2.3.4}$$

then, $\left(Y_t^{t,x} \triangleq \upsilon\left(s,X_s^{t,x}\right), Z_s^{t,x} \triangleq \nabla_x \upsilon\left(s,X_s^{t,x}\right)\sigma\left(s,X_s^{t,x}\right)\right)_{t\leq s\leq T}$ is the unique solution of the BSDE (2.3.1).

See [EKPQ97] for a the seminal reference on BSDEs in finance.

To approximate numerically the solution of (2.3.1), several algorithms are addressed in the literature : Ma, Protter and Young [MPY94], Bally and Pagès [BP03], Bouchard and Touzi [BT04], Zhang [Zha04], Gobet, Lemor and Warin [GLW$^+$05], Delarue and Menozzi [DM06], Bender and Denk [BD07], Gobet and Labart [GL10]. These methods are based principally on the Dynamic Programming Principle, except Gobet and Labart's algorithm which uses the Picard's contraction principle.

In the following, we refer to the pioneer work of Crépey [Cré15b, Cré15a] and Crépey *et al.* [CGGN13] on pricing XVA adjustments using BSDEs.

As stated previously, T denotes the time terminal maturity of the contract. We assume that the contract delivers a cash flows rate dD_t from I to her counterparty resulting in an effective dividend stream $dC_t = 1_{\tau_I \wedge \tau_C > t} dD_t$. We denote by $\tau = \tau_I \wedge \tau_C$ and by $\bar{\tau} = T \wedge \tau$. From the point of view of I (the bank), a collateral, heding and funding portfolios are set-up from inception and unwound at $\bar{\tau}$. I can rely on a third party as a liquidity provider (or funder).

The *OIS* rate is denote by r_t while $\tilde{r}_t = r_t + \lambda(t)$ denotes the credit-risk adjusted rate where λ is the hazard intensity of τ. We refer to the corresponding discount factors by $\beta_t = \exp\left(-\int_0^t r_s ds\right)$ and $\tilde{\beta}_t = \exp\left(-\int_0^t \tilde{r}_s ds\right)$. $\mathbb{E}_t$ and $\tilde{\mathbb{E}}_t$ stand for the $t-$conditional expectation given $\mathscr{G}_t$ and $\mathscr{F}_t$ respectively (see chapter 1).

The value process $\hat{V}_t$ of the contract is defined for $t \in [0,\bar{\tau}]$ by

$$\beta_t \hat{V}_t \triangleq \tilde{\mathbb{E}}_t\left[\int_t^T \beta_s dD_s\right] = \mathbb{E}_t\left[\int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} \hat{V}_{\bar{\tau}}\right] \tag{2.3.5}$$

A time τ of default, a terminal cash-flow R is paid by I in order to close all her positions with C and potentially the funder. In particular, R is decomposed into a close-out cash-flow R^i from I to the counterparty C, minus, in case of default of I, a cash-flow R^f from C (the funder) to I. Both R^i and R^f depend on the wealth of the hedging portfolio π. We denote by V the risk-free (CSA) value of the contract and Γ the value process of the collateralization scheme. If $\tau < T$, the total close-out cashflow is then modeled as $R(\pi) = R^i - 1_{\tau=\tau_I} R^f(\pi)$ where

$$\begin{aligned} R^i &= \Gamma_\tau + 1_{\tau=\tau_C}\left(R_C \chi^+ - \chi^-\right) - 1_{\tau=\tau_I}\left(R_I \chi^- - \chi^+\right) - 1_{\tau_I=\tau_C}\chi \\ R^f(\pi) &= (1-\mathfrak{r})\,\mathfrak{X}^-(\pi) \end{aligned}$$

with

- R_I and R_C the recovery rates of I and C
- $\mathfrak{r}$ the recovery rate of I to her funder
- $\chi = V_\tau - \gamma_\tau$
- $\mathfrak{X} = -(\pi - \Gamma_{\tau^-})$

The exposure at default is then defined by

$$\begin{aligned}\xi(\pi) &\triangleq \hat{V}_\tau - R(\pi) \\ &= \hat{V}_\tau - V_\tau + LGD_C 1_{\tau=\tau_C}\chi^+ - 1_{\tau=\tau_I}\left(LGD_I\chi^- + (1-\mathfrak{r})\mathfrak{X}^-(\pi)\right)\end{aligned} \tag{2.3.6}$$

We now consider the cash-flows required for funding the positions of the bank I. We introduce the bases b_t and $\bar{b}_t$ related to the collateral posted and received by the I and $\mathfrak{b}_t$ and $\bar{\mathfrak{b}}_t$ related to the external lending and borrowing. The funding coefficient $g_t(\pi)$ is defined by

$$g_t(\pi) = \left(b_t\Gamma_t^- - \bar{b}_t\Gamma_t^+\right) + \mathfrak{b}_t(\pi - \Gamma_t)^+ - \bar{\mathfrak{b}}_t(\pi - \Gamma_t)^- \tag{2.3.7}$$

It readily follows that $(r_t\pi + g_t(\pi))\,dt$ represents funding cost of I over the period $(t, t+dt)$.

Once χ and g are specified, the XVA process Θ is defined on $[0,\bar{\tau}]$ as the solution of the BSDE: For $t \in [0,\bar{\tau}]$,

$$\beta_t\Theta_t = \mathbb{E}_t\left[\beta_{\bar{\tau}}1_{\tau<T}\xi\left(\hat{V}_\tau - \Theta_{\tau^-}\right) + \int_t^{\bar{\tau}}\beta_s g_s(V_s - \Theta_s)\,ds\right]. \tag{2.3.8}$$

Following [Cré15b], defaultability of both counterparties can be represented through their default intensities:

$$\xi(\pi) \triangleq \hat{V}_\tau - V_\tau + LGD_C p_C(t)\chi^+ - p_I(t)\left(LGD_I\chi^- + (1-\mathfrak{r})\mathfrak{X}^-(\pi)\right) \tag{2.3.9}$$

with

$$p_I(t) = \mathbb{Q}(\tau_I = t|\mathscr{G}_{t^-})\ ,\ p_C(t) = \mathbb{Q}(\tau_C = t|\mathscr{G}_{t^-}) \tag{2.3.10}$$

This yields a reduced form representation of the total adjustment XVA. For $\Theta = \tilde{\Theta}$ on $[0,\bar{\tau})$ and $\Theta_{\bar{\tau}} = 1_{\tau<T}\xi$,

$$\tilde{\beta}_t\tilde{\Theta}_t = \tilde{\mathbb{E}}_t\left[\int_t^{\bar{\tau}}\tilde{\beta}_s\left(g_s\left(V_s - \tilde{\Theta}_s\right) + \lambda_s\tilde{\xi}_s\left(\hat{V}_s - \tilde{\Theta}_s\right)\right)ds\right] \tag{2.3.11}$$

for $t \in [0,T]$.

2.4 Conclusion

In this chapter, we gave a brief description of major numerical methods recently addressed for solving the XVA valuation problem. From a general perspective, XVA valuation imposes several constraints:

- Dimensionality: XVA metrics are computed either on the netting-set or the trade levels. The stock calculations usually suffer from limitations due to dimensionality of the underlying risk factors

- Consistency with trade level valuation: In order to resolve the dimensionality problem and to maintain the accuracy of the product valuation, banks commonly choose to implement a third party valuation system for XVA. This usually results in pricing discrepancies between the XVA system and the original trade level valuation.
- Performance constraints: The critical constraint on the choice of XVA valuation methods is performance w.r.t. live calculations of prices and Greeks.

Numerical methods for the exposure valuation and Wrong-Way risk

"In the land of the blind, the one-eyed man is king."

(Michael Lewis, Liar's Poker)

Chapter 3

A forward equation for computing derivatives exposure

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We derive a forward equation for computing the expected exposure of financial derivatives. Under general assumptions about the underlying diffusion process, we give an explicit decomposition of the exposure into the sum of an intrinsic value which can be directly deduced from the term structure of the forward mark-to-market and of a time value part which expresses the variability of the future mark-to-market values. Our approach is inspired by Dupire's equation for local volatility and leads to an ordinary differential equation qualifying the evolution of the expected exposure with respect to the observation dates and so to a representation as a very simple one dimensional integral. We show how this approach can be linked with local times theory in dimension one and to the co-area formula otherwise.

As for numerical considerations, we show how this approach leads to numerical method especially efficient when dimension equal 1 or 2. The accuracy and time-efficiency of this forward representation in small dimension are of special interest in benchmarking Xva valuation adjustments at trade level.

Keywords: Expected exposure, co-area formula, local times, Dupire's equation

3.1 Introduction

The Financial crisis of 2008 has shown that counterparties default and uncertain funding represent significant components in determining the fair value of a financial derivative. The key function of the valuation and the management of these contingent risks is the quantification of the exposure. Under the independence assumption between the derivative's future mark-to-market and the risk factors that drive funding and defaults, the exposure is a pure market risk measure that depends only on the underlying payoff and the valuation model.

Quantifying the derivative's exposure is an essential element for the approval of a trade with regard to credit and funding lines, as well as pricing and hedging. In the case of measuring CVA and DVA adjustments, it is intended to give an overview of the contract's loss profile in case of default. The exposure is expressed under a risk probability measure as the expectation of

entering (positive) cash flows at a predefined set of observation dates. The debate about choosing the risk-neutral or the physical risk measures is becoming heated among practitioners and academics, since the choice relies on the validity of replication-based arguments in the context of contingent credit risks. As widely posited, the standard pricing measure is based on market observations and thus is more appropriate to valuate pricing adjustments. However, regulatory-compliant adjustments such as capital requirements and clearing represent an exception to this rule ([KG14]).

Numerical methods for pricing CVA/DVA/FVA adjustments is becoming an area of extensive research in quantitative finance. This intensity is predominantly explained by the need of the financial industry to satisfy simultaneously several requirement such as global portfolio pricing, incremental charging, and the management of hybrid cross-assets positions.

The fulfillment of these requirements involves necessarily a successful implementation of fast and accurate numerical methods. In order to avoid the nested Monte Carlo schemes, explicit Xva pricing and hedging are formulated in terms of semilinear PDEs in a Markov setup. To the best of our knowledge, there exist 3 major classes of algorithms dedicated to this problem: backward Monte Carlo ([CAC$^+$09]), backward SDEs ([CGGN13]) and marked branching diffusions ([HL12]). These approaches are mutually consistent and lead to an approximate probabilistic solution that requires a lot of computational power (see [CGGN13],[HLTT14]).

Our main objective is to give a forward representation of the term structure of a derivative's expected exposure. The key idea is to consider the expected exposure as the price of a compound option. In absence of collateral, this option corresponds to an European Call on the expected value of residual cash flows with a zero strike and expiring at the observation date of the default. More generally, deterministic collateral specifications such as collateral threshold and initial margins are easily taken into account through a combination of European Call and Put options with specific strikes.

The solution presented here use a Dupire like (see [Dup94]) forward equation. From a theoretical point of view, our approach gives a payoff-free analytical representation of the expected exposure, and we formulate it in terms of an *ordinary differential equation* (ODE) depending on the contract's forward *delta* and the underlying local volatility function. The solution conveniently describes the expected exposure as the deterministic exposure (based on the forward value of the contract), to which is added a volatility term that expresses the variability of future mark-to-markets. In a numerical perspective, the forward solution allows an incremental implementation between consecutive observation dates. By dealing only with increments, our formula leads to highly accurate computation of the expected exposure and its Greeks, and shows very good numerical performances. However, these results are obtained at the cost of necessary conditions and auxiliary calculations that we determine and discuss.

Outline of the chapter

The chapter is organized as follows. After introducing the concept of the expected exposure and the pricing framework in Section 3.2, we fully derive in Section 3.3 our new integral formula in the case of 1 risk factor. We exhibit the underlying assumptions and we illustrate this case with examples. In Section 3.4, we give a multidimensional extension and we detail the case of dimension 2 for numerical purposes. Owing to the favorable analytical properties of our solution, we dedicate Section 3.5 to the derivation of Greeks formulas. In Section 3.6, we discuss the

algorithmic implementation and we present some numerical results. The mathematical lemmas on which are based our approach are given in Section 3.7. Our conclusions are finally addressed in Section 3.8.

3.2 Pricing framework

We consider a standard pricing setup using a risk neutral measure or any equivalent one, and we denote it by $\mathbb{Q}$.

We consider a financial contract where cash flows are paid at discrete times $(T_i)_{i=1,\ldots,N} \in [0,T]$ between pricing time 0 and the maturity $T > 0$. We define the valuation times $s,t \in [0,T]$ where $s \leq t$. $D(s,t)$ denotes the discount factor referring to the value at s of one monetary unit received at t. We introduce $\Pi(t,T)$ as the sum of cash flows of the underlying contract exchanged between t and T and discounted to time t. It is well known that its arbitrage-free value at observation s is given by

$$M(s;t) \triangleq D(s,t)\,\mathbb{E}_s\left[\Pi(t,T)\right] \tag{3.2.1}$$

where $\mathbb{E}_s$ ($\mathbb{E}_0 = \mathbb{E}$) denotes the $\mathbb{Q}-$expectation based on market information up to time s. The value of the expected exposure to a default occurring at time t, $\mathbf{EE}(t)$, is given by

$$\mathbf{EE}(t) \triangleq D(0,t)\,\mathbb{E}\left[(M(t;t))^+\right] \tag{3.2.2}$$

This quantity expresses the loss on favorable mark-to-market scenarios in case of default of the counterparty. This takes into account the amount of collateral held as well as the recovery rate. In addition, the exposure is calculated on the basis of *risk-free closeout*, i.e. risk-free mark-to-market at default. In presence of continuous collateralization triggered at $H > 0$, the expected exposure given by

$$\mathbf{EE}(t) = D(0,t)\,\mathbb{E}\left[(M(t;t))^+ - (M(t;t)-H)^+\right] \tag{3.2.3}$$

Similar expressions can be derived in presence of additional deterministic collateral specifications such as *Independent Amounts*.

We assume that $\Pi(t,T)$ depends on the realization at time t of a multidimensional Markov process $(X_u)_{u\geq 0}$ such that

$$M(s;t) = D(s,t)\,\mathbb{E}_s\left[\Pi(t,T)\,|X_s\right] \tag{3.2.4}$$

and there exists a regular function $m(s,x;t)$ with $M(s;t) = m(s,X_s;t)$. For sake of simplicity, we assume that discounting rates are zero, leading to $D(s,t) = 1$. In this case $(m(s,X_s;t))_{s\in[0,t]}$ is then a martingale under the pricing probability $\mathbb{Q}$. Moreover, we suppose that $X \in \mathbb{R}^d$ evolves under $\mathbb{Q}$ according to the stochastic differential equation

$$\forall u \in [0,T],\, X_u = (X_{i,u})_{i=1\ldots d} = X_0 + \int_0^u \mu(v,X_v)\,dv + \int_0^u \sigma(v,X_v)\,dW_v \tag{3.2.5}$$

where $W = \left(W^1,\ldots,W^r\right)^T$ is a $\mathbb{R}^r-$valued standard Brownian motion, $X_0 = \left(X_{1,0},\ldots,X_{d,0}\right)^T \in \mathbb{R}^d$, $\mu : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times r}$.

In order to obtain our formula, we need to ensure that the law of X_t admit a regular probability density. This is obtained by the means of the following assumption.

Assumption 3.2.1 ($\mathscr{H}_1$)**.** *The derivatives μ and σ w.r.t. the space variable X exist at any order and are bounded. This ensures the existence and uniqueness of $(X_t)_{t\in[0,T]}$as the strong solution of Equation* (3.2.5)*. In addition, σ fulfills the uniform ellipticity condition; i.e. there exits $K_{\min}$ and $K_{\max}$ such that $0 < K_{\min} \leq K_{\max} < +\infty$ and*

$$K_{\min} Id \leq \sigma(u,x)\,\sigma(u,x)^T \leq K_{\max} Id \tag{3.2.6}$$

for all $u \in [0,T]$ and $x \in \mathbb{R}^d$. This implies the existence of a smooth transition density $q(s,x)$. Additional regularity assumptions (see [Fri75], theorem 4.5 p. 141 for instance) implies the existence of constants c and K such that for $s \leq t$ and $x \in \mathbb{R}^n$

$$|q(s,x)| \leq \frac{K}{\sqrt{s}} \exp\left(-c\frac{|x-x_0|^2}{s}\right),$$

$$\left|\frac{\partial q}{\partial x_i}(s,x)\right| \leq \frac{K}{s} \exp\left(-c\frac{|x-x_0|^2}{s}\right)$$

and

$$\left|\frac{\partial^2 q}{\partial x_i \partial x_j}(s,x)\right| \leq \frac{K}{s\sqrt{s}} \exp\left(-c\frac{|x-x_0|^2}{s}\right).$$

We introduce smoothness requirements on the value function m:

Assumption 3.2.2 ($\mathscr{H}_2$)**.** *We assume that the value function $(s,x) \mapsto m(s,x;t)$ is $\mathscr{C}_b^{1,2}\left([0,t]\times\mathbb{R}^d \to \mathbb{R}\right)$ for any $t \in [0,T[$ with bounded derivatives. Moreover, there exist positive constants A, B and $0 \leq \beta \leq \frac{c}{s}$ such that*

$$|x| \geq A \Rightarrow |m(s,x)|\, e^{\beta|x|^2} \geq B.$$

c is defined in Assumption 3.2.1.

Under ($\mathscr{H}_1$), q satisfies the Kolmogorov forward equation:

$$\partial_s q(s,x) + \sum_{i=1}^{d} \partial_{x_i}\left[\mu_i(s,x)\, q(s,x)\right] - \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{x_i x_j}\left[\left(\sigma\sigma^T(s,x)\right)_{i,j} q(s,x)\right] = 0 \tag{3.2.7}$$

Given ($\mathscr{H}_2$), m satisfies the Kolmogorov backward equation:

$$\partial_s m(s,x;t) + \sum_{i=1}^{d} \mu_i(s,x)\,\partial_{x_i} m(s,x;t) + \frac{1}{2}\sum_{i,j=1}^{d} \left(\sigma\sigma^T(s,x)\right)_{i,j} \partial^2_{x_i x_j} m(s,x;t) = 0 \tag{3.2.8}$$

Assumption 3.2.3 ($\mathscr{H}_3$)**.** *For $t \in [0,T[$, there exists a finite set of continuous functions*

$$\mathscr{L}(t) \triangleq \left\{ s \in [0,t] \mapsto l_j(s,t) \in \mathbb{R}^d, j = 1\ldots n \right\},\ n \geq 1 \tag{3.2.9}$$

where for $j \in [\![1,n]\!]$ and $s \in [0,t]$

1. $m\left(s, l_j(s,t); t\right) = 0$
2. $\partial_x m(s,x;t)\,|_{x=l_j(s,t)} \neq 0_{\mathbb{R}^d}$
3. $l_j \neq l_k$ *if* $k \neq j$

3.3 The forward exposure representation for $X \in \mathbb{R}$

3.3.1 Analytical derivation

We now present our main result in the uni-dimensional case with $d = r = 1$, relying on the previous assumptions:

Theorem 3.3.1. *Under $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$, the expected exposure at time t, $\mathbf{EE}(t)$, satisfies the following ordinary differential equation*

$$\mathbf{EE}(t) = (m(0,t))^+ + \frac{1}{2}\int_0^t \sum_{j=1}^{n} \sigma^2\left(s, l_j(s,t)\right) \left|\partial_x m\left(s, l_j(s,t); t\right)\right| q\left(s, l_j(s,t)\right) ds \tag{3.3.1}$$

Remark 3.3.2. Theorem 3.3.1 presents a convenient decomposition of the expected exposure $\mathbf{EE}(t)$. The first term $(m(0,t))^+ = (\mathbb{E}[\Pi(t,T)])^+$ corresponds the positive part of the forward mark-to-market which can be assimilated to the intrinsic value of the exposure. The second term depends on the volatility function σ and represents consequently the time value part.

To prove the theorem, we introduce an approximation Φ_ε of function $x \mapsto \max(x,0)$. For this, we introduce

$$\delta_\varepsilon : x \mapsto \frac{1}{2\varepsilon} 1_{\{|x| \le \varepsilon\}},$$

and

$$\begin{cases} \phi_\varepsilon(x) & = \int_{-\infty}^{x} \delta_\varepsilon(u)\, du \\ \Phi_\varepsilon(x) & = \int_{-\infty}^{x} \phi_\varepsilon(u)\, du \end{cases}$$

For $\varepsilon > 0$, we approximate $(m(s,X_s,t))^+$ by $\Phi_\varepsilon(m(s,X_s,t))$. The following lemma can be obtained using classical results.

Lemma 3.3.3. *Under $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, we have*

$$\begin{aligned} \partial_s \mathbb{E}[\Phi_\varepsilon(m(s,X_s;t))] &= \frac{1}{2}\mathbb{E}\left[\sigma^2(s,X_s)\left(\partial_x m(s,X_s;t)\right)^2 \delta_\varepsilon(m(s,X_s;t))\right] \\ &= \frac{1}{2}\int_{\mathbb{R}} dx \sigma^2(s,x)\left(\partial_x m(s,x;t)\right)^2 q(s,x)\, \delta_\varepsilon(m(s,x;t)) \end{aligned} \tag{3.3.2}$$

Proof. We have, using the density q

$$\mathbb{E}[\Phi_\varepsilon(m(s,X_s;t))] = \int_{\mathbb{R}} m(s,x;t)\, \phi_\varepsilon(m(s,x;t))\, q(s,x)\, dx.$$

Using regularity of q, its exponential decrease and boundedness, we get

$$\partial_s \mathbb{E}[\Phi_\varepsilon(m(s,X_s;t))] = \int_{\mathbb{R}} \partial_s m(s,x;t)\, \phi_\varepsilon(m(s,x;t))\, q(s,x) + \Phi_\varepsilon(m(s,x;t))\, \partial_s q(s,x)\, dx$$

We recall that given $(\mathscr{H}_2)$, the time and space evolution of q is described by (3.2.7), for $x \in \mathbb{R}$ and $s > 0$

$$\partial_s q(s,x) = -\partial_x\left[\mu(s,x)\, q(s,x)\right] + \frac{1}{2}\partial_{xx}^2\left[\sigma(s,x)^2 q(s,x)\right]$$

From the above equations, it follows that

$$\partial_s \mathbb{E}\left[\Phi_\varepsilon\left(m\left(s,X_s,t\right)\right)\right] = \int_{\mathbb{R}} dx \partial_s m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right) q\left(s,x\right) - \Phi_\varepsilon\left(m\left(s,x;t\right)\right) \partial_x\left[\mu\left(s,x\right) q\left(s,x\right)\right] + \frac{1}{2}\int_{\mathbb{R}} dx \Phi_\varepsilon\left(m\left(s,x;t\right)\right) \partial_{xx}^2\left[\sigma\left(s,x\right)^2 q\left(s,x\right)\right]$$

We perform successive integration by part w.r.t. ∂_x and ∂_{xx} terms on the right-hand side. Owing to the exponential decay of q, one has

$$\begin{aligned}
&\int_{\mathbb{R}} dx \Phi_\varepsilon\left(m\left(s,x;t\right)\right) \partial_x\left[\mu\left(s,x\right) q\left(s,x\right)\right] \\
&= \left[\Phi_\varepsilon\left(m\left(s,x;t\right)\right) \mu\left(s,x\right) q\left(s,x\right)\right]_{x=-\infty}^{x=+\infty} - \int_{\mathbb{R}} dx \mu\left(s,x\right) q\left(s,x\right) \partial_x m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right) \\
&= -\int_{\mathbb{R}} dx \mu\left(s,x\right) q\left(s,x\right) \partial_x m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} \Phi_\varepsilon\left(m\left(s,x;t\right)\right) \partial_{xx}^2\left[\left(\sigma^2 q\right)\left(s,x\right)\right] \\
&= \left[\Phi_\varepsilon\left(m\left(s,x;t\right)\right) \partial_x\left(\left(\sigma^2 q\right)\left(s,x\right)\right)\right]_{x=-\infty}^{x=+\infty} - \int_{\mathbb{R}} dx \partial_x\left(\left(\sigma^2 q\right)\left(s,x\right)\right) \partial_x m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right) \\
&= \left[\left(\sigma^2 q\right)\left(s,x\right) \partial_x m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right)\right]_{x=-\infty}^{x=+\infty} + \int_{\mathbb{R}} dx \left(\sigma^2 q\right)\left(s,x\right) \partial_x\left(\partial_x m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right)\right) \\
&= \int_{\mathbb{R}} dx \left(\sigma^2 q\right)\left(s,x\right) \partial_{xx}^2 m\left(s,x;t\right) \phi_\varepsilon\left(m\left(s,x;t\right)\right) + \int_{\mathbb{R}} dx \left(\sigma^2 q\right)\left(s,x\right) \left(\partial_x m\left(s,x;t\right)\right)^2 \delta_\varepsilon\left(m\left(s,x;t\right)\right)
\end{aligned}$$

Equation (3.3.2) is obtained by summing up all the previous terms together and using the Kolmogorov backward equation (3.2.8)

$$\partial_s m\left(s,x;t\right) + \mu\left(s,x\right) \partial_x m\left(s,x;t\right) + \frac{1}{2}\sigma^2\left(s,x\right) \partial_{xx}^2 m\left(s,x;t\right) = 0$$

□

In order to prove Theorem 3.3.1, it remains to show that we can let ε go to 0 in (3.3.2). This is done, assuming $(\mathscr{H}_3)$, and using Lemma 3.7.4 with

$$V\left(s,x\right), = m\left(s,x;t\right), \; f\left(s,x\right) = \sigma^2\left(s,x\right)\left(\partial_x m\left(s,x;t\right)\right)^2 q\left(s,x\right)$$

for $s,t \in [0,T[$, $t \geq s$. Details of this technical part of the proof are given page 69.

Note that as $\left(m\left(0,t\right)\right)^+$ remains constant between two successive coupon payment dates, an incremental reformulation of the expected exposure is then possible. It is given in the following corollary.

Corollary 3.3.4 (Incremental exposure)**.** *Given Theorem 3.3.1, one has* $\forall i \in [[1 \ldots N]]$,

$$\mathbf{EE}\left(T_{i-1}\right) = \left(m\left(0,T_{i-1}\right)\right)^+ + \frac{1}{2}\int_0^{T_{i-1}} \sum_{j=1}^{n} \sigma^2\left(s, l_j\left(s,T_{i-1}\right)\right) \left|\partial_x m\left(s, l_j\left(s,T_{i-1}\right), T_{i-1}\right)\right| q\left(s, l_j\left(s,T_{i-1}\right)\right) ds \quad (3.3.3)$$

and $\forall t \in [T_{i-1}, T_i[$

$$\mathbf{EE}(T_i) = \mathbf{EE}(T_{i-1}) + \frac{1}{2}\int_{T_{i-1}}^{t} \sum_{j=1}^{n} \sigma^2\left(s, l_j(s,t)\right) \left|\partial_x m\left(s, l_j(s,T_{i-1}); t\right)\right| q\left(s, l_j(s,t)\right) ds \quad (3.3.4)$$

Proof. The proof is straightforward using (3.3.1) and by separating cash flows dates. □

Remark 3.3.5. Different choices of the state variable X, and implicitly q, are possible for pricing a contingent claim. In the purpose of using formula (3.3.1), an optimal choice of X should satisfy $(\mathscr{H}_3)$, and in particular, an explicit calculation of $\mathscr{L}$ set.

Remark 3.3.6. Assumption 3.2.3 can be extended to the case where the number of root functions in $\mathscr{L}$ depends on t.

3.3.2 Valuation examples in dimension one

We illustrate the result (3.3.1) with two pricing examples. The implementation aspects and numerical results are addressed in Section 3.5. Representation (3.3.1) mainly requires an explicit knowledge of the density function q, as well as the set $\mathscr{L}$. The density q function is required only for terminal transitions, which is given explicitly for an important variety of diffusion processes such as the normal and log-normal diffusions. q can also be implied numerically from market prices of vanilla options.

As stated earlier, we emphasize that the $(l_i)_{i\leq n}$ terms depend on the choice of the risk variable. For instance, in an interest rate swap, one may consider to model the underlying swap rate instead of taking a short rate model. The $\{l_i\}$ terms would not have then the same significance. Moreover, they are not generally given in an explicit expression and their computation would require the use of a root-finding algorithm. Therefore, one should take into account a modeling criteria that render straightforward the calculation of $\mathscr{L}$.

Equity forward contract:

We consider an Equity forward contract for which the expected exposure in given by the prices of European call options. We refer by S the price of an Equity asset and by X its logarithm. We assume that X has the following dynamics

$$\begin{cases} dX_s &= -\frac{1}{2}\sigma(s,X_s)^2 ds + \sigma(s,X_s) dW_s \\ X_0 &= x \end{cases} \quad (3.3.5)$$

where σ the local volatility function. The price of the forward contract is given at time s by

$$m(s,x;t) = \mathbb{E}_s\left[e^{X_T^{s,x}} - K\right] = e^x - K$$

and the expected exposure by the call option price

$$EE(t) = \mathbb{E}\left[m(t,X_t,t)^+\right] = \mathbb{E}\left[\left(e^{X_t} - K\right)^+\right]$$

$EE(t)$ is given explicitly in case of the classical Black model or some special cases of local volatility models such as the *Constant Elasticity of Variance (CEV)*. For any local volatility function σ, $EE(t)$ is given explicitly using our forward representation with:

- $n = 1$
- $l_1(s,t) = \ln(K)$ such that $m(s, l_1(s,t), t) = 0$
- $\partial_x m(s,x;t)|_{x=l_1(s,t)} = e^{l_1(s,t)} = K$
- $q(s,x)$ is the density function X_s, either given explicitly or constructed numerically.

Interest rate swap:

We consider an Interest rate swap as an example of a contract with multiple payments. The typical example is where the Libor rate, fixed in advance, is swapped at the same frequency for a pre-agreed fixed rate. We recall that $\{T_i\}_{i=1,\dots,N}$ denotes the swap payment dates and we denote the accrual periods by $\tau_i = \Delta T_i$. The Libor rate is refereed by L, the corresponding zero-coupon bond by P and the fixed rate by K.

The price of the vanilla swap is

$$m(s,t) = \sum_{i|T_i \geq t}^{N} \left(P(s,T_{i-1}) - P(s,T_i)(1+\tau_i K)\right) \tag{3.3.6}$$

We use the 1-factor Hull&White short rate model that states that

$$P(s,T) := P(s,r_s,T) = \mathbb{E}_s\left[e^{-\int_s^T r_u du}\right] = A(s,T)\exp\left(-B(s,T)\,r_s\right) \tag{3.3.7}$$

where $(r_u)_u$ is driven by an affine diffusion process so that functions A and B are given explicitly.

Using the forward representation (3.3.1), the expected exposure $EE(t)$ is computed by taking for $s \leq t$

- $\partial_x m(s,r_s,t) = \sum_{i|T_i \geq t}^{N} B(s,T_i)P(s,r_s,T_i)(1+\tau K) - B(s,T_{i-1})P(s,r_s,T_{i-1})$
- $\sigma(s,r_s) = \sigma$
- q is the density function of the Gaussian distribution.

Due to multiple payments, there is no explicit formulation of the $l_j(s,t)$ points. However, a very accurate approximation can be performed using a polynomial expansion of the exponential function in $P(s,r_s,T)$.

3.4 The forward exposure representation for $d \geq 2$

We propose a generalization of Theorem 3.3.1 to the case where X is an $\mathbb{R}^d-$valued process. We introduce the *co-area formula* in order to handle the limit $\varepsilon \to 0$. The derivation arguments invoked in Section 3.3 (before going to the limit in ε) remain valid in the multidimensional case. However, we take advantage of the co-area formula in order to perform a simpler derivation.

Our main result expresses the expected exposure in terms of the $(d-1)$-Hausdorff measure (see [EG15] for a definition of this measure) on the set $\Upsilon_{s,t}(0)$ defined by

$$\Upsilon_{s,t}(z) = \left\{x \in \mathbb{R}^d | m(s,x;t) = z\right\} \tag{3.4.1}$$

We denote by $\nu_{\Upsilon_{s,t}(z)}$ this Hausdorff measure defined on $\Upsilon_{s,t}(z)$. Note that this measure coincided with the trace of the Lebesgue measure on $\Upsilon_{s,t}$ in the case $d = 1$, see Lemma 3.7.6 and references for details.

Assumption 3.4.1 ($\mathscr{H}_4$). *For any $s \leq t$, we assume that $\nu_{\Upsilon_{s,t}(z)}(dx)$ is continuous at $z = 0$ in the sense of the narrow convergence of measure and that, there exist $\varepsilon > 0$, $\alpha > \frac{d}{2} - 1$ and $K > 0$, such that for all $s \leq t$ and $z \in [-\varepsilon, \varepsilon]$*

$$\int_{\Upsilon_{s,t}(z)} |\nabla_x m(s,x;t)| \, d\nu_{\Upsilon_{s,t}(z)}(x) \leq K s^{\alpha} < +\infty \tag{3.4.2}$$

We now state our main result in a multidimensional framework. We give a derivation that is based on the co-area formula as addressed in Lemma 3.7.6. We use in particular the conditional expectation formula (3.7.21) that results from the *Conditional law* expression in Lemma 3.7.7.

Theorem 3.4.2. *Let*

$$h(s,X_s;t) \triangleq \sum_{i,j=1}^{d} \sum_{k=1}^{r} \nabla_{x_i} m(s,X_s;t) \, \sigma_{i,k}(s,X_s) \, \nabla_{x_j} m(s,X_s;t) \, \sigma_{j,k}(s,X_s) \tag{3.4.3}$$

Assuming $(\mathscr{H}_i)_{i=1\ldots4}$, $\frac{h(s,x;t)}{|\nabla_x m(s,x;t)|}$ and $q(s,x)$ are continuous and bounded. The expected exposure satisfies then the following equation

$$\mathbf{EE}(t) = (m(0,t))^{+} + \frac{1}{2} \int_0^t ds \int_{\Upsilon_{s,t}(0)} \frac{h(s,x;t)}{|\nabla_x m(s,x;t)|} q(s,x) \, d\nu_{\Upsilon_{s,t}(0)}(x) \tag{3.4.4}$$

Proof. We recall that we approximate $(m(s,X_s;t))^{+}$ by $\Phi_\varepsilon(m(s,X_s;t))$. Owing to the Itô lemma, one has

$$\begin{aligned} \Phi_\varepsilon(m(s,X_s;t)) = \Phi_\varepsilon(m(0,X_0;t)) &+ \int_0^s \phi_\varepsilon(m(u,X_u;t)) \, dm(u,X_u;t) \\ &+ \frac{1}{2} \int_0^s \delta_\varepsilon(m(u,X_u;t)) \, h(u,X_u;t) \, du \end{aligned}$$

where $h(s,X_s;t)$ is the quadratic variation of $m(s,X_s;t)$

$$\begin{aligned} h(s,X_s;t)ds &= \langle m(.,X_.;t), m(.,X_.;t) \rangle_s \\ &= \sum_{i,j=1}^{d} \sum_{k=1}^{r} \nabla_{x_i} m(s,X_s;t) \, \sigma_{i,k}(s,X_s) \, \nabla_{x_j} m(s,X_s;t) \, \sigma_{j,k}(s,X_s) ds \\ &= \left\langle \nabla_x m(s,X_s;t), \sigma(s,X_s) \, \sigma^T(s,X_s) \, \nabla_x m(s,X_s;t) \right\rangle ds. \end{aligned}$$

given that

$$dm(s,X_s;t) = \sum_{i=1}^{d} \nabla_{x_i} m(s,X_s;t) \sum_{k=1}^{d} \sigma_{i,k}(s,X_s) \, dW_s^k,$$

It readily follows that

$$\mathbb{E}[\Phi_\varepsilon(m(t,X_t;t))] = \Phi_\varepsilon(m(0,X_0;t)) + \frac{1}{2} \int_0^t \mathbb{E}[\delta_\varepsilon(m(s,X_s;t)) \, h(s,X_s;t)] \, ds$$

The next step consists in applying the co-area Lemma 3.7.7, in the form given by Equation (3.7.21) with $g = \delta_\varepsilon$ and $f = h$:

$$\mathbb{E}\left[\delta_\varepsilon\left(m\left(s,X_s;t\right)\right)h\left(s,X_s;t\right)\right] = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} dz \int_{\Upsilon_{s,t}(z)} \frac{h\left(s,x;t\right)}{\left|\nabla_x m\left(s,x;t\right)\right|} q\left(s,x\right) d\nu_{\Upsilon_{s,t}(z)}\left(x\right)$$

Given the ellipticity assumption $(\mathscr{H}_1)$, $\sigma\left(s,x\right)\sigma^T\left(s,x\right)$ is assumed to be bounded by $K_{\max} Id$ leading to

$$\left|h\left(s,x;t\right)\right| = \left|\left\langle \nabla_x m\left(s,x;t\right), \sigma\left(s,x\right)\sigma^T\left(s,x\right)\nabla_x m\left(s,x;t\right)\right\rangle\right| \leq K_{\max}\left|\nabla_x m\left(s,x;t\right)\right|^2,$$

so $\frac{h(s,x;t)}{|\nabla_x m(s,x;t)|}$ is bounded by $K_{\max}\left|\nabla_x m\left(s,x;t\right)\right|$ and so can be prolonged by continuity on the set $\left|\nabla_x m\left(s,x;t\right)\right| = 0$. Under continuity assumption $(\mathscr{H}_3)$, the limit $\varepsilon \to 0$ is

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\delta_\varepsilon\left(m\left(s,X_s;t\right)\right)h\left(s,X_s;t\right)\right] = \int_{\Upsilon_{s,t}(0)} \frac{h\left(s,x;t\right)}{\left|\nabla_x m\left(s,x;t\right)\right|} q\left(s,x\right) d\nu_{\Upsilon_{s,t}(0)}\left(x\right)$$

We conclude the proof using Lebesgue theorem in s using Assumption $(\mathscr{H}_4)$. □

3.4.1 Case $X \in \mathbb{R}^2$

For $0 \leq s \leq t$, $\Upsilon_{s,t}\left(0\right)$ corresponds to the set of points (x_0,y_0) where $m\left(s,\left(x_0,y_0\right);t\right) = 0$ and $\left|\nabla_{(x,y)} m\left(s,\left(x_0,y_0\right);t\right)\right| \neq 0$. The case $d = 2$ can be made more explicit by the means of the implicit function theorem.

Proposition 3.4.3 (Implicit function theorem)**.** *Let $(x_0,y_0) \in \Upsilon_{s,t}\left(0\right)$. We admit that $\partial_y m\left(s,\left(x_0,y_0\right);t\right) \neq 0$. There exists a function $x \mapsto \psi_t\left(s,x\right)$ defined on an open interval W containing x_0, and an open set $\mathscr{V}_t$ in $W \times \mathbb{R}$, such that*

$$\forall\left(x,y\right) \in \mathscr{V},\ \left(x,y\right) \in \Upsilon_{s,t}\left(0\right) \Leftrightarrow \psi_t\left(s,x\right) = y$$

The condition $\partial_y m\left(s,\left(x_0,y_0\right);t\right) \neq 0$ is not restrictive with regard to $(\mathscr{H}_3)$. In fact, one has to consider an explicit function $x = \bar{\psi}_t\left(s,y\right)$ if $\partial_y m\left(s,\left(x_0,y_0\right);t\right) = 0$ and hence $\partial_x m\left(s,\left(x_0,y_0\right);t\right) \neq 0$.

The implicit function theorem allows to derive an explicit representation of the surface integral on $\Upsilon_{s,t}\left(0\right)$. A simple situation consists in having $\mathscr{V}_t = \Upsilon_{s,t}\left(0\right)$ and uniqueness of the implicit function ψ_t. It follows then that the Hausdorff surface measure $\nu_{\Upsilon_{s,t}(0)}\left(dx\right)$ is induced by the Lebesgue real measure dx. Equation (3.4.4) becomes

$$\mathbf{EE}\left(t\right) = \left(m\left(0,t\right)\right)^+ + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} \frac{h\left(s,\left(x,\psi_t\left(s,x\right)\right);t\right)}{\left|\nabla_x m\left(s,\left(x,\psi_t\left(s,x\right)\right);t\right)\right|} q\left(s,\left(x,\psi_t\left(s,x\right)\right)\right) dx$$

The existence of ψ_t is guaranteed by the implicit function theorem for any $(x_0,y_0) \in \Upsilon_{s,t}\left(0\right)$. We denote the set of implicit functions characterizing $\Upsilon_{s,t}\left(0\right)$ by Ψ_t. In the case where Ψ_t is a denumerable and is explicit, one has

$$\mathbf{EE}\left(t\right) = \left(m\left(0,t\right)\right)^+ + \frac{1}{2}\sum_{\psi_t\in\Psi_t}\int_0^t ds \int_{\mathbb{R}} \frac{h\left(s,\left(x,\psi_t\left(s,x\right)\right);t\right)}{\left|\nabla_x m\left(s,\left(x,\psi_t\left(s,x\right)\right);t\right)\right|} q\left(s,\left(x,\psi_t\left(s,x\right)\right)\right) dx \quad (3.4.5)$$

It is worth mentioning that, thanks to the implicit function theorem, the expected exposure formula (3.4.4) can be run at the cost of numerical integration in $\mathbb{R}$. The same argument can be adapted to $d \geq 3$, and shows that a numerical integration in $\mathbb{R}^{d-1}$ is required to compute the expected exposure in $\mathbb{R}^d$.

3.4.2 A valuation example in dimension 2

We consider a FX swap under the joint Garman-Kolhagen/Hull-White model. The swap corresponds to receiving a float rate in the domestic currency and paying a fixed rate in the foreign currency. We denote by P and $\tilde{P}$ the zero-coupon bond price functions in respectively the domestic and foreign currencies, and $\left(X_t^{f/d}\right)$ the price of 1 monetary unit of the foreign currency expressed in the domestic currency.

The mark-to-market function m reads

$$\begin{aligned} m(s,t) &= \sum_{i|T_i \geq t}^{N} \mathbb{E}_s \left[P(s,T_i)\, \tau_i \left(L(T_{i-1},T_i) - X_{T_i} K \right) \right] \\ &= \sum_{i|T_i \geq t}^{N} \mathbb{E}_s \left[P(s,T_{i-1}) - P(s,T_i) \right] - \sum_{i|T_i \geq t}^{N} \tau_i K \mathbb{E}_s \left[P(s,T_i) X_{T_i}^{f/d} \right] \\ &= P\left(s, T_{\beta(t)-1}\right) - P(s,T_N) - X_s^{f/d} \sum_{i|T_i \geq t}^{N} \tau_i K \tilde{P}(s,T_i) \end{aligned}$$

where $\beta(t) = \inf\{i \in [1..N] \,|\, T_i \geq t\}$.

As for the IR swap, we consider the 1-factor Hull&White short rate model that states that

$$\begin{cases} r_u &= \phi(u) + \sigma \int_t^u e^{-a(u-\upsilon)} dW_\upsilon^r \\ \tilde{r}_u &= r_u + s(u) \end{cases}$$

where a and σ are respectively the mean-reversion and the volatility parameters and ϕ is a deterministic spread function between domestic and foreign short rates r and $\tilde{r}$. Consequently, the prices P and $\tilde{P}$ are explicit functions of r

$$\begin{cases} P(s,T) &\triangleq P(s,r_s,T) = \mathbb{E}_s \left[e^{-\int_s^T r_u du} \right] = A(s,T) \exp\left(-B(s,T) r_s\right) \\ \tilde{P}(s,T) &\triangleq \tilde{P}(s,\tilde{r}_s,T) = \mathbb{E}_s \left[e^{-\int_s^T \tilde{r}_u du} \right] = \tilde{A}(s,T) \exp\left(-\tilde{B}(s,T) r_s\right) \end{cases} \tag{3.4.6}$$

Finally, the exchange rate is driven by a log-normal dynamics such that

$$d \ln\left(X_u^{f/d}\right) = \left[r_u - \tilde{r}_u - \frac{1}{2} \left(\sigma^X \left(u, X_u^{f/d} \right) \right)^2 \right] du + \sigma^X \left(u, X_u^{f/d} \right) dW_u^X$$

The Brownian motions W^r and W^X are correlated, i.e. $d\left\langle W^r, W^X \right\rangle_t = \rho dt$.

In order to compute the expected exposure $EE(t)$, we specify quantities that are involved in the forward representation (3.3.1). For $s \leq t$,

- $\partial_r m\left(s, r_s, X_s^{f/d}; t\right) = -(B.P)\left(s, T_{\beta(t)-1}\right) + (B.P)(s,T_N) + X_t^{f/d} \sum_{i|T_i \geq t}^{N} \tau_i K \left(\tilde{B}.\tilde{P}\right)(s,T_i)$. In particular $|\, \nabla_{(r,X)} m \,| \neq 0$ since $\partial_X m\left(s, r_s, X_s^{f/d}; t\right) \neq 0$.

- $\partial_X m\left(s, r_s, X_s^{f/d}; t\right) = -\sum_{i|T_i \geq t}^{N} \tau_i K \tilde{P}(s,T_i)$

- $\sigma(s,(r_s,X_s))$ is given through the marginal densities of r_s and X_s, as well as the density of the Gaussian copula. In case where $\sigma^X\left(u,X_u^{f/d}\right)=\sigma^X(u)$, the derivation of the joint density corresponds to the bivariate Gaussian density.
- q is the transition density function of the Gaussian distribution.

Finally, the set $\Upsilon_{s,t}(0)$ is entirely specified using the implicit function theorem. In fact, one can easily find that $m\left(s,r_s,X_s^{f/d},t\right)=0$ implies that

$$X_s^{f/d}=\frac{P\left(s,T_{\beta(t)-1}\right)-P(s,T_N)}{\sum_{i|T_i\geq t}^{N}\tau_i K\tilde{P}(s,T_i)}=\psi_t(r_s)$$

and the implicit theorem function ψ_t is given explicitly in terms of A, $\tilde{A}$, B and $\tilde{B}$.

3.5 Computation of sensitivities

One can use the integral representation (3.3.1) in order to retrieve analytical formulas for the exposure sensitivities. For instance, we take the derivative w.r.t. x_0:

$$\begin{aligned}\partial_{x_0}\mathbf{EE}(t)=\partial_{x_0}m(0,x_0,t)\,1_{m(0,x_0,t)>0}\\ +\frac{1}{2}\int_0^t\sum_{i=1}^{n}\sigma^2\left(s,l_j(s,t)\right)\left|\partial_x m\left(s,l_j(s,t);t\right)\right|\partial_{x_0}q\left(s,l_j(s,t)\right)ds\end{aligned}\quad(3.5.1)$$

where $q\left(s,l_j(s,t)\right)\triangleq q\left(s,l_j(s,t);0,x_0\right)$ and $\partial_{x_0}q\left(s,l_j(s,t)\right)$ is its partial derivative w.r.t. to the initial value of x_0. The strong Markov property retained through m ensures that $\partial_{x_0}l_j(s,t)=0$. We notice that $t\mapsto\partial_{x_0}m(0,x_0,t)$ corresponds to sensitivity profile of the forward mark-to-market.

The second-order sensitivity (or Gamma) is obtained by a straightforward derivation. It reads

$$\begin{aligned}\partial_{x_0}^2\mathbf{EE}(t)=\partial_{x_0}^2m(0,x_0,t)\,1_{m(0,x_0,t)>0}\\ +\frac{1}{2}\int_0^t\sum_{i=1}^{n}\sigma^2\left(s,l_j(s,t)\right)\left|\partial_x m\left(s,l_j(s,t);t\right)\right|\partial_{x_0}^2q\left(s,l_j(s,t)\right)ds\end{aligned}\quad(3.5.2)$$

From an algorithmic perspective, q, $\partial_{x_0}q(|0,x_0)$ and $\partial_{x_0}^2q(|0,x_0)$ can be computed simultaneously and injected into (3.3.1), (3.5.1) and (3.5.2). For additional sensitivities, the same derivation remains applicable and leads to analog formulas.

3.6 Numerical experiments

We present in this section numerical results based on the benchmark examples that we described in Sections 3.3 and 3.4. The computation of the expected exposure relies on the incremental scheme that we introduced in Corollary 3.3.4. We consider a Riemann approximation of the integral with a arbitrarily small time step in order to achieve convergence.

Our simulations are proceeded on a system with Intel(R) Duo-Core 2.13 GHz processors and 4 GB RAM. We use Monte Carlo simulations when closed formulas of the exposure are not available. We report the elapsed time taken of each simulation.

3.6.1 Equity forward contract under the CEV model

Under the *Constant Elasticity of Variance*, we assume that log-spot process X has the following dynamics

$$dX_s = -\frac{1}{2}\sigma^2 e^{2X_s(\gamma-1)}ds + \sigma e^{X_s(\gamma-1)}dW_s \ , \ s \leq t$$

with constant parameters $\sigma \geq 0$ and $\gamma \geq 0$. The parameter γ is the central feature of the CEV model since it controls the so-called leverage effect.

The expected exposure of an Equity forward contract is given by the price of vanilla call option. Under the CEV model, the pricing function is explicit.

$$Call_{CEV}(\tau,K) = \begin{cases} e^{X_0}\left[1-\chi^2(a,b+2,c)\right] - K\chi^2(c,b,a) & \text{when } 0<\gamma<1 \\ e^{X_0}\left[1-\chi^2(c,-b,a)\right] - K\chi^2(a,2-b,c) & \text{when } \gamma>1 \end{cases}$$

with $a = \frac{K^{2(1-\gamma)}}{\nu(1-\gamma)^2}$, $b = \frac{1}{1-\gamma}$, $c = \frac{e^{X_0(1-\gamma)}}{\sigma\nu(1-\gamma)}$, and $\nu = \sigma^2 T$. $\chi^2(z,k,v)$ is the cumulative probability that a variable with a non-central χ^2 distribution with non-centrality parameter v and k degrees of freedom is less that z.

The CEV transition density is obtained from the well known expression for the transition density of the Bessel process (see [RY94], p. 446). It is given by

$$q(s,x) = \frac{e^{x\left(-2\gamma+\frac{3}{2}\right)+\frac{1}{2}x_0}}{\sigma^2 s \mid \gamma-1 \mid} I_{\mid\frac{1}{2(\gamma-1)}\mid}\left(\frac{e^{-(\gamma-1)(x+x_0)}}{\sigma^2 t(\gamma-1)^2}\right) e^{-\frac{e^{-2x(\gamma-1)}+e^{-2x_0(\gamma-1)}}{2\sigma^2 t(\gamma-1)^2}}$$

where I_ν is the modified Bessel function of the first kind of order ν.

The simulation parameters are taken as follow:

X_0	K	T	T_i	σ
0	1	10 Y	$T \times \frac{i}{10^6}$	50%

Figure 3.6.1 shows the results of the forward solution versus the exact one obtained through the above closed formula for $\gamma = 0.75$. The execution times of both solutions are as the following:

Exact solution	640 ms
Forward solution	170 ms

3.6.2 Interest Rates Swap under the Hull-White model

Under the Hull-White model, q is the transition density function of the Gaussian distribution.

Our forward representation is compared to Monte Carlo estimation. The simulation parameters are taken as follow:

T	K	τ	a	σ	T_i	ϕ	M (Nb MC)
5 Y	ATM	0.5	1%	1%	$T \times \frac{i}{10^4}$	Arbitrary input	10^5

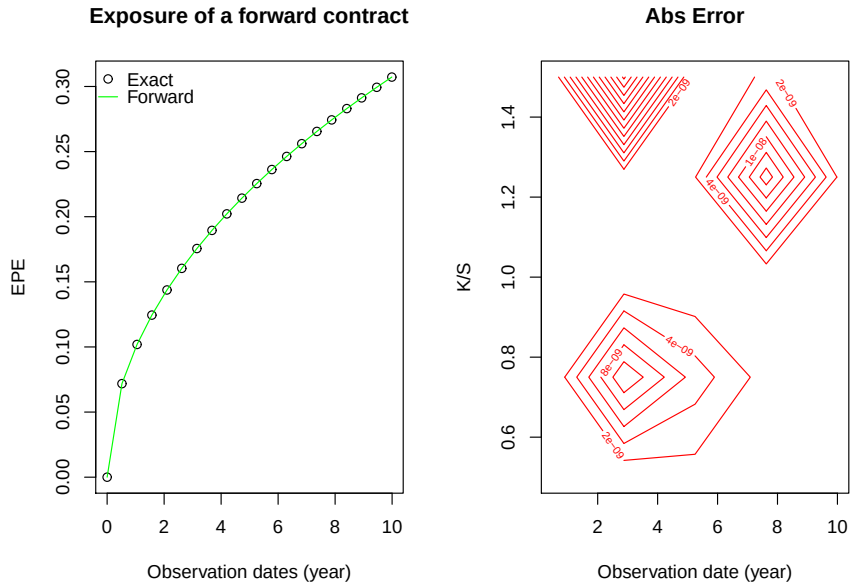


Figure 3.6.1: Expected exposure of an Equity Forward: **EE** profile (left), Absolute Error *vs.* closed-form solution (right)

The results are reported in Figure 3.6.2. The following table gives the execution times for both methods:

Exact solution	5.68 s
Forward solution	1.32 s

3.6.3 FX-Swap under the Garman-Kolhagen/Hull-White models

We consider constant IR and FX volatilities. This leads to simple expression of the density function q using the bivariate Gaussian distribution.

We compare our result to a crude Monte Carlo estimation. The simulation parameters are taken as follow:

T	K	τ	a	σ	T_i	$r-\tilde{r}$	σ^X	ϕ	M (Nb MC)
5 Y	ATM	0.5	1%	1%	$T \times \frac{i}{10^4}$	0.5%	20%	Arbitrary input	10^5

The results are reported in Figure 3.6.3. The following table gives the execution times for both methods:

Exact solution	9.43 s
Forward solution	1.72 s

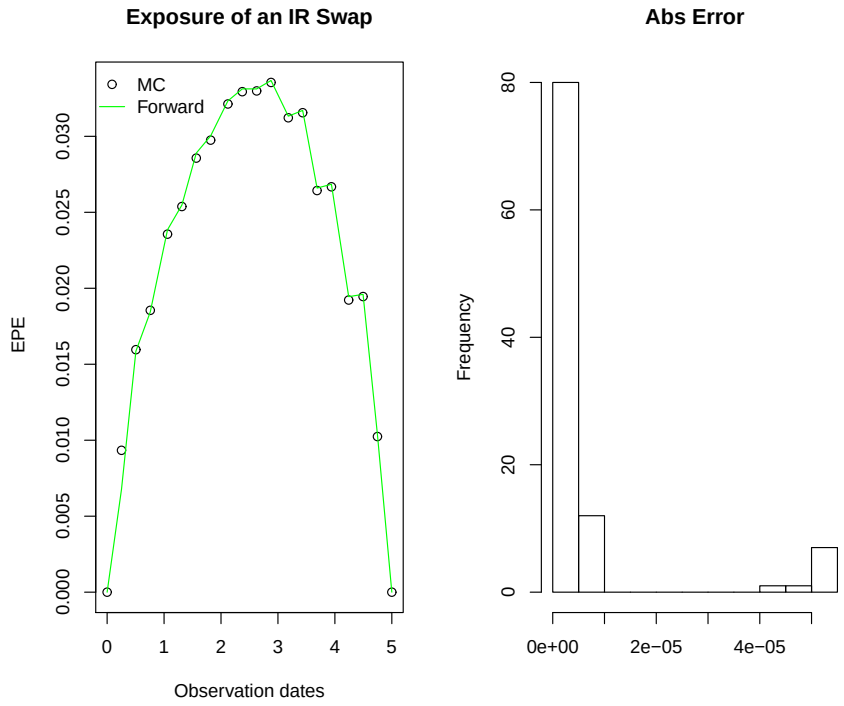


Figure 3.6.2: Expected exposure of a Vanilla Swap: **EE** profile (left), Absolute Error *vs.* MC (right)

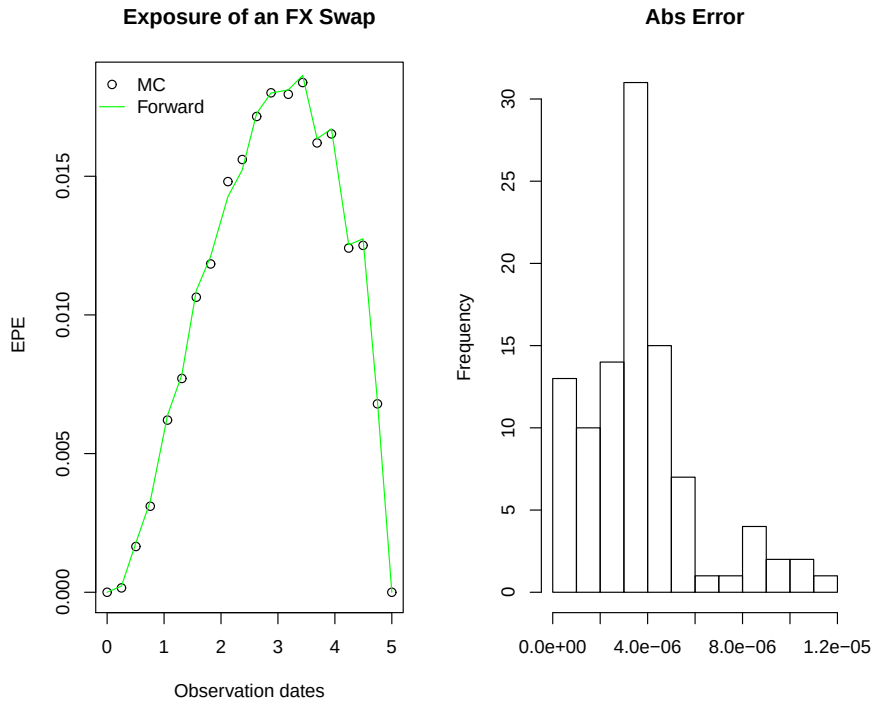


Figure 3.6.3: Expected Exposure of an FX Swap: **EE** profile (left), Absolute Error *vs.* MC (right)

3.7 Mathematical aspects of the forward representation

We address in this section the mathematical aspects on which is based the formal derivation of the forward representation of the exposure. Our proof mainly relies on an analytical arguments.

3.7.1 The unidimensional case

We begin the proof by stating a lemma that deals with a pricing function V which does not depend on time and is restrained on a compact interval in space.

Lemma 3.7.1. *Let $V : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function $\left(V \in \mathscr{C}^1(\mathbb{R} \to \mathbb{R})\right)$, such that there exists a unique $l \in \mathbb{R}$ where $V(l) = 0$ and $V'(l) \neq 0$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. One has for every M such that $l \in]-M, M[$*

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{-M}^{M} 1_{\{|V(x)|<\varepsilon\}} f(x)\, dx = \frac{f(l)}{|V'(l)|} \tag{3.7.1}$$

Proof. We begin by assuming that $V'(l) = \eta > 0$. As V' is continuous, their exists α such that, on $]l-\alpha, l+\alpha[$, $V'(x) \geq \eta/2$. This proves that V is a strictly-increasing bijection from $]l-\alpha, l+\alpha[$ to $]V(l-\alpha), V(l+\alpha)[$. We denote by V^{-1} its inverse. Note that V^{-1} is differentiable on $]V(l-\alpha), V(l+\alpha)[$ and that $\left(V^{-1}\right)'(0) = \dfrac{1}{V'(l)}$. Now, using compacity and the continuity of V, we have

$$\inf_{[-M,M]\setminus]l-\alpha, l+\alpha[} |V(x)| = \varepsilon_0 > 0$$

So, for $\varepsilon < \varepsilon_0$ we have

$$\{x \in [-M, M], |V(x)| < \varepsilon\} =]V^{-1}(-\varepsilon), V^{-1}(\varepsilon)[$$

and

$$\int_{-M}^{M} 1_{\{|V(x)|<\varepsilon\}} f(x)\, dx = \int_{V^{-1}(-\varepsilon)}^{V^{-1}(\varepsilon)} f(x)\, dx.$$

For ε small enough, we have

$$\frac{1}{2\varepsilon} \int_{-M}^{M} 1_{|V(x)|<\varepsilon} f(x)\, dx = f(l) \left(\frac{V^{-1}(\varepsilon) - V^{-1}(-\varepsilon)}{2\varepsilon} \right) + \frac{1}{2\varepsilon} \int_{V^{-1}(-\varepsilon)}^{V^{-1}(\varepsilon)} (f(x) - f(l))\, dx$$

The second term of the right-hand term is bounded by

$$\frac{1}{2\varepsilon} \left| \int_{V^{-1}(-\varepsilon)}^{V^{-1}(\varepsilon)} (f(x) - f(l))\, dx \right| \leq \max_{x \in \left[V^{-1}(-\varepsilon), V^{-1}(\varepsilon)\right]} |f(x) - f(l)| \left| \frac{V^{-1}(\varepsilon) - V^{-1}(-\varepsilon)}{2\varepsilon} \right|.$$

Using continuity of V^{-1} at 0 and of f at l, we get

$$\lim_{\varepsilon \to 0} \max_{x \in \left[V^{-1}(-\varepsilon), V^{-1}(\varepsilon)\right]} |f(x) - f(l)| = 0.$$

Now, as V^{-1} is differentiable at 0 with a derivative given by $\frac{1}{V'(l)}$, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-M}^{M}1_{\{|V(x)|<\varepsilon\}}f(x)\,dx=\frac{f(l)}{V'(l)}$$

The case where $V'(l)<0$ is treated similarly: as $V^{-1}(\varepsilon)<V^{-1}(-\varepsilon)$ one has

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-M}^{M}1_{\{|V(x)|<\varepsilon\}}f(x)\,dx=-\frac{f(l)}{V'(l)},$$

and we finally end with the limit (3.7.1). □

We now derive a time-integrated version of Lemma 3.7.1. We let $t\in\mathbb{R}_+$ and $(t,x)\mapsto V(t,x)$ and $(t,x)\mapsto f(t,x)$ be two functions of respectively $\mathscr{C}^{1,2}([0,t]\times\mathbb{R}\to\mathbb{R})$ and $\mathscr{C}^{1,1}([0,t]\times\mathbb{R}\to\mathbb{R})$. Following Assumption $(\mathscr{H}_2)$, $\partial_x V$ and $\partial_x^2 V$ are bounded.

Lemma 3.7.2. *We assume that, $V(s,x)$ is a C^1 function on $[0,t]\times\mathbb{R}$ and that $f(s,x)$ is a continuous function on $]0,t]\times\mathbb{R}$ such that $|f(s,x)|\leq\frac{K}{\sqrt{s}}e^{-c\frac{x^2}{s}}$ with $c,K>0$ (cf. $(\mathscr{H}_1)$). $\forall s\in[0,t]$, we assume that there exists a unique $l(s,t)\in\mathbb{R}$ such that $V(s,l(s,t))=0$ and $\partial_x V(s,l(s,t))\neq 0$. We also assume that $s\in[0,t]\mapsto l(s,t)$ is continuous. Then, for M large enough, one has*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t ds\int_{-M}^{M}dx1_{\{|V(s,x)|<\varepsilon\}}f(s,x)=\int_0^t ds\frac{f(s,l(s,t))}{|\partial_x V(s,l(s,t))|}\tag{3.7.2}$$

Proof. We firstly note that the sign of $s\in[0,t]\mapsto\partial_x V(s,l(s,t))$ must be constant since $\partial_x V(s,l(s))$ is continuous on $[0,t]$ and non zero. We will assume that $\partial_x V(s,l(s))$ is positive on $[0,t]$, so there exist $\eta>0$ such that $\partial_x V(s,l(s))\geq\eta$ for $s\in[0,t]$.

On the other hand, $(s,x)\in[0,t]\times[-M,M]\mapsto\partial_x V(s,l(s,t)+x)$ is uniformly continuous as a continuous function on a compact set. It follows that there exists $\alpha,\eta>0$ such that, for $s\in[0,t]$

$$|x|\leq\alpha\Rightarrow|\partial_x V(s,l(s,t)+x)-\partial_x V(s,l(s,t))|\leq\frac{\eta}{2}\tag{3.7.3}$$

and we can deduce that, for every $s\in[0,t]$ and for $|x|\leq\alpha$

$$|\partial_x V(s,l(s,t)+x)|\geq\frac{\eta}{2}.\tag{3.7.4}$$

$\forall s\leq t$, $x\to V(s,x)$ is then an increasing bijection from $]l(s,t)-\alpha,l(s,t)+\alpha[$ to the interval $]V(s,l(s,t)-\alpha),V(s,l(s,t)+\alpha)[$. For a fixed s, we denote by $V^{-1}(s,x)$ the inverse of this function for $x\in]l(s,t)-\alpha,l(s,t)+\alpha[$. Moreover, $V(s,x)>0$ on the compact set $\{(s,x),s\in[0,t]\times[-M,M],|x-l(s,t)|\geq\alpha\}$. By continuity and compacity, we have

$$\varepsilon_0\triangleq\inf_{s\in[0,t]\times[-M,M],|x-l(s,t)|\geq\alpha}|V(s,x)|>0,$$

and for $\varepsilon<\varepsilon_0$, for every time $s\in[0,t]$

$$\{x\in[-M,M],|V(s,x)|<\varepsilon\}=]V^{-1}(s,-\varepsilon),V^{-1}(s,\varepsilon)[,$$

and

$$\int_0^t ds \int_{-M}^{M} 1_{\{|V(s,x)|<\varepsilon\}} f(s,x)\, dx = \int_0^t ds \int_{V^{-1}(s,-\varepsilon)}^{V^{-1}(s,\varepsilon)} f(s,x)\, dx.$$

Let $s \in [0,t]$. Using the previous lemma 3.7.1 we have the simple convergence for every time s

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{-M}^{M} 1_{|V(s,x)|<\varepsilon} f(s,x)\, dx = \frac{f(s,l(s,t))}{|\partial_x V(s,l(s,t))|}.$$

Moreover we have, for $\varepsilon < \varepsilon_0$

$$\frac{1}{2\varepsilon} \left| \int_{-M}^{M} 1_{\{|V(s,x)|<\varepsilon\}} f(s,x)\, dx \right| \leq \frac{\sup_{x\in\left[V^{-1}(s,-\varepsilon),V^{-1}(s,\varepsilon)\right]} |f(s,x)|}{\inf_{x\in\left[V^{-1}(s,-\varepsilon),V^{-1}(s,\varepsilon)\right]} |\partial_x V(s,x)|}$$

But using the boundedness hypothesis on f, one has $\sup_{x\in\left[V^{-1}(s,-\varepsilon),V^{-1}(s,\varepsilon)\right]} |f(s,x)| < \frac{K}{\sqrt{s}}$. Now, using (3.7.4), we get $|V(s,x)| = |V(s,x) - V(s,l(s,t))| \geq |x - l(s,t)|\frac{\eta}{2}$. It follows that $V^{-1}(s,\varepsilon) \leq l(s,t) + \frac{2\varepsilon}{\eta}$ and $V^{-1}(s,-\varepsilon) \geq l(s,t) - \frac{2\varepsilon}{\eta}$. Then for ε small enough, and owing to Equation (3.7.3) , one has

$$\inf_{s\in[0,t]} \inf_{x\in\left[V^{-1}(s,-\varepsilon),V^{-1}(s,\varepsilon)\right]} |\partial_x V(s,x)| \geq \inf_{s\in[0,t]} \inf_{x\in\left[-\frac{2\varepsilon}{\eta},-\frac{2\varepsilon}{\eta}\right]} |\partial_x V(s,l(s,t)+x)| \geq \frac{\eta}{2}$$

We end-up with

$$\frac{1}{2\varepsilon} \int_{-M}^{M} 1_{|V(s,x)|<\varepsilon} |f(s,x)|\, dx \leq \frac{2K}{\eta\sqrt{s}}.$$

As $1/\sqrt{s}$ is integrable near 0, we obtain (3.7.2) using dominated convergence theorem. A slight variation of the previous arguments apply when $\partial_x V(s,l(s,t)) < 0$. □

We now need an additional assumption on the asymptotic behavior of the function V at infinity in order to extend the previous result with a space integral on $\mathbb{R}$.

Lemma 3.7.3. *Given Lemma 3.7.2, we assume there exist K and c positive constants such that for $s \leq t$ $|f(s,x)| \leq \frac{K}{\sqrt{s}} e^{-\frac{cx^2}{s}}$, and there exist β, $0 \leq \beta \leq \frac{c}{s}$, and $A, B > 0$, such that $|x| \geq A \Rightarrow |V(s,x)|\, e^{\beta x^2} \geq B$ (cf. $(\mathscr{H}_2)$).Then*

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t ds \int_{\mathbb{R}} 1_{\{|V(s,x)|<\varepsilon\}} f(s,x)\, dx = \int_0^t \frac{f(s,l(s,t))}{|\partial_x V(s,l(s,t))|} ds. \tag{3.7.5}$$

Proof. First note that, using the first hypothesis that $|V(t,x)| \leq \varepsilon$ and $|x| \geq A$ implies that $|x| \geq A(\varepsilon) = \sqrt{\frac{1}{\beta}\left|\log\left(\frac{\varepsilon}{B}\right)\right|}$. So for M large enough

$$\int_M^{+\infty} 1_{\{|V(s,x)|\leq\varepsilon\}} |f(s,x)|\, dx \leq \int_{A(\varepsilon)}^{+\infty} |f(s,x)|\, dx.$$

Now using the classical inequality $\int_x^{+\infty} e^{-\alpha x^2} dx \leq \frac{1}{2\alpha x} e^{-\alpha x^2}$ we obtain

$$\int_M^{+\infty} 1_{\{|V(s,x)|\leq\varepsilon\}} |f(s,x)| dx \leq \frac{K\sqrt{s}}{2cA(\varepsilon)} e^{-cA(\varepsilon)/s} = \frac{K\sqrt{s}}{2cA(\varepsilon)} \left(\frac{\varepsilon}{B}\right)^{\frac{c}{\beta s}}.$$

But, for all $s \leq t$, $\frac{c}{\beta s} \geq 1$ (cf. $(\mathscr{H}_2)$) and then , for ε smaller than B

$$\int_M^{+\infty} 1_{\{|V(s,x)|\leq\varepsilon\}} |f(s,x)| dx \leq \frac{K\sqrt{s}}{2cA(\varepsilon)} \frac{\varepsilon}{B}.$$

This prove that $\lim_{\varepsilon\to 0} \int_0^t ds \frac{1}{\varepsilon} \int_M^{+\infty} 1_{\{|V(s,x)|\leq\varepsilon\}} |f(s,x)| dx = 0$. We obtain the same zero limit for

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{+\infty}^{-M} 1_{\{|V(s,x)|\leq\varepsilon\}} |f(s,x)| dx$$

Put with (3.7.2), this allows to conclude the proof. □

For sake of simplicity, we have assumed in Lemmas 3.7.1 and 3.7.2 the uniqueness of $l(s,t)$ for $s \in [0,t]$. We can relax this assumption by showing that our statements hold if there exists a finite set of distinct roots $l_i(s,t)$, as assumed in $(\mathscr{H}_3)$.

Lemma 3.7.4. *Given $(\mathscr{H}_i)_{i=1\ldots 3}$, one has*

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t ds \int_{\mathbb{R}} dx 1_{|V(s,x)|<\varepsilon} f(s,x) = \int_0^t ds \sum_{i=1}^n \frac{f(s,l_i(s))}{|\partial_x V(s,l_i(s))|} \tag{3.7.6}$$

Proof. Using the asymptotic behavior of V and f we only have to prove that, for M large

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t ds \int_{-M}^{M} dx 1_{|V(s,x)|<\varepsilon} f(s,x) = \int_0^t ds \sum_{i=1}^n \frac{f(s,l_i(s,t))}{|\partial_x V(s,l_i(s,t))|}.$$

Because all continuous functions $(l_i(s,t), 0 \leq s \leq t)_{1\leq i\leq n}$ are never equal, we can find an $\alpha > 0$ such that $]l_i(s,t) - \alpha, l_i(s,t) + \alpha[$ are disjoint for all time $s \in [0,t]$. Now, using $V_x(s,l_i(s,t)) \neq 0$ as in the previous lemmas, we can assume that, for $i = 1,\ldots,n$, $V(s,x)$ is bijective from $]l_i(s,t) - \alpha, l_i(s,t) + \alpha[$ to its image. We denote by $V_i^{-1}(s,x)$ the local inverse of V on this i-th interval. Now by choosing ε small enough, we have (using $\pm$ or $\mp$ in function of the sign of $V_x'(s,l_i(s))$)

$$\{x \in [-M,M], |V(s,x)| < \varepsilon\} = \cup_{i=1}^n]V_i^{-1}(s,\pm\varepsilon), V_i^{-1}(s,\mp\varepsilon)[,$$

and it follows that

$$\frac{1}{2\varepsilon} \int_{\mathbb{R}} 1_{|V(s,x)|<\varepsilon} f(s,x) dx = \frac{1}{2\varepsilon} \sum_{i=1}^n \int_{V_i^{-1}(s,\pm\varepsilon)}^{V_i^{-1}(s,\mp\varepsilon)} f(s,x) dx$$

The remaining steps are based on the same same arguments of previous Lemmas 3.7.1 and 3.7.2. □

Link with Local Times

Theorem 3.3.1 is tightly linked to local times theory (we refer to chapter 6 of [RY94] for background). This paragraph informally outline this link.

Let X_s be a local martingale as defined before. Then, by applying Ito-Meyer formula, one has:

$$X_s^+ = X_0^+ + \int_O^t \operatorname{sign}(X_s)\, dX_s + \frac{1}{2} L_s^0(X) \tag{3.7.7}$$

where $L_s^0(X)$ is the local time of X at 0 up to time s. Under additional hypothesis on integrability of X, we have $\mathbb{E}\left(\int_O^t \operatorname{sign}(X_s)\, dX_s\right) = 0$, and $\mathbb{E}(X_s^+) = \mathbb{E}\left(X_0^+\right) + \frac{1}{2}\mathbb{E}\left(L_s^0(X)\right)$. Moreover, using the occupation time formula, one has:

$$\int_0^t \delta_\varepsilon(X_s)\, d\langle X_s, X_s\rangle = \int_{-\infty}^{\infty} \delta_\varepsilon(a)\, L_t^a(X)\, da \tag{3.7.8}$$

with δ_ε the regularization function defined previously. Using the regularity assumptions on X and q continuity for $L_t^a(X)$ and the regularity of the density $q(t,x)$, the limit $\varepsilon \to 0$ yields

$$\mathbb{E}\left(L_t^0\right) = \int_0^t q(s,0)\, \sigma^2(s,0)\, ds \tag{3.7.9}$$

In order to derive (3.3.1), we need to express the local time of $Y_t = V(t, X_t)$ in term of the locals times of X. In the time-homogeneous case, i.e. $V(s,X_s) = V(X_s)$, this link is given in [CO00] (extending [ÉY98]). We reformulate a corollary of their results:

Corollary 3.7.5 ([CO00])**.** *Let X be a continuous semimartingale, and V denote a positive function which is the difference of two convex functions, with respective left and right derivatives V_g' and V_d' and distribution second derivative f''. Moreover, we define $Y_t = V(X_t)$ and we set*

$$B = V^{-1}(0) \cap \left\{a, \mid V_g'(a) \mid + \mid V_d'(a) \mid > 0\right\} \tag{3.7.10}$$

Then B is denumerable and

$$L_s(Y) = \sum_{a \in B} \left(\left(V_d'(a)\right)^+ L_s^a(X) + \left(V_g'(a)\right)^- L_s^{a-}(X) \right). \tag{3.7.11}$$

Under regularity assumptions $(\mathscr{H}_1)$ and $(\mathscr{H}_3)$, the elements of B correspond to the roots $(l_i)_i$, and this result can be rewritten as

$$L_s^0(Y) = \sum_{i=0}^{n} \left|V'(l_i)\right| L_s^{l_i}(X) \tag{3.7.12}$$

Now using $d\mathbb{E}\left[L_u^{l_i}(X)\right] = \sigma^2(u,l_i)\, q(u,l_i)\, du$, a time-homogenous expression, analog to equation (3.3.1), is obtained:

$$\mathbb{E}\left[(Y_s)^+\right] = (Y_0)^+ + \frac{1}{2} \sum_{l_i \in B} \mid V(l_i) \mid \int_0^s \sigma^2(u,l_i)\, q(u,l_i)\, du \tag{3.7.13}$$

The natural extension of (3.7.12) to a time-non-homogeneous case is

$$L_s^0(Y) = \sum_{i=0}^{n} \left|V'(l_i(s,t))\right| L_s^{l_i(s,t)}(X) \tag{3.7.14}$$

can be achieved at the cost of additional formalism and leads, by taking expectation, to equation (3.3.1).

3.7.2 The co-area formula for the multidimensional case

We introduce here the co-area formula which is useful to achieve the proof of our main result in the multidimensional case.

Lemma 3.7.6 (Co-area formula)**.** *Let V be a $\mathscr{C}^1\left(\mathbb{R}^d\right)$ function with bounded derivatives, we denote by $\Upsilon(z)$ its z-level surface:*

$$\Upsilon_t(z) = \left\{x \in \mathbb{R}^d, V(x) = z\right\} \tag{3.7.15}$$

and we denote by $\nu_{\Upsilon(z)}(dx)$ the $d-1$ Hausdorff measure on $\Upsilon(z)$. For a positive and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\int_{\mathbb{R}} dz \int f(x)\, d\nu_{\Upsilon_t(z)}(x) = \int_{\mathbb{R}^d} f(x)\, |\nabla_x V(x)|\, dx. \tag{3.7.16}$$

The regular case ($V \in \mathscr{C}^1\left(\mathbb{R}^d\right)$) can be obtained using a change of variable in $\mathbb{R}^d$. See for instance [LSR10] for a proof in this case and [Fed14], [EG15] for irregular generalizations (V Lipschitz) and background on Hausdorff measures.

Assuming that $|\nabla_x V(x)| \neq 0\ \forall x \in \mathbb{R}^d$, the co-area formula allows to define an adequate generalization of the "Dirac measure", $\Gamma_z(dx)$, on $\Upsilon(z)$ by setting:

$$d\Gamma_z(x) = \frac{d\nu_{\Upsilon(z)}(x)}{|\nabla_x V(x)|} \tag{3.7.17}$$

With this definition, we obviously have (using the co-area formula), for every positive measurable function f

$$\int f(x)\, dx = \int_{\mathbb{R}} dz \int_{\Upsilon(z)} f(x)\, d\Gamma_z(x) \tag{3.7.18}$$

This equation can be formally rewritten as "$dx = \int_{\mathbb{R}} dz d\Gamma_z(x)$".

Assume now that X is a random variable taking its values in $\mathbb{R}^d$ and follows a law with density $q(x)$. As a corollary of the co-area formula we will have an explicit representation of the law of $V(X)$, as well as the law of X conditionally to $V(X)$.

Lemma 3.7.7 (Conditional laws)**.** *$V(X)$ follows a law with density $q_V(z)$ where*

$$q_V(z) = \int_{\Upsilon_t(z)} q(x)\, \frac{d\nu_{\Upsilon_t(z)}(x)}{|\nabla_x V(x)|} \tag{3.7.19}$$

and the law of X conditionally to $V(X) = z$ is given by

$$\frac{q(x)}{q_V(z)} \frac{d\nu_{\Upsilon_t(z)}(x)}{|\nabla_x V(x)|} \tag{3.7.20}$$

This lemma readily follows that, for two arbitrary positive and measurable functions, f and g, one has, using the co-area formula

$$\begin{aligned} \mathbb{E}\left[g(V(X))\, f(X)\right] &= \int_{\mathbb{R}} g(V(x))\, f(x)\, q(x)\, dx \\ &= \int_{\mathbb{R}} g(z)\, q_V(z)\, dz \int_{\Upsilon(z)} f(x)\, \frac{q(x)}{q_V(z)} \frac{d\nu_{\Upsilon_t(z)}(x)}{|\nabla_x V(x)|}. \end{aligned} \tag{3.7.21}$$

3.8 Conclusion

In this chapter, we introduced a new result for computing derivatives expected exposures. The forward representation is an exact decomposition of the expected exposure into an intrinsic exposure, i.e. positive part of the current mark-to-market, and time value exposure which depends on the forward delta sensitivity as well as the local volatility function of the underlying risk factors.

The unidimensional case is derived analytically, although very close links with the co-area formula and the local times are highlighted. Our result can be interpreted as an extension of the occupation time formula to the non-homogenous (in time) case. In the multidimensional extension, we use the co-area formula and show that, in $\mathbb{R}^d$, an integration in $\mathbb{R}^{d-1}$ is needed to compute the expected exposure.

Through numerical examples, we show that our representation leads to an efficient computation of the expected exposure in cases where $d = 1$ and $d = 2$. More generally, the use of the forward representation requires first the calculation of auxiliary quantities such as the derivative's delta and the density function q along the set $\Upsilon(0)$ which needs be specified too. This is restrictive for massive multi-asset portfolio computations. However, our solution represents a suitable benchmark approach for standard Monte Carlo based methods, since it is exact and allows to handle a large variety of benchmark examples. The genericity of our result is also of special interest from the perspective of model risk assessment and model comparison in the context of Xva valuation.

Chapter 4

An expansion approach for stochastic intensity default models

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We address in this paper new developments in pricing derivatives subject to the occurrence of a default event. Based on stochastic expansion arguments, the pricing is made under general diffusion dynamics of the default intensity. The derivative's price is approximated by the sum of a proxy value and a explicit combination of adjustment terms, all computed under the deterministic intensity model and involve the derivative's Greeks and the default intensity parameters. The implications of our methodology are twofold: Firstly, we derive analytic formulas for conditional survival probabilities under arbitrary intensity diffusion models. Secondly, the sensitivity of the derivative's value w.r.t credit parameters, such as the intensity's volatility and the correlation between the default and the remaining risk factors, become explicit and can be directly computed and analyzed. In addition, we show that the implementation and the accuracy of our pricing formulas are minimally dependent on the smoothness of the payoff function. This makes our approach easily embeddable in pricing credit-hybrid derivatives as well as Xva adjustments.

Keywords: Credit-contingent derivatives, Hazard rate process, Stochastic expansion, Malliavin calculus

4.1 Introduction

The pricing of derivatives including a defaultable entity has gained much attention since the emergence of credit valuation topics in the practice of the financial markets. Several works have already addressed the problem in the fixed-income market where both the hazard and interest rates are allowed to be stochastic. We refer for instance to the work of Jarrow and Turnbull (1995), Kim and Kunitomo (1999), Kunitomo and Takahashi (2001).

The present article addresses the pricing in a more general framework where the derivative pays a generic payoff contingent to the default of a reference entity that can be the counterparty itself. Due to the dimensionality curse, the pricing of default-contingent derivatives with multiple risk factors is commonly achieved by Monte Carlo simulations. This usually results in a

significant computational cost and a numerical error that may hardly affect the understanding of each parameter contribution. Another preferable feature in a pricing methodology is the ability to easily achieve stress-tests. This is even more desired when unobservable model parameters are at play. Particularly, we refer to the credit spread volatility and mean reversion, as well as the correlations between the credit spread and the remaining risk factors.

We present in this work new pricing formulas that result from an asymptotic expansion and we justify that our methodology is not restrictive with regard to the model calibration, as can be some singular perturbation approaches. The accuracy of our approach depends only on the smoothness of the hazard rate process and not on the regularity of the payoff function. A characteristic that is of special interest for financial applications.

Outline of the chapter

The chapter is organized as follow: in Section 4.2, we introduce the framework of Muroi for the fixed-income market ([Mur05, Mur12]) and we comment the singular perturbation methodology adopted. We introduce our generic pricing framework in Section 4.3. In contrast with the singular perturbation approach, we follow the work of Benhamou and *al.* ([BGM09, BGM10a, BGM10b]) and we justify our expansion in the case of cumulative hazard rate with small diffusion. Our pricing formulas and error estimates are given at the second and third approximation orders. In Section 4.4, we define some credit derivatives and show how they can be priced using our approximations. Section 4.5 is dedicated to numerical experiments. Finally, we bring together in Section 4.6 all mathematical results needed to make our expansion and error analysis explicit.

4.2 Muroi's asymptotic expansions

Muroi [Mur05, Mur12] and Muroi and Takino [MT11] present a joint model for pricing in the defaultable bond market. The model supports a positive stochastic process r_t^ε for the spot interest rate, and a second positive stochastic process λ_t^ε that is interpreted as the hazard rate.

Under a risk-neutral martingale measure $\mathbb{Q}$, the non-negative valued spot interest and hazard rate processes r_t^ε and λ_t^ε satisfy the following stochastic differential equations

$$d\begin{pmatrix} r_t^\varepsilon \\ \lambda_t^\varepsilon \end{pmatrix} = \begin{pmatrix} \mu_1 \times (\bar{r} - r_t^\varepsilon) \\ \mu_2 \times (\bar{\lambda} - \lambda_t^\varepsilon) \end{pmatrix} dt + \varepsilon \sigma (r_t^\varepsilon, \lambda_t^\varepsilon)\, dW_t \tag{4.2.1}$$

where (W_t) is a two-dimensional Brownian motion and ε a small parameter.

Credit derivatives involve usually a recovery payment when the default occurs. Duffie and Singleton [DS99] introduced several recovery rules for the credit derivatives, some of which are listed below:

Definition 4.2.1 (Recovery rules)**.** For a recovery ratio $\delta \in [0,1)$, we define:

1. Recovery of Market Value (RMV): A defaultable zero coupon bond pays 1 monetary unit at the maturity date if the issuer is solvent or pays δ times the defaultable bond price just before the time of bankruptcy.

2. Recover Treasury (RT): A defaultable zero coupon pays 1 monetary unit at the maturity date if the issuer is solvent or pays δ units at the maturity date if bankruptcy has occurred before maturity.

Under both rules, the risky zero-coupon bond is priced by using a singular perturbation technique in the context of a small diffusion $\varepsilon \times \sigma$ for either r and λ. The accuracy of the method is ensured by imposing ε small, which can be restrictive choice with regard to the market calibration. The obtained formulas of zero-coupon bonds are used to valuate fixed income derivatives such as CDS and defaultable-bond options.

4.3 New formulas for pricing credit derivatives

We place ourselves in a probability space $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbb{Q})$. The filtration $\mathscr{G}_t$ models the flow of information of the whole market, including credit, and $\mathbb{Q}$ is the risk neutral measure. This space is endowed also with a right-continuous and complete sub-filtration $\mathscr{F}_t$ representing all the observable market quantities except the default event, $\mathscr{F}_t \subseteq \mathscr{G}_t \triangleq \mathscr{F}_t \vee \mathscr{H}_t$. Here, $\mathscr{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$ is the right-continuous filtration generated by the default event. In the latter, prices are expressed at time $t = 0$ and we simply denote the risk-neutral expectation by $\mathbb{E}[.] = \mathbb{E}^{\mathbb{Q}}[.]$.

In the framework of credit contingent derivatives, we present a new methodology to derive a closed form pricing solution. The default time τ is modeled under the reduced form approach and defined by

$$\tau = \inf\left\{ t > 0 \text{ such that } \int_0^t \lambda_s^{\varepsilon} ds > \xi \right\} \tag{4.3.1}$$

for ξ a unit exponential random variable independent of $\mathscr{F}$, and λ^{ε} the default's intensity. The intensity process is denoted by $(\lambda_t^{\varepsilon})_{t \geq 0}$. It is assumed to be a stochastic diffusion that satisfies the following SDE

$$d\lambda_t^{\varepsilon} = \kappa(t)(\psi(t) - \lambda_t^{\varepsilon})\, dt + \varepsilon \nu(t, \lambda_t^{\varepsilon})\, dW_t, \qquad \lambda_0^{\varepsilon} = \lambda_0 > 0 \tag{4.3.2}$$

where (W_t) is an $\mathbb{R}-$valued Brownian motion. In addition, we assume that:

Assumption 4.3.1. *The default intensity $(\lambda_t^{\varepsilon})$ is strictly positive almost everywhere.*

The mean-reversion is defined w.r.t. a long run ψ and a speed of adjustment driven by $\kappa > 0$. Time-dependency of the mean-reversion parameters κ and ψ is intended to provide a suitable calibration to market data. λ^{ε} 's volatility ν may depend on time and space variables and, therefore, it allows to cover a broad range of stochastic processes (Ornstein-Uhlenbeck, Lognormal[1], CIR, ...). We show in the following that ν and its derivatives (w.r.t. λ) play a key role in developing our pricing formulas. As for the remaining parameter ε, notice that $\varepsilon = 0$ leads to a deterministic intensity λ, while our goal is to take $\varepsilon = 1$ in order to recover the *target* credit model. This parametrization is similar to the one followed in [BGM10b].

[1]The resulting mean-reverting lognormal dynamics have already been addressed in the Brennan-Schwartz 2-factors model

We denote by T the maturity of the contract. We choose a *generic* multidimensional risk factor represented by the $\mathbb{R}^n-$valued diffusion process $X_t = \left(X_t^i\right)_{i\in[\![1,n]\!]}$ governed by the SDE

$$dX_t = (\Phi(t) + \Theta(t) X_t)\, dt + \Sigma(t)\, dB_t, \qquad X_0 \in \mathbb{R}^n \tag{4.3.3}$$

with $\Phi : [0,T] \to \mathbb{R}^n$, $\Theta : [0,T] \to \mathbb{R}^{n\times n}$, $\Sigma : [0,T] \to \mathbb{R}^{n\times d}$ and $(B_t)_{t\geq 0}$ a $\mathbb{R}^d-$Brownian motion. Moreover, we define the instantaneous correlations $\rho = (\rho_i)_{i=1\ldots d}$ by

$$d\langle W, B_i\rangle_t = \rho_i dt \ , \ 1 \leq i \leq d \tag{4.3.4}$$

Our main objective is to give an accurate analytic approximation for the expected value of the payoff function $h(X_T)$ subject to the default event $1_{\{\tau\leq T\}}$. Such a pricing can be dealt with under different situations:

1. Under the RMV rule, the price of the contingent claim with a given fractional recovery rate δ is represented by $u_\delta^\varepsilon(T)$:

$$u_{h,\delta}^\varepsilon(T) = \mathbb{E}\left[h(X_T)\, 1_{\{\tau>T\}} + (1-\delta)\, u_{h,\delta}^\varepsilon\left(\tau^-, T\right) 1_{\{\tau\leq T\}}\right] \tag{4.3.5}$$

   An application of Duffie and Singleton's result (cf. [DS99]) leads to the following convenient expression

$$u_{h,\delta}^\varepsilon(T) = \mathbb{E}\left[\exp\left(-(1-\delta)\int_0^T \lambda_t^\varepsilon dt\right) h(X_T)\right] \tag{4.3.6}$$

2. In a slightly different setting, $u_{h,\delta}^\varepsilon$ is the price of a contingent claim $h(X_T)$ paid only if the default τ does not occur before maturity T. Under this situation, the derivative's price is $u_{h,0}^\varepsilon$ corresponding to

$$u_{h,0}^\varepsilon(T) = \mathbb{E}\left[h(X_T)\, 1_{\{\tau>T\}}\right] = \mathbb{E}\left[\exp\left(-\int_0^T \lambda_t^\varepsilon dt\right) h(X_T)\right] \tag{4.3.7}$$

   The second equality is obtained by conditioning w.r.t. $(\lambda_t^\varepsilon)_{0\leq t\leq T}$ and $(X_t)_{0\leq t\leq T}$.

In the perspective of financial applications that we address in the sequel, we separate the survival horizon S from the contract's maturity T by considering the pricing function

$$u_{h,\delta}^\varepsilon(S,T) \triangleq \mathbb{E}\left[\exp\left(-(1-\delta)\int_0^S \lambda_s^\varepsilon ds\right) h(X_T)\right]. \tag{4.3.8}$$

subject to the condition $S \leq T$.

The approach consists in expanding the price $u_{h,\delta}^\varepsilon$ with respect to ε. However, instead of restricting the model to small values of ε, we rely on expansion arguments in the spirit of Benhamou *et al.* [BGM09, BGM10a, BGM10b] and we set $\varepsilon = 1$. In fact, ε is used as a parametrization tool to derive a convenient formula for $u_{h,\delta}^{\varepsilon=1}$ in terms of $u_{h,\delta}^{\varepsilon=0}$ and a combination of correction terms. The accuracy of the expansion does not rely on the smallness of ε. Rather, it is based on the fact that the hazard rate $\left(\int_0^t \lambda_s^\varepsilon ds\right)_{t\geq 0}$ shows more regularity properties, as a time-integrated (finite variation) process, in contrast with the intensity process λ^ε. We stress that the quantity at play in the pricing function $u_{h,\delta}^\varepsilon$ (Equation (4.3.7)) is $\left(\int_0^t \lambda_s^\varepsilon ds\right)_t$ and not λ^ε. Using an asymptotic expansion in the context of small diffusion of the hazard rate, we can establish estimates of the derivatives. This allows us to make explicit the contributions at given order and to control the error by the means of the infinite dimensional analysis of Malliavin calculus.

Methodology targets:

The technique allows to express the price $u_{h,\delta}^{\varepsilon=1}$ under a stochastic intensity $\lambda^{\varepsilon=1}$ in terms of the price $u_{h,\delta}^{\varepsilon=0}$ that is computed under a deterministic intensity and is generally given explicitly. This can be assimilated to choosing a proxy model where the pricing of $u_{h,\delta}^{\varepsilon=0}$ is explicit. The distance to the *target* price $u_{h,\delta}^{\varepsilon=1}$ is corrected by a series of additional terms that, in the spirit of [BGM09, BGM10a, BGM10b], is a summation of Greeks plus an error term. Formally, we can write

$$u_{h,\delta}^{\varepsilon=1}(S,T) = u_{h,\delta}^{\varepsilon=0}(S,T) + \text{weighted sum of Greeks of } \mathbb{E}[h(X_T)] + \text{Error} \tag{4.3.9}$$

The Greeks are computed in the proxy model and have to be explicit as well. In the sequel, we explain that Malliavin weights appear naturally for each Greek and we show a clear dependence on the cross-correlations between the credit intensity λ and the market risk factors $(X_i)_{1\leq i\leq n}$.

In the following, we formally introduce the mathematical setting in order to establish the pricing methodology and clarify the regularity assumptions on which depend the accuracy of the method. In particular, our assumptions are addressed with the regard to the regularity of the volatility function v.

Comparison with [Mur05, Mur12]:

Muroi's setting ([Mur05, Mur12]) is dedicated to credit derivatives (defaultable bonds, options on defaultable bonds) as functions of r^{ε} and λ^{ε} ((4.2.1)), and possibly multiple default intensities λ_i^{ε} ([Mur12]). If the payoff function Φ is smooth enough, then it can be approximated by the Taylor expansion

$$\Phi\left(r^{\varepsilon}(T), \lambda^{\varepsilon}(T)\right) \approx g_0 + \varepsilon\left(g_1 + \frac{\varepsilon}{2} g_2\right)$$

The density function of $g_1 + \frac{\varepsilon}{2} g_2$ is approached using an asymptotic development of the characteristic function, following Takahashi *et al.* [Tak99]

$$\begin{aligned}
\phi(u) &= \mathbb{E}\left[e^{iu(g_1+\varepsilon g_2)}\right] \\
&\approx \mathbb{E}\left[e^{iug_1}(1+\varepsilon iug_2)\right] \\
&= \mathbb{E}\left[e^{iug_1}(1+\varepsilon iu\mathbb{E}[g_2|g_1])\right]
\end{aligned}$$

An explicit representation of the conditional expectation $\mathbb{E}[g_2|g_1]$ gives the asymptotic pricing formula.

Our framework can be considered as a generalization of Muroi's setting. Our payoff function allows a generic representation $(X_t)_{t\geq 0}$ of additional risk factors ((4.3.3)). In addition, our asymptotic expansion is based on infinitesimal stochastic analysis, which is fundamentally different of the characteristic function expansion technique. Finally, our use of ε as an interpolation parameter contrasts with Muroi's interpretation where ε is intended to be small.

Comparison with [BGM09, BGM10a, BGM10b]:

From a technical perspective, we follow the work of Benhamou *et al.* in order to achieve our analysis. Meanwhile, our setting is fundamentally different. In fact, Benhamou *et al.* perform

a stochastic expansion of the pricing function $\mathbb{E}\left[h\left(X_T^{\varepsilon}\right)\right]$ through the parametrization of the process X underlying to the payoff h. In addition to expanding X^{ε} w.r.t. ε, a Taylor approximation is performed for the payoff function h at the first order (second order approximation) or at the second order (third order approximation). As a consequence, the error analysis involves assumptions about the regularity of function h and the law of X. Moreover, the error estimates depend on the magnitude of the Greek terms, implied by the derivatives of h. In our case, the Taylor expansions are applied to λ^{ε} and the exponential function in (4.3.8), while the payoff h and its underlying process X remain unchanged. In order to get accurate approximations, only smoothness conditions about λ's volatility function are needed. Finally, we emphasize that our approximation formulas include lower order Greeks and the error estimate does only depend on the payoff magnitude.

4.3.1 Notations and assumptions

The following notations will be used extensively throughout the chapter.

Notation 4.3.2 (Product operators). We define the element-wise product operator $(.)$ respectively as:

1. $X.Y = \sum_{i=1}^{n} X_i Y_i$, if $X, Y \in \mathbb{R}^n$ (scalar product)
2. $X.Y = \sum_{i,j=1}^{n} X_{i,j} Y_{i,j}$, if $X, Y \in \mathbb{R}^{n,n}$ (*Frobenius* inner product)
3. $X.Y = \sum_{i,j,k=1}^{n} X_{i,j,k} Y_{i,j,k}$, if $X, Y \in \mathbb{R}^{n,n,n}$ (triadic tensor product)

Notation 4.3.3 (Differentiation and Residuals). If these derivatives have a meaning, we write

- $h^{(1)}(x) = \left[\frac{\partial}{\partial x_i} h(x)\right]_{1\leq i\leq n}, x \in \mathbb{R}^n$
- $h^{(2)}(x) = \left[\frac{\partial^2}{\partial x_i \partial x_j} h(x)\right]_{1\leq i,j\leq n}, x \in \mathbb{R}^n$
- $h^{(3)}(x) = \left[\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} h(x)\right]_{1\leq i,j,k\leq n}, x \in \mathbb{R}^n$
- $\nu(t) = \nu\left(t, \lambda_t^{\varepsilon=0}\right)$ and $\nu^{(i)}(t) = \frac{\partial^i \nu(t+s)}{\partial s^i}|_{s=0}$
- $\lambda_{i,t}^{\varepsilon} = \frac{\partial^i \lambda_t^{\varepsilon}}{\partial \varepsilon^i}$ is the i^{th} derivative of the parametrized process w.r.t. ε
- $\lambda_{i,t} = \frac{\partial^i \lambda_t^{\varepsilon}}{\partial \varepsilon^i}|_{\varepsilon=0}$
- $R_{i,t}^{Z^{\varepsilon}} = Z_t^{\varepsilon} - \sum_{j=0}^{i} \frac{\varepsilon^j}{j!} Z_{j,t}$ for every parametrized process Z^{ε}. For instance, $R_{i,t}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} = \int_0^t \lambda_s^{\varepsilon} ds - \sum_{j=0}^{i} \frac{\varepsilon^j}{j!} \int_0^t \lambda_{i,s} ds$.

The following Greeks notations provides a convenient representation for the correction terms

Notation 4.3.4 (Greeks). Let Z be a random variable. Given the payoff function h, we define i^{th} order Greek, $i = 1,2,3$, for the variable Z by the quantity (when it has a meaning)

$$\begin{cases} Greek^{(1)}(Z) &= \left[\frac{\partial \mathbb{E}[h(Z+x)]}{\partial x_i}|_{x=0_{\mathbb{R}^n}}\right]_{1\leq i\leq n} \\ Greek^{(2)}(Z) &= \left[\frac{\partial^2 \mathbb{E}[h(Z+x)]}{\partial x_i \partial_{x_j}}|_{x=0_{\mathbb{R}^n}}\right]_{1\leq i,j\leq n} \\ Greek^{(3)}(Z) &= \left[\frac{\partial^3 \mathbb{E}[h(Z+x)]}{\partial x_i \partial x_j \partial x_k}|_{x=0_{\mathbb{R}^n}}\right]_{1\leq i,j,k\leq n} \end{cases}$$

Under the appropriate smoothness and boundedness assumptions on h, one also has

$$Greek^{(i)}(Z) = \mathbb{E}\left[h^{(i)}(Z)\right], \ i = 1,2,3$$

In addition, we define i^{th} order Greek in presence of a default risk and under the proxy model, i.e. with deterministic hazard rate, by

$$\widetilde{Greek}_{\delta}^{(i)}(S,Z) = e^{-(1-\delta)\int_0^S \lambda_s ds} Greek^{(i)}(Z) \tag{4.3.10}$$

In order to derive a tight error control, we introduce the following notations and assumptions

Assumption 4.3.5 ($\mathscr{R}_{\mathbf{N}}$). *The functions κ, ψ and ν are bounded. The volatility function ν is of class $\mathscr{C}^N$ w.r.t. λ^{ε} and its derivatives up to order N are bounded.*

This assumption is made explicit in order to clear up our analysis. An additional Lipschitz condition is usually assumed on ν (with the boundedness of κ and ψ) ensuring the existence and uniqueness of a strong solution $(\lambda_t^{\varepsilon})$ of (4.3.2). In our numerical experiments, we will challenge the accuracy of the expansion methodology by partially relaxing some of these requirements.

The L_p norm of a random variable Z is denoted, as usually, by $\|Z\|_p = \mathbb{E}\left[|Z|^p\right]^{1/p}$. In addition, in order to derive a conducive error analysis, we use the following constants

$$\begin{cases} M_0 &= \max_{0\leq i\leq 3}\left(\left\|\nu^{(i)}\right\|_{\infty}\right) \\ M_1 &= \max_{1\leq i\leq 3}\left(\left\|\nu^{(i)}\right\|_{\infty}\right) \end{cases} \tag{4.3.11}$$

as well as the following simplified notation for upper bounds:

Notation 4.3.6 (Upper bounds constants). For the upper bounds, we use the unique notation

$$A \leq_c B$$

to assert that $A \leq cB$ where c is a positive constant depending on the modeling parameters and other pricing constants. The constant c remains bounded when the model parameters go to 0, and it is uniform w.r.t. the parameter $\varepsilon \in [0,1]$. In particular, c depends on the time parameter S. We denote by $\leq_{c(S)}$ the comparison operator where this dependence is made explicit.

Definition 4.3.7. We define $\mathscr{H}$ as the space of functions with exponential growth. In other words, a function $h \in \mathscr{H}$ fulfills $|h(x)| \leq c_1 e^{c_2\|x\|_{\infty}}$ for any $x \in \mathbb{R}^n$ and for two constants c_1 and c_2.

As mentioned previously, we require that $h \in \mathscr{H}$ which represent a reasonable assumption with regard to financial applications.

Assumption 4.3.8 ($\mathscr{H}$). *h belongs to $\mathscr{H}$.*

4.3.2 Stochastic expansion development

The perturbation approach is based on the Taylor expansion of the parametrized process (λ_t^ε) and the real-valued exponential function.

4.3.2.1 Second order approximation

Under the condition $(\mathscr{R}_3)$, λ_t^ε is $\mathscr{C}^2$ w.r.t. ε almost surely for any t (see Benhamou *et al.* [BGM09] for e.g.). From the definition $\lambda_{i,t} = \frac{\partial^i \lambda_t^\varepsilon}{\partial^i \varepsilon^i}|_{\varepsilon=0}$, we expand the process λ_t^ε as follows:

$$\lambda_t^\varepsilon = \lambda_t^\varepsilon|_{\varepsilon=0} + \varepsilon\lambda_{1,t}^\varepsilon|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\lambda_{2,t}^\varepsilon|_{\varepsilon=0} + R_{2,t}^{\lambda^\varepsilon} \tag{4.3.12}$$

To obtain an approximation formula for $u_{h,\delta}^\varepsilon(t,T)$, we use the Taylor formula twice. First, we apply the above expansion at $\varepsilon = 0$ up to order 2. The diffusion dynamics of $(\lambda_{i,t})_{t\geq 0}$ is obtained by a straightforward differentiation of the parameters of the diffusion equation of λ_t^ε. This is achieved in Proposition 4.6.3 that gives:

$$d\lambda_t = -\kappa(t)(\lambda_t - \psi(t))\,dt \tag{4.3.13}$$

$$d\lambda_{1,t} = -\kappa(t)\lambda_{1,t}dt + \nu(t)\,dW_t \tag{4.3.14}$$

$$d\lambda_{2,t} = -\kappa(t)\lambda_{2,t}dt + 2\lambda_{1,t}\nu^{(1)}(t)\,dW_t \tag{4.3.15}$$

with $\lambda_0 = \lambda_0^{\varepsilon=0} = \lambda_0^{\varepsilon=1}$ and $\lambda_{i,0} = 0$ for $i \geq 1$. λ_t is interpreted as a deterministic proxy for λ_t^ε. In particular, an explicit expression for λ_t is given by

$$\lambda_t = e^{-\int_0^t \kappa(s)ds}\left(\lambda_0 + \int_0^t e^{\int_0^s \kappa(u)du}\kappa(s)\,\psi(s)\,ds\right) \tag{4.3.16}$$

which yields

$$\int_0^S \lambda_t dt = \lambda_0 \int_0^S e^{-\int_0^t \kappa(s)ds}dt + \int_0^S \int_t^S e^{-\int_0^s \kappa(u)du}ds e^{\int_0^t \kappa(u)du}\kappa(t)\,\psi(t)\,dt \tag{4.3.17}$$

One may easily argue that pricing under this proxy model for the default intensity is convenient since $u_{h,\delta}^{\varepsilon=0}(S,T)$ is computed with no extra cost as in a default-free setting. More precisely, one has

$$u_{h,\delta}^{\varepsilon=0}(S,T) = e^{-(1-\delta)\int_0^S \lambda_t dt}\mathbb{E}[h(X_T)] \tag{4.3.18}$$

where $\mathbb{E}[h(X_T)]$ corresponds to the default-free price. Secondly, we use a Taylor formula in (4.3.8), at the second order for function $x \mapsto e^x$ around $\int_0^S \lambda_t dt$. Bringing all terms together, the second order approximation reads

$$\begin{aligned} u_{h,\delta}^\varepsilon(S,T) = u_{h,\delta}^{\varepsilon=0}(S,T) &- (1-\delta)e^{-(1-\delta)\int_0^S \lambda_s ds}\mathbb{E}\left[\int_0^S \left(\varepsilon\lambda_{1,t} + \frac{\varepsilon^2}{2}\lambda_{2,t}\right)dt\,h(X_T)\right] \\ &+ \frac{1}{2}(1-\delta)^2 e^{-(1-\delta)\int_0^S \lambda_s ds}\mathbb{E}\left[\left(\int_0^S \varepsilon\lambda_{1,t}dt\right)^2 h(X_T)\right] + \text{Error}^\varepsilon \end{aligned} \tag{4.3.19}$$

To attain an explicit formula, it remains to show that correction terms involving $\int_0^S \lambda_{1,t}dt$ and $\int_0^S \lambda_{2,t}dt$ can be transformed into a weighted sum of Greeks computed in the proxy model. This is addressed in the following theorem.

Theorem 4.3.9 (Second order approximation price formula). *Suppose that the intensity process (λ_t) fulfills $(\mathscr{R}_3)$ and that the payoff function h satisfies $(\mathscr{H})$. Then, one has*

$$u_{h,\delta}^{\varepsilon=1}(S,T) = u_{h,\delta}^{\varepsilon=0}(S,T) + (1-\delta)^2 C_{0,1}(S)\, u_{h,\delta}^{\varepsilon=0}(S,T) - (1-\delta) C_{1,1}(S,T).\widetilde{Greek}_{\delta}{}^{(1)}(S,X_T) - (1-\delta)\left(C_{2,1}(S,T) - (1-\delta) C_{2,2}(S,T)\right).\widetilde{Greek}_{\delta}{}^{(2)}(S,X_T) + Error_2^{\varepsilon=1} \tag{4.3.20}$$

where

$$C_{0,1}(S) = \frac{1}{2}\int_0^S \left(\int_t^S e^{-\int_0^s \kappa(u)du} ds \tilde{\nu}(t)\right)^2 dt \tag{4.3.21}$$

$$C_{1,1}(S,T) = \int_0^S \left(\int_t^S e^{-\int_0^s \kappa(u)du} ds\right) \tilde{\nu}(t)\tilde{\Sigma}(t,T)\rho dt \tag{4.3.22}$$

$$C_{2,1}(S,T) = \int_0^S \left(\int_t^S \left(\int_s^S e^{-\int_0^u \kappa(v)dv} du\right) \nu^{(1)}(s)\tilde{\Sigma}(s,T)\rho ds\right) \tilde{\nu}(t)\tilde{\Sigma}(t,T)\rho dt \tag{4.3.23}$$

$$C_{2,2}(S,T) = \int_0^S \left(\int_t^S \left(\int_s^S e^{-\int_0^u \kappa(v)dv} du\right) \tilde{\nu}(s)\tilde{\Sigma}(s,T)\rho ds\right) \left(\int_t^S e^{-\int_0^s \kappa(u)du} ds\right) \tilde{\nu}(t)\tilde{\Sigma}(t,T)\rho dt \tag{4.3.24}$$

with the modified volatility functions

$$\tilde{\Sigma}(t,T) = e^{\int_t^T \Theta(s)ds}\Sigma(t) \tag{4.3.25}$$

$$\tilde{\nu}(t) = e^{\int_0^t \kappa(s)ds}\nu(t) \tag{4.3.26}$$

$x \mapsto e^x$ refers to either the real exponential (4.3.26) *or to the matrix exponential in* (4.3.25)*. In addition, an estimate of the error term is given by:*

$$|Error_2^{\varepsilon}| \leq_{c(S)} S^2 M_0 (M_1 + S M_0) \|h(X_T)\|_2 \tag{4.3.27}$$

Proof. The price $u^{\varepsilon=1}$ is approximated by Equation (4.3.19), following a second order expansion. We show that it is equal to the summation (4.3.20). This is achieved in Proposition 4.6.5 using Malliavin calculus techniques. The error estimate is accomplished by inspecting the error terms in the Taylor expansions:

$$\begin{aligned}
\mathrm{Error}_2^{\varepsilon} = &-(1-\delta) e^{-(1-\delta)\int_0^S \lambda_s ds} \mathbb{E}\left[R_{2,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} h(X_T)\right] \\
&+ \frac{1}{2}(1-\delta)^2 e^{-(1-\delta)\int_0^S \lambda_s ds} \mathbb{E}\left[\left(2\varepsilon\left(\int_0^S \lambda_{1,t} dt\right) R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} + \left(R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t}\right)^2\right) h(X_T)\right] \\
&- \frac{1}{2}(1-\delta)^3 \mathbb{E}\left[\int_0^{\varepsilon} (1-\xi)^2 e^{-(1-\delta)\left(\int_0^S \xi\lambda_t^{\xi} + (1-\xi)\lambda_t dt\right)} d\xi \left(R_{0,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t}\right)^2 h(X_T)\right]
\end{aligned}$$

Observe that λ_t^{ξ} is bounded almost surely by a constant that depends on S (i.e. $\sup_{t\in[0,S]}\sup_{\gamma\in[0,1]} \left|\lambda_t^{\gamma}\right| <$

∞), and that $t \mapsto \lambda_t$ is a continuous function. it readily follows from the Hölder inequality [2] that

$$
\begin{aligned}
|\text{Error}_2^{\varepsilon}|^2 &\leq_c \left(\left\| R_{2,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_2^2 + \left\| \left(\int_0^S \lambda_{1,t} dt \right) R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_2^2 + \left\| R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 + \left\| R_{0,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 \right) \|h(X_T)\|_2^2 \\
&\leq_c \left(\left\| R_{2,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_2^2 + \left\| R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 + \left\| R_{0,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 + \left\| \int_0^S \lambda_{1,t} dt \right\|_4^4 \right) \|h(X_T)\|_2^2
\end{aligned}
$$

where we have used the inequality

$$
\mathbb{E}\left[\left| \left(\int_0^S \lambda_{1,t} dt \right) R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right|^2 \right] \leq \frac{1}{2} \left(\mathbb{E}\left[\left| \int_0^S \lambda_{1,t} dt \right|^4 \right] + \mathbb{E}\left[\left| R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right|^4 \right] \right)
$$

We now use the L_2upper-bound estimates established in Proposition 4.6.8 and Corollary 4.6.10. We end-up with the following estimate:

$$
\begin{aligned}
|\text{Error}_2^{\varepsilon}|^2 &\leq_c \left(S^4 M_0^2 M_1^2 + S^8 M_0^4 M_1^4 + S^6 M_0^4 \right) \|h(X_T)\|_2^2 \\
&\leq_c S^4 M_0^2 \left(M_1^2 + S^2 M_0^2 \left(1 + S^2 M_1^4 \right) \right) \|h(X_T)\|_2^2 \\
&\leq_c S^4 M_0^2 \left(M_1^2 + S^2 M_0^2 \right) \left(1 + S^2 M_1^2 \right) \|h(X_T)\|_2^2
\end{aligned}
$$

□

Remark 4.3.10. The obtained price is the summation of three order terms:

- $u_{h,\delta}^{\varepsilon=0}(S,T)$: The leading term corresponds to the price of the contingent claim with a deterministic intensity λ of the default event. We know that in this case, the pricing of $u_{h,\delta}^{\varepsilon=0}(S,T)$ consists in estimating $\mathbb{E}[h(X_T)]$, and thus, it can be made with no extra cost in comparison with the default-free setting.
- $(1-\delta)^2 C_{0,1}(S)\, u_{h,\delta}^{\varepsilon=0}(S,T)$: A 0-order correction that corresponds to a convexity adjustment stemming from $\left(\int_0^S \varepsilon \lambda_{1,t} dt \right)^2$ and depends only on the volatility ν.
- $(1-\delta) C_{1,1}(S,T)\, \widetilde{Greek}_{\delta}^{(1)}(S,X_T)$: A first order correction that depends on the first-order Greeks as well as the volatility ν and the correlation ρ.
- $(1-\delta)\left(C_{2,1}(S,T) + C_{2,2}(S,T)\right) \widetilde{Greek}_{\delta}^{(2)}(S,X_T)$: Combination of second order Greeks that depends on the products $\tilde{\nu} \times \tilde{\nu}$, $\tilde{\nu} \times \nu^{(1)}$ and $\left(\rho_i \times \rho_j\right)_{1 \leq i,j \leq n}$.

This approximation shows explicitly how the correction weights $C_{i,j}$ depend on ν and ρ. For instance, taking $\nu = 0$ leads to $u_{h,\delta}^{\varepsilon=1} = u_{h,\delta}^{\varepsilon=0}$ with a zero error. If $\rho = 0$ then

$$
u_{h,\delta}^{\varepsilon=1}(S,T) = u_{h,\delta}^{\varepsilon=0}(S,T) + (1-\delta)^2 C_{0,1}(S)\, u_{h,\delta}^{\varepsilon=0}(S,T) + \text{Error}_2^{\varepsilon=1}. \tag{4.3.28}
$$

Notice that coefficients $C_{i,j}$ depend on the magnitude of Σ as well. Finally, when the total market value is recovered at default ($\delta = 1$) one has $u_{h,\delta}^{\varepsilon=1} = u_{h,\delta}^{\varepsilon=0}$ corresponding to a default-free price.

[2] Using $|X+Y|^p \leq 2^{p-1}\left(|X|^p + |Y|^p\right), p \geq 1$

The zero order correction term $(1-\delta)^2 C_{0,1}(S)\, u_{h,\delta}^{\varepsilon=0}(S,T)$ corresponds to a convexity adjustment that is due to the volatility of λ^ε and intends to correct $e^{-(1-\delta)\int_0^S \lambda_t dt}$ in approximating $\mathbb{E}\left[e^{-\int_0^S (1-\delta)\lambda_t^\varepsilon dt}\right]$:

$$\mathbb{E}\left[e^{-\int_0^S (1-\delta)\lambda_t^\varepsilon dt}\right] \approx e^{-(1-\delta)\int_0^S \lambda_t dt}\left(1+(1-\delta)^2 C_{0,1}(S)\right) \tag{4.3.29}$$

Using the definition (4.3.18), this leads to

$$\begin{aligned} u_{h,\delta}^{\varepsilon=0}(S,T) + (1-\delta)^2 C_{0,1}(S,T)\, u_{h,\delta}^{\varepsilon=0}(S,T) &= \mathbb{E}[h(X_T)]\, e^{-(1-\delta)\int_0^S \lambda_t dt}\left(1+(1-\delta)^2 C_{0,1}(S)\right) \\ &\approx \mathbb{E}[h(X_T)]\,\mathbb{E}\left[e^{-\int_0^S (1-\delta)\lambda_t^\varepsilon dt}\right] \end{aligned}$$

We expect that market calibration allows to obtain $\mathbb{E}\left[e^{-\int_0^S (1-\delta)\lambda_t^\varepsilon dt}\right]$ from market data. Usually, one obtains the survival probabilities $\mathbb{Q}^{market}(\tau > S)$ implied from market CDS spread quotation (for $\delta = 0$). Hence, one may directly use the market-based estimate of $\mathbb{E}\left[e^{-\int_0^S (1-\delta)\lambda_t^\varepsilon dt}\right]$ instead of the zero order correction expression. Nevertheless, this zero-order approximation is an explicit representation of the survival probability $\mathbb{Q}(\tau > S) = \mathbb{E}\left[e^{-\int_0^S \lambda_t^\varepsilon dt}\right]$ in terms of the survival probability in the deterministic intensity proxy $e^{-\int_0^S \lambda_t dt}$.

Corollary 4.3.11. *Under the stochastic intensity model* (4.3.2)*, the survival probability* $\mathbb{Q}(\tau > S)$ *can be approximated by*

$$\mathbb{Q}(\tau > S) \approx e^{-\int_0^S \lambda_t dt}\left(1 + C_{0,1}(S)\right) \tag{4.3.30}$$

where the approximation error is bounded by

$$\left|\mathbb{Q}(\tau > S) - e^{-\int_0^S \lambda_t dt}\left(1 + C_{0,1}(S)\right)\right| \leq_c S^2 M_0 (M_1 + S M_0) \tag{4.3.31}$$

Numerical evidences about the accuracy of approximation (4.3.30) are given in Section 4.5.

4.3.2.2 Third order approximation

We show in this section that one can obtain higher accuracy by adding third order correction terms. In order to have a convenient representation of the solution, we address, in the following, definition a modification of iterated time integration operator introduced in [BGM10b].

Definition 4.3.12 (Integral Operator)**.** The integral operator ω_ρ^T is defined as follows: for $t \in [0,S]$

1. For any integrable function l, we set

$$\omega_\rho^T(l)_t^S = \left[\int_t^S l(u)\left[\tilde{\Sigma}(u,T)\,\rho\right]_i du\right]_{1\leq i\leq n} \tag{4.3.32}$$

$$\omega_{\rho,i}^T(l)_t^S = \left[\omega_\rho^T(l)_t^S\right]_i,\ 1 \leq i \leq n \tag{4.3.33}$$

a) Its m-times iteration is defined analogously: for any integrable functions $(l_1,\ldots,l_m)$, we set

$$\omega_\rho^T(l_1,\ldots,l_m)_t^S = \left[\omega_\rho^T\left(\left[\omega_\rho^T(l_2,\ldots,l_m)_\cdot^S\right]_{i_2,\ldots,i_m} l_1\left[\tilde{\Sigma}(u,T)\rho\right]_{i_1}\right)_t^S\right]_{1\le i_1,i_2,\ldots,i_m\le n} \tag{4.3.34}$$

$$\omega_{\rho,i_1,i_2,\ldots,i_m}^T(l_1,\ldots,l_m)_t^S = \left[\omega_\rho^T(l)_t^S\right]_{i_1,i_2,\ldots,i_m},\ 1\le i_1,i_2,\ldots,i_m\le n \tag{4.3.35}$$

Remark 4.3.13. In terms of the above integration operator, the coefficients of the second order approximation can be rewritten as follow:

$$C_{1,1}(S,T) = \omega_\rho^T\left(\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)\tilde{\nu}(.)\right)_0^S \tag{4.3.36}$$

$$C_{2,1}(S,T) = \omega_\rho^T\left(\tilde{\nu},\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)\nu^{(1)}\right)_0^S \tag{4.3.37}$$

$$C_{2,2}(S,T) = \omega_\rho^T\left(\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)\tilde{\nu},\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)\tilde{\nu}\right)_0^S \tag{4.3.38}$$

Theorem 4.3.14 (Third order approximation price formula). *Suppose that the intensity process (λ_t) fulfills $(\mathscr{R}_4)$ and that the payoff function h satisfies $(\mathscr{H})$. Then, one has*

$$\begin{aligned}
u_{h,\delta}^{\varepsilon=1}(S,T) &= u_{h,\delta}^{\varepsilon=0}(S,T) + (1-\delta)^2 C_{0,1}(S)\,u_{h,\delta}^{\varepsilon=0}(S,T)\\
&-(1-\delta)\left(C_{1,1}+C_{1,2}-(1-\delta)(C_{1,3}+C_{1,4})+(1-\delta)^2\left(C_{1,5}+C_{1,6}\right)\right)(S,T).\widetilde{Greek}_\delta{}^{(1)}(X_T)\\
&-(1-\delta)\left(C_{2,1}-(1-\delta)C_{2,2}\right)(S,T).\widetilde{Greek}_\delta{}^{(2)}(X_T)\\
&-(1-\delta)\left(C_{3,1}+C_{3,2}-(1-\delta)\left(C_{3,3}+C_{3,4}+C_{3,5}\right)+(1-\delta)^2 C_{3,6}\right)(S,T).\widetilde{Greek}_\delta{}^{(3)}(X_T)\\
&+Error_3^{\varepsilon=1}
\end{aligned} \tag{4.3.39}$$

where the weights $C_{i,j}$ are partially defined in Theorem 4.3.9 and by

$$C_{1,2}(S,T) = \frac{1}{2}\omega_\rho^T\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds e^{-\int_0^\cdot\kappa(s)ds}\nu^{(2)}\left(\int_0^\cdot(\tilde{\nu}(s))^2ds\right)\right)_0^S \tag{4.3.40}$$

$$C_{1,3}(S,T) = \omega_\rho^T\left(\left(\int_\cdot^S\left(\int_s^S e^{-\int_0^u\kappa(v)dv}du\right)^2\nu^{(1)}(s)\,\tilde{\nu}(s)\,ds\right)\tilde{\nu}\right)_0^S \tag{4.3.41}$$

$$C_{1,4}(S,T) = \omega_\rho^T\left(\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)\left(\int_0^\cdot\left(\int_s^S e^{-\int_0^u\kappa(v)dv}du\right)\tilde{\nu}^2(s)\,ds\right)\nu^{(1)}\right)_0^S \tag{4.3.42}$$

$$C_{1,5}(S,T) = \frac{1}{3}C_{0,1}(S)\,\omega_\rho^T\left(\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)\tilde{\nu}(.)\right)_0^S \tag{4.3.43}$$

$$C_{1,6}(S,T) = \frac{2}{3}\omega_\rho^T\left(\left(\int_\cdot^S e^{-\int_0^s\kappa(u)du}ds\right)C_{0,1}(.)\,\tilde{\nu}(.)\right)_0^S \tag{4.3.44}$$

$$C_{3,1}(S,T) = \omega_\rho^T \left(\tilde{\nu}, \nu^{(1)}, \int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \nu^{(1)} \right)_0^S \tag{4.3.45}$$

$$C_{3,2}(S,T) = \omega_\rho^T \left(\tilde{\nu}, \tilde{\nu}, \int_\cdot^S e^{-\int_0^s \kappa(u)du} ds e^{-\int_0^\cdot \kappa(s)ds} \nu^{(2)} \right)_0^S \tag{4.3.46}$$

$$C_{3,3}(S,T) = \omega_\rho^T \left(\tilde{\nu}, \left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \nu^{(1)}, \left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \tilde{\nu} \right)_0^S \tag{4.3.47}$$

$$C_{3,4}(S,T) = \omega_\rho^T \left(\tilde{\nu}, \left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \tilde{\nu}, \int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \nu^{(1)} \right)_0^S \tag{4.3.48}$$

$$C_{3,5}(S,T) = \omega_\rho^T \left(\left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \tilde{\nu}, \tilde{\nu}, \int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \nu^{(1)} \right)_0^S \tag{4.3.49}$$

$$C_{3,6}(S,T) = \omega_\rho^T \left(\left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \tilde{\nu}(.), \left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \tilde{\nu}(.), \left(\int_\cdot^S e^{-\int_0^s \kappa(u)du} ds \right) \tilde{\nu}(.) \right)_0^S \tag{4.3.50}$$

Additionally, the error term satisfies:

$$|Error_3^\varepsilon| \leq_{c(S)} S^2 M_0 \left(\sqrt{S} M_0 M_1 + S M_0 \right) \|h(X_T)\|_2 \tag{4.3.51}$$

Proof. The price $u_{h,\delta}^\varepsilon(S,T)$ is approximated at the third order by

$$\begin{aligned} u_{h,\delta}^\varepsilon(S,T) = u_{h,\delta}^{\varepsilon=0}(S,T) &- (1-\delta) e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\int_0^S \left(\varepsilon \lambda_{1,t} + \frac{\varepsilon^2}{2} \lambda_{2,t} + \frac{\varepsilon^3}{6} \lambda_{3,t} \right) dt h(X_T) \right] \\ &+ \frac{1}{2}(1-\delta)^2 e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\varepsilon^2 \left(\left(\int_0^S \lambda_{1,t} dt \right)^2 + \left(\int_0^S \lambda_{1,t} dt \right) \left(\int_0^S \lambda_{2,t} dt \right) \right) h(X_T) \right] \\ &- \frac{1}{6}(1-\delta)^3 e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\varepsilon^3 \left(\int_0^S \lambda_{1,t} dt \right)^3 h(X_T) \right] + \text{Error}_3^\varepsilon \end{aligned}$$

Additional the third order coefficients are given in Proposition 4.6.6. The error term estimation is an extension of the error analysis of Theorem 4.3.9:

$$\begin{aligned} \text{Error}_3^\varepsilon \propto &- e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[R_{3,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} h(X_T) \right] \\ &+ \frac{1}{2} e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\left(\left(\frac{\varepsilon^2}{2} \int_0^S \lambda_{2,t} dt \right)^2 + 2 \left(\varepsilon \int_0^S \lambda_{1,t} dt + \frac{\varepsilon^2}{2} \int_0^S \lambda_{2,t} dt \right) R_{2,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right) h(X_T) \right] \\ &- \frac{1}{6} e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\left(3 \left(\varepsilon \int_0^S \lambda_{1,t} dt \right)^2 R_{1,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} + 3\varepsilon \int_0^S \lambda_{1,t} dt \left(R_{1,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right)^2 \right) h(X_T) \right] \\ &+ \frac{1}{2} e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\left(R_{2,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right)^2 h(X_T) \right] - \frac{1}{6} e^{-(1-\delta)\int_0^S \lambda_t dt} \mathbb{E}\left[\left(R_{1,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right)^3 h(X_T) \right] \\ &- \frac{1}{6} \mathbb{E}\left[\int_0^\varepsilon (1-\xi)^2 e^{-(1-\delta)\int_0^S \xi \lambda_t^\varepsilon + (1-\xi)\lambda_t dt} d\xi \left(R_{0,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right)^2 h(X_T) \right] \end{aligned}$$

withdrawing all terms in $(1-\delta)$. It readily follows that

$$\begin{aligned}
|\text{Error}_3^{\varepsilon}|^2 \leq_c & \left(\left\| R_{3,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_2^2 + \left\| R_{2,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 + \left\| \int_0^S \lambda_{2,t} dt \right\|_4^4 + \left\| \int_0^S \lambda_{1,t} dt \right\|_4^4 \right) \|h\|_2^2 \\
& + \left(\left\| R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 + \left\| R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_8^8 + \left\| \int_0^S \lambda_{1,t} dt \right\|_8^8 \right) \|h\|_2^2 \\
& + \left(\left\| R_{0,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_4^4 + \left\| R_{1,S}^{\left(\int_0^t \lambda_s^{\varepsilon} ds\right)_t} \right\|_6^6 \right) \|h\|_2^2
\end{aligned}$$

Using Propositions 4.6.8 , 4.6.9 and Corollaries 4.6.10, 4.6.12, we end-up with the following estimate:

$$\begin{aligned}
|\text{Error}_3^{\varepsilon}|^2 \leq_c & \left(S^5 M_0^4 M_1^2 + S^5 M_0^2 M_1^4 + S^5 M_0^3 M_1^3 + S^6 M_0^4 + S^{12} M_0^8 + S^9 M_0^6 \right) \|h(X_T)\|_2^2 \\
\leq_c & \left(S^5 M_0^4 M_1^2 \left(1 + S^3 M_1^2\right) + S^6 M_0^4 \left(1 + S^6 M_0^4 + S^3 M_0^2\right) \right) \|h(X_T)\|_2^2 \\
\leq_c & \left(S^5 M_0^4 M_1^2 + S^6 M_0^4 \right) \left(1 + S^6 M_0^4 + S^3 M_0^2\right) \|h(X_T)\|_2^2
\end{aligned}$$

□

The third order approximation includes a combination of third order Greeks. The weights of additional corrections are iterated integrals involving multiplications and powers of ν, $\nu^{(1)}$ and $\nu^{(2)}$. From a numerical perspective, the contribution of the third order terms, in comparison with the second order ones, may be small if the volatility function ν is chosen such that the magnitude of its derivatives is null or small. We emphasize that in the particular case of an Ornstein-Uhlenbeck dynamics ($\nu(t,\lambda_t) = \nu(t)$), major terms of the third order approximation are nil, depending on $\nu^{(1)}$ and $\nu^{(2)}$:

$$\begin{aligned}
C_{1,i}(S,T) &= 0_{\mathbb{R}} \text{ , } i = 2 \dots 4 \\
C_{3,i}(S,T) &= 0_{\mathbb{R}^3} \text{ , } i = 1 \dots 5
\end{aligned}$$

Remark 4.3.15 (Model comparison). Approximation formulas (4.3.20) and (4.3.39) represent a parsimonious way to achieve a model comparison, based on choosing different volatility functions ν. Considering two credit models for λ^{ε}, i.e. two volatility functions ν_1 and ν_2, only correction terms have to be computed in order to evaluate the price in each model and to have a fast measurement of model impact. While the main proxy price $u_{h,\delta}^{\varepsilon=0}$ remains unchanged, as well as the Greek terms, one only has to compute coefficients $C_{i,j}$ for each model. This can be efficiently achieved using the Gauss-Legendre quadrature rule.

4.3.3 Expansion w.r.t. correlation ρ

We now examine the possibility to assign the $\varepsilon-$parametrization to the correlation ρ, instead of the intensity volatility ν. We define the instantaneous correlations $\rho = (\rho_i)_{i=1\dots d}$ such as

$$d\langle W^{\varepsilon}, B_i \rangle_t = \varepsilon \rho_i dt \text{ , } 1 \leq i \leq d \tag{4.3.52}$$

We recall that, by the means of a regular expansion of λ^ε w.r.t. ε and a Taylor expansion of function $x \mapsto e^x$, the second order approximation of

$$u^\varepsilon_{h,\delta}(S,T) = \mathbb{E}\left[\exp\left(-(1-\delta)\int_0^S \lambda^\varepsilon_s ds\right) h(X_T)\right]$$

reads

$$\begin{aligned} u^\varepsilon_{h,\delta}(S,T) = u^{\varepsilon=0}_{h,\delta}(S,T) - (1-\delta)\, e^{-(1-\delta)\int_0^S \lambda_s ds}\mathbb{E}\left[\left(\varepsilon\int_0^S \lambda_{1,t} + \frac{\varepsilon^2}{2}\lambda_{2,t}dt\right) h(X_T)\right] \\ + \frac{1}{2}(1-\delta)^2 e^{-(1-\delta)\int_0^S \lambda_s ds}\mathbb{E}\left[\left(\varepsilon\int_0^S \lambda_{1,t}dt\right)^2 h(X_T)\right] + \text{Error}^\varepsilon \end{aligned}$$

Let us examine now the tangent flows of λ^ε w.r.t. ε. One can write $W^\varepsilon_t = \varepsilon\rho B_t + \sqrt{1-(\varepsilon\rho)^2}dB^\perp_t$ and

$$\begin{aligned} d\lambda^\varepsilon_t &= \kappa(t)\left(\psi(t) - \lambda^\varepsilon_t\right)dt + \nu\left(t,\lambda^\varepsilon_t\right)\left(\varepsilon\rho dB_t + \sqrt{1-(\varepsilon\rho)^2}dB^\perp_t\right) \\ d\lambda_t &= \kappa(t)\left(\psi(t) - \lambda_t\right)dt + \nu\left(t,\lambda_t\right)dB^\perp_t \end{aligned}$$

Hence, from the definition $\lambda^\varepsilon_{i,t} = \frac{\partial^i \lambda^\varepsilon_t}{\partial^i \varepsilon^i}$, one has

$$d\lambda^\varepsilon_{1,t} = -\kappa(t)\lambda^\varepsilon_{1,t}dt + \lambda^\varepsilon_{1,t}\partial_\lambda \nu\left(t,\lambda^\varepsilon_t\right)\left(\varepsilon\rho dB_t + \sqrt{1-(\varepsilon\rho)^2}dB^\perp_t\right) \tag{4.3.53}$$

$$+\nu\left(t,\lambda^\varepsilon_t\right)\left(\rho dB_t - \frac{\varepsilon\rho^2}{\sqrt{1-(\varepsilon\rho)^2}}dB^\perp_t\right) \tag{4.3.54}$$

Similar SDEs are obtained for $\lambda^\varepsilon_{2,t}$ and $\lambda^\varepsilon_{3,t}$. The following lemma is a similar version to Lemma 4.6.1, where the $\varepsilon-$parametrization is performed w.r.t. ρ.

Lemma 4.3.16. *Let (u_t) be a square-integrable and predictable $\mathbb{R}-$valued process. One has for $t \le T$*

$$\mathbb{E}\left[\left(\int_0^S u_t dW^\varepsilon_t\right) h(X_T)\right] = \varepsilon\sum_{i=1}^n \mathbb{E}\left[\int_0^S u_t \left[e^{\int_\cdot^T \Theta(s)ds}\Sigma\rho\right]_i dt \left[h^{(1)}(X_T)\right]_i\right]$$

Proof. The proof is similar to Lemma 4.6.1. We set

$$\Pi^\varepsilon = \frac{1}{dt}d\left\langle \begin{pmatrix} W^\varepsilon \\ B \end{pmatrix}, \begin{pmatrix} W^\varepsilon \\ B \end{pmatrix}\right\rangle_t = \begin{pmatrix} 1 & \varepsilon\rho_1 & \dots & \varepsilon\rho_d \\ \varepsilon\rho_1 & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{1,1} & \dots & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\rho_d & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{d,1} & \dots & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{d,d} \end{pmatrix} \tag{4.3.55}$$

□

From technical perspective, it is possible to derive an analogous expansion approach performed w.r.t. correlation ρ. However, it is crucial to mention that, when $\varepsilon = 0$, the intensity process λ is stochastic. One can still perform the duality formula order to express $\mathbb{E}\left[\int_0^S \lambda_{i,t} dt h(X_T)\right]$ as a combination of $\mathbb{E}[h(X_T)]$ Greeks. The combination weights can no longer be expressed in a closed form way, since they depend on the stochastic volatility $\nu(t,\lambda_t)$ and its derivatives w.r.t. the variable λ. This can be handled without extra effort in the case of normal or log-normal dynamics, i.e. $\nu(t,\lambda_t) = \nu_0(t)$ or $\nu(t,\lambda_t) = \nu_0(t)\lambda_t$.

On the other hand, we believe that preserving the stochastic intensity when $\varepsilon = 0$ would result in fewer correction terms, since no convexity adjustment of survival probabilities are needed. This topic represents a major perceptive for a future work.

4.4 Pricing credit derivatives

We use the approximation formulas developed in the previous section in order to express the price of some reference credit derivatives. We refer to r as an instantaneous-short-rate process under the risk neutral measure. We assume its dynamics to be given by

$$r_t = y_t + \phi_r(t;a,\sigma_t)$$

where y is a $\mathscr{F}_t-$adapted process satisfying

$$dy_t = -ay_t dt + \sigma_t dB_t^r, \qquad y_0 = 0$$

a and σ_t are and positive parameter and function, and B^r a standard Brownian motion such that $d\langle B^r, W\rangle_t = \rho_{r,\lambda} dt$.

4.4.1 Defaultable coupon bond

The zero coupon bond price $u^{\varepsilon=1}_{h_{ZCB},\delta}$ is immediately given by taking $X_T = \int_0^T r_s ds$ and $h_{ZCB}(X_T) = \exp\left(-\int_0^T r_s ds\right)$. Notice that h_{ZCB} belongs to the class of smooth payoffs. Following this basic example, the price of a coupon bond which pays a coupon rate $b_i > 0$ at each payment date T_i, $(0 \le T_1 \le \cdots \le T_p \le T)$, is given by

$$v^{\varepsilon=1}_{\delta}(T) = \sum_{i=1}^{p} b_i u^{\varepsilon=1}_{h_{ZCB},\delta}(T_i, T_i)$$

4.4.2 Credit default swap (CDS)

Two parties, the protection buyer (PB) and the protection seller (PS), agree that if a reference entity (RE) defaults at a time τ, $0 < \tau \le T$, then PS pays to PB $(1-\delta)$of the notional. In exchange PB pays at regular times $(T_i)_{i=1\ldots p}$, usually every three months, a spread $\mathscr{S}$, interrupting payments at $\tau \le T$. The CDS is priced from the point of view of the protection seller as the expectation of its discounted cash flows:

$$CDS(0,T) = \mathbb{E}\left[\sum_{i=1}^{p} \mathscr{S}\alpha_i e^{-\int_0^{T_i} r_s ds} 1_{\{\tau > T_i\}} - (1-\delta) e^{-\int_0^{T_i} r_s ds} 1_{\{T_{i-1} \le \tau \le T_i\}}\right]$$

where we have discretized the default event and have neglected the accrual payment at default. Finally, rewriting the CDS price using $u^{\varepsilon=1}$, one has

$$\begin{aligned} CDS(0,T) &= \sum_{i=1}^{p} \mathscr{S}\alpha_i u_{\delta}^{\varepsilon=1}(0,T_i) - (1-\delta)\,\mathbb{E}\left[e^{-\int_0^{T_i} r_s ds}\left(1_{\tau>T_{i-1}} - 1_{\tau>T_i}\right)\right] \\ &= \sum_{i=1}^{p} \mathscr{S}\alpha_i u_{h_{ZCB},\delta}^{\varepsilon=1}(0,T_i) - (1-\delta)\left(u_{h_{ZCB},\delta}^{\varepsilon=1}(T_{i-1},T_i) - u_{h_{ZCB},\delta}^{\varepsilon=1}(T_i,T_i)\right) \end{aligned}$$

4.4.3 Contingent Credit Default Swap (CCDS)

A CCDS is a CDS that, upon the default of the reference credit, pays the loss given default ($LGD = 1-\delta$) on the residual net present value of a given portfolio if this is positive. Thanks to the mathematical shape of its payoff, a CCDS represents a convenient instrument to hedge counterparty risk. Given the price function $h(t,X_t)$ of the underlying portfolio, it readily follows that the price of the CCDS reads

$$\begin{aligned} CCDS(0,T) &= LGD\mathbb{E}\left[1_{\tau<T}\left(h(\tau,X_\tau)\right)^+\right] \\ &\approx LGD\sum_{i=0}^{n-1}\mathbb{E}\left[1_{T_i\le\tau<T_{i+1}}\left(h\left(T_{i+1},X_{T_{i+1}}\right)\right)^+\right] \\ &\approx LGD\sum_{i=0}^{n-1}\left(u_{h,0}^{\varepsilon=1}(T_i,T_{i+1}) - u_{h,0}^{\varepsilon=1}(T_{i+1},T_{i+1})\right) \end{aligned}$$

where $u_{h,\delta=0}^{\varepsilon=1}$ corresponds definition (4.3.8).

4.5 Numerical experiments

In this section, we discuss the details for the implementation of approximation formulas (4.3.20) and (4.3.39). We illustrate the performance of this approach by taking different examples for the payoff function h and the credit volatility function ν.

The coefficients $C_{i,j}$ are given explicitly when the parameters are time homogeneous. More generally, we use a Gauss-Legendre quadrature integration, after ensuring that the integrated functions and data are smooth enough. From an algorithmic point of view, we recommend a recursive implementation of the integration operator $\omega_{\Sigma\rho}$, in order to have a parsimonious computation of the third order adjustment terms.

The Greeks used in our pricing formulas are defined with respect to the terminal variable X_T (see Notation 4.3.4). This is not the usual representation of Greeks since price sensitivities are commonly computed w.r.t. initial data, the initial value X_0 here. In our context, the computation of $Greek^{(i)}(X_T)$ can still be efficiently performed, by the means of the following transformation.

Proposition 4.5.1 (Greeks transformation)**.** *Using Notation 4.3.4, one has:* $\forall i,j,k \in [[1\dots n]]$

$$\left[Greek^{(1)}(X_T)\right]_i = \sum_{l=1}^{n} \frac{\partial \mathbb{E}[h(X_T)]}{\partial X_0^l} \left[e^{-\int_0^T \Theta(s)ds}\right]_{l,i} \tag{4.5.1}$$

$$\left[Greek^{(2)}(X_T)\right]_{i,j} = \sum_{l,p=1}^{n} \frac{\partial^2 \mathbb{E}[h(X_T)]}{\partial X_0^l \partial X_0^p} \left[e^{-\int_0^T \Theta(s)ds}\right]_{l,i} \left[e^{-\int_0^T \Theta(s)ds}\right]_{p,j} \tag{4.5.2}$$

$$\left[Greek^{(3)}(X_T)\right]_{i,j,k} = \sum_{l,p,q=1}^{n} \frac{\partial^3 \mathbb{E}[h(X_T)]}{\partial X_0^l \partial X_0^p \partial X_0^q} \left[e^{-\int_0^T \Theta(s)ds}\right]_{l,i} \left[e^{-\int_0^T \Theta(s)ds}\right]_{p,j} \left[e^{-\int_0^T \Theta(s)ds}\right]_{q,k} \tag{4.5.3}$$

Proof. We admit that the required smoothness and boundedness conditions allowing to substitute the derivation and expectation operators are met. Owing to the linear dynamics (4.3.3) of X, X_t is given explicitly by

$$X_t = e^{\int_0^t \Theta(s)ds}\left(X_0 + \int_0^t e^{-\int_0^s \Theta(u)du}\Phi(s)\,ds\right) + e^{\int_0^t \Theta(s)ds}\int_0^t e^{-\int_0^s \Theta(u)du}\Sigma(s)\,dB_s$$

one obtains then the following tangent flows $\frac{\partial X_T^i}{\partial X_0^l} = \left[e^{\int_0^t \Theta(s)ds}\right]_{i,l}$ while $\frac{\partial^2 X_T^i}{\partial X_0^l X_0^p} = \frac{\partial^3_{X_0^l, X_0^p, X_0^q}\left(X_T^i\right)}{\partial X_0^l X_0^p X_0^q} = 0$ for $i,j,k,l \in [\![1,n]\!]$. It readily follows that

$$\left[Greek^{(1)}(X_T)\right]_i = \mathbb{E}\left[\frac{\partial h(X_T)}{\partial X_T^i}\right] = \mathbb{E}\left[\sum_{l=1}^{n} \frac{\partial h(X_T)}{\partial X_0^l}\frac{\partial X_0^l}{\partial X_T^i}\right] = \sum_{l=1}^{n} \frac{\partial \mathbb{E}[h(X_T)]}{\partial X_0^l}\left[e^{-\int_0^T \Theta(s)ds}\right]_{l,i}$$

where we have used the fact that $\left(e^{\int_0^T \Theta(s)ds}\right)^{-1} = e^{-\int_0^T \Theta(s)ds}$. The second and third order Greeks are treated similarly:

$$\begin{aligned}
\left[Greek^{(2)}(X_T)\right]_{i,j} &= \mathbb{E}\left[\frac{\partial^2 h(X_T)}{\partial X_T^i \partial X_T^j}\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial X_T^j}\left(\sum_{l=1}^{n} \frac{\partial h(X_T)}{\partial X_0^l}\frac{\partial X_0^l}{\partial X_T^i}\right)\right] \\
&= \mathbb{E}\left[\sum_{l=1}^{n}\left(\frac{\partial}{\partial X_0^l}\left(\frac{\partial h(X_T)}{\partial X_T^j}\right)\right)\frac{\partial X_0^l}{\partial X_T^i}\right] + \mathbb{E}\left[\sum_{l=1}^{n} \frac{\partial h(X_T)}{\partial X_0^l}\frac{\partial^2 X_0^l}{\partial X_T^j \partial X_T^i}\right] \\
&= \mathbb{E}\left[\sum_{l,p=1}^{n} \frac{\partial^2 h(X_T)}{\partial X_0^l \partial X_0^p}\frac{\partial X_0^l}{\partial X_T^i}\frac{\partial X_0^p}{\partial X_T^j}\right] \\
&= \sum_{l,p=1}^{n} \frac{\partial^2 \mathbb{E}[h(X_T)]}{\partial X_0^l \partial X_0^p}\left[e^{-\int_0^T \Theta(s)ds}\right]_{l,i}\left[e^{-\int_0^T \Theta(s)ds}\right]_{p,j}
\end{aligned}$$

using $\frac{\partial}{\partial X_T^j}\left(\frac{\partial X_0^l}{\partial X_T^i}\right)=0$. Finally, a similar derivation gives

$$\begin{aligned}\left[Greek^{(3)}(X_T)\right]_{i,j,k} &= \mathbb{E}\left[\sum_{l,p,q=1}^{n}\frac{\partial^3 h(X_T)}{\partial X_0^l\partial X_0^p\partial X_0^q}\frac{\partial X_0^l}{\partial X_T^i}\frac{\partial X_0^p}{\partial X_T^j}\frac{\partial X_0^q}{\partial X_T^k}\right]\\ &= \sum_{l,p,q=1}^{n}\frac{\partial^3\mathbb{E}[h(X_T)]}{\partial X_0^l\partial X_0^p\partial X_0^q}\left[e^{-\int_0^T\Theta(s)ds}\right]_{l,i}\left[e^{-\int_0^T\Theta(s)ds}\right]_{p,j}\left[e^{-\int_0^T\Theta(s)ds}\right]_{q,k}\end{aligned}$$

□

We consider the case of derivative contracts written on a stock S that is assumed to follow the standard univariate Black-Scholes dynamics with time-homogeneous parameters

$$\begin{cases} S_t & = \exp(X_t)\\ dX_t & = \left(r-\frac{1}{2}\Sigma^2\right)dt+\Sigma dB_t, \qquad X_0=\ln(S_0)\\ d\langle W,B\rangle_t & = \rho dt\end{cases}$$

The terms $Greek^{(i)}(X_T)$, $i=1,2,3$ are expressed w.r.t the log-variable X_0. They are explicitly obtained in terms of usual Black-Scholes Greeks by

$$\begin{pmatrix} Greek^{(1)}(X_T)\\ Greek^{(2)}(X_T)\\ Greek^{(3)}(X_T)\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\1&3&1\end{pmatrix}\begin{pmatrix} S_0\partial_S Price^{BS}(0,S_0)\\ S_0^2\partial_S^2 Price^{BS}(0,S_0)\\ S_0^3\partial_S^3 Price^{BS}(0,S_0)\end{pmatrix}$$

where $Price^{BS}(0,S_0)$ refers to the explicit Black-Scholes price function.

Throughout our numerical experiments, we look forward to emphasize the robustness of the pricing methodology w.r.t. two criteria:

- **The smoothness of the volatility function** ν: Our error estimates have a clear dependence on the magnitude of ν and its derivatives up to order 2 and 3 for respectively the second and third order approximations. An appropriate accuracy test is to consider different volatility functions ν:
  - $(\mathscr{V})$: $\nu(t,\lambda_t^\varepsilon)=\bar{\nu}$. This corresponds to the Vasicek dynamics for λ_t^ε. One has $\nu^{(i)}=0$, $i\geq 1$. Notice that this case does not guarantee the positivity of λ_t^ε.
  - $(\mathscr{LN})$: $\nu(t,\lambda_t^\varepsilon)=\bar{\nu}\lambda_t^\varepsilon$ which corresponds to a lognormal model for λ_t^ε. The model preserves the positivity of λ^ε and one has $\nu^{(1)}=\bar{\nu}$ and $\nu^{(i)}=0$, $i\geq 2$.
  - $(\mathscr{C})$: $\nu(t,\lambda_t^\varepsilon)=\bar{\nu}\sqrt{\lambda_t^\varepsilon}$. This corresponds to a CIR dynamics for λ_t^ε. It does not necessarily satisfy the boundedness condition required throughout our theoretical analysis.
- **The smoothness of the payoff function** h: There are few regularity conditions required from the payoff function. In fact, only the existence of the three order Greeks is needed, a requirement that we consider as minimalist with regard to financial applications. Additionally, our error estimates show our approximation errors are proportional to the (L_2) magnitude of the payoff $h(X_T)$. In spite of these theoretical elements, we separate our analysis in order to test different payoff functions with different smoothness traits.

- $(\mathscr{H}_1)$: h belongs to $\mathscr{C}_0^\infty(\mathbb{R})$. This case corresponds to smooth payoffs and we treat it by considering a forward contract. Formally, $h(X_T) = \left(K - e^{X_T}\right)$ paid at maturity T for a strike price K.
- $(\mathscr{H}_2)$: h is almost everywhere differentiable. This case corresponds to vanilla options such as a European Put option $h(X_T) = \left(K - e^{X_T}\right)^+$.
- $(\mathscr{H}_3)$: h simply belongs to $\mathscr{H}$. We consider in this case binary options like a Cash-or-nothing Put $h(X_T) = 1_{X_T < \ln K}$.

Parameters κ, ψ and Σ are assumed to be constant and positive. We study the numerical accuracy w.r.t. the strike K, the maturity $S = T$ as well as $\bar{\nu}$, ρ and Σ. We focus on two different sets for the credit model parameters that we name hereafter *Mid* and *High* risk settings. They are respectively given in Tables 4.5.1a and 4.5.1b.

λ_0	κ	ψ	$\bar{\nu}(\mathscr{V})$	$\bar{\nu}(\mathscr{LN})$	$\bar{\nu}(\mathscr{C})$	$\rho = \rho^{\lambda,X}$
1%	1	2%	1%	50%	20%	30%

(a) Mid risk

λ_0	κ	ψ	$\bar{\nu}(\mathscr{V})$	$\bar{\nu}(\mathscr{LN})$	$\bar{\nu}(\mathscr{C})$	$\rho = \rho^{\lambda,X}$
3%	1.6	8%	3%	100%	50%	90%

(b) High risk

Table 4.5.1: Credit spread parameters

Notice that the high risk setting represents an extreme situation. It is intended to reveal the accuracy limits of our formulas, as well as the magnitude of the numerical error one may encounter in case of severe stress-tests. Finally, default parameters of the contract and diffusion process X are reported in Table 4.5.2.

S_0	r	Σ	K	$S = T$
100%	0.5%	25%	80%	10

Table 4.5.2: Default parameters of the diffusion X and the payoff

We take $S = T$ and $\delta = 0$. A benchmark of our approximation formulas is based on a Monte Carlo valuation method with $M = 5 \times 10^5$ simulations and $N = T \times 24$ .i.e a bi-weekly time discretization. Monte Carlo errors (Confidence Error, CE hereafter) are reported at the confidence level 97.5%.

Retrieving market's survival probabilities

We first analyze the accuracy by comparing the survival probabilities obtained by the approximation (4.3.30) w.r.t to Monte Carlo. These probabilities are usually fitted to the market probabilities taken as input. The comparison is achieved considering a payoff function h constantly equal to 1. The second and third approximation formulas coincide in this case and give Equation 4.3.30. The approximated probabilities and the absolute errors w.r.t. to Monte Carlo (and confidence interval) are reported in Table 4.5.3. We observe that the error does not exceed 0.03% (**3 bps**) for the mid risk parameters and 0.12% (**12 bps**) for the high risk parameters. The absolute error is significantly lower if one considers the Vasicek or the Lognormal settings: 3 bps for the Vasicek and the Lognormal models in the mid risk setting, and 9 bps in the high-risk setting. These results are concordant with our error analysis, in particular with the observation made on the regularity of the volatility function ν.

As previously highlighted, we emphasize that the approximation formula (4.3.30) work as a zero order correction to adjust the default probabilities. One can circumvent this zero-order approximation by directly using market survival probabilities $\mathbb{E}\left[e^{-(1-\delta)\int_0^T \lambda_t^{\varepsilon} dt}\right]$ instead of $e^{-(1-\delta)\int_0^T \lambda_t dt}\left(1+(1-\delta)^2 C_{0,1}(T)\right)$ after calibration.

Model	T (y)	*Mid Risk*						*High Risk*					
		MC	**CE**[3]	**Proxy**	**|Err|**	**App.** (4.3.30)	**|Err|**	**MC**	**CE**	**Proxy**	**|Err|**	**App.** (4.3.30)	**|Err|**
($\mathscr{V}$)	1	98.66%	**0.00%**	98.67%	**0.01%**	98.67%	**0.01%**	94.73%	**0.00%**	94.81%	**0.08%**	94.80%	**0.08%**
	5	91.39%	**0.01%**	91.44%	**0.05%**	91.43%	**0.03%**	69.21%	**0.01%**	69.36%	**0.14%**	69.31%	**0.09%**
	10	82.70%	**0.01%**	82.77%	**0.06%**	82.73%	**0.03%**	46.43%	**0.01%**	46.53%	**0.10%**	46.46%	**0.03%**
	20	67.70%	**0.01%**	67.80%	**0.09%**	67.73%	**0.03%**	20.93%	**0.01%**	20.94%	**0.02%**	20.87%	**0.05%**
	30	55.45%	**0.01%**	55.54%	**0.09%**	55.46%	**0.01%**	9.42%	**0.00%**	9.43%	**0.01%**	9.38%	**0.04%**
($\mathscr{LN}$)	1	98.66%	**0.00%**	98.67%	**0.01%**	98.67%	**0.01%**	94.73%	**0.01%**	94.80%	**0.08%**	94.82%	**0.09%**
	5	91.41%	**0.01%**	91.43%	**0.02%**	91.44%	**0.03%**	69.55%	**0.06%**	69.31%	**0.25%**	69.60%	**0.05%**
	10	82.74%	**0.02%**	82.73%	**0.01%**	82.76%	**0.02%**	47.00%	**0.06%**	46.46%	**0.54%**	46.95%	**0.05%**
	20	67.79%	**0.03%**	67.73%	**0.05%**	67.79%	**0.01%**	21.43%	**0.04%**	20.87%	**0.56%**	21.36%	**0.08%**
	30	55.54%	**0.03%**	55.46%	**0.08%**	55.53%	**0.01%**	9.79%	**0.02%**	9.38%	**0.41%**	9.71%	**0.08%**
($\mathscr{C}$)	1	98.66%	**0.01%**	98.67%	**0.01%**	98.67%	**0.01%**	94.76%	**0.03%**	94.80%	**0.04%**	94.87%	**0.11%**
	5	91.51%	**0.04%**	91.43%	**0.09%**	91.54%	**0.03%**	70.18%	**0.10%**	69.31%	**0.87%**	70.30%	**0.12%**
	10	82.98%	**0.06%**	82.73%	**0.24%**	83.00%	**0.02%**	47.90%	**0.10%**	46.46%	**1.44%**	48.02%	**0.12%**
	20	68.21%	**0.07%**	67.73%	**0.48%**	68.22%	**0.01%**	22.35%	**0.07%**	20.87%	**1.48%**	22.40%	**0.05%**
	30	56.08%	**0.07%**	55.46%	**0.63%**	56.08%	**0.00%**	10.43%	**0.04%**	9.38%	**1.05%**	10.43%	**0.00%**

Table 4.5.3: Survival Probabilities $\mathbb{Q}(\tau > T)$ Monte Carlo vs. Expansion (4.3.30)

Impact of the maturity T

In view of error bounds in (4.3.27) and (4.3.51), the approximation errors are bigger with longer maturities. Nevertheless, we show in Tables 4.5.4a, 4.5.4b and 4.5.4c that the approximation

prices are still accurate enough for long maturities (up to 30y). As expected, the regularity of the underlying intensity model is a key element for the methodology. In the high risk setting, the results obtained for the Vasicek and the Lognormal models are excellent. However, only the third order approximation for the CIR model is satisfactory. Our results also show that the numerical accuracy is not altered when we consider payoffs with different regularity characteristics, as it is the case for the binary option.

We emphasize that the impacts of the volatility of the stochastic intensity λ and the hybrid correlation $\rho^{\lambda,X}$ are significant for long-term portfolios as in the case of counterparty risk management.

Impact of the moneyness K

In view of the error analysis, approximation errors are relative to the magnitude of the risk-free price $\mathbb{E}[h(X_T)]$. This is a relevant consequence in comparison to Monte Carlo techniques where numerical errors are important for out the money strikes. This observation is illustrated numerically in Tables 4.5.5a, 4.5.5b and 4.5.5c. We notice that the approximation errors do not depend on the moneyness, neither on the regularity of the payoff h. For realistic settings (mid risk), the accuracy is excellent. It remains very good for the high risk parameters. As expected, our approach is less accurate in the framework of CIR credit model $(\mathscr{C})$ but remains very acceptable when we compare the absolute error with the magnitude of the prices.

Impact of the underlying's volatility Σ

We show in Tables 4.5.6a, 4.5.6b and 4.5.6c the results for different levels of volatility Σ (10%, 50% and 80%). The absolute errors increase with increasing volatility. This is also related to the increase of the price magnitude $\mathbb{E}[h(X_T)]$ w.r.t. Σ. In the mid credit risk setting, the accuracy remains excellent under all the considered models. However, coupling high annual volatility $\Sigma \geq 50\%$ over 10 years with *high* risk credit parameters generates results with quite lower accuracy. Nonetheless, 2^{nd} and 3^{rd} order approximations remain acceptable in this high volatility regime when we take into account the confidence error of the Monte Carlo prices.

Impact of the hybrid correlation $\rho^{\lambda,X}$

We finally report in Table 4.5.7 an example of correlation analysis of a derivative that is contingent to the survival of a reference entity. For illustration, we take the example of the vanilla put option. This kind of analysis can be performed straightforwardly w.r.t $\rho^{\lambda,X}$ or the intensity volatility ν since these parameters are present only in the calculation of the adjustment coefficients $C_{i,j}$. The dependence of $C_{i,j}$ on ρ and $\bar{\nu}$ is quadratic for the second order approximation, and cubic for the third order one.

Model	T (y)	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	10	-0.20%	**0.34%**	-0.22%	**0.02%**	-0.22%	**0.02%**	1.21%	**0.18%**	1.09%	**0.12%**	1.09%	**0.12%**
	20	-0.67%	**0.48%**	-0.71%	**0.04%**	-0.71%	**0.04%**	1.22%	**0.13%**	1.11%	**0.11%**	1.12%	**0.11%**
	30	-0.68%	**0.62%**	-0.74%	**0.05%**	-0.73%	**0.05%**	0.74%	**0.08%**	0.67%	**0.07%**	0.67%	**0.07%**
($\mathscr{L}$)	10	0.48%	**0.33%**	0.45%	**0.03%**	0.45%	**0.03%**	4.60%	**0.15%**	4.41%	**0.19%**	4.56%	**0.03%**
	20	0.88%	**0.45%**	0.82%	**0.06%**	0.82%	**0.06%**	4.08%	**0.10%**	3.91%	**0.17%**	4.08%	**0.00%**
	30	0.51%	**0.53%**	0.43%	**0.08%**	0.43%	**0.08%**	2.73%	**0.05%**	2.61%	**0.12%**	2.70%	**0.03%**
($\mathscr{C}$)	10	0.19%	**0.33%**	0.30%	**0.12%**	0.29%	**0.10%**	6.70%	**0.15%**	6.92%	**0.22%**	6.79%	**0.09%**
	20	0.72%	**0.45%**	0.89%	**0.17%**	0.88%	**0.17%**	6.14%	**0.09%**	5.89%	**0.25%**	6.20%	**0.07%**
	30	1.17%	**0.54%**	1.32%	**0.15%**	1.33%	**0.16%**	4.06%	**0.05%**	3.52%	**0.54%**	4.14%	**0.08%**

(a) Contingent Forward contract

Model	T (y)	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	10	25.49%	**0.11%**	25.49%	**0.00%**	25.49%	**0.00%**	14.72%	**0.06%**	14.70%	**0.02%**	14.70%	**0.02%**
	20	28.62%	**0.10%**	28.62%	**0.00%**	28.62%	**0.00%**	9.28%	**0.03%**	9.26%	**0.02%**	9.26%	**0.02%**
	30	28.05%	**0.09%**	28.05%	**0.00%**	28.05%	**0.00%**	5.00%	**0.02%**	4.99%	**0.01%**	4.99%	**0.01%**
($\mathscr{L}$)	10	25.51%	**0.11%**	25.52%	**0.00%**	25.52%	**0.00%**	15.84%	**0.07%**	15.72%	**0.12%**	15.78%	**0.05%**
	20	28.78%	**0.10%**	28.77%	**0.00%**	28.77%	**0.00%**	10.28%	**0.04%**	10.14%	**0.14%**	10.18%	**0.10%**
	30	28.15%	**0.09%**	28.14%	**0.01%**	28.14%	**0.01%**	5.65%	**0.02%**	5.55%	**0.11%**	5.56%	**0.09%**
($\mathscr{C}$)	10	25.59%	**0.11%**	25.64%	**0.05%**	25.63%	**0.05%**	16.91%	**0.08%**	17.12%	**0.21%**	17.08%	**0.16%**
	20	29.14%	**0.11%**	29.21%	**0.07%**	29.21%	**0.07%**	11.30%	**0.05%**	11.34%	**0.04%**	11.40%	**0.10%**
	30	28.67%	**0.10%**	28.73%	**0.05%**	28.73%	**0.06%**	6.40%	**0.03%**	6.35%	**0.05%**	6.42%	**0.02%**

(b) Contingent Put Option

Model	T (y)	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	10	54.35%	**0.17%**	54.36%	**0.01%**	54.36%	**0.01%**	30.94%	**0.10%**	30.94%	**0.00%**	30.94%	**0.00%**
	20	48.12%	**0.14%**	48.12%	**0.00%**	48.12%	**0.00%**	15.32%	**0.04%**	15.31%	**0.01%**	15.31%	**0.01%**
	30	41.79%	**0.11%**	41.79%	**0.00%**	41.79%	**0.00%**	7.29%	**0.02%**	7.28%	**0.01%**	7.28%	**0.01%**
($\mathscr{L}$)	10	54.11%	**0.17%**	54.12%	**0.01%**	54.12%	**0.01%**	32.73%	**0.11%**	32.55%	**0.18%**	32.64%	**0.10%**
	20	48.48%	**0.14%**	48.47%	**0.01%**	48.47%	**0.01%**	16.55%	**0.05%**	16.37%	**0.19%**	16.40%	**0.15%**
	30	41.93%	**0.11%**	41.92%	**0.02%**	41.92%	**0.02%**	8.03%	**0.02%**	7.89%	**0.14%**	7.90%	**0.12%**
($\mathscr{C}$)	10	54.29%	**0.18%**	54.35%	**0.06%**	54.35%	**0.06%**	34.33%	**0.12%**	34.44%	**0.10%**	34.37%	**0.04%**
	20	48.85%	**0.14%**	48.91%	**0.06%**	48.92%	**0.06%**	17.76%	**0.06%**	17.67%	**0.09%**	17.73%	**0.04%**
	30	42.55%	**0.11%**	42.58%	**0.03%**	42.58%	**0.04%**	8.81%	**0.03%**	8.67%	**0.13%**	8.75%	**0.06%**

(c) Contingent Cash-or-Nothing Put

Table 4.5.4: Impact of the maturity T

Model	K	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	80%	-16.41%	**0.34%**	-16.43%	**0.03%**	-16.43%	**0.03%**	-8.15%	**0.18%**	-8.28%	**0.13%**	-8.28%	**0.13%**
	100%	0.22%	**0.33%**	0.20%	**0.02%**	0.20%	**0.02%**	1.44%	**0.18%**	1.32%	**0.12%**	1.32%	**0.12%**
	120%	17.17%	**0.33%**	17.15%	**0.02%**	17.15%	**0.02%**	10.67%	**0.18%**	10.57%	**0.10%**	10.57%	**0.10%**
($\mathscr{L}$)	80%	-16.42%	**0.33%**	-16.46%	**0.04%**	-16.46%	**0.04%**	-5.17%	**0.15%**	-5.33%	**0.16%**	-5.19%	**0.02%**
	100%	-0.46%	**0.34%**	-0.50%	**0.04%**	-0.50%	**0.04%**	4.28%	**0.16%**	4.09%	**0.19%**	4.23%	**0.06%**
	120%	16.50%	**0.33%**	16.48%	**0.03%**	16.47%	**0.03%**	13.94%	**0.16%**	13.74%	**0.20%**	13.90%	**0.04%**
($\mathscr{C}$)	80%	-15.91%	**0.33%**	-15.81%	**0.11%**	-15.82%	**0.09%**	-2.48%	**0.14%**	-2.27%	**0.22%**	-2.40%	**0.09%**
	100%	0.55%	**0.33%**	0.64%	**0.09%**	0.62%	**0.07%**	7.05%	**0.15%**	7.24%	**0.20%**	7.12%	**0.08%**
	120%	17.56%	**0.33%**	17.65%	**0.09%**	17.63%	**0.07%**	16.18%	**0.15%**	16.38%	**0.19%**	16.25%	**0.07%**

(a) Contingent Forward contract

Model	K	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	80%	15.46%	**0.08%**	15.45%	**0.00%**	15.45%	**0.00%**	9.00%	**0.05%**	8.98%	**0.02%**	8.98%	**0.02%**
	100%	25.46%	**0.11%**	25.47%	**0.00%**	25.47%	**0.00%**	14.78%	**0.06%**	14.76%	**0.02%**	14.76%	**0.02%**
	120%	37.06%	**0.13%**	37.06%	**0.00%**	37.06%	**0.00%**	21.41%	**0.08%**	21.40%	**0.01%**	21.40%	**0.01%**
($\mathscr{L}$)	80%	15.44%	**0.08%**	15.44%	**0.00%**	15.44%	**0.00%**	9.68%	**0.05%**	9.60%	**0.08%**	9.64%	**0.04%**
	100%	25.16%	**0.11%**	25.16%	**0.00%**	25.16%	**0.00%**	15.81%	**0.07%**	15.69%	**0.12%**	15.76%	**0.05%**
	120%	36.90%	**0.13%**	36.91%	**0.01%**	36.91%	**0.01%**	22.72%	**0.09%**	22.57%	**0.15%**	22.65%	**0.07%**
($\mathscr{C}$)	80%	15.61%	**0.08%**	15.66%	**0.04%**	15.65%	**0.04%**	10.68%	**0.06%**	10.86%	**0.18%**	10.82%	**0.15%**
	100%	25.60%	**0.11%**	25.64%	**0.04%**	25.63%	**0.04%**	17.07%	**0.08%**	17.28%	**0.21%**	17.23%	**0.17%**
	120%	37.42%	**0.14%**	37.48%	**0.06%**	37.47%	**0.05%**	24.09%	**0.10%**	24.31%	**0.23%**	24.26%	**0.17%**

(b) Contingent Put Option

Model	K	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	80%	45.10%	**0.18%**	45.10%	**0.01%**	45.10%	**0.01%**	25.91%	**0.11%**	25.89%	**0.01%**	25.89%	**0.01%**
	100%	54.12%	**0.17%**	54.13%	**0.01%**	54.13%	**0.01%**	30.99%	**0.10%**	30.99%	**0.00%**	30.99%	**0.00%**
	120%	61.12%	**0.16%**	61.13%	**0.01%**	61.13%	**0.01%**	34.73%	**0.09%**	34.74%	**0.02%**	34.74%	**0.02%**
($\mathscr{L}$)	80%	45.11%	**0.18%**	45.12%	**0.00%**	45.12%	**0.00%**	27.55%	**0.11%**	27.37%	**0.18%**	27.47%	**0.08%**
	100%	53.71%	**0.17%**	53.71%	**0.01%**	53.71%	**0.01%**	32.64%	**0.11%**	32.46%	**0.19%**	32.55%	**0.09%**
	120%	60.50%	**0.16%**	60.51%	**0.01%**	60.51%	**0.01%**	36.34%	**0.10%**	36.17%	**0.18%**	36.25%	**0.09%**
($\mathscr{C}$)	80%	45.66%	**0.18%**	45.75%	**0.09%**	45.74%	**0.08%**	29.68%	**0.12%**	29.86%	**0.19%**	29.78%	**0.11%**
	100%	54.34%	**0.18%**	54.39%	**0.05%**	54.38%	**0.04%**	34.64%	**0.12%**	34.75%	**0.11%**	34.68%	**0.04%**
	120%	61.15%	**0.16%**	61.20%	**0.05%**	61.19%	**0.04%**	37.78%	**0.11%**	37.81%	**0.04%**	37.76%	**0.01%**

(c) Contingent Cash-or-Nothing Put

Table 4.5.5: Impact of the strike K

Model	Σ	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	10%	-0.22%	**0.12%**	-0.23%	**0.01%**	-0.23%	**0.01%**	0.33%	**0.06%**	0.28%	**0.05%**	0.28%	**0.05%**
	50%	-0.12%	**1.12%**	-0.17%	**0.04%**	-0.17%	**0.04%**	3.07%	**0.58%**	2.86%	**0.21%**	2.86%	**0.21%**
	80%	1.64%	**3.67%**	1.58%	**0.07%**	1.58%	**0.07%**	4.50%	**4.13%**	4.21%	**0.29%**	4.21%	**0.29%**
($\mathscr{L}$)	10%	-0.46%	**0.12%**	-0.47%	**0.01%**	-0.47%	**0.01%**	1.40%	**0.06%**	1.32%	**0.08%**	1.38%	**0.01%**
	50%	0.68%	**1.11%**	0.58%	**0.10%**	0.58%	**0.10%**	9.60%	**0.42%**	9.36%	**0.24%**	9.63%	**0.03%**
	80%	2.00%	**4.36%**	1.90%	**0.10%**	1.90%	**0.10%**	17.44%	**1.01%**	17.35%	**0.09%**	17.37%	**0.06%**
($\mathscr{C}$)	10%	0.11%	**0.12%**	0.16%	**0.04%**	0.15%	**0.04%**	2.51%	**0.06%**	2.69%	**0.17%**	2.63%	**0.12%**
	50%	2.97%	**1.05%**	3.09%	**0.13%**	3.07%	**0.10%**	14.70%	**0.32%**	14.58%	**0.13%**	14.38%	**0.32%**
	80%	6.53%	**4.50%**	6.76%	**0.23%**	6.70%	**0.17%**	21.78%	**1.41%**	20.72%	**1.06%**	20.80%	**0.98%**

(a) Contingent Forward contract

Model	Σ	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	10%	10.26%	**0.03%**	10.26%	**0.00%**	10.26%	**0.00%**	5.98%	**0.07%**	5.97%	**0.01%**	5.97%	**0.01%**
	50%	47.27%	**0.07%**	47.28%	**0.01%**	47.28%	**0.01%**	27.26%	**0.16%**	27.26%	**0.08%**	27.26%	**0.08%**
	80%	65.69%	**0.06%**	65.71%	**0.02%**	65.71%	**0.02%**	37.42%	**0.14%**	37.46%	**0.07%**	37.46%	**0.07%**
($\mathscr{L}$)	10%	10.15%	**0.03%**	10.16%	**0.00%**	10.16%	**0.00%**	6.42%	**0.07%**	6.36%	**0.04%**	6.39%	**0.03%**
	50%	47.38%	**0.07%**	47.38%	**0.00%**	47.38%	**0.00%**	28.72%	**0.17%**	28.55%	**0.09%**	28.63%	**0.08%**
	80%	65.84%	**0.06%**	65.86%	**0.02%**	65.86%	**0.02%**	38.98%	**0.15%**	38.82%	**0.08%**	38.87%	**0.11%**
($\mathscr{C}$)	10%	10.48%	**0.03%**	10.51%	**0.03%**	10.50%	**0.03%**	7.03%	**0.08%**	7.18%	**0.15%**	7.15%	**0.04%**
	50%	47.92%	**0.07%**	47.98%	**0.06%**	47.97%	**0.05%**	30.48%	**0.19%**	30.64%	**0.17%**	30.58%	**0.10%**
	80%	66.22%	**0.06%**	66.27%	**0.05%**	66.26%	**0.05%**	40.53%	**0.17%**	40.55%	**0.02%**	40.50%	**0.09%**

(b) Contingent Put Option

Model	Σ	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**	**MC**	**CE**	**2^{nd} order App**	**|Err|**	**3^{rd} order App**	**|Err|**
($\mathscr{V}$)	10%	46.05%	**0.18%**	46.05%	**0.00%**	46.05%	**0.00%**	26.60%	**0.10%**	26.59%	**0.01%**	26.59%	**0.01%**
	50%	65.01%	**0.15%**	65.03%	**0.02%**	65.02%	**0.02%**	37.12%	**0.09%**	37.14%	**0.03%**	37.15%	**0.03%**
	80%	74.29%	**0.11%**	74.32%	**0.02%**	74.32%	**0.02%**	42.09%	**0.06%**	42.15%	**0.06%**	42.15%	**0.06%**
($\mathscr{L}$)	10%	45.86%	**0.18%**	45.86%	**0.01%**	45.86%	**0.01%**	28.07%	**0.11%**	27.89%	**0.18%**	27.99%	**0.08%**
	50%	65.06%	**0.15%**	65.06%	**0.01%**	65.06%	**0.01%**	38.65%	**0.09%**	38.47%	**0.17%**	38.54%	**0.11%**
	80%	74.31%	**0.11%**	74.33%	**0.02%**	74.33%	**0.02%**	43.42%	**0.07%**	43.27%	**0.15%**	43.30%	**0.12%**
($\mathscr{C}$)	10%	46.68%	**0.18%**	46.75%	**0.07%**	46.74%	**0.06%**	30.06%	**0.12%**	30.27%	**0.21%**	30.19%	**0.14%**
	50%	65.68%	**0.15%**	65.72%	**0.05%**	65.72%	**0.04%**	40.33%	**0.10%**	40.36%	**0.03%**	40.31%	**0.02%**
	80%	74.62%	**0.11%**	74.66%	**0.04%**	74.66%	**0.04%**	44.68%	**0.08%**	44.63%	**0.05%**	44.60%	**0.08%**

(c) Contingent Cash-or-Nothing Put

Table 4.5.6: Impact of the underlying volatility's Σ

Model	$\rho^{\lambda,X}$	*Mid Risk*						*High Risk*					
		MC	**CE**	**2^{nd}order App**	**|Err|**	**3^{rd}order App**	**|Err|**	**MC**	**CE**	**2^{nd}order App**	**|Err|**	**3^{rd}order App**	**|Err|**
($\mathscr{V}$)	−90%	24.88%	0.10%	24.91%	0.03%	24.91%	0.03%	13.50%	0.06%	13.57%	0.07%	13.57%	0.07%
	−50%	25.02%	0.11%	25.05%	0.02%	25.05%	0.02%	13.89%	0.06%	13.94%	0.05%	13.94%	0.05%
	50%	25.55%	0.11%	25.55%	0.01%	25.54%	0.01%	14.50%	0.06%	14.51%	0.00%	14.51%	0.00%
	90%	25.80%	0.11%	25.78%	0.02%	25.78%	0.02%	14.67%	0.06%	14.65%	0.02%	14.65%	0.02%
($\mathscr{L}$)	−90%	24.72%	0.10%	24.76%	0.04%	24.76%	0.04%	12.77%	0.05%	12.81%	0.04%	12.74%	0.03%
	−50%	24.93%	0.10%	24.95%	0.03%	24.95%	0.03%	13.40%	0.06%	13.41%	0.00%	13.37%	0.03%
	50%	25.35%	0.11%	25.35%	0.00%	25.35%	0.00%	15.12%	0.07%	15.04%	0.08%	15.07%	0.04%
	90%	25.58%	0.11%	25.59%	0.01%	25.59%	0.00%	15.90%	0.07%	15.78%	0.12%	15.85%	0.05%
($\mathscr{C}$)	−90%	23.75%	0.10%	23.66%	0.10%	23.68%	0.07%	12.12%	0.05%	11.96%	0.16%	12.01%	0.10%
	−50%	24.51%	0.10%	24.45%	0.06%	24.46%	0.05%	13.11%	0.06%	13.00%	0.10%	13.03%	0.08%
	50%	25.75%	0.11%	25.84%	0.09%	25.83%	0.07%	15.90%	0.07%	16.01%	0.11%	15.98%	0.08%
	90%	26.65%	0.11%	26.83%	0.18%	26.80%	0.15%	17.27%	0.08%	17.50%	0.22%	17.44%	0.17%

Table 4.5.7: Contingent Put Option: Impact of the correlation $\rho^{\lambda,X}$

4.6 Technical results and proofs

This section brings together the technical results and proofs used to derive the extension formulas.

4.6.1 Technical lemmas

Lemma 4.6.1. *For any continuous (or piecewise continuous) function f, any continuous semi-martingale Z vanishing at $t = 0$ one has:*

$$\int_0^t f_s Z_s ds = \int_0^t \left(\int_s^t f_u du \right) dZ_s$$

Proof. This lemma is given in [BGM09]. It follows from the integration by part formula applied on $\left(\int_s^t f_u du Z_s\right)_s$. □

We recall the definitions of Θ and Σ given in (4.3.3).

Lemma 4.6.2. *Let (u_t) be a square-integrable and predictable $\mathbb{R}-$valued process. For X being the solution of* (4.3.3)*, one has for $S \leq T$*

$$\mathbb{E}\left[\left(\int_0^S u_t dW_t \right) h\left(X_T\right) \right] = \sum_{i=1}^{n} \mathbb{E}\left[\int_0^S u_t \left[e^{\int_t^T \Theta(s) ds} \Sigma\left(t\right) \rho \right]_i dt \left[h^{(1)}\left(X_T\right) \right]_i \right]$$

where the integration on the exponential matrix argument is applied element-wisely.

Proof. The proof follows an application of Malliavin's duality formula in the context of a multidimensional Brownian motion. In fact, let us define the correlation matrix Π

$$\Pi = \frac{1}{dt} d \left\langle \begin{pmatrix} W \\ B \end{pmatrix}, \begin{pmatrix} W \\ B \end{pmatrix} \right\rangle_t = \begin{pmatrix} 1 & \rho_1 & \dots & \rho_d \\ \rho_1 & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{1,1} & \dots & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_d & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{d,1} & \dots & \left(\frac{d\langle B,B\rangle_t}{dt}\right)_{d,d} \end{pmatrix}$$

and L a Cholesky matrix, such that $\Pi = LL'$ and $L_{j+1,1} = \rho_j$ for $j = 1 \dots d$. Additionally, we set

$$\bar{\Sigma}(t) = \begin{pmatrix} 0 & \Sigma_{1,1}(t) & \dots & \Sigma_{1,d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \Sigma_{n,1}(t) & \dots & \Sigma_{n,d}(t) \end{pmatrix}$$

and we let $\bar{W} = \left(W, W^{1,\perp}, \dots, W^{d,\perp}\right)$ be a multidimensional Brownian motion. One have then

$$\Sigma(t)\, dB_t = \bar{\Sigma}(s) \begin{pmatrix} dW_t \\ dB_t^1 \\ \vdots \\ dB_t^d \end{pmatrix} \underset{=}{\mathscr{D}} \bar{\Sigma}(t) L \begin{pmatrix} dW_t \\ dW_t^{1,\perp} \\ \vdots \\ dW_t^{d,\perp} \end{pmatrix}$$

where $\underset{=}{\mathscr{D}}$ refer to equality in distributions. Taking the Malliavin derivative of X_T w.r.t. $\bar{W}$, we obtain (cf. [Nua06])

$$\mathscr{D}_t^{\bar{W}} X_T = Y_T Z_t \bar{\Sigma}(t) L 1_{\{t \le T\}}$$

where (Y_t) and $(Z)_t$ satisfy

$$Y_0 = I_{\mathbb{R}^{n,n}},\ dY_t = \Theta(t) Y_t dt,\ Z_t = Y_t^{-1}$$

This leads to $Y_t = \exp\left(\int_0^t \Theta(s)\, ds\right)$ and $Z_t = \exp\left(-\int_0^t \Theta(s)\, ds\right)$ where function exp refers to the matrix exponential function [4]. Finally, the Malliavin derivatives of X w.r.t. W corresponds to the first column of the derivative matrix $\mathscr{D}_t^{\bar{W}} X_T$. Consequently, for $1 \le i \le n$, and $t \le T$

$$\mathscr{D}_t^W X_T^i = \left[e^{\int_t^T \Theta(u) du} . \Sigma(t) L \right]_{i,1} = \sum_{j=1}^{d} \left[e^{\int_t^T \Theta(u) du} \Sigma(t) \right]_{i,j} \rho_j$$

An application of the chain rule concludes the proof. □

4.6.2 Explicit corrections

Proposition 4.6.3. *Assume $(\mathscr{R}_3)$. Then, using Notation 4.3.3, $(\lambda_{1,t})$ and $(\lambda_{2,t})$ are the solutions of the following linear SDEs:*

$$d\lambda_{1,t}^{\varepsilon} = -\kappa(t)\, \lambda_{1,t}^{\varepsilon} dt + \nu\left(t, \lambda_t^{\varepsilon}\right) dW_t + \varepsilon \lambda_{1,t}^{\varepsilon} \partial_\lambda \nu\left(t, \lambda_t^{\varepsilon}\right) dW_t \tag{4.6.1}$$

$$d\lambda_{2,t}^{\varepsilon} = -\kappa(t)\, \lambda_{2,t}^{\varepsilon} dt + 2\lambda_{1,t}^{\varepsilon} \partial_\lambda \nu\left(t, \lambda_t^{\varepsilon}\right) dW_t + \varepsilon \left(\lambda_{2,t}^{\varepsilon} \partial_\lambda \nu\left(t, \lambda_t^{\varepsilon}\right) + \left(\lambda_{1,t}^{\varepsilon}\right)^2 \partial_\lambda^2 \nu\left(t, \lambda_t^{\varepsilon}\right) \right) dW_t \tag{4.6.2}$$

[4] It is defined by the power series $e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ where X^0 corresponds to the identity matrix

where $\lambda^{\varepsilon}_{1,0} = \lambda^{\varepsilon}_{2,0} = 0$. *Moreover, under* $(\mathscr{R}_4)$, $(\lambda_{3,t})$ *is the solution of :*

$$\begin{aligned} d\lambda^{\varepsilon}_{3,t} &= -\kappa(t)\,\lambda^{\varepsilon}_{3,t}dt + 3\left(\lambda^{\varepsilon}_{2,t}\partial_\lambda \nu(t,\lambda^{\varepsilon}_t) + \left(\lambda^{\varepsilon}_{1,t}\right)^2 \partial^2_\lambda \nu(t,\lambda^{\varepsilon}_t)\right) dW_t \\ &\quad + \varepsilon\left(\lambda^{\varepsilon}_{3,t}\partial_\lambda \nu(t,\lambda^{\varepsilon}_t) + 3\lambda^{\varepsilon}_{1,t}\lambda^{\varepsilon}_{2,t}\partial^2_\lambda \nu(t,\lambda^{\varepsilon}_t) + \left(\lambda^{\varepsilon}_{1,t}\right)^3 \partial^3_\lambda \nu(t,\lambda^{\varepsilon}_t)\right) dW_t \end{aligned} \tag{4.6.3}$$

with $\lambda^{\varepsilon}_{3,0} = 0$.

Proof. Assuming $(\mathscr{R}_N)$ for $N \geq 1$, $(\lambda^{\varepsilon}_t)$ is $\mathscr{C}^{N-1}$ w.r.t. ε. We apply a successive differentiation to the dynamics of $(\lambda^{\varepsilon}_t)$ of (4.3.2) w.r.t ε at orders 1, 2 and 3. See Theorem 2.3 in [Kun84] for more details. □

Proposition 4.6.4. *Given Proposition 4.6.3, one has*

$$\int_0^S \lambda_{1,t}dt = \int_0^S \int_t^S e^{-\int_0^s \kappa(u)du} ds e^{\int_0^t \kappa(s)ds} \nu(t)\, dW_t \tag{4.6.4}$$

$$\int_0^S \lambda_{2,t}dt = 2\int_0^S \int_t^S e^{-\int_0^s \kappa(u)du} ds \nu^{(1)}(t) \left(\int_0^t e^{\int_0^s \kappa(u)du} \nu(s)\, dW_s\right) dW_t \tag{4.6.5}$$

and

$$\begin{aligned} \int_0^S \lambda_{3,t}dt &= 3\int_0^S \int_t^S e^{-\int_0^s \kappa(u)du} ds e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t \left(e^{\int_0^s \kappa(u)du}\nu(s)\right)^2 ds\right) dW_t \\ &\quad + 6\int_0^S \int_t^S e^{-\int_0^s \kappa(u)du} ds \nu^{(1)}(t) \left(\int_0^t \nu^{(1)}(s) \left(\int_0^s e^{\int_0^u \kappa(v)dv} \nu(u)\, dW_u\right) dW_s\right) dW_t \\ &\quad + 6\int_0^S \int_t^S e^{-\int_0^s \kappa(u)du} ds e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t e^{\int_0^s \kappa(u)du} \nu(s) \left(\int_0^s e^{\int_0^u \kappa(v)dv} \nu(u)\, dW_u\right) dW_s\right) dW_t \end{aligned} \tag{4.6.6}$$

Proof. One has

$$\begin{aligned} d\lambda_t &= \kappa(t)\,(\psi(t) - \lambda_t)\,dt, \\ d\lambda_{1,t} &= -\kappa(t)\,\lambda_{1,t}dt + \nu(t)\,dW_t,\ \lambda_{1,0} = 0 \\ d\lambda_{2,t} &= -\kappa(t)\,\lambda_{2,t}dt + 2\lambda_{1,t}\nu^{(1)}(t)\,dW_t,\ \lambda_{2,0} = 0 \\ d\lambda_{3,t} &= -\kappa(t)\,\lambda_{3,t}dt + 3\left(\nu^{(1)}(t)\,\lambda_{2,t} + \nu^{(2)}(t)\,(\lambda_{1,t})^2\right) dW_t,\ \lambda_{3,0} = 0 \end{aligned}$$

Solving these linear SDEs leads to

$$\begin{aligned} \lambda_t &= e^{-\int_0^t \kappa(s)ds}\left(\lambda_0 + \int_0^t \kappa(s)\, e^{\int_0^s \kappa(u)du} \psi(s)\, ds\right) \\ \lambda_{1,t} &= e^{-\int_0^t \kappa(s)ds} \int_0^t e^{\int_0^s \kappa(u)du} \nu(s)\, dW_s \\ \lambda_{2,t} &= 2e^{-\int_0^t \kappa(s)ds} \int_0^t e^{\int_0^s \kappa(u)du} \nu^{(1)}(s)\, \lambda_{1,s} dW_s \\ \lambda_{3,t} &= 3e^{-\int_0^t \kappa(s)ds} \int_0^t e^{\int_0^s \kappa(u)du} \left(\nu^{(1)}(s)\,\lambda_{2,s} + \nu^{(2)}(s)\,(\lambda_{1,s})^2\right) dW_s \end{aligned}$$

We develop further the last equality: Observe that $(\lambda_{1,t})^2$ solve the following SDE

$$\begin{aligned} d(\lambda_{1,t})^2 &= 2\lambda_{1,t}d\lambda_{1,t} + d\langle\lambda_{1,.}\rangle_t \\ &= 2\lambda_{1,t}(-\kappa(t)\lambda_{1,t}dt + \nu(t)dW_t) + \nu^2(t)dt \\ &= -2\kappa(t)(\lambda_{1,t})^2 dt + \nu^2(t)dt + 2\lambda_{1,t}\nu(t)dW_t \end{aligned}$$

giving

$$\begin{aligned} (\lambda_{1,t})^2 &= e^{-2\int_0^t \kappa(s)ds}\int_0^t e^{2\int_0^s \kappa(u)du}\left(\nu^2(s)ds + 2\lambda_{1,s}\nu(s)dW_s\right) \\ &= e^{-2\int_0^t \kappa(s)ds}\int_0^t \left(e^{\int_0^s \kappa(u)du}\nu(s)\right)^2 ds \\ &+ 2e^{-2\int_0^t \kappa(s)ds}\int_0^t e^{\int_0^s \kappa(u)du}\nu(s)\left(\int_0^s e^{\int_0^u \kappa(v)dv}\nu(u)dW_u\right)dW_s \end{aligned}$$

One has then,

$$\begin{aligned} \lambda_{3,t} &= 3e^{-\int_0^t \kappa(s)ds}\int_0^t e^{\int_0^s \kappa(u)du}\nu^{(2)}(s)(\lambda_{1,s})^2 dW_s + 3e^{-\int_0^t \kappa(s)ds}\int_0^t e^{\int_0^s \kappa(u)du}\nu^{(1)}(s)\lambda_{2,s}dW_s \\ &= 3e^{-\int_0^t \kappa(s)ds}\int_0^t e^{-\int_0^s \kappa(u)du}\nu^{(2)}(s)\left(\int_0^s \left(e^{\int_0^u \kappa(v)dv}\nu(u)\right)^2 du\right)dW_s \\ &+ 6e^{-\int_0^t \kappa(s)ds}\int_0^t \nu^{(1)}(s)\left(\int_0^s \nu^{(1)}(u)\left(\int_0^u e^{\int_0^v \kappa(w)dw}\nu(v)dW_v\right)dW_u\right)dW_s \\ &+ 6e^{-\int_0^t \kappa(s)ds}\int_0^t e^{-\int_0^s \kappa(u)du}\nu^{(2)}(s)\left(\int_0^s e^{\int_0^u \kappa(v)dv}\nu(u)\left(\int_0^u e^{\int_0^v \kappa(w)dw}\nu(v)dW_v\right)dW_u\right)dW_s \end{aligned}$$

At the end, we use repeatedly Lemma 4.6.1 in order to express $\int_0^S \lambda_{i,t}dt$, $i = 1,2,3$ as an iterated Wiener integrals. □

Proposition 4.6.5. *One has*

$$\mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)h(X_T)\right] = \sum_{i=1}^n [C_{1,1}(S,T)]_i \mathbb{E}\left[h_i^{(1)}(X_T)\right] \tag{4.6.7}$$

$$\mathbb{E}\left[\frac{1}{2}\left(\int_0^S \lambda_{2,t}dt\right)h(X_T)\right] = \sum_{i,j=1}^n [C_{2,1}(S,T)]_{i,j} \mathbb{E}\left[h_{i,j}^{(2)}(X_T)\right] \tag{4.6.8}$$

$$\mathbb{E}\left[\frac{1}{2}\left(\int_0^S \lambda_{1,t}dt\right)^2 h(X_T)\right] = C_{0,1}(S,T)\mathbb{E}[h(X_T)] + \sum_{i,j=1}^n [C_{2,2}(S,T)]_{i,j} \mathbb{E}\left[h_{i,j}^{(2)}(X_T)\right] \tag{4.6.9}$$

where $C_{0,1}$, $C_{1,1}$ and $C_{2,1}$ and $C_{2,2}$ are given in the Theorem 4.3.9.

Proof. Owing to Proposition 4.6.4 and Lemma 4.6.2, we have :

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)h(X_T)\right] &= \sum_{i=1}^n \int_0^S \int_t^S e^{-\int_0^s \kappa(u)du}ds\tilde{\nu}(t)\left[\tilde{\Sigma}(t,T)\rho\right]_i dt\mathbb{E}\left[h_i^{(1)}(X_T)\right] \\ &= \sum_{i=1}^n [C_{1,1}(S,T)]_i Greek_i^{(1)}(T) \end{aligned}$$

We proceed similarly in order to establish the remaining equalities. First,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{2}\left(\int_0^S \lambda_{2,t}dt\right)h(X_T)\right] &= \sum_{i=1}^n \mathbb{E}\left[\int_0^S\int_t^S e^{-\int_0^s \kappa(u)du}ds\nu^{(1)}(t)\left[\tilde{\Sigma}(t,T)\rho\right]_i\left(\int_0^t \tilde{\nu}(s)dW_s\right)dth_i^{(1)}(X_T)\right]\\ &= \sum_{i=1}^n \mathbb{E}\left[\int_0^S\left(\int_t^S\left(\int_s^S e^{-\int_0^u \kappa(v)dv}du\right)\nu^{(1)}(s)\left[\tilde{\Sigma}(s,T)\rho\right]_i ds\right)\tilde{\nu}(t)dW_t h_i^{(1)}(X_T)\right]\\ &= \sum_{i,j=1}^n [C_{2,1}(S,T)]_{i,j}\mathbb{E}\left[h_{i,j}^{(2)}(X_T)\right]\end{aligned}$$

Second, by the Itô lemma,

$$\begin{aligned}\left(\int_0^S \lambda_{1,t}dt\right)^2 &= \left(\left(\int_0^S e^{-\int_0^s \kappa(u)du}ds\right)\int_0^S \tilde{\nu}(t)dW_t - \int_0^S\int_0^t e^{-\int_0^s \kappa(u)du}ds\tilde{\nu}(t)dW_t\right)^2\\ &= \int_0^S\left(\int_t^S e^{-\int_0^s \kappa(u)du}ds\tilde{\nu}(t)\right)^2 dt\\ &+ 2\int_0^S\int_t^S e^{-\int_0^s \kappa(u)du}ds\tilde{\nu}(t)\left(\int_0^t\left(\int_s^S e^{-\int_0^u \kappa(v)dv}du\right)\tilde{\nu}(s)dW_s\right)dW_t\end{aligned}$$

Thanks to Lemma 4.6.2, it follows that

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)^2 h(X_T)\right]\\ &= \frac{1}{2}\int_0^S\left(\int_t^S e^{-\int_0^s \kappa(u)du}ds\tilde{\nu}(t)\right)^2 dt\mathbb{E}[h(X_T)]\\ &+ \left(\int_0^S e^{-\int_0^s \kappa(u)du}ds\right)\sum_{i,j=1}^n\int_0^S\left(\int_t^S\int_s^S e^{-\int_0^u \kappa(v)dv}du\tilde{\nu}(s)\left[\tilde{\Sigma}(s,T)\rho\right]_i ds\right)\tilde{\nu}(t)\left[\tilde{\Sigma}(t,T)\rho\right]_j dt\mathbb{E}\left[h_{i,j}^{(2)}(X_T)\right]\\ &- \sum_{i,j=1}^n\int_0^S\left(\int_t^S\int_s^S e^{-\int_0^u \kappa(v)dv}du\tilde{\nu}(s)\left[\tilde{\Sigma}(s,T)\rho\right]_i ds\right)\left(\int_0^t e^{-\int_0^s \kappa(u)du}ds\right)\tilde{\nu}(t)\left[\tilde{\Sigma}(t,T)\rho\right]_j dt\mathbb{E}\left[h_{i,j}^{(2)}(X_T)\right]\\ &= C_{0,1}(S)\,\mathbb{E}[h(X_T)] + \sum_{i,j=1}^n [C_{2,2}(S,T)]_{i,j}\mathbb{E}\left[h_{i,j}^{(2)}(X_T)\right]\end{aligned}$$

□

Proposition 4.6.6. *One has*

$$\frac{1}{6}\mathbb{E}\left[\left(\int_0^S \lambda_{3,t}dt\right)h(X_T)\right] = C_{1,2}(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right] + (C_{3,1}+C_{3,2})(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right] \tag{4.6.10}$$

$$\frac{1}{6}\mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)^3 h(X_T)\right] = \left(C_{1,5}+C_{1,6}\right)(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right] + C_{3,6}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right] \tag{4.6.11}$$

$$\frac{1}{2}\mathbb{E}\left[\left(\int_0^S \lambda_{2,t}dt\int_0^S \lambda_{1,t}dt\right)h(X_T)\right] = (C_{1,3}+C_{1,4})(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right] + \left(\sum_{j=3}^5 C_{3,j}(S,T)\right).\mathbb{E}\left[h^{(3)}(X_T)\right]$$

where coefficients $C_{i,j}$ are given in the Theorem 4.3.14.

Proof. The proof is performed using equations (4.6.4), (4.6.5), (4.6.6) and repeatedly applying Lemmas 4.6.1 and 4.6.2. We refer to 4.6.3 for the detailed proof. □

4.6.3 Error Analysis

Lemma 4.6.7. *Let A be a measurable set and $|A|$ its Lebesgue measure. For any $q \geq 1$, one has*

$$\left| \int_A f(x)\, dx \right|^q \leq |A|^{q-1} \int_A |f(x)|^q\, dx$$

Proof. The claim is obvious if $|A| \in \{0, +\infty\}$. Otherwise, it is an immediate consequence of Jensen's inequality $\left(\frac{1}{|A|} \int_A |f(x)|\, dx \right)^q \leq \frac{1}{|A|} \int_A |f(x)|^q\, dx$. □

Proposition 4.6.8. *Under $(\mathscr{R}_3)$ one has, for every $p \geq 2$*

$$\mathbb{E}\left[\left| \int_0^S \lambda_{1,t} dt \right|^p \right] \leq S^{\frac{3p}{2}} \|\nu\|_\infty^p \tag{4.6.12}$$

$$\mathbb{E}\left[\left| \int_0^S \lambda_{2,t} dt \right|^p \right] \leq S^{2p} \|\nu\|_\infty^p \left\| \nu^{(1)} \right\|_\infty^p \tag{4.6.13}$$

Proof. By Lemma 4.6.7, $\mathbb{E}\left[\left| \int_0^S \lambda_{i,t} dt \right|^p \right] \leq S^{p-1} \int_0^S \mathbb{E}\left[|\lambda_{i,t}|^p \right] dt$, $i = 1, 2$. Using the Burkholder-Davis-Gundy (BDG) inequality and (4.6.1), one has:

$$\mathbb{E}\left[\left| \lambda_{1,t} \right|^p \right] \leq_c t^{\frac{p}{2}} \|\nu\|_\infty$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left[\left| \lambda_{2,t} \right|^p \right] &\leq_c \left\| \nu^{(1)} \right\|_\infty^p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[\left| \left(\int_0^s e^{\int_0^u \kappa(v) dv} \nu(u)\, dW_u \right) \right|^p \right] ds dt \\
&\leq_c \|\nu\|_\infty^p \left\| \nu^{(1)} \right\|_\infty^p t^{\frac{p}{2}-1} \int_0^t s^{\frac{p}{2}} ds dt \\
&\leq_c t^p \|\nu\|_\infty^p \left\| \nu^{(1)} \right\|_\infty^p
\end{aligned}$$

□

Proposition 4.6.9. *Under $(\mathscr{R}_3)$, the processes $\left(R_{i,t}^{\lambda^\varepsilon} = \lambda_t^\varepsilon - \sum_{j=0}^i \frac{\varepsilon^j}{j!} \lambda_{j,t} \right)_{0 \leq t \leq S}$ are given by*

$$R_{0,t}^{\lambda^\varepsilon} = \varepsilon e^{-\int_0^t \kappa(s) ds} \int_0^t e^{\int_0^s \kappa(u) du} \nu(s, \lambda_s^\varepsilon)\, dW_s \tag{4.6.14}$$

$$R_{1,t}^{\lambda^\varepsilon} = \varepsilon e^{-\int_0^t \kappa(s) ds} \int_0^t e^{\int_0^s \kappa(u) du} \left(\nu(s, \lambda_s^\varepsilon) - \nu(s) \right) dW_s \tag{4.6.15}$$

$$R_{2,t}^{\lambda^\varepsilon} = \varepsilon e^{-\int_0^t \kappa(s) ds} \int_0^t e^{\int_0^s \kappa(u) du} \left(\nu(s, \lambda_s^\varepsilon) - \nu(s) - \varepsilon \lambda_{1,s} \nu^{(1)}(s) \right) dW_s \tag{4.6.16}$$

Moreover, for every $p \geq 2$, one has the following upper-bounds w.r.t the L_p norm

$$\left\| R^{\lambda^\varepsilon}_{0,t} \right\|_p \leq_c \sqrt{t}\, \|\nu\|_\infty \tag{4.6.17}$$

$$\left\| R^{\lambda^\varepsilon}_{1,t} \right\|_p \leq_c t\, \|\nu\|_\infty \left\| \nu^{(1)} \right\|_\infty \tag{4.6.18}$$

$$\left\| R^{\lambda^\varepsilon}_{2,t} \right\|_p \leq_c t\, \|\nu\|_\infty \left(\left\| \nu^{(1)} \right\|^{2p}_\infty t^{\frac{p}{2}} + \left\| \nu^{(2)} \right\|^p_\infty \right)^{1/p} \tag{4.6.19}$$

Proof. From (4.3.2), (4.3.13), (4.3.14), and (4.3.15), the SDEs of $\left(R^{\lambda^\varepsilon}_{i,t} \right)_{i=0,1,2}$ are given by

$$\begin{aligned}
dR^{\lambda^\varepsilon}_{0,t} &= -\kappa(t) R^{\lambda^\varepsilon}_{0,t} dt + \varepsilon \nu(t, \lambda^\varepsilon_t)\, dW_t \\
dR^{\lambda^\varepsilon}_{1,t} &= -\kappa(t) R^{\lambda^\varepsilon}_{1,t} + \varepsilon \left(\nu(t, \lambda^\varepsilon_t) - \nu(t) \right) dW_t \\
dR^{\lambda^\varepsilon}_{2,t} &= -\kappa(t) R^{\lambda^\varepsilon}_{2,t} + \varepsilon \left(\nu(t, \lambda^\varepsilon_t) - \nu(t) - \varepsilon \lambda_{1,t} \nu^{(1)}(t) \right) dW_t
\end{aligned}$$

with $\left(R^{\lambda^\varepsilon}_{i,0} \right)_{i=0,1,2} = 0$. Explicit representations of $\left(R^{\lambda^\varepsilon}_{i,0} \right)_{i=0,1,2}$ are then easily deduced. The calculation of the L_p upper-bounds is mainly based on the BDG inequality:

$$\begin{aligned}
\mathbb{E}\left[\left| R^{\lambda^\varepsilon}_{0,t} \right|^p \right] &\leq \mathbb{E}\left[\left(\int_0^t e^{\int_0^s \kappa(u) du} \nu(s, \lambda^\varepsilon_s)\, dW_s \right)^p \right] \\
&\leq_c t^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[|\nu(s, \lambda^\varepsilon_s)|^p \right] ds \\
&\leq_c t^{\frac{p}{2}} \|\nu\|^p_\infty
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\left| R^{\lambda^\varepsilon}_{1,t} \right|^p \right] &\leq \mathbb{E}\left[\left| \int_0^t e^{\int_0^s \kappa(u) du} \left(\nu(s, \lambda^\varepsilon_s) - \nu(s) \right) dW_s \right|^p \right] \\
&\leq_c t^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[\left| \nu(s, \lambda^\varepsilon_s) - \nu\left(s, \lambda^{\varepsilon=0}_s\right) \right|^p \right] ds \\
&\leq_c t^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[\left| \int_0^\varepsilon \nu^{(1)}\left(s, \lambda^\xi_s\right) R^{\lambda^\xi}_{0,s} d\xi \right|^p \right] ds \\
&\leq_c t^p \|\nu\|^p_\infty \left\| \nu^{(1)} \right\|^p_\infty
\end{aligned}$$

Finally, for $\mathbb{E}\left[\left| R^{\lambda^\varepsilon}_{2,t} \right|^p \right]$, first observe that

$$\begin{aligned}
&\nu(t, \lambda^\varepsilon_t) - \nu(t) - \varepsilon \lambda_{1,t} \nu^{(1)}(t) \\
&= \nu(t, \lambda^\varepsilon_t) - \nu(t) - \left(\lambda^\varepsilon_t - \lambda_t \right) \nu^{(1)}(s) + \left(\lambda^\varepsilon_t - \lambda_t - \varepsilon \lambda_{1,t} \right) \nu^{(1)}(s) \\
&= \int_0^\varepsilon \nu^{(2)}\left(t, \lambda^\xi_t\right) R^{\lambda^\xi}_{0,t} d\xi + R^{\lambda^\varepsilon}_{1,t} \nu^{(1)}(s)
\end{aligned}$$

using Taylor's integral remainder. It readily follows that,

$$\begin{aligned}
\mathbb{E}\left[\left|R_{2,t}^{\lambda^\varepsilon}\right|^p\right] &\leq_c \mathbb{E}\left[\left|\int_0^t e^{\int_0^s \kappa(u)du}\left(\nu\left(s,\lambda_s^\varepsilon\right)-\nu\left(s\right)-\varepsilon\lambda_{1,s}\nu^{(1)}\left(s\right)\right)dW_s\right|^p\right] \\
&\leq_c t^{\frac{p}{2}-1}\int_0^t\left(\mathbb{E}\left[\left|R_{1,s}^{\lambda^\varepsilon}\nu^{(1)}\left(s\right)+\int_0^\varepsilon \nu^{(2)}\left(s,\lambda_s^\xi\right)R_{0,s}^{\lambda^\xi}d\xi\right|^p\right]\right)ds \\
&\leq_c t^{\frac{p}{2}-1}\left\|\nu\right\|_\infty^p\int_0^t\left(\left\|\nu^{(1)}\right\|_\infty^{2p}s^p+\left\|\nu^{(2)}\right\|_\infty^p s^{\frac{p}{2}}\right)ds \\
&\leq_c t^p\left\|\nu\right\|_\infty^p\left(\left\|\nu^{(1)}\right\|_\infty^{2p}t^{\frac{p}{2}}+\left\|\nu^{(2)}\right\|_\infty^p\right)
\end{aligned}$$

where we have used the triangle inequality and the L_p bound estimates of the lower order residuals. □

Corollary 4.6.10. *The residuals for the integrated intensity satisfy the following inequalities: for $p \geq 2$*

$$\left\|R_{0,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t}\right\|_p \leq_c S\sqrt{S}\left\|\nu\right\|_\infty \tag{4.6.20}$$

$$\left\|R_{1,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t}\right\|_p \leq_c S^2\left\|\nu\right\|_\infty\left\|\nu^{(1)}\right\|_\infty \tag{4.6.21}$$

$$\left\|R_{2,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t}\right\|_p \leq_c S^2\left\|\nu\right\|_\infty\left(\left\|\nu^{(1)}\right\|_\infty^{2p}S^{\frac{p}{2}}+\left\|\nu^{(2)}\right\|_\infty^p\right)^{1/p} \tag{4.6.22}$$

Proof. We use Lemma 4.6.7 and the estimates (4.6.17), (4.6.18) and (4.6.19). □

Proposition 4.6.11. *Under $(\mathscr{R}_4)$, the process $\left(R_{3,t}^{\lambda^\varepsilon}=\lambda_t^\varepsilon-\lambda_t-\varepsilon\lambda_{1,t}-\frac{\varepsilon^2}{2}\lambda_{2,t}-\frac{\varepsilon^3}{3!}\lambda_{3,t}\right)_{0\leq t\leq S}$ fulfills the equality*

$$R_{3,t}^{\lambda^\varepsilon}=\varepsilon e^{-\int_0^t\kappa(s)ds}\int_0^t e^{\int_0^s\kappa(u)du}\left(\nu\left(s,\lambda_s^\varepsilon\right)-\nu\left(s\right)-\left(\varepsilon\lambda_{1,s}+\frac{\varepsilon^2}{2}\lambda_{2,s}\right)\nu^{(1)}\left(t\right)-\frac{\varepsilon^2}{2}\left(\lambda_{1,s}\right)^2\nu^{(2)}\left(s\right)\right)dW_s \tag{4.6.23}$$

In addition, the $L_{p\geq 2}$ upper-bound estimate is given by

$$\begin{aligned}
\left\|R_{3,t}^{\lambda^\varepsilon}\right\|_p^p \leq_c\ & t^{\frac{3p}{2}}\left\|\nu\right\|_\infty^{2p}\left(\left\|\nu^{(2)}\right\|_\infty^p\left(t^p\left\|\nu^{(1)}\right\|_\infty^{2p}+1\right)+\left\|\nu^{(3)}\right\|_\infty^p\right) \\
&+t^{\frac{3p}{2}}\left\|\nu\right\|_\infty^p\left\|\nu^{(1)}\right\|_\infty^p\left(\left\|\nu^{(1)}\right\|_\infty^{2p}t^{\frac{p}{2}}+\left\|\nu^{(2)}\right\|_\infty^p\right)
\end{aligned} \tag{4.6.24}$$

Proof. We establish the linear SDE satisfied by $\left(R_{3,t}^{\lambda^\varepsilon}\right)_{0\leq t\leq S}$

$$\begin{aligned}
dR_{3,t}^{\lambda^\varepsilon} &= d\left(\lambda_t^\varepsilon-\lambda_t-\varepsilon\lambda_{1,t}-\frac{\varepsilon^2}{2}\lambda_{2,t}-\frac{\varepsilon^3}{3!}\lambda_{3,t}\right) \\
&= -\kappa\left(t\right)R_{3,t}^{\lambda^\varepsilon}+\varepsilon\left(\nu\left(t,\lambda_t^\varepsilon\right)-\nu\left(t\right)-\varepsilon\lambda_{1,t}\nu^{(1)}\left(t\right)-\frac{\varepsilon^2}{2}\left(\lambda_{2,t}\nu^{(1)}\left(t\right)+\left(\lambda_{1,t}\right)^2\nu^{(2)}\left(t\right)\right)\right)dW_t \\
&= -\kappa\left(t\right)R_{3,t}^{\lambda^\varepsilon}+\varepsilon\left(\nu\left(t,\lambda_t^\varepsilon\right)-\nu\left(t\right)-\left(\varepsilon\lambda_{1,t}+\frac{\varepsilon^2}{2}\lambda_{2,t}\right)\nu^{(1)}\left(t\right)-\frac{\varepsilon^2}{2}\left(\lambda_{1,t}\right)^2\nu^{(2)}\left(t\right)\right)dW_t
\end{aligned}$$

The estimate of the error L_p norm is established as for $\left(R_t^{\lambda^\varepsilon}\right)_t$. We first write

$$\begin{aligned}
&\nu\left(t,\lambda_t^\varepsilon\right)-\nu\left(t\right)-\left(\varepsilon\lambda_{1,t}+\frac{\varepsilon^2}{2}\lambda_{2,t}\right)\nu^{(1)}\left(t\right)-\frac{\varepsilon^2}{2}\left(\lambda_{1,t}\right)^2\nu^{(2)}\left(t\right)\\
&=\nu\left(t,\lambda_t^\varepsilon\right)-\nu\left(t\right)-\left(\lambda_t^\varepsilon-\lambda_t\right)\nu^{(1)}\left(t\right)-\frac{1}{2}\left(\lambda_t^\varepsilon-\lambda_t\right)^2\nu^{(2)}\left(t\right)\\
&+\left(\lambda_t^\varepsilon-\lambda_t-\varepsilon\lambda_{1,t}-\frac{\varepsilon^2}{2}\lambda_{2,t}\right)\nu^{(1)}\left(t\right)+\frac{1}{2}\left(\left(\lambda_t^\varepsilon-\lambda_t\right)^2-\left(\varepsilon\lambda_{1,t}\right)^2\right)\nu^{(2)}\left(t\right)\\
&=\frac{1}{2}\int_0^\varepsilon\nu^{(3)}\left(t,\lambda_t^\xi\right)\left(R_{0,t}^{\lambda^\xi}\right)^2d\xi+R_{2,t}^{\lambda^\varepsilon}\nu^{(1)}\left(t\right)+\frac{1}{2}\left(\left(R_{1,t}^{\lambda^\varepsilon}\right)^2+2\varepsilon\lambda_{1,t}R_{1,t}^{\lambda^\varepsilon}\right)\nu^{(2)}\left(t\right)
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|R_{3,t}^{\lambda^\varepsilon}\right|^p\right]\\
&\leq_c t^{\frac{p}{2}-1}\int_0^t\mathbb{E}\left[\left|\frac{1}{2}\int_0^\varepsilon\nu^{(3)}\left(s,\lambda_s^\xi\right)\left(R_{0,s}^{\lambda^\xi}\right)^2d\xi+R_{2,s}^{\lambda^\varepsilon}\nu^{(1)}\left(s\right)+\left(\frac{1}{2}\left(R_{1,s}^{\lambda^\varepsilon}\right)^2+\varepsilon\lambda_{1,s}R_{1,s}^{\lambda^\varepsilon}\right)\nu^{(2)}\left(s\right)\right|^p\right]ds\\
&\leq_c t^{\frac{p}{2}-1}\int_0^t\left(\left\|\nu^{(3)}\right\|_\infty^p\mathbb{E}\left[\left|\int_0^\varepsilon\left(R_{0,s}^{\lambda^\xi}\right)^2d\xi\right|^p\right]+\left\|\nu^{(1)}\right\|_\infty^p\mathbb{E}\left[\left|R_{2,s}^{\lambda^\varepsilon}\right|^p\right]\right)ds\\
&+t^{\frac{p}{2}-1}\left\|\nu^{(2)}\right\|_\infty^p\int_0^t\left(\mathbb{E}\left[\left|R_{1,s}^{\lambda^\varepsilon}\right|^{2p}\right]+\mathbb{E}\left[\left|\lambda_{1,t}^\varepsilon R_{1,s}^{\lambda^\varepsilon}\right|^p\right]\right)ds\\
&\leq_c t^{\frac{p}{2}-1}\int_0^t\left(\left\|\nu^{(3)}\right\|_\infty^p\int_0^\varepsilon\mathbb{E}\left[\left|R_{0,s}^{\lambda^\xi}\right|^{2p}\right]d\xi+\left\|\nu^{(1)}\right\|_\infty^p\mathbb{E}\left[\left|R_{2,s}^{\lambda^\varepsilon}\right|^p\right]\right)ds\\
&+t^{\frac{p}{2}-1}\left\|\nu^{(2)}\right\|_\infty^p\int_0^t\left(\mathbb{E}\left[\left|R_{1,s}^{\lambda^\varepsilon}\right|^{2p}\right]+\mathbb{E}\left[\left|\lambda_{1,t}^\varepsilon\right|^{2p}\right]\right)ds
\end{aligned}$$

where we have used the inequality

$$\mathbb{E}\left[\left|\varepsilon\lambda_{1,t}^\varepsilon R_{1,s}^{\lambda^\varepsilon}\right|^p\right]\leq_c\frac{1}{2}\left(\mathbb{E}\left[\left|\lambda_{1,t}^\varepsilon\right|^{2p}\right]+\mathbb{E}\left[\left|R_{1,s}^{\lambda^\varepsilon}\right|^{2p}\right]\right)$$

Using the estimates of the lower order residuals, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left|R_{3,t}^{\lambda^\varepsilon}\right|^p\right]&\leq_c t^{\frac{p}{2}-1}\int_0^t\left(s^p\left\|\nu\right\|_\infty^{2p}\left\|\nu^{(3)}\right\|_\infty^p+\left\|\nu\right\|_\infty^p\left\|\nu^{(1)}\right\|_\infty^p\left(\left\|\nu^{(1)}\right\|_\infty^{2p}s^{\frac{3p}{2}}+\left\|\nu^{(2)}\right\|_\infty^p s^p\right)\right)ds\\
&+t^{\frac{p}{2}-1}\left\|\nu^{(2)}\right\|_\infty^p\left\|\nu\right\|_\infty^{2p}\int_0^t\left(s^{2p}\left\|\nu^{(1)}\right\|_\infty^{2p}+s^p\right)ds\\
&\leq_c t^{\frac{3p}{2}}\left\|\nu\right\|_\infty^{2p}\left(\left\|\nu^{(2)}\right\|_\infty^p\left(t^p\left\|\nu^{(1)}\right\|_\infty^{2p}+1\right)+\left\|\nu^{(3)}\right\|_\infty^p\right)\\
&+t^{\frac{3p}{2}}\left\|\nu\right\|_\infty^p\left\|\nu^{(1)}\right\|_\infty^p\left(\left\|\nu^{(1)}\right\|_\infty^{2p}t^{\frac{p}{2}}+\left\|\nu^{(2)}\right\|_\infty^p\right)
\end{aligned}$$

which ends the proof. □

Corollary 4.6.12. *The residuals for the integrated intensity satisfy the following inequalities for $p \geq 2$*

$$\begin{aligned}\left\| R_{3,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right\|_p^p \leq_c S^{\frac{5p}{2}} \|\nu\|_\infty^{2p} \left(\left\| \nu^{(2)} \right\|_\infty^p \left(S^p \left\| \nu^{(1)} \right\|_\infty^{2p} + 1 \right) + \left\| \nu^{(3)} \right\|_\infty^p \right) \\ + S^{\frac{5p}{2}} \|\nu\|_\infty^p \left\| \nu^{(1)} \right\|_\infty^p \left(\left\| \nu^{(1)} \right\|_\infty^{2p} t^{\frac{p}{2}} + \left\| \nu^{(2)} \right\|_\infty^p \right) \end{aligned} \tag{4.6.25}$$

Proof. A simple application of Lemma 4.6.7 gives the above claim. □

We end this technical section by summarizing the main upper-bound inequalities using the constants M_0 and M_1 of (4.3.11):

$$\left\| \int_0^S \lambda_{1,t} dt \right\|_p \leq_c S\sqrt{S} M_0 \tag{4.6.26}$$

$$\left\| \int_0^S \lambda_{2,t} dt \right\|_p \leq_c S^2 M_0 M_1 \tag{4.6.27}$$

$$\left\| R_{0,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right\|_p \leq_c S\sqrt{S} M_0 \tag{4.6.28}$$

$$\left\| R_{1,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right\|_p \leq_c S^2 M_0 M_1 \tag{4.6.29}$$

$$\left\| R_{2,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right\|_p \leq_c S^2 M_0 M_1 \tag{4.6.30}$$

$$\left\| R_{3,S}^{\left(\int_0^t \lambda_s^\varepsilon ds\right)_t} \right\|_p \leq_c S^{\frac{5}{2}} M_0 M_1 \left(M_0 + M_1 \right) \tag{4.6.31}$$

Appendix A

A.1 An explicit approximation of the CEV mean-reverting process

The expansion formulas developed for pricing credit derivatives can also be used to approximate spot and forward survival probabilities for a default intensity model with an arbitrary volatility function $\nu(t,\lambda_t)$. Under the technical requirements for ν, the approximation formulas are of special interest for:

- Pricing contingent claim indexed on forward CDS spreads. XVA negotiations can also embed collateral calculation formulas involving CDS spreads.
- Model extension to include a deterministic shift function such as in the the Hull-White or the CIR++ short rate models (see [BM07]).

We consider the following stochastic process for the default intensity

$$d\lambda_t = \kappa(\theta-\lambda_t)\,dt + \varepsilon\nu_0(\lambda_t)^{\beta}\,dW_t, \qquad \lambda_0 > 0 \tag{4.6.32}$$

The dynamics correspond to a mean-reverting CEV process with the constant coefficients $\nu_0,\kappa,\theta > 0$ and $\beta > \frac{1}{2}$ since the process λ_t has to be non-negative. In particular, having $\beta > \frac{1}{2}$ implies that λ is positive and non explosive, i.e. 0 and ∞ are unattainable (see [HS15]). We recover the CIR model when $\beta = \frac{1}{2}$, while $\beta = 1$ gives the Lognormal (Brennan-Schwartz) model. This family of stochastic processes play an important role in financial mathematics. For instance, it has been considered for the dynamics of the short-term interest rates in [CKLS92]. [HS15, Hal15] address numerical issues related to preserving the positivity of the process λ under usual SDE discretization schemes like Euler and Milstein. We emphasize that there is no analytical expressions for λ with $\beta \in \left[\frac{1}{2},1\right]$.

Following our expansion approach, we showed that the survival probability can be approximated by

$$\mathbb{Q}(\tau > S) \approx e^{-\int_0^S \lambda_t^0 dt}\left(1 + C_{0,1}(S)\right) \tag{4.6.33}$$

where

$$C_{0,1}(S) = \frac{\nu_0^2}{2\kappa^2}\int_0^S \left(1-e^{-\kappa(S-t)}\right)^2 \left(\lambda_t^0\right)^{2\beta} dt \tag{4.6.34}$$

and

$$\begin{cases} \lambda_t^0 & = e^{-\kappa t}(\lambda_0-\theta)+\theta \\ \int_0^S \lambda_t dt & = \theta S + \frac{1}{\kappa}(\theta-\lambda_0)\left(e^{-\kappa S}-1\right) \end{cases} \tag{4.6.35}$$

In particular, one has

- $\beta = \frac{1}{2}$:

$$C_{0,1}(S) = \frac{\nu_0^2}{2\kappa^3}e^{-2\kappa S}\left(e^{2\kappa S}(2\kappa\theta S - 5\theta + 2\lambda_0) + 4e^{\kappa S}(\theta\kappa S - \kappa\lambda_0 S + \theta) + \theta - 2\lambda_0\right) \tag{4.6.36}$$

- $\beta = 1$:

$$C_{0,1}(S) = \frac{\nu_0^2}{4\kappa^3} e^{-2\kappa S}\left(2\left((S\kappa-3)e^{2S\kappa}+4S\kappa e^{S\kappa}+S\kappa+3\right)\theta^2+\lambda_0^2\left(e^{2\kappa S}-4e^{\kappa S}+2S\kappa+3\right)\right) + \frac{\nu_0^2}{2\kappa^3} e^{-2\kappa S}\left(e^{2S\kappa}+4(1-S\kappa)e^{S\kappa}-2S\kappa-5\right)\lambda_0\theta \tag{4.6.37}$$

The conditional survival probability can be approximated by an explicit function of the intensity

$$\mathbb{Q}_t(\tau > S|\lambda_t) \triangleq \mathbb{E}[1_{\tau>S}|\tau > t, \lambda_t] \approx e^{-\int_t^S \lambda_s ds}\left(1+\tilde{C}_{0,1}\left(t,S;\lambda_t^0\right)\right) \tag{4.6.38}$$

where

$$\tilde{C}_{0,1}\left(t,S;\lambda_t^0\right) = \frac{\nu_0^2}{2\kappa^2}\int_t^S \left(1-e^{-\kappa(S-s)}\right)^2 \left(\lambda_s^0\right)^{2\beta} ds \tag{4.6.39}$$

and

$$\lambda_s^0 = \kappa\theta\left(e^{-\kappa s}-e^{-\kappa t}\right)+\left(\lambda_t^0-\kappa\theta\right) \tag{4.6.40}$$

A.2 An explicit fitting of the initial hazard rate term structure

We consider an extended version of the hazard rate model (4.6.32). Let us set

$$\begin{cases} \lambda_t &= \psi(t)+y_t \\ dy_t &= \kappa(\theta-y_t)\,dt+\varepsilon\nu(t,y_t)\,dW_t, \qquad y_0>0 \end{cases} \tag{4.6.41}$$

The deterministic shift function ψ is determined given an explicit knowledge of $\mathbb{E}\left[e^{-\int_0^S y_s ds}\right]$. More precisely, ψ should satisfy

$$\mathbb{Q}^{market}(\tau > S) = \mathbb{Q}^{model}(\tau > S) = e^{-\int_0^S \psi(t)dt}\mathbb{E}\left[e^{-\int_0^S y_t dt}\right] \tag{4.6.42}$$

for $t > 0$ leading to

$$\psi(t) = \frac{\partial}{\partial S}\left(-\ln\left(\mathbb{Q}^{mkt}(\tau > S)\right)+\ln\left(\mathbb{E}\left[e^{-\int_0^S y_t dt}\right]\right)\right)|_{S=t} \tag{4.6.43}$$

One can then approximate ψ by

$$\psi(t) \approx h(t) - y_t^0 + \frac{\partial_t C_{0,1}(t)}{1+C_{0,1}(t)} \tag{4.6.44}$$

with

$$\begin{cases} h(t) &= -\frac{\partial}{\partial t}\left(\ln\left(\mathbb{Q}^{mkt}(\tau > t)\right)\right) \\ y_t^0 &= e^{-\kappa t}(y_0-\theta)+\theta \\ C_{0,1}(S) &= \frac{1}{2\kappa^2}\int_0^S \left(\left(1-e^{-\kappa(S-t)}\right)\nu\left(t,y_t^0\right)\right)^2 dt \end{cases} \tag{4.6.45}$$

The derivative of $C_{0,1}$ corresponds to:

$$\partial_t C_{0,1}(t) = \frac{1}{\kappa}\int_0^S \left(1-e^{-\kappa(S-t)}\right)e^{-\kappa(S-t)}\left(\nu\left(t,y_t^0\right)\right)^2 dt \tag{4.6.46}$$

It can be given explicitly in case of $\beta = \frac{1}{2}$ or $\beta = 1$ using (4.6.36) and (4.6.37). We show in Figures 4.6.1a and 4.6.1b the accuracy of fitting market survival probabilities using (4.6.43) for different values of β and credit risk parameters. The errors are reported w.r.t. the right-hand axis.

Finally, the conditional survival probability $\mathbb{Q}_t(\tau > S|\lambda_t) \triangleq \mathbb{E}[1_{\tau>S}|\tau > t, \lambda_t]$ is given by

$$\begin{aligned}\mathbb{Q}_t(\tau > S|\lambda_t) &= e^{-\int_t^S \psi(s)ds}\mathbb{E}\left[e^{-\int_t^S y_s ds}\right] \\ &= \frac{\mathbb{Q}^{mkt}(\tau > S)}{\mathbb{Q}^{mkt}(\tau > t)}\frac{\mathbb{E}\left[e^{-\int_0^t y_s ds}\right]}{\mathbb{E}\left[e^{-\int_0^S y_s ds}\right]}\mathbb{E}\left[e^{-\int_t^S y_s ds}\right]\end{aligned} \tag{4.6.47}$$

It can be approximated by

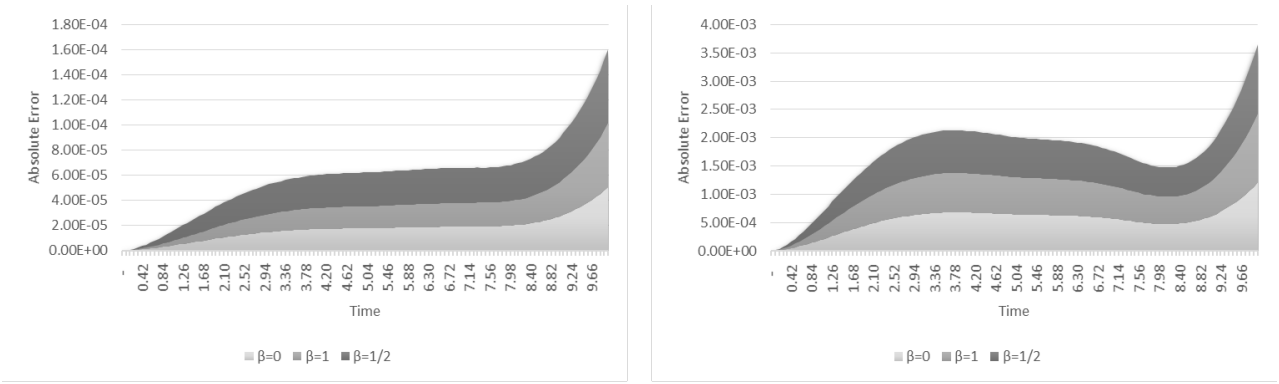
$$\mathbb{Q}_t(\tau > S|y_t) \approx \frac{\mathbb{Q}^{mkt}(\tau > S)}{\mathbb{Q}^{mkt}(\tau > t)}\frac{(1+C_{0,1}(t))}{(1+C_{0,1}(S))}\left(1+\tilde{C}_{0,1}(t,S;y_t)\right) \tag{4.6.48}$$

with

$$\tilde{C}_{0,1}\left(t,S;y_t^0\right) = \frac{1}{2\kappa^2}\int_t^S \left(\left(1-e^{-\kappa(S-s)}\right)\nu\left(s,y_s^0\right)\right)^2 ds \tag{4.6.49}$$

and

$$y_s^0 = \kappa\theta\left(e^{-\kappa s} - e^{-\kappa t}\right) + \left(y_t^0 - \kappa\theta\right), s > t \tag{4.6.50}$$



(a) Mid risk parameters

(b) High risk parameters

Figure 4.6.1: Fitting market survival probabilities

A.3 A detailed proof of Proposition 4.6.6

For sake of notation ease, let us define $f(t) = \left(\int_0^t e^{-\int_0^s \kappa(u)du}ds\right)$. From Proposition 4.6.4, we recall the main equations

$$\int_0^S \lambda_{1,t} dt = \int_0^S (f(S) - f(t))\, \tilde{\nu}(t)\, dW_t$$
$$\int_0^S \lambda_{2,t} dt = 2\int_0^S (f(S) - f(t))\, \nu^{(1)}(t) \left(\int_0^t \tilde{\nu}(s)\, dW_s \right) dW_t$$
$$\begin{aligned}\int_0^S \lambda_{3,t} dt = {} & 3\int_0^S (f(S) - f(t))\, e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t (\tilde{\nu}(s))^2 ds \right) dW_t \\ & + 6\int_0^S (f(S) - f(t))\, \nu^{(1)}(t) \left(\int_0^t \nu^{(1)}(s) \left(\int_0^s \tilde{\nu}(u)\, dW_u \right) dW_s \right) dW_t \\ & + 6\int_0^S (f(S) - f(t))\, e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t \tilde{\nu}(s) \left(\int_0^s \tilde{\nu}(u)\, dW_u \right) dW_s \right) dW_t\end{aligned}$$

Expression $\frac{1}{6}\mathbb{E}\left[\left(\int_0^S \lambda_{3,t} dt\right) h(X_T)\right]$:

As previously, we use repeatedly technical lemmas 4.6.1 and 4.6.2. One has then

$$\begin{aligned}& \frac{1}{2}\mathbb{E}\left[\left(\int_0^S (f(S) - f(t))\, e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t (\tilde{\nu}(s))^2 ds \right) dW_t \right) h(X_T) \right] \\ & = \frac{1}{2}\sum_{i=1}^n \int_0^S (f(S) - f(t))\, e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t (\tilde{\nu}(s))^2 ds \right) \left[\tilde{\Sigma}(t,T)\rho\right]_i dt \mathbb{E}\left[h_i^{(1)}(X_T)\right] \\ & = \frac{1}{2}\sum_{i=1}^n \omega_i^T \left((f(S) - f(.))\, e^{-\int_0^{\cdot} \kappa(s)ds} \nu^{(2)}(.) \left(\int_0^{\cdot} (\tilde{\nu}(s))^2 ds \right) \right)_0^S \mathbb{E}\left[h_i^{(1)}(X_T)\right] \\ & = (C_{1,2}(S,T)).\mathbb{E}\left[h^{(1)}(X_T)\right]\end{aligned}$$

The two remaining terms are derived analogously:

$$\begin{aligned}& \mathbb{E}\left[\left(\int_0^S (f(S) - f(t))\, \nu^{(1)}(t) \left(\int_0^t \nu^{(1)}(s) \left(\int_0^s \tilde{\nu}(u)\, dW_u \right) dW_s \right) dW_t \right) h(X_T) \right] \\ & = \rho \sum_{i=1}^n \mathbb{E}\left[\left(\int_0^S \int_t^S (f(S) - f(s))\, \nu^{(1)}(s) \left[\tilde{\Sigma}(s,T)\right]_i ds \nu^{(1)}(t) \left(\int_0^t \tilde{\nu}(s)\, dW_s \right) dW_t \right) h_i^{(1)}(X_T) \right] \\ & = \rho^2 \sum_{i,j=1}^n \mathbb{E}\left[\left(\int_0^S \int_t^S \int_s^S (f(S) - f(u))\, \nu^{(1)}(u) \left[\tilde{\Sigma}(u,T)\right]_i du \nu^{(1)}(s) \left[\tilde{\Sigma}(s,T)\right]_j ds \tilde{\nu}(t)\, dW_t \right) h_{i,j}^{(2)}(X_T) \right] \\ & = \ldots \\ & = \sum_{i,j,k=1}^n \omega_{i,j,k}^T \left(\tilde{\nu}, \nu^{(1)}, ((f(S) - f(.)))\, \nu^{(1)} \right)_0^S .\mathbb{E}\left[h_{i,j,k}^{(3)}(X_T)\right] \\ & = C_{3,1}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]\end{aligned}$$

and similarly:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^S (f(S)-f(t))\, e^{-\int_0^t \kappa(s)ds} \nu^{(2)}(t) \left(\int_0^t \tilde{\nu}(s) \left(\int_0^s \tilde{\nu}(u)\, dW_u\right) dW_s\right) dW_t\right) h(X_T)\right] \\
&= \sum_{i,j,k=1}^{n} \omega_{i,j,k}^T \left(\tilde{\nu}, \tilde{\nu}, (f(S)-f(.))\, e^{-\int_0^{\cdot} \kappa(s)ds} \nu^{(2)}\right)_0^S .\mathbb{E}\left[h_{i,j,k}^{(3)}(X_T)\right] \\
&= C_{3,2}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]
\end{aligned}$$

Bringing all terms together, we finally obtain

$$\frac{1}{6}\mathbb{E}\left[\left(\int_0^S \lambda_{3,t} dt\right) h(X_T)\right] = (C_{1,2}(S,T)).\mathbb{E}\left[h^{(1)}(X_T)\right] + (C_{3,1}(S,T) + C_{3,2}(S,T)).\mathbb{E}\left[h^{(3)}(X_T)\right]$$

Expression $\frac{1}{2}\mathbb{E}\left[\left(\int_0^S \lambda_{1,t} dt\right)\left(\int_0^S \lambda_{2,t} dt\right) h(X_T)\right]$:

We start by developing the product $\left(\int_0^S \lambda_{1,t} dt\right)\left(\int_0^S \lambda_{2,t} dt\right)$ using some Itô calculus:

$$\begin{aligned}
&\frac{1}{2}\left(\int_0^S \lambda_{1,t} dt\right)\left(\int_0^S \lambda_{2,t} dt\right) \\
&= \int_0^S \int_t^S (f(S)-f(s))^2 \tilde{\nu}(s) \nu^{(1)}(s)\, ds \tilde{\nu}(t)\, dW_t \\
&+ \int_0^S (f(S)-f(t)) \nu^{(1)}(t) \left(\int_0^t (f(S)-f(s)) (\tilde{\nu}(s))^2 ds\right) dW_t \\
&+ \int_0^S (f(S)-f(t)) \tilde{\nu}(t) \left(\int_0^t (f(S)-f(s)) \nu^{(1)}(s) \left(\int_0^s \tilde{\nu}(u)\, dW_u\right) dW_s\right) dW_t \\
&+ \int_0^S (f(S)-f(t)) \nu^{(1)}(t) \left(\int_0^t \tilde{\nu}(s)\, dW_s \left(\int_0^s (f(S)-f(u)) \tilde{\nu}(u)\, dW_u\right)\right) dW_t \\
&+ \int_0^S (f(S)-f(t)) \nu^{(1)}(t) \left(\int_0^t (f(S)-f(s)) \tilde{\nu}(s) \left(\int_0^s \tilde{\nu}(u)\, dW_u\right) dW_s\right) dW_t
\end{aligned}$$

The coefficients of the adjustment terms are deduced using immediately the integration operator ω^T:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^S \int_t^S (f(S)-f(s))^2 \tilde{\nu}(s) \nu^{(1)}(s)\, ds \tilde{\nu}(t)\, dW_t\right) h(X_T)\right] \\
&= \omega^T \left(\int_{\cdot}^S (f(S)-f(s))^2 \tilde{\nu}(s) \nu^{(1)}(s)\, ds \tilde{\nu}\right)_0^S .\mathbb{E}\left[h^{(1)}(X_T)\right] \\
&= C_{1,3}(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right]
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^S (f(S)-f(t)) \nu^{(1)}(t) \left(\int_0^t (f(S)-f(s)) (\tilde{\nu}(s))^2 ds\right) dW_t\right) h(X_T)\right] \\
&= \omega^T \left((f(S)-f(.)) \left(\int_0^{\cdot} (f(S)-f(s)) (\tilde{\nu}(s))^2 ds\right) \nu^{(1)}\right)_0^S .\mathbb{E}\left[h^{(1)}(X_T)\right] \\
&= C_{1,4}(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right]
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\left(\int_0^t (f(S)-f(s))\,\nu^{(1)}(s)\left(\int_0^s \tilde{\nu}(u)\,dW_u\right)dW_s\right)dW_t\right)h(X_T)\right]\\
&=\omega^T\left(\tilde{\nu},(f(S)-f(.))\,\nu^{(1)},(f(S)-f(.))\,\tilde{\nu}\right)_0^S.\mathbb{E}\left[h^{(3)}(X_T)\right]\\
&=C_{3,3}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^S (f(S)-f(t))\,\nu^{(1)}(t)\left(\int_0^t (f(S)-f(s))\,\tilde{\nu}(s)\left(\int_0^s \tilde{\nu}(u)\,dW_u\right)dW_s\right)dW_t\right)h(X_T)\right]\\
&=\omega^T\left(\tilde{\nu},(f(S)-f(.))\,\tilde{\nu},(f(S)-f(.))\,\nu^{(1)}\right)_0^S.\mathbb{E}\left[h^{(3)}(X_T)\right]\\
&=C_{3,4}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^S (f(S)-f(t))\,\nu^{(1)}(t)\left(\int_0^t \tilde{\nu}(s)\left(\int_0^s (f(S)-f(u))\,\tilde{\nu}(u)\,dW_u\right)dW_s\right)dW_t\right)h(X_T)\right]\\
&=\omega^T\left((f(S)-f(.))\,\tilde{\nu},\tilde{\nu},(f(S)-f(.))\,\nu^{(1)}\right)_0^S.\mathbb{E}\left[h^{(3)}(X_T)\right]\\
&=C_{3,5}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]
\end{aligned}$$

Bringing all terms together, we get

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)\left(\int_0^S \lambda_{2,t}dt\right)h(X_T)\right] &= (C_{1,3}+C_{1,4})(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right]\\
&\quad+\left(C_{3,3}+C_{3,4}+C_{3,5}\right)(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]
\end{aligned}$$

Expression $\frac{1}{6}\mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)^3 h(X_T)\right]$:

$$\begin{aligned}
\left(\int_0^S \lambda_{1,t}dt\right)^3 &= 2C_{0,1}(S)\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\,dW_t\right)\\
&\quad+2\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\left(\int_0^t (f(S)-f(s))\,\tilde{\nu}(s)\left(\int_0^s (f(S)-f(u))\,\tilde{\nu}(u)\,dW_u\right)dW_s\right)dW_t\\
&\quad+2\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\left(\int_0^t (f(S)-f(s))\,\tilde{\nu}(s)\,dW_s\right)^2 dW_t\right)\\
&= 2C_{0,1}(S)\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\,dW_t\right)+4\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\,C_{0,1}(t)\,dW_t\right)\\
&\quad+6\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\left(\int_0^t (f(S)-f(s))\,\tilde{\nu}(s)\left(\int_0^s (f(S)-f(u))\,\tilde{\nu}(u)\,dW_u\right)dW_s\right)dW_t
\end{aligned}$$

We deduce the following adjustment terms:

$$\frac{1}{6}\mathbb{E}\left[2C_{0,1}(S)\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\,dW_t\right)h(X_T)\right] = \frac{C_{0,1}(S)}{3}\omega^T\left((f(S)-f(.))\,\tilde{\nu}(.)\right)_0^S.\mathbb{E}\left[h^{(1)}(X_T)\right]$$
$$= C_{1,5}(S,T)\,\mathbb{E}\left[h^{(1)}(X_T)\right]$$

$$\frac{1}{6}\mathbb{E}\left[4\left(\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\,C_{0,1}(t)\,dW_t\right)h(X_T)\right] = \frac{2}{3}\omega^T\left((f(S)-f(.))\,C_{0,1}(.)\,\tilde{\nu}(.)\right)_0^S.\mathbb{E}\left[h^{(1)}(X_T)\right]$$
$$= C_{1,6}(S,T)\,\mathbb{E}\left[h^{(1)}(X_T)\right]$$

$$\frac{1}{6}\mathbb{E}\left[6\int_0^S (f(S)-f(t))\,\tilde{\nu}(t)\left(\int_0^t (f(S)-f(s))\,\tilde{\nu}(s)\left(\int_0^s (f(S)-f(u))\,\tilde{\nu}(u)\,dW_u\right)dW_s\right)dW_t h(X_T)\right]$$
$$= \omega^T\left((f(S)-f(.))\,\tilde{\nu}(.),(f(S)-f(.))\,\tilde{\nu}(.),(f(S)-f(.))\,\tilde{\nu}(.)\right)_0^S.\mathbb{E}\left[h^{(3)}(X_T)\right]$$
$$= C_{3,6}(S,T)\,\mathbb{E}\left[h^{(3)}(X_T)\right]$$

Bringing all terms together, we have

$$\frac{1}{6}\mathbb{E}\left[\left(\int_0^S \lambda_{1,t}dt\right)^3 h(X_T)\right] = \left(C_{1,5}+C_{1,6}\right)(S,T).\mathbb{E}\left[h^{(1)}(X_T)\right] + C_{3,6}(S,T).\mathbb{E}\left[h^{(3)}(X_T)\right]$$

Chapter 5

Expansion formulas for CVA valuation with *WWR*

This chapter extends the expansion technique developed in Chapter 4 to the context of credit valuation adjustments. This extension aims to derive analytical formulas for credit-linked adjustments such as CVA and DVA in presence of the *Wrong-way risk*. Given a stochastic intensity model for the default events, we present our analytical approximation of the unilateral CVA that can be easily described as a proxy UCVA^0 to which we add a combination of analytically tractable terms involving exposure Greeks.

Our methodology represents a fast and parsimonious way to compute *Wrong-way risk* impacts and to achieve CVA stress-testing w.r.t. the credit spread's volatility and *WWR* correlations. In addition, we show how to adapt easily the pricing formulas in order to account for the funding *Wrong-way risk.* The correlation risk, stemming either from credit or from funding, is embedded into a correlation-adjusted exposure that can be used in the calculation other XVA metrics.

Keywords: CVA, FVA, *Wrong-way risk*, credit spread volatility, asymptotic expansion, XVA stress-testing.

5.1 Introduction

Right and *Wrong-way risks,* commonly called *WWR,* can be an important source of counterparty and funding uncertainty for financial institutions. This risk tends to be poorly quantified and managed by XVA practitioners. The main reason being that measuring this risk is very calculation demanding. It usually requires the adaption of pricing functions in order to account for a hybrid (cross-asset) correlation. In addition, it introduces a methodological question with regard to the estimation of the credit and funding spread volatilities, as well as the measurement of their correlation with the exposure.

There are several manifestations of Wrong-way risk: It can be specific to the structure of a transaction or to the nature of the collateral pledged. In all cases, it represents a correlation risk that requires a careful analysis. The most representative example of *WWR* consists of buying a protection in a CDS contract with a strong relationship between the credit quality of

the reference entity and the CDS counterparty. This example was encountered during 2008 and 2009. For instance, *WWR* loses were caused by high leverage of monoline insurers and an unprecedented increase of credit risk, leading to the widening of the credit spreads of protection sellers. The credit spreads of monolines widened from 5–10 to several hundred basis points. Banks that had bought insurance from monolines had to realize substantial losses due to the increased counterparty risk.

From a regulatory perspective, the Basel committee uses a simple approach to account for *Wrong-way risk*, while Basel's Accords ignore *Right-way risk*. *WWR* is accounted for by increasing the exposure metric by an arbitrary constant factor $\alpha = 1.4$,or greater that 1.2 if estimated by the financial institution using an internal model. [BMP13] shows the weakness of this rule in several common cases.

The standard approach in quantifying the impact of *WWR* consists of crude Monte Carlo simulations. Generally, this approach offers a good setup to measure and manage *WWR*. However, it requires significant computation power for portfolio valuation and incremental billing. Cesari *et al.* [CAC$^+$09] propose to change the risk measure towards the measure implied by the credit spread evolution, the so call Risk Factor Evolution measure. For this purpose, they use the distribution of transactions future values, calculated by *American Monte Carlo* algorithm. The change of measure requires then changing the drift of the diffusion models.

We introduce in this work a new approach to tackle the *WWR* valuation problem. Our approach is based on asymptotic expansion of the credit spread diffusion model. The resulting XVA adjustment corresponds to the standard adjustment without *WWR* to which is added an explicit combination of correction terms. The *WWR* components are made explicit, and expressed in terms of the expected exposure Greeks. In addition, our pricing formulas offer a convenient way to run XVA stress-testing w.r.t. the credit spread volatility and *WWR* correlation.

Outline of the chapter

This chapter is organized as follows. In Section 5.2, we define the credit valuation adjustments and highlight the main features in computing unilateral CVA/DVA. We also discuss different examples of *Wrong-Way risk*. In Section 5.3, we present our stochastic intensity model for the defaulting counterparty used to introduce correlation risk between credit and market risk factors. The expansion pricing formulas are developed in Section 5.4 which also presents extensions to valuate derivatives under the funding *WWR*. Section 5.5 is dedicated to the calculation of the expected exposure and its sensitivities. In particular, we present a sensitivity-based Gaussian framework that allows to compute the exposure Greeks analytically. Our numerical results are presented in Section 5.6. Finally, our conclusions are addressed in Section 5.7.

5.2 Arbitrage-free valuation, closeout conventions and *Wrong-way risk*

We refer to a financial transaction in which two names are involved and are subject to default risk. We will address the valuation as seen from the point of view of an investor I, so that cash-flows received by I are accounted positively whereas cash-flows received by its counterparty C are accounted negatively.

We denote by τ_I and τ_C respectively the default times of the investor and its counterparty, and LGD_I (resp. LGD_C) the deterministic loss given default of I (resp. C). We call T the final maturity of the payoff which we need to evaluate and $D(t,T)$ the discount factor function. We define $\Pi_I(t,T)$ (resp. $\Pi_C(t,T)$) to be the discounted sum of cash-flows of the derivative from t to T seen from the point of view of I (resp. C) and free from from all credit assumptions. Namely, one has

$$\Pi_I(t,T) = -\Pi_C(t,T)$$

Π_B^D and Π_C^D are the equivalent discounted claims with counterparty default risk.

5.2.1 An arbitrage-free framework for valuation

Under default-free assumptions, the net present value of the derivative at time t reads

$$V_i(t) := \mathbb{E}_t\left[\Pi_i(t,T)\right] \text{ for party } i \in \{I,C\}$$

where $\mathbb{E}_t$ ($\mathbb{E} = \mathbb{E}_0$) indicates the risk-neutral expectation based on market information up to time t. We denote by $\mathbb{Q}_t$ ($\mathbb{Q}_0 = \mathbb{Q}$) the risk neutral measure conditional to the same flow of information.

In this chapter, we focus on the unilateral risk of default. Only the default of one of the two parties is taken into account, and the surviving party is treated as risk-free. The resulting adjustment is called *Unilateral Credit Valuation Adjustment* ($UCVA$) and, formally, it corresponds to

$$UCVA(t) := \mathrm{LGD}_C \mathbb{E}_t\left[1_{t\leq\tau_C\leq T} D(t,\tau_C)\left(V_I(\tau_C)\right)^+\right] \tag{5.2.1}$$

from the point of view of I. Analogously, C is treated as default-free while the default of the investor I is taken into account. Always from the point of view of I, the *Unilateral Debt Valuation Adjustment* ($UDVA$) formally writes

$$UDVA(t) := \mathrm{LGD}_I \mathbb{E}_t\left[1_{t\leq\tau_I\leq T} D(t,\tau_I)\left(-V_I(\tau_I)\right)^+\right] \tag{5.2.2}$$

We refer to Chapter 1 and to [BMP13] for a detailed derivation of Equations (5.2.1) and (5.2.2).

A more realistic framework consists of considering simultaneously the default of both counterparties. The resulting adjustment, known as BVA for *Bilateral Valuation Adjustment*, brings more complexity since it implies that the CVA alone depends on the credit quality of the surviving party. This captures the "first to default" nature of a contract with respect to the default of both counterparties and then avoids double counting. Practitioners usually approach BVA by the difference UCVA-UDVA. Whilst this approximation can be shown very inaccurate, it has the advantage of using only the unilateral adjustments. It is immediate to see that the approximation ignores the correlation risk standing behind the default of counterparties and it is no longer realistic after the credit events of 2008. [BBM11] highlights the dangers of using of the unilateral default approximation of BVA.

Institutions are allowed to account for the bilateral risk of default in determining the *closeout amount*. The latter corresponds to the cost of replacing the transaction with a third counterparty when the default occurs. A standard assumption is to determine the closeout value of transactions based on risk-free valuation. This approach, refereed by *risk-free closeout* allows a straightforward derivation of CVA and DVA formulas (as given previously in (5.2.1) and

(5.2.2)). However, a more reasonable choice, as pointed out by ISDA (2009) protocol, suggests that an institution may consider its own DVA in determining the amount to be settled. This *risky close-out* (also known as *replacement close-out*) includes the DVA of the surviving party since this would correspond to the CVA charged on a replacement trade in the market. Under this close-out convention, the unilateral credit and debt adjustments become

$$UCVA(t) \to UCVA(t) = \mathrm{LGD}_C \mathbb{E}_t \left[1_{t \le \tau_C \le T} D(t, \tau_C) (V_I(\tau_C) + UDVA(\tau_C))^+ \right] \tag{5.2.3}$$

$$UDVA(t) \to UDVA(t) = \mathrm{LGD}_I \mathbb{E}_t \left[1_{t \le \tau_I \le T} D(t, \tau_I) (-V_I(\tau_I) - UCVA(\tau_I))^+ \right] \tag{5.2.4}$$

From an algorithmic perspective, an appropriate modeling of close-out conventions as well as pricing under the bilateral default setting add a huge complexity to the problem of counterparty risk valuation. This topic is addressed thoroughly in [BM10].

Finally, it is commonly used to estimate the unilateral valuation adjustments by assuming that the default times (τ_I, τ_C) and V_I are independent in (5.2.1) and (5.2.2). This allows to separate the pure market exposure from the credit default risk events:

$$UCVA(t) = \mathrm{LGD}_C \int_t^T \mathbb{E}_t \left[D(t,s) (V_I(s))^+ \right] d\mathbb{Q}_t (\tau_C \le s) \tag{5.2.5}$$

$$UDVA(t) = \mathrm{LGD}_I \int_t^T \mathbb{E}_t \left[D(t,s) (-V_I(s))^+ \right] d\mathbb{Q}_t (\tau_I \le s) \tag{5.2.6}$$

While this appears oversimplified when compared to the bilateral CVA framework, Gregory and German show in [GG12] that, at a first order approximation, the method is actually close to the (far more complex) risky close-out convention. In particular, the unconditional unilateral CVA/DVA formulas disentangle the pricing from solving recursive stochastic system, addressed in expectations (5.2.3) and (5.2.4).

5.2.2 *Wrong-way risk*

Pricing with *Wrong-way risk* (or *Right-way risk*) consists of taking into account the hybrid correlation between counterparties defaults and the exposure (driven by the mark-to-market). Whilst it may often be a reasonable assumption to ignore *WWR*, its impacts can be subtle, model-dependent, hard to catch objectively, and sometimes very harmful. In the opposite, favorable correlation risk or *Right-way risk* reduces counterparty risk and CVA. The market events of 2007 and 2008 have illustrated that *WWR* consequences can be extremely severe. One of the simplest examples is subprime mortgage providers who face both falling property prices and higher default rates by homeowners. They led insurers to bankruptcy in their wake and caused the critical events of the credit crunch.

From an operational perspective, trades that obviously contain *WWR* indications due to the payoff structure and dependency on counterparties credit quality are identified as *Specific WWR* trades. For the remaining trades, the area of *WWR* is probably limited. They are denoted as *General WWR* trades. Of course, this classification requires a careful analysis of payoffs w.r.t. involved counterparties, as well as non-modeled systemic risk factors. We give a short list of specific and general *WWR* examples hereafter:

- ***Specific Wrong-way risk***:

  - **Equity Put option**s: Buying a put option on a stock of the counterparty. The put option is more valuable when the stock goes down, in which case the counterparty's credit quality is worsening. The call option should be then a right-way product.
  - **Equity Total Return swap**s: Receiving a fixed premium against paying the return of the counterparty's stock. In practice, nominal reset mechanisms allow to mitigate the total exposure.
  - **Cross-currency products**: A potential weakening of the currency received by the counterparty *versus* the paid currency represents a source of *Wrong-way risk*. Think about trading a cross-currency swap contracted with an emergent sovereign, paying their local currency and receiving EUR. The devaluation of the emergent currency vs. EUR, is closely related to the credit worthiness of the sovereign counterparty.
  - **Commodity swaps**: In an commodity swap, one party pays cash-flows based on a fixed commodity price and receives cash-flows based on an average spot price of the same commodity over a period. The exposure of the payer (fixed leg) swap will be high when the price of commodity increases. When the counterparty is one of the commodity producers, high commodity prices is expected to improve the credit worthiness of the counterparty.
  - **Credit default swaps**: The classical *WWR* situation occurs when buying protection in a CDS product on a reference entity to which the CDS counterparty is closely related such as in a legal affiliation.

- General ***Wrong-way risk***: This refers to situations where the relationship between counterparties credit worthiness and the risk factors that drive the exposure are not explicit. We encounter such situations in case of interest rates products majorly. The relationship between interest rates and credit spreads still persists since low interest rates are generally indicative of recession where default are more likely to occur.

From the above list, pricing under specific *WWR* necessitates particular modeling choices that should meet the specificity of the trade from the counterparty risk point of view. Non-cash collateral represents a major alternative source of *WWR*. Here, the link between the credit quality of the counterparty and the collateral value impacts directly the netted amount of exposure. Concretely, if a downgrade of the counterparty's rating (or an increase of it CDS spreads) causes a decrease of the collateral value then the exposure and CVA increase. The distinction between Specific and General collateral *WWR* can also be adopted. In the following, we describe few situations where collateral induces *W/R-WR:*

- Collateral ***Wrong-way risk***:
  - **Counterparty's Equity stock posted as collateral**: This is a classical situation when the counterparty is allowed to pledge its own equity stock as collateral. The counterparty's stock is generally excluded form the scope of eligible collateral assets.
  - **Asset collateral (Equity/Bond) closely correlated to the counterparty**: In contrast with the first situation, eligible assets such as stocks or bonds can be closely related to the counterparty's credit quality. For instance, if the counterparty is a

an emerging supranational corporate and the posted collateral is the counterparty's government bond, then the widening of the counterparty's credit spreads are more likely to be followed by an increase of the bonds spreads, and, as a result, a decrease of the collateral value. The situation can get even worse when the value of the underlying trade is also affected by these moves (general *WWR*). Very famous and recent examples are associated with energy swaps traded with the major oil producers of oil-emerging countries. A deterioration of counterparty's credit spread would lead at the same time to a decrease of the collateral value and an increase of the trades mark to market. Such concomitant systemic risks were observed during oil market recession and prices flops during 2015 (see [McC15]).

– **Collateral agreement with rating triggers**: Some collateral agreements obligate one or both counterparties to follow margin rules that explicitly depend on their rating migrations. Usually, additional collateral is requested when the counterparty is downgraded. This situation represents a Right-way risk exposure.

General *WWR* is encountered much more frequently since it is associated to a large variety of trades. In spite of its limited impact on the trade level, the sum of impacts over a netting set level can be very significant. Due to the large spectrum of calculation, the general form of *WWR* requires a generic modeling and pricing approaches. A joint hybrid credit diffusion coupled with Monte-Carlo is usually adopted, in spite of important costs due to computation time. Our present work addresses a new solution, based on *standard (WWR*-free) pricers and exposure Greeks that practitioners are familiar with.

5.3 A parametrized model for the default intensity

We present in this section a valuation framework for pricing the unilateral adjustments formulated in equations (5.2.1) and (5.2.2) where correlations between default times $\tau_{i\in\{I,C\}}$ and the market-to-market V_I are fully specified. We focus only on UCVA formulas, while UDVA formulas are derived similarly. We always consider the investor's point of view, and for notation ease, we denote $V_I = V$ and take $t = 0$.

In a pricing setting, default probabilities $s \mapsto \mathbb{Q}_t(\tau_i \leq s)$, $i \in \{I,C\}$ are implied from market quotes of credit derivatives, and this is usually achieved quite well by considering a piece-wise constant default intensity term structure. Moreover, we admit that the default-free *net present value* V of the contingent claim is a deterministic function of time t and a stochastic state variable X_t, i.e.

$$V(t) = V(t, X_t) \tag{5.3.1}$$

In full generality, X is a $\mathbb{R}^n-$valued diffusion process which dynamics are specified by the stochastic differential equation

$$dX_t = (\Phi(t) + \Theta(t) X_t)\, dt + \Sigma(t)\, dB_t, \qquad X_0 \in \mathbb{R}^n \tag{5.3.2}$$

with $\Phi : [0,T] \to \mathbb{R}^n$, $\Theta : [0,T] \to \mathbb{R}^{n\times n}$, $\Sigma : [0,T] \to \mathbb{R}^{n\times d}$ and $(B_t)_{t\geq 0}$ a $\mathbb{R}^d-$Brownian motion such that $\forall i, j \in [\![1,d]\!]$, $d\left\langle [B]_i, [B]_j \right\rangle dt = \rho_{i,j}^{X,X} dt$.

The default time τ_C is modeled under the reduced form approach, i.e.

$$\tau_C = \inf\left\{ t > 0 \text{ such that } \int_0^t \lambda_s^{\varepsilon} ds > \xi \right\} \tag{5.3.3}$$

where ξ is an independent unit-mean exponential random variate[1] and $(\lambda_t^{\varepsilon})_t$ the default's intensity. To model *Wrong-way risk*, λ is assumed to be a diffusion process satisfying

$$\lambda_t^{\varepsilon} = y_t^{\varepsilon} + \psi^{\varepsilon}(t) \tag{5.3.4}$$

where y_t^{ε} is a stochastic risk factor governed by the SDE

$$\begin{cases} dy_t^{\varepsilon} & = \kappa(\mu - y_t^{\varepsilon})\,dt + \varepsilon \nu(t, y_t^{\varepsilon})\,dW_t, \qquad y_0^{\varepsilon} = y_0 > 0 \\ d\langle W, B_i \rangle_t & = \rho_i^{X,\lambda} dt \ , 1 \leq i \leq d. \end{cases} \tag{5.3.5}$$

$\rho^{X,\lambda} = \left(\rho_i^{X,\lambda}\right)_{i=1\ldots d}$ being a vector of instantaneous correlations. ψ^{ε} is a deterministic shift function whose role is to calibrate the generated CDS spreads to the market term structure. In fact, the survival probabilities given by the model correspond to

$$\mathbb{Q}(\tau_C > t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s^{\varepsilon} ds\right)\right] = \exp\left(-\int_0^t \psi^{\varepsilon}(s)\,ds\right)\mathbb{E}\left[\exp\left(-\int_0^t y_s^{\varepsilon} ds\right)\right] \tag{5.3.6}$$

and must be equal to probabilities bootstrapped from market instruments such as Credit Default Swaps (CDS)

$$\mathbb{Q}(\tau_C > t) = \mathbb{Q}^{market}(\tau_C > t) \tag{5.3.7}$$

leading to

$$\exp\left(-\int_0^t \psi^{\varepsilon}(s)\,ds\right) = \frac{\mathbb{Q}^{market}(\tau_C > t)}{\mathbb{E}\left[\exp\left(-\int_0^t y_s^{\varepsilon} ds\right)\right]} \tag{5.3.8}$$

In particular, ψ^{ε} is given by

$$\psi^{\varepsilon}(t) = h(t) + \frac{\partial}{\partial t} \ln\left(\mathbb{E}\left[\exp\left(-\int_0^t y_s^{\varepsilon} ds\right)\right]\right) \tag{5.3.9}$$

with $h(t) \triangleq -\frac{\partial}{\partial t} \ln\left(\mathbb{Q}^{market}(\tau_C > t)\right)$ the market deterministic hazard rate. Once the model is calibrated to CDS data through (5.3.9), we are left with parameters κ, μ and function ν to specify in order to calibrate further products such as options on CDS. In practice, options on CDS are very illiquid, or not available for a given reference entity. The determination of parameters κ, μ and ν is usually based then on historical estimations.

5.4 Explicit pricing formulas of CVAs with *Wrong-way risk*

We follow the approach presented in the last chapter as we consider the $\varepsilon-$parametrization of the intensity process λ. For $\varepsilon = 0$, the credit spread volatility ν and the *WWR* correlations $\rho^{X,\lambda}$

[1] Independent from any other source of randomness

are offset. The resultant intensity λ^0 is deterministic and the adjustment formulas ((5.2.5) and (5.2.6)) are obtained under the independence assumption between V and default events. We rename $UCVA$ and $UDVA$ of equations (5.2.5) and (5.2.6) respectively $UCVA^0$ and $UDVA^0$ as they represent our proxy adjustments.

Notation 5.4.1. For any parametrized process Z^ε, we denote $Z^0 = Z^{\varepsilon=0}$ and $Z = Z^{\varepsilon=1}$.

We rely then on expansion arguments and we set $\varepsilon = 1$ in order to recover a stochastic intensity diffusion $\lambda^{\varepsilon=1}$, as well as the correlation between default events and future mark-to-markets. ε is a parametrization tool that we use in order to derive a convenient representation of $UCVA$ ($\varepsilon = 1$) as an expression of the proxy $UCVA^0$ ($\varepsilon = 0$) to which is added a combination of correction terms. As discussed earlier, the accuracy of this approximation does not depend on the smallness of ε but rather on the volatility magnitude of the integrated hazard rate process $\int_0^t \lambda_s^{\varepsilon=1} ds$.

5.4.1 *WWR*-adjusted exposure

Formally, we consider a time discretization $\mathscr{T}_M = \{0 = T_0, T_1, \ldots, T_M\}$ such that the default time τ_C is observed on time intervals $]T_{i-1}, T_i]$, T_M matches the maturity of the contract. $UCVA$ is then approximated by

$$UCVA(0) = LGD_C \sum_{i=1}^{M} D(0,T_i)\,\mathbb{Q}(\tau_C \in]T_{i-1},T_i])\,\mathbb{E}\left[(V(T_i,X_{T_i}))^+\right] \tag{5.4.1}$$

where LGD_C is the *loss given default* of counterparty C. The number of discretization steps M is chosen such that the numerical convergence to (5.2.5) is satisfactory. Given the parametrized hazard rate λ^ε for τ^C, one has

$$\begin{aligned} UCVA(0) &= LGD_C \sum_{i=1}^{M} D(0,T_i)\,\mathbb{E}\left[1_{\tau_C \in]T_{i-1},T_i]}\,(V(T_i,X_{T_i}))^+\right] \\ &= LGD_C \sum_{i=1}^{M} D(0,T_i)\,\mathbb{E}\left[\left(1_{T_{i-1}<\tau_C} - 1_{T_i<\tau_C}\right)(V(T_i,X_{T_i}))^+\right] \end{aligned} \tag{5.4.2}$$

Owing to the *expectations tower property* and filtration conditioning, it is well known that the following equality holds

$$UCVA(0) = LGD_C \sum_{i=1}^{M} D(0,T_i)\,\mathbb{E}\left[\left(e^{-\int_0^{T_{i-1}} \lambda_s^\varepsilon ds} - e^{-\int_0^{T_i} \lambda_s^\varepsilon ds}\right)(V(T_i,X_{T_i}))^+\right] \tag{5.4.3}$$

Let us now define

$$u_{V^+}^\varepsilon : (s,t) \mapsto \frac{\mathbb{E}\left[e^{-\int_0^s \lambda_\omega^\varepsilon d\omega}\,(V(t,X_t))^+\right]}{\mathbb{Q}(\tau_C > s)} = \frac{\mathbb{E}\left[e^{-\int_0^s y_\omega^\varepsilon d\omega}\,(V(t,X_t))^+\right]}{\mathbb{E}\left[e^{-\int_0^s y_\omega^\varepsilon d\omega}\right]} \tag{5.4.4}$$

Using (5.3.8), we can write

$$UCVA(0) = LGD_C \sum_{i=1}^{M} D(0,T_i)\left[\mathbb{Q}(\tau_C > T_{i-1})\,u_{V^+}^\varepsilon(T_{i-1},T_i) - \mathbb{Q}(\tau_C > T_i)\,u_{V^+}^\varepsilon(T_i,T_i)\right] \tag{5.4.5}$$

This formulation gives a convenient decomposition of *UCVA* in terms of price functions $u_{V^+}(t,s)$ taken at different maturities t and survival horizons s. One can assimilate this representation to the price of a replication portfolio comprising credit contingent contracts paying V^+ conditionally on the survival of counterparty C. The price of each of these replicating components is given by $u^{\varepsilon}_{V^+}$, their weights are explicitly given in (5.4.5), and correspond to market survival probabilities on times T_{i-1} and T_i.

By analogy to the expected positive exposure involved in equation (5.4.1), i.e.

$$EPE(T_i) = \mathbb{E}\left[(V(T_i, X_{T_i}))^+\right] \tag{5.4.6}$$

one can define the *WWR-adjusted* expected positive exposure by

$$WWREPE^{\varepsilon}(T_{i-1}, T_i) = \frac{\mathbb{Q}(\tau_C > T_{i-1})\, u^{\varepsilon}_{V^+}(T_{i-1}, T_i) - \mathbb{Q}(\tau_C > T_i)\, u^{\varepsilon}_{V^+}(T_i, T_i)}{\mathbb{Q}(\tau_C \in]T_{i-1}, T_i])} \tag{5.4.7}$$

Hence, we recover an analogous formulation for both (5.4.1) and (5.4.5)

$$UCVA(0) = LGD_C \sum_{i=1}^{M} D(0,T_i)\, \mathbb{Q}(\tau_C \in]T_{i-1}, T_i])\, WWREPE^{\varepsilon}(T_{i-1}, T_i) \tag{5.4.8}$$

WWR-adjusted expected exposures can be used to compute further XVA adjustments.

5.4.2 Approximation formulas for the *WWR*-adjusted exposure

Approximation formulas at the second and third orders are provided in Chapter 4 for the price function $u^{\varepsilon}_{V^+}$. These approximations allow us to express the price $u^{\varepsilon=1}_{V^+}$ in terms of $u^{\varepsilon=0}_{V^+}$ and a combination of terms that depend mainly on the Greeks of $u^{\varepsilon=0}_{V^+}$ w.r.t. to X_0:

$$u^{\varepsilon=1}_{V^+}(s,t) = u^{\varepsilon=0}_{V^+}(s,t) + \text{weighted sum of Greeks of } \mathbb{E}\left[(V(t,X_t))^+\right] + \text{Error}^{\varepsilon=1} \tag{5.4.9}$$

Notice that $u^{\varepsilon=0}_{V^+}$ is explicitly given by the expected positive exposure equation (5.4.6):

$$u^{\varepsilon=0}_{V^+}(s,t) = \mathbb{E}\left[(V(t,X_t))^+\right]. \tag{5.4.10}$$

In addition, as highlighted in our previous work, the weighted sum of Greeks exhibits a clear dependence on the credit spread volatility function ν and the *WWR* correlations $\rho^{X,\lambda}$. In this chapter, these results are extended to the context of CVA by summing the approximated prices of all its replicating components.

In order to derive our approximation formulas, we introduce the following notations for the expected exposure Greeks. These quantities will be important for interpreting the expansion terms:

Notation 5.4.2 (Expected Exposure Greeks). If they exist, we denote the first 3 order Greeks for the expected positive exposures as seen from time 0 by:

$$\begin{cases} Greek^{(1)}_{V^+}(t,X_t) &= \left[\frac{\partial \mathbb{E}\left[(V(t,X_t+x))^+\right]}{\partial x_i}|_{x=0_{R^n}}\right]_{1\le i\le n} \\ Greek^{(2)}_{V^+}(t,X_t) &= \left[\frac{\partial^2 \mathbb{E}\left[(V(t,X_t+x))^+\right]}{\partial x_i \partial_{x_j}}|_{x=0_{R^n}}\right]_{1\le i,j\le n} \\ Greek^{(3)}_{V^+}(t,X_t) &= \left[\frac{\partial^3 \mathbb{E}\left[(V(t,X_t+x))^+\right]}{\partial x_i \partial x_j \partial x_k}|_{x=0_{R^n}}\right]_{1\le i,j,k\le n} \end{cases}$$

The Greeks for the expected negative exposure are defined similarly, by taking $V^- = \max(-V, 0)$ instead of V^+.

Assumption 5.4.3. *We assume the appropriate smoothness and boundedness conditions on* $(t,x) \mapsto (V(t,x))^+$ *such that one can substitute the expectation and differentiation operators in Notation 5.4.2.*

Notation 5.4.4 (Expansion notations). We recall that operator . (dot) stands for the element-wise multiplication between real-valued vectors, matrices or triadic tensors.

We define the modified mean reversion speed $\tilde{\kappa}$ and volatility functions $\tilde{\Sigma}$ and $\tilde{\nu}$ such that

$$\tilde{\kappa}(t,T) = \frac{1}{\kappa}\left(e^{-\kappa t} - e^{-\kappa T}\right) \tag{5.4.11}$$

$$\tilde{\Sigma}(t,T) = e^{\int_t^T \Theta(\omega) d\omega} \Sigma(t) \tag{5.4.12}$$

$$\tilde{\nu}(t) = e^{\kappa t} \nu(t) \tag{5.4.13}$$

Finally, we introduce the volatility upper-bounds

$$\begin{cases} M_0 &= \max_{0 \le i \le 3} \left(\left\| \nu^{(i)} \right\|_\infty \right) \\ M_1 &= \max_{1 \le i \le 3} \left(\left\| \nu^{(i)} \right\|_\infty \right) \end{cases}$$

We refer by ω_ρ^T the integration operator introduced in the last chapter, and we recall that $\nu(t) = \nu\left(t, y_t^{\varepsilon=0}\right)$, and $\nu^{(i)}(t) = \frac{\partial^i \nu(t+s)}{\partial s^i}|_{s=0}$. The following theorem gives a second order approximation formula for $UCVA(0)$. It is obtained by a straightforward application of Theorem 4.3.9 to functions $u_{V^+}^\varepsilon$.

Theorem 5.4.5 (***Second order approximation price formula***)**.** *Suppose that the intensity state process* (y_t^ε) *fulfills the regularity conditions of Theorem 4.3.9. Then, one has*

$$u_{V^+}^{\varepsilon=1}(S,T) = \mathbb{E}\left[(V(t,X_t))^+\right] - \sum_{i=1}^{2} \left(\frac{C_{2,i}(S,T)}{1 + C_{0,1}(S)} \right) . Greek^{(i)}(T, X_T) + Error_2^{\varepsilon=1} \tag{5.4.14}$$

with

$$C_{0,1}(S) = \frac{1}{2} \int_0^S \left(\tilde{\kappa}(t,S)\, \tilde{\nu}(t)\right)^2 dt \tag{5.4.15}$$

$$C_{2,1}(S,T) = \omega_\rho^T \left(\tilde{\kappa}(.,S)\, \tilde{\nu}(.)\right)_0^S \tag{5.4.16}$$

$$C_{2,2}(S,T) = \omega_\rho^T \left(\tilde{\nu}, \tilde{\kappa}(.,S)\, \nu^{(1)}\right)_0^S - \omega_\rho^T \left(\tilde{\kappa}(.,S)\, \tilde{\nu}, \tilde{\kappa}(.,S)\, \tilde{\nu}\right)_0^S \tag{5.4.17}$$

In addition, an estimate of the error term $Error_2^{\varepsilon=1}$ *is given by*

$$\left| Error_2^{\varepsilon=1} \right| \le \mathfrak{C}(S) \times S^2 M_0 (M_1 + S M_0) \left\| V(T, X_T) \right\|_2 \tag{5.4.18}$$

for some constant $\mathfrak{C}(S)$ *that depends on S,* M_0 *and* M_1*.*

Proof. We first apply 4.3.9 for $\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\left(V\left(T,X_T\right)\right)^+\right]$:

$$\frac{\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\left(V\left(T,X_T\right)\right)^+\right]}{e^{-\int_0^S y_\omega^0 d\omega}\left(1+C_{0,1}\left(S\right)\right)} = \mathbb{E}\left[\left(V\left(T,X_T\right)\right)^+\right] - \frac{C_{2,1}\left(S,T\right)}{\left(1+C_{0,1}\left(S\right)\right)} Greek^{(1)}\left(T,X_T\right) - \frac{C_{2,2}\left(S,T\right)}{\left(1+C_{0,1}\left(S\right)\right)}.Greek^{(2)}\left(T,X_T\right) + \text{Error}_2^\varepsilon$$

where $\left|\text{Error}_2^\varepsilon\right| \leq \mathscr{C}\left(S\right) \times S^2 M_0 \sqrt{M_1^2 + S^2 M_0^2}\left\|V\left(T,X_T\right)\right\|_2$. On the other hand,we write

$$\begin{aligned}
&\left| u_{V^+}^{\varepsilon=1}\left(S,T\right) - \frac{\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\left(V\left(T,X_T\right)\right)^+\right]}{e^{-\int_0^S y_\omega^0 d\omega}\left(1+C_{0,1}\left(S\right)\right)} \right| \\
&= \frac{\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\left(V\left(T,X_T\right)\right)^+\right]}{e^{-\int_0^S y_\omega^0 d\omega}\left(1+C_{0,1}\left(S\right)\right)\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\right]} \left|\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\right] - e^{-\int_0^S y_\omega^0 d\omega}\left(1+C_{0,1}\left(S\right)\right)\right| \\
&\leq \mathscr{C}\left(S\right) \times S^2 M_0 \sqrt{M_1^2 + S^2 M_0^2}\left\|V\left(T,X_T\right)\right\|_2
\end{aligned}$$

where we used Corollary 4.3.11:

$$\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\right] = e^{-\int_0^S y_\omega^0 d\omega}\left(1+C_{0,1}\left(S\right)\right) + \mathscr{C}\left(S\right)\left(S^2 M_0 \sqrt{\left(M_1^2 + S^2 M_0^2\right)}\right)$$

We take $e^{-\int_0^S y_\omega^0 d\omega}\left(1+C_{0,1}\left(S\right)\right)\mathbb{E}\left[\left(V\left(T,X_T\right)\right)^+\right]$ instead of $\mathbb{E}\left[e^{-\int_0^S y_\omega^\varepsilon d\omega}\left(V\left(T,X_T\right)\right)^+\right]$ to complete the proof. The magnitude of the error term is proportional to $S^2 M_0\left(M_1 + SM_0\right)\left\|V\left(T,X_T\right)\right\|_2$. □

Remark 5.4.6. In contrast with Chapter 4, the credit model that we chose here is, by definition of ψ^ε, fitted to the market-implied default probabilities. Therefore, there is no zero-order approximation term representing a convexity adjustment stemming from the volatility of the state process y_t. In other words, when the correlation $\rho^{X,\lambda} = 0$, we obtain $u_{V^+}^{\varepsilon=1}\left(S,T\right) = \mathbb{E}\left[\left(V\left(t,X_t\right)\right)^+\right]$, without any expansion error.

Corollary 5.4.7. *We deduce the following second order approximation for the WWR-adjusted exposure of* (5.4.7)*:* $\forall i \in [\![1,\ldots,M]\!]$,

$$\text{WWR}EPE^\varepsilon\left(T_{i-1},T_i\right) \approx \mathbb{E}\left[\left(V\left(T_i,X_{T_i}\right)\right)^+\right] - \sum_{j=1}^{2} \tilde{C}_{2,j}\left(T_{i-1},T_i\right).Greek^{(j)}\left(T_i,X_{T_i}\right) \tag{5.4.19}$$

where

$$\tilde{C}_{2,1}\left(T_{i-1},T_i\right) = \frac{\mathbb{Q}\left(\tau_C > T_{i-1}\right)\frac{C_{2,1}(T_{i-1},T_i)}{1+C_{0,1}(T_{i-1})} - \mathbb{Q}\left(\tau_C > T_i\right)\frac{C_{2,1}(T_i,T_i)}{1+C_{0,1}(T_i)}}{\mathbb{Q}\left(\tau_C \in]T_{i-1},T_i]\right)} \tag{5.4.20}$$

$$\tilde{C}_{2,1}\left(T_{i-1},T_i\right) = \frac{\mathbb{Q}\left(\tau_C > T_{i-1}\right)\frac{C_{2,2}(T_{i-1},T_i)}{1+C_{0,1}(T_{i-1})} - \mathbb{Q}\left(\tau_C > T_i\right)\frac{C_{2,2}(T_i,T_i)}{1+C_{0,1}(T_i)}}{\mathbb{Q}\left(\tau_C \in]T_{i-1},T_i]\right)} \tag{5.4.21}$$

Then, the resulting unilateral CVA is approximated by

$$UCVA(0) \approx UCVA(0) - LGD_C \sum_{i=1}^{M} D(0,T_i)\,\mathbb{Q}(\tau_C \in]T_{i-1},T_i])\sum_{j=1}^{2} \tilde{C}_{2,j}(T_{i-1},T_i)\,.Greek^{(j)}(T_i,X_{T_i}) \tag{5.4.22}$$

Proof. From Theorem 5.4.5, we sum-up the approximation terms of $u_{V^+}(T_{i-1},T_i)$ and $u_{V^+}(T_i,T_i)$. We then apply respectively (5.4.7), and (5.4.8). □

In Chapter 4, we have also paved the way to a third order approximation formula.

Theorem 5.4.8 (***Third order approximation price formula***)**.** *Suppose that the intensity process* (y_t^ε) *fulfills the regularity conditions of Theorem 4.3.14. Then, one has*

$$u_{V^+}^{\varepsilon=1}(S,T) = \mathbb{E}\left[(V(t,X_t))^+\right] - \sum_{i=1}^{3}\left(\frac{C_{3,i}(S,T)}{1+C_{0,1}(S)}\right).Greek^{(i)}(T,X_T) + Error_3^{\varepsilon=1} \tag{5.4.23}$$

where the weights $C_{i,j}$ *are partially defined in Theorem 5.4.5 and by*

$$C_{3,1}(S,T) = C_{2,1}(S,T) + \left(c_{1,1} - c_{1,2} - c_{1,3} + c_{1,4} + c_{1,5}\right)(S,T) \tag{5.4.24}$$
$$C_{3,2}(S,T) = C_{2,2}(S,T) \tag{5.4.25}$$
$$C_{3,3}(S,T) = \left(c_{3,1} + c_{3,2} - c_{3,3} - c_{3,4} - c_{3,5} + c_{3,6}\right)(S,T) \tag{5.4.26}$$

where

$$c_{1,1}(S,T) = \frac{1}{2}\omega_\rho^T\left(\tilde{\kappa}(.,S)\,e^{-\kappa.}\nu^{(2)}\left(\int_0^. (\tilde{\nu}(s))^2\,ds\right)\right)_0^S \tag{5.4.27}$$
$$c_{1,2}(S,T) = \omega_\rho^T\left(\left(\int_.^S \tilde{\kappa}^2(s,S)\,\nu^{(1)}(s)\,\tilde{\nu}(s)\,ds\right)\tilde{\nu}\right)_0^S \tag{5.4.28}$$
$$c_{1,3}(S,T) = \omega_\rho^T\left(\tilde{\kappa}(.,S)\left(\int_0^. \tilde{\kappa}(s,S)\,\tilde{\nu}^2(s)\,ds\right)\nu^{(1)}\right)_0^S \tag{5.4.29}$$
$$c_{1,4}(S,T) = \frac{1}{3}C_{0,1}(S)\,\omega_\rho^T\left(\tilde{\kappa}(.,S)\,\tilde{\nu}(.)\right)_0^S \tag{5.4.30}$$
$$c_{1,5}(S,T) = \frac{2}{3}\omega_\rho^T\left(\tilde{\kappa}(.,S)\,C_{0,1}(.)\,\tilde{\nu}(.)\right)_0^S \tag{5.4.31}$$
$$c_{3,1}(S,T) = \omega_\rho^T\left(\tilde{\nu},\nu^{(1)},\tilde{\kappa}(.,S)\,\nu^{(1)}\right)_0^S \tag{5.4.32}$$
$$c_{3,2}(S,T) = \omega_\rho^T\left(\tilde{\nu},\tilde{\nu},\tilde{\kappa}(.,S)\,e^{-\kappa.}\nu^{(2)}\right)_0^S \tag{5.4.33}$$
$$c_{3,3}(S,T) = \omega_\rho^T\left(\tilde{\nu},\tilde{\kappa}(.,S)\,\nu^{(1)}\tilde{\kappa}(.,S)\,\tilde{\nu}\right)_0^S \tag{5.4.34}$$
$$c_{3,4}(S,T) = \omega_\rho^T\left(\tilde{\nu},\tilde{\kappa}(.,S)\,\tilde{\nu},\tilde{\kappa}(.,S)\,\nu^{(1)}\right)_0^S \tag{5.4.35}$$
$$c_{3,5}(S,T) = \omega_\rho^T\left(\tilde{\kappa}(.,S)\,\tilde{\nu},\tilde{\nu},\tilde{\kappa}(.,S)\,\nu^{(1)}\right)_0^S \tag{5.4.36}$$
$$c_{3,6}(S,T) = \omega_\rho^T\left(\tilde{\kappa}(.,S)\,\tilde{\nu}(.),\tilde{\kappa}(.,S)\,\tilde{\nu}(.),\tilde{\kappa}(.,S)\,\tilde{\nu}(.)\right)_0^S \tag{5.4.37}$$

In addition, an estimate of the error term $Error_3^{\varepsilon=1}$ *is bounded by*

$$\left|Error_3^{\varepsilon=1}\right| \leq \mathfrak{C}(S) \times S^{\frac{5}{2}} M_0^2 \left(M_1 + \sqrt{S}\right) \|V(T, X_T)\|_2 \tag{5.4.38}$$

where $\mathfrak{C}(S)$ *is a constant that depends on S.*

Proof. The proof is based on an application of Theorem 4.3.14, as well as, similar arguments given in the proof of Theorem (5.4.5). □

Corollary 5.4.9. *We deduce the following third order approximation for the* WWR*-adjusted exposure of* (5.4.7)*:* $\forall i \in [\![1, \ldots, M]\!]$*,*

$$\text{WWR}EPE^{\varepsilon}(T_{i-1}, T_i) \approx \mathbb{E}\left[\left(V(T_i, X_{T_i})\right)^+\right] - \sum_{j=1}^{3} \tilde{C}_{3,j}(T_{i-1}, T_i) . Greek^{(j)}(T_i, X_{T_i}) \tag{5.4.39}$$

where

$$\tilde{C}_{i,j}(T_{i-1}, T_i) = \frac{\mathbb{Q}(\tau_C > T_{i-1}) \frac{C_{i,j}(T_{i-1}, T_i)}{1 + C_{0,1}(T_{i-1}, T_i)} - \mathbb{Q}(\tau_C > T_i) \frac{C_{i,j}(T_i, T_i)}{1 + C_{0,1}(T_i, T_i)}}{\mathbb{Q}(\tau_C \in]T_{i-1}, T_i])} \tag{5.4.40}$$

Then, the resulting unilateral CVA reads

$$UCVA(0) \approx UCVA(0) - LGD_C \sum_{i=1}^{M} D(0, T_i) \mathbb{Q}(\tau_C \in]T_{i-1}, T_i]) \sum_{j=1}^{3} \tilde{C}_{1,j}(T_{i-1}, T_i) . Greek^{(j)}(T_i, X_{T_i}) \tag{5.4.41}$$

CVA formulas are given by (5.4.22) and (5.4.41). DVA formulas are obtained similarly. From a general perspective, we can derive any XVA adjustment with credit *Wrong-way risk* using the *WWR-adjusted* exposures (5.4.19) and (5.4.39). For instance, FVA[2] is obtained by considering forward funding spread $\left(t \mapsto f^{FVA}(0,t)\right)$ instead of the credit default probabilities, and ColVA[3]:

$$FCA(0) = \sum_{i=1}^{M} D(0, T_i) \left(\int_{T_{i-1}}^{T_i} f^{FVA}(0, s)\, ds\right) WRWEPE^{\varepsilon}(T_{i-1}, T_i) \tag{5.4.42}$$

$$ColCA^{\varepsilon}(0) = \sum_{i=1}^{M} D(0, T_i) \left(\int_{T_{i-1}}^{T_i} f^{ColVA}(0, s)\, ds\right) WRWEPE^{\varepsilon}(T_{i-1}, T_i) \tag{5.4.43}$$

where *FCA* and *ColCA* are respectively the funding (resp. collateral) cost adjustments; $\int_{T_{i-1}}^{T_i} f^{FVA}(0,s)\, ds$ (resp. $\int_{T_{i-1}}^{T_i} f^{ColVA}(0,s)\, ds$) their funding cost associated to the risk period $]T_{i-1}, T_i]$.

Finally, we emphasize that XVA sensitivities w.r.t. to credit spreads volatility ν and *WWR* correlation $\rho^{X,\lambda}$ are obtained analytically.

[2]Funding Valuation Adjustment: accounts for discount dissymmetry w.r.t. CSA hedging

[3]Collateral Valuation Adjustment: accounts for collateral remuneration spread over CSA (e.g. Cheapest to Deliver cash)

5.4.3 XVA adjustments with funding *Wrong-way risk*

Wrong-way risk in funding of derivatives is critical as well, due to liquidity requirement in posting collateral on long term trades. In the context of funding, the *WWR* expresses the tendency between the bank's funding spreads and the amount of cash collateral to be posted. For instance, during the credit crisis of 2008, or the sovereign debt turmoil of 2011, major banks faced significantly high funding spreads, as well as large collateral margin calls due to downgrades. Funding *Wrong-way risk* is mainly measured by the liquidity coverage ratio (LCR) and it is calculated as the impact of stress scenario (e.g. rates 100 bps sell off). In a valuation perspective, our expansion approach can be easily adapted in order to handle the *funding* correlation risk.

Without loss of generality, given the instantaneous forward curve of the funding spread, $t \mapsto f^{FVA}(0,t)$, the FVA adjustment is approximated by

$$FVA(0) = \sum_{i=1}^{M} D(0,T_i)\left(e^{-\int_0^{T_{i-1}} f^{FVA}(0,s)ds} - e^{-\int_0^{T_i} f^{FVA}(0,s)ds}\right) EPE(T_i) \tag{5.4.44}$$

where the funding spread is approximated at a first order by the difference of spot discounts factors

$$\int_{T_{i-1}}^{T_i} f^{FVA}(0,s)\,ds \approx e^{-\int_0^{T_{i-1}} f^{FVA}(0,s)ds} - e^{-\int_0^{T_i} f^{FVA}(0,s)ds} \approx D(0,T_{i-1}) - D(0,T_i) \tag{5.4.45}$$

We assume now the funding spread short rate $r_t^{FVA} = f^{FVA}(t,t)$ is stochastic and is driven by the process z_t^{ε} such that

$$r_t^{FVA,\varepsilon} = \phi^{\varepsilon}(t) + z_t^{\varepsilon} \tag{5.4.46}$$

where z_t^{ε} satisfies an SDE analogous to (5.3.5) and ϕ^{ε} is a deterministic shift function designed to fit the initial forward curve.

$$\exp\left(-\int_0^t \phi^{\varepsilon}(s)\,ds\right) = \frac{e^{-\int_0^t f^{FVA}(0,s)ds}}{\mathbb{E}\left[\exp\left(-\int_0^t z_s^{\varepsilon} ds\right)\right]} \tag{5.4.47}$$

In particular, the *funding WWR* is determined by a correlation vector $\rho^{X,r^{FVA}}$. Finally, we defined, the funding *WWR-adjusted* exposure by

$$FWRWEPE^{\varepsilon}(T_{i-1},T_i) = \frac{D(0,T_{i-1})\,u^{\varepsilon}_{V^+,FVA}(T_{i-1},T_i) - D(0,T_{i-1})\,u^{\varepsilon}_{V^+,FVA}(T_{i-1},T_i)}{D^{FVA}(0,T_{i-1}) - D^{FVA}(0,T_i)} \tag{5.4.48}$$

such that

$$FVA(0) = \sum_{i=1}^{M} D(0,T_i)\, e^{\int_{T_{i-1}}^{T_i} f^{FVA}(0,s)ds} FWWREPE(T_{i-1},T_i) \tag{5.4.49}$$

$u^{\varepsilon}_{V^+,FVA}(T_{i-1},T_i)$ is the risky valuation function analog to (5.4.4):

$$u^{\varepsilon}_{V^+,FVA} : (s,t) \mapsto \frac{\mathbb{E}\left[e^{-\int_0^s z^{\varepsilon}_{\omega} d\omega}\left(V(t,X_t)\right)^+\right]}{\mathbb{E}\left[e^{-\int_0^s z^{\varepsilon}_{\omega} d\omega}\right]} \tag{5.4.50}$$

The setup is identical to the credit *WWR* framework we developed earlier. We emphasize the analogy between credit and funding *Wrong-way risk* adjusted exposures in (5.4.7) and (5.4.48). Pricing formulas (5.4.22) and (5.4.41) can then be applied. The pricing approach can be adapted to be generically used to assess any XVA metric of the form

$$XVA(0) = \sum_{i=1}^{M} D(0,T_i) \left(e^{-\int_0^{T_{i-1}} f^{XVA}(0,s)ds} - e^{-\int_0^{T_i} f^{XVA}(0,s)ds} \right) EPE(T_i) \tag{5.4.51}$$

and to quantify *WWR risk* add-on due to correlation between the exposure and the XVA cost.

Regarding to the possible interactions between DVA (banks credit) and FVA (funding), it is interesting to superimpose funding and credit *Wrong-way risk* and to analyze their mutual impacts. This can be achieved by a doubly-stochastic process for a joint diffusion of credit intensity and funding spreads. We believe that our expansion approach is suited for deriving convenient pricing expressions. We leave this topic as a future research project.

5.5 The expected exposure and its Greeks

5.5.1 Monte Carlo valuation

To evaluate the expected exposure at a given time t, Monte Carlo engines are usually used because of their flexibility for pricing new payoffs, such as an option on the portfolio's mark-to-market. The Monte Carlo simulations are evolved to the exposure date, and then the portfolio value is calculated on every path, by calling the pricer of each underlying trade. The critical constraint about the Monte Carlo approach is performance. Some long-term complex trades can require minutes to be valuated (with their appropriate complex models). Monte Carlo within a Monte Carlo is clearly something that should be avoided. Sp should be PDEs within Monte Carlo.

Least-Square Monte Carlo techniques are likely to be the only viable option available. The approach is based on the principle that at each time step of the Monte Carlo simulations, a function can be identified, providing the value of the derivative for the state variable past values of the current scenario. $V(t,X_t)$ is then computed as the conditional expectation of the sum of future cash-flows. Greeks are computed through a finite difference scheme, or using more advanced techniques such as algorithmic differentiation ([LSG15]). We refer to Chapter 2 for a brief survey about numerical methods applied to the calculation of the expected exposure and its Greeks.

5.5.2 Expected exposure under a normal approximation

It is also useful to be able to estimate the expected exposure and its Greeks outside of a heavy Monte Carlo system. Analytical formulas for the expected exposure are derived in the case when the future mark-to-market are normally distributed. An analog approach has already been followed by [PR10] and [Gre12]. The expected exposure can then be analytically derived in a Bachelier setup.

From an operational point of view, the normal distribution of future mark-to-market is an admissible assumption for linear or, to some extent, vanilla trades. This distribution of future

prices is based on assumptions that we exhibit by going back the portfolio replication pricing equations. In fact, the variation of the price process $V(t,X_t)$ is driven by the SDE

$$dV(t,X_t) = \sum_{i=1}^{n} \frac{\partial V(t,X_t+x)}{\partial x_i}|_{x=0_{R^n}} dX_t^i + \left(\partial_t V(t,X_t)\,dt + \frac{1}{2} \frac{\partial^2 V(t,X_t+x)}{\partial x_i \partial x_j}|_{x=0_{R^n}} d\left\langle X^i, X^j \right\rangle_t \right) \tag{5.5.1}$$

The perfect-replication approach is based on a hedging portfolio R that is dynamically re-adjusted and satisfies

$$\begin{cases} dR(t,X_t) & = \sum_{i=1}^{n} H_i(t,X_t)\,dX_t^i \\ d\left(V(t,X_t) - R(t,X_t)\right) & = r\left(V(t,X_t) - R(t,X_t)\right)dt \end{cases} \tag{5.5.2}$$

where r is a reference risk-free rate, corresponding to the rate of return of the residual position $(V-R)$. The composition of replication portfolio is related to the deltas w.r.t. to each component X_t^i taken locally at (t,X_t), i.e.

$$H_i(t,X_t) = \frac{\partial V(t,X_t+x)}{\partial x_i}|_{x=0_{R^n}} \tag{5.5.3}$$

$V(0,X_0) = R(0,X_0)$ ensures that the hedge is perfect, and hence leads to $V(t,X_t) = R(t,X_t) = \sum_{i=1}^{n} H_i(t,X_t) X_t^i$. H_i is a surface of future first order Greeks that are usually difficult to compute in a portfolio pricing framework. Therefore, the standard approximation consists of substituting H_t^i by its expectation. Namely, we define

$$H_i(0,t) = \mathbb{E}\left[H_i(t,X_t)\right] \tag{5.5.4}$$

$H^i(0,t)$ refers to the forward sensitivity of the future price V at t. This expected sensitivity is rather less complex to calculate. One can push the simplification a step further by assuming that the sensitivity $H^i(0,t)$ is a deterministic function of the spot sensitivity $H^i(0,0)$, maturity T and time t:

$$H^i(0,t) = \mathscr{A}\left(t,T,H^i(0,0)\right) \tag{5.5.5}$$

The *aging* function $\mathscr{A}$ is defined empirically, although examples can be easily derived from linear products such as vanilla swaps.

Under the forward sensitivity assumption, $V(t,X_t)$ is approximated by

$$V(t,X_t) \approx \sum_{i=1}^{n} H^i(0,t) X_t^i \tag{5.5.6}$$

The approximation is exact when the underlying portfolio is exclusively composed of linear instrument ($H^i(0,t) = H_t^i$). In practice the approach remains admissible for an optional portfolio as a first order approximation. Finally, it easily follows that $V(t,X_t)$ is normally distributed if X_t is a Gaussian vector.

5.5.2.1 Expected exposure valuation

Based on a approximate replication arguments, we assume that for $t \in \mathscr{T}_M$,

$$V(t,X_t) = \sum_{j=1}^{n} H^j(0,t) X_t^j = H(0,t).X_t \tag{5.5.7}$$

We assume that X_t is a Gaussian random vector. $V(t,X_t)$ is then normally distributed with a mean and a standard deviation given by

$$\mu_V(t) = H(0,t).\mathbb{E}[X_t] \tag{5.5.8}$$

$$\sigma_V(t) = \frac{1}{\sqrt{t}}\sqrt{\sum_{j,j'=1}^{n} H^j(0,t)H^{j'}(0,t)\mathbb{E}\left[\left\langle X^j,X^{j'}\right\rangle_t\right]} \tag{5.5.9}$$

where $\mathbb{E}[X_t]$ and $\mathbb{E}\left[\left\langle X^j,X^{j'}\right\rangle_{T_i}\right]$ have is given from (5.3.2). In fact, (5.3.2) has an explicit solution that corresponds to

$$X_t = e^{\int_0^t \Theta(s)ds}\left(X_0 + \int_0^t e^{-\int_0^s \Theta(u)du}\Phi(s)\,ds\right) + e^{\int_0^t \Theta(s)ds}\int_0^t e^{-\int_0^s \Theta(u)du}\Sigma(s)\,dB_s \tag{5.5.10}$$

where the integral $\int_0^t \Theta(s)\,ds$ is calculated element-wisely and $e^{\int_0^t \Theta(s)ds}$ refers to the matrix exponential of $\int_0^t \Theta(s)\,ds$. It readily follows that

$$\mathbb{E}[X_t] = e^{\int_0^t \Theta(s)ds}\left(X_0 + \int_0^t e^{-\int_0^s \Theta(u)du}\Phi(s)\,ds\right) \tag{5.5.11}$$

In addition, each component X_t^j, $j \in [\![1,n]\!]$, satisfies

$$\begin{aligned}
[X_t]_j &= \mathbb{E}[X_t]_j + \left[e^{\int_0^t \Theta(s)ds}\int_0^t \tilde{\Sigma}(0,t)\,dB_s\right]_j \\
&= \mathbb{E}[X_t]_j + \sum_{k=1}^{n}\left[e^{\int_0^t \Theta(s)ds}\right]_{j,k}\left[\int_0^t \tilde{\Sigma}(0,s)\,dB_s\right]_k \\
&= \mathbb{E}[X_t]_j + \sum_{k=1}^{n}\sum_{l=1}^{d}\left[e^{\int_0^t \Theta(s)ds}\right]_{j,k}\int_0^t \left[\tilde{\Sigma}(0,s)\right]_{k,l} d[B_s]_l
\end{aligned}$$

Hence, for $j,j' \in [\![1,n]\!]$, one has

$$\mathbb{E}\left[\left\langle X^j,X^{j'}\right\rangle_t\right] = \sum_{k,k'=1}^{n}\sum_{l,l'=1}^{d}\left[e^{\int_0^t \Theta(s)ds}\right]_{j,k}\left[e^{\int_0^t \Theta(s)ds}\right]_{j',k'}\int_0^t \left[\tilde{\Sigma}(0,s)\right]_{k,l}\left[\tilde{\Sigma}(0,s)\right]_{k',l'}\rho_{l,l'}^{X,X}\,ds \tag{5.5.12}$$

From Equations (5.5.11) and (5.5.12), we compute the values of $\mu_V(t)$ and $\sigma_V(t)$. $\mu_V(t) = \mathbb{E}[V(t,X_t)]$ can also be refereed to as the forward value of the future mark-to-market of time t. It can be obtained directly from pricing systems by aggregating for the portfolio's underlying trades.

We recall the Bachelier formulas for Call and Put vanilla options with strike K, maturity t, forward μ and volatility σ:

$$C^B(t,\mu,\sigma,K) \triangleq (\mu-K)N(d(\mu-K,\sigma)) + \sigma\sqrt{t}n(d(\mu-K,\sigma)) \tag{5.5.13}$$

$$P_V^B(t,\mu,\sigma,K) \triangleq -(\mu-K)N(-d(\mu-K,\sigma)) + \sigma\sqrt{t}n(d(\mu-K,\sigma)) \tag{5.5.14}$$

$$d(m,\sigma) \triangleq \frac{m}{\sigma} \tag{5.5.15}$$

with $n(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ and $N(x) = \int_{-\infty}^{x} n(u)\,du$. $d(m,\sigma)$ refers to the absolute *moneyness* by volatility unit. It readily follows that the expected positive and negative exposures satisfy

$$EPE(t) \triangleq \mathbb{E}\left[(V(t,X_t))^+\right] = C^B(t,\mu_V(t),\sigma_V(t),0) \tag{5.5.16}$$

$$ENE(t) \triangleq \mathbb{E}\left[(-V(t,X_t))^+\right] = P^B(t,\mu_V(t),\sigma_V(t),0) \tag{5.5.17}$$

Similar formulas can be derived for the expected exposure in presence of collateral margin threshold H [4]:

$$\begin{aligned} EPE(t,H) &\triangleq \mathbb{E}\left[(V(t,X_t))^+\right] - \mathbb{E}\left[(V(t,X_t)-H)^+\right] \\ &\triangleq C^B(t,\mu_V(t),\sigma_V(t),0) - C^B(t,\mu_V(t),\sigma_V(t),H) \end{aligned} \tag{5.5.18}$$

$$\begin{aligned} ENE(t,H) &= \mathbb{E}\left[(H-V(t,X_t))^+\right] - \mathbb{E}\left[(-V(t,X_t))^+\right] \\ &= P^B(\mu_V(t),\sigma_V(t);t,H) - P^B(\mu_V(t),\sigma_V(t);t,0) \end{aligned} \tag{5.5.19}$$

The expected exposures are computed analytically at each observation date each observation date $t \in \mathscr{T}_M$. The zero-order term $UCVA^0$ is computed then using (5.4.1).

5.5.2.2 Expected exposure Greeks

A significant advantage of the Gaussian framework presented above is the ability to derive closed formulas of the exposure's sensitivities. We first recall that the sensitivities involves in approximation formulas (5.4.22) and (5.4.41) are expressed w.r.t to the terminal variable. Proposition 4.5.1 allows to transform standard Greeks w.r.t. X_0 into $Greek^{(1)}(X_t)$, $Greek^{(2)}(X_t)$ and $Greek^{(3)}(X_t)$, as required in our approximation formulas. The resulting expressions correspond to

$$\left[Greek^{(1)}_{V^+}(t,X_t)\right]_i = \sum_{l=1}^{n} \frac{\partial \mathbb{E}\left[(V(t,X_t))^+\right]}{\partial [X_0]_l} \left[e^{-\int_0^t \Theta(s)ds}\right]_{l,i} \tag{5.5.20}$$

$$\left[Greek^{(2)}_{V^+}(t,X_t)\right]_{i,j} = \sum_{l,p=1}^{n} \frac{\partial^2 \mathbb{E}\left[(V(t,X_t))^+\right]}{\partial [X_0]_l \partial [X_0]_p} \left[e^{-\int_0^t \Theta(s)ds}\right]_{l,i} \left[e^{-\int_0^t \Theta(s)ds}\right]_{p,j} \tag{5.5.21}$$

$$\left[Greek^{(3)}_{V^+}(t,X_t)\right]_{i,j,k} = \sum_{l,p,q=1}^{n} \frac{\partial^3 \mathbb{E}\left[(V(t,X_t))^+\right]}{\partial [X_0]_l \partial [X_0]_p \partial [X_0]_q} \left[e^{-\int_0^t \Theta(s)ds}\right]_{l,i} \left[e^{-\int_0^t \Theta(s)ds}\right]_{p,j} \left[e^{-\int_0^t \Theta(s)ds}\right]_{q,k} \tag{5.5.22}$$

In addition, the sensitivities of the mark-to-market expectation μ_V and volatility σ_V w.r.t. X_0 are given by

$$\begin{cases} \frac{\partial \mu_V(t)}{\partial X_0^l} & = \sum_{i=1}^{n} H_i(0,t) \left[e^{\int_0^t \Theta(s)ds}\right]_{i,l} = \left[e^{\int_0^t \Theta^t(s)ds} H(0,t)\right]_l \\ \frac{\partial \sigma_V(t)}{\partial X_0^l} & = 0 \end{cases} \tag{5.5.23}$$

where we have used the transpose property of matrix exponential $\left(e^A\right)^t = e^{A^t}$. Higher order derivatives w.r.t. X_0 are identically nil.

[4] Counterparties post collateral when the exposure to their respective defaults exceeds H

Remark 5.5.1. We consciously neglect the sensitivity of $H^i(0,t)$ w.r.t. X_0^l. Otherwise, $\frac{\partial \mu_V(t)}{\partial X_0^l}$ should embed second order Greeks of V.

The expected exposure Greeks w.r.t. X_0 are derived analytically in terms of Bachelier standard sensitivities, as well as the sensitivities of the forward mark-to-market. For $l,p,q \in [\![1,n]\!]$, one has

- First order sensitivity $\frac{\partial EPE(t)}{\partial X_0^l}$:

$$\frac{\partial EPE(t)}{\partial X_0^l} = \frac{\partial \mu_V(t)}{\partial [X_0]_l} \Delta_C^B(t, \mu_V(t), \sigma_V(t), 0) \tag{5.5.24}$$

- Second order sensitivity $\frac{\partial^2 EPE(t)}{\partial X_0^l \partial X_0^p}$:

$$\frac{\partial^2 EPE(t)}{\partial X_0^l \partial X_0^p} = \left(\frac{\partial \mu_V(t)}{\partial [X_0]_l} \frac{\partial \mu_V(t)}{\partial [X_0]_p} \right) \Gamma_C^B(t, \mu_V(t), \sigma_V(t), 0) \tag{5.5.25}$$

- Third order sensitivity $\frac{\partial^3 EPE(t)}{\partial X_0^l \partial X_0^p \partial X_0^q}$:

$$\frac{\partial^3 EPE(t)}{\partial X_0^l \partial X_0^p \partial X_0^q} = \left(\frac{\partial \mu_V(t)}{\partial [X_0]_l} \frac{\partial \mu_V(t)}{\partial [X_0]_p} \frac{\partial \mu_V(t)}{\partial [X_0]_q} \right) \mathscr{S}_C^B(\mu_V(t), \sigma_V(t); t, 0) \tag{5.5.26}$$

Δ^B, Γ^B and $\mathscr{S}^B$ denote respectively the *Delta*, *Gamma* and *Speed* in the Bachelier setting. Their expressions are given in Appendix B. Finally, by regrouping the chain rule components, we end-up with

$$\left[Greek_{V^+}^{(1)}(t, X_t) \right]_i = \left(\sum_{l=1}^n \omega_{i,l}(t) \right) \Delta_C^B(t, \mu_V(t), \sigma_V(t), 0) \tag{5.5.27}$$

$$\left[Greek_{V^+}^{(2)}(t, X_t) \right]_{i,j} = \left(\sum_{l,p=1}^n \omega_{i,l}(t) \omega_{j,p}(t) \right) \Gamma_C^B(t, \mu_V(t), \sigma_V(t), 0) \tag{5.5.28}$$

$$\left[Greek_{V^+}^{(3)}(t, X_t) \right]_{i,j,k} = \left(\sum_{l,p,q=1}^n \omega_{i,l}(t) \omega_{j,p}(t) \omega_{k,q}(t) \right) \mathscr{S}_C^B(t, \mu_V(t), \sigma_V(t), 0) \tag{5.5.29}$$

with

$$\omega_{i,l}(t) = \left[e^{\int_0^t \Theta^t(s) ds} H(0,t) \right]_l \left[e^{-\int_0^t \Theta(s) ds} \right]_{l,i} \tag{5.5.30}$$

Similar Greeks are easily derived for the expected negative exposure, and including additional collateral features such as margin thresholds.

5.6 Numerical Experiments

We devote this section to numerical experiments in order to illustrate the impact of the intensity volatility and the WWR correlation on the expected exposure profiles. We refer to the last chapter for the error analysis related to the expansion approach used to derive Theorems 5.4.5 and 5.4.8. We showed that the 2^{nd} order approximation is accurate enough for financial applications, and we underline that it requires much fewer terms to compute. In the sequel, we report CVA and DVA results using the 3^{rd} order approximation.

We focus on counterparty risk for a netted portfolio of interest rates swaps in multiple currencies. This represents a standard setting for counterparty risk analysis and General *Wrong-way risk* (see [BMP13] and [CGGN13]). Long-termed swaps from other asset classes such as Oil swaps or Equity return swaps are also analyzed in [BMP13] and can be treated similarly. Conversely, a more specific approach has to be adapted for Credit default swaps (CDS) since the future value $V(t,X_t)$ depends on the early termination event triggered by the default of the CDS underlying.

Notations and valuation of IR swaps

An interest-rate swap between counterparties I and C is an exchange trade where fixed/floating payments are performed at times $T_{a+1},\ldots,T_b$. T_a and T_b are respectively the start and end dates of the contracts. We denote by $D(t,T_i)$ the risk-free zero coupon bond price at time t for maturity T_i, and β_i the year fraction between T_{i-1} and T_i.

From the point of view of I, a standard payer swap requires to I to pay a fixed rate K against receiving a floating rate L tied to an interest rate index such as the Libor which is the reference rate of monetary policies and mortgages related to US dollar, GB pound, Swiss franc, ect. The Euribor, compiled by the European Banking Federation, is the reference rate for Euro denominated interest rate products. Cashflow exchanges occur between I and C until a default time of one counterparty $\tau=\tau_I\wedge\tau_C$, or the final maturity T_b.

Single IR Swap

The discounted sum of the exchanged cash-flows is

$$\sum_{i=a+1}^{b} D(t,T_i)\,\beta_i\,(L(T_{i-1},T_i)-K)$$

with a unit notional. The forward swap rate rendering the contract fair is determined using

$$K=S_{\gamma(t),b}(t)=\sum_{i=a+1}^{b}1_{t\leq T_i}w_i(t)\,F_i(t)$$

with

$$\begin{cases} F_i(t) & =\mathbb{E}_t\left[L(T_{i-1},T_i)\right]\\ w_i(t) & =\frac{D(t,T_i)\beta_i}{\mathscr{A}_{\gamma(t),b}(t)}\\ \mathscr{A}_{\gamma(t),b}(t) & =\sum_{j=a+1}^{b}1_{t\leq T_j}D\left(t,T_j\right)\beta_j \end{cases}$$

$\gamma(t)$ refers to the index of the latest accrual period start date before t. We assume here that fixed and floating rates are paid simultaneously, and that the floating rate L is determined at T_{i-1} and paid naturally at T_i. This contrast with In-arrears swaps where floating cashflows are determined using the interest rate fixed and paid at the same payment date. The price in t of this swap is given by

$$\begin{aligned} V\left(t, S_{\gamma(t),b}(t) - K\right) &= \sum_{k=a+1}^{b} 1_{t \le T_i} D(t,T_i)\, \beta_i \left(\mathbb{E}_t\left[L(T_{i-1},T_i)\right] - K\right) \\ &= \mathscr{A}_{\gamma(t),b}(t)\left(S_{\gamma(t),b}(t) - K\right) \end{aligned}$$

Under the forward swap measure $\mathbb{Q}^{a,b}$, $\left(S_{a,b}(t)\right)_{t \le T_a}$ evolves according to a martingale. We assume $S_{a,b}(t)$ follows a normal dynamics with $\sigma^B_{a,b}(t)$ its average absolute volatility given by market quotation of European Swaptions at time $t \le T_a$. The superscript B is an indication to Bachelier's implied volatility. The swap measure $\mathbb{Q}^{a,b}$ is directly associated to the annuity $\mathscr{A}_{a,b}(t)$ with the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{a,b}}{d\mathbb{Q}}|_{\mathscr{F}_t} = \frac{B(0)}{B(t)} \frac{\mathscr{A}_{a,b}(t)}{\mathscr{A}_{a,b}(0)}$$

where B is the cash numeraire associated to risk neutral measure $\mathbb{Q}$. From a practical perspective, numerical studies show that the variability of the annuity $\mathscr{A}$ is small compared to the variability of the swap rate (see Chapter 6 of [BM07]). By freezing $\mathscr{A}$ to its forward value seen from 0, we assume that under the usual $\mathbb{Q}$ usual measure, $S_{a,b}(t)$ follows a normal distribution with mean the forward swap rate and volatility $\sigma^B_{a,b}(t)$. With regard to the Gaussian framework presented previously, we set $n = 1$, $X_t = S_{a,b}(t) - K$, and

$$\begin{cases} \mathscr{A}_{\gamma(t),b}(0,t) & = \sum_{i=a+1}^{b} 1_{t \le T_i} \beta_i D(0,T_i) \\ H(0,t) & = \mathscr{A}_{\gamma(t),b}(0,t) \\ \mu_V(t) & = \mathscr{A}_{\gamma(t),b}(0,t)\left(S_{\gamma(t),b}(0,t) - K\right) \end{cases}$$

$\sigma_V(t)$ is obtained by interpolation in the Swaption ATM volatility surface.

Single-currency IR Swaps portfolio

We consider the case of dealing with a portfolio of N IR swaps facing a single counterparty and under a netting agreement. The total portfolio discounted payoff at time t may be written as

$$\sum_{j=1}^{N} \phi_j \left(\sum_{i=a_j+1}^{b_j} 1_{T_i \ge t} D(t,T_i)\, \beta_i \left(L(T_{i-1},T_i) - K_j\right) \right)$$

ϕ_j is the receiver/payer fixed rate flag with respective values $\{-1,1\}$, K_j are the fixed rates, a_j and b_j are resp. the start and end dates. We denote by $S_{\gamma_j(t),b_j}(t)$ the fair swap rate of the swap j satisfying

$$S_{\gamma_j(t),b_j}(t) = \sum_{i=a_j+1}^{b_j} 1_{t \le T_i} w_{i,j}(t) F_i(t); \;\; w_{i,j}(t) = \frac{D(t,T_i)\,\beta_i}{\mathscr{A}_{\gamma(t),b_j}(t)}$$

The portfolios values at time t is given by

$$V\left(t,\left(S_{\gamma_j(t),b_j}(t)-K_j\right)_j\right)=\sum_{j=1}^{N}\phi_j\mathscr{A}_{\gamma_j(t),b_j}(t)\left(S_{\gamma_j(t),b_j}(t)-K_j\right)$$

For the portfolio analysis, we set $X_t=\left(S_{\gamma_j(t),b_j}(t)-K_j\right)_j$, and

$$\begin{cases} H_j(0,t) & =\phi_j\mathscr{A}_{\gamma_j(t),b_j}(0,t)=\sum_{i=a_j+1}^{b_j}1_{t\leq T_i}D(0,T_i)\beta_i \\ \mu_V(t) & =\sum_{j=1}^{N}\phi_j\mathscr{A}_{\gamma_j(t),b_j}(t)\left(S_{\gamma_j(t),b_j}(0,t)-K_j\right) \\ \sigma_V(t) & =\sqrt{\sum_{j,j'=1}^{N}\phi_j\phi_{j'}\mathscr{A}_{\gamma_j(t),b_j}(0,t)\mathscr{A}_{\gamma_{j'}(t),b_{j'}}(0,t)\sigma^B_{\gamma_j(t),b_j}(t)\sigma^B_{\gamma_{j'}(t),b_{j'}}(t)\rho\left(t,S_{\gamma_j(t),b_j},S_{\gamma_{j'}(t),b_{j'}}\right)} \end{cases}$$

$\rho\left(t,S_{\gamma_j(t),b_j},S_{\gamma_j(t),b_{j'}}\right)$ denotes the terminal correlation between swap rates $S_{\gamma_j(t),b_j},S_{\gamma_{j'}(t),b_{j'}}$ at time t.

Two-currencies IR Swaps portfolio

We finally consider the case of a portfolio of IR swaps labeled in two currencies, i.e. the Euro and US dollar. The portfolio's value at time t denominated in EUR is given by

$$\begin{aligned} &V\left(t,\left(\left(S^{EUR}_{\gamma_j(t),b_j}(t)-K^{EUR}\right)_{j=1,N_1},\left(S^{USD}_{\gamma_j(t),b_j}(t)-K^{USD}\right)_j,X_t^{USDEUR}\right)\right) \\ &=\sum_{j=1}^{N_1}\phi_j^{EUR}\mathscr{A}^{EUR}_{\gamma_j(t),b_j}(t)\left(S^{EUR}_{\gamma_j(t),b_j}(t)-K_j^{EUR}\right)+X_t^{USDEUR}\sum_{j=1}^{N_2}\phi_j^{USD}\mathscr{A}^{USD}_{\gamma_j(t),b_j}(t)\left(S^{USD}_{\gamma_j(t),b_j}(t)-K_j^{USD}\right) \end{aligned}$$

For the portfolio analysis, we set $X_t=\left(\left(S^{EUR}_{\gamma_j(t),b_j}(t)-K^{EUR}\right)_{j=1,N_1},\left(S^{USD}_{\gamma_j(t),b_j}(t)-K^{USD}\right)_j,X_t^{USDEUR}\right)$. One has

$$V(t,X_t)=\sum_{j=1}^{N_1}H_j^{EUR}(0,t)\left(S^{EUR}_{\gamma_j(t),b_j}(t)-K^{EUR}\right)+X_t^{USDEUR}\sum_{j=1}^{N_2}H_j^{USD}(0,t)\left(S^{USD}_{\gamma_j(t),b_j}(t)-K^{USD}\right)$$

with $H_j^{EUR}(0,t)=\phi_j^{EUR}\mathscr{A}^{EUR}_{\gamma_j(t),b_j}(0,t)$ and $H_j^{USD}(0,t)=\phi_j^{USD}\mathscr{A}^{USD}_{\gamma_j(t),b_j}(0,t)$. Let us also define

$$H^{USDEUR}(0,t)=\sum_{j=1}^{N_2}\phi_j^{USD}H_j^{USD}(0,t)\left(S^{USD}_{\gamma_j(t),b_j}(0,t)-K^{USD}\right)$$

the forward value of the foreign (USD) leg and correspond to the portfolio's value delta w.r.t. X_t^{USDEUR}. The portfolio's volatility involved the EUR and USD swap rates volatilities, the USDEUR FX rate volatility, intra-currency swap rates correlations, cross-currency swap rates

correlations, and finally, the hybrid FX-Rates correlations:

$$\begin{aligned}
\sigma_V^2(t) = &\sum_{j,j'=1}^{N^{EUR}} H_j^{EUR}(0,t)\, H_{j'}^{EUR}(0,t)\, \sigma_{\gamma_j(t),b_j}^{EUR,B}(t)\, \sigma_{\gamma_{j'}(t),b_{j'}}^{EUR,B}(t)\, \rho\left(t, S_{\gamma_j(t),b_j}^{EUR}, S_{\gamma_{j'}(t),b_{j'}}^{EUR}\right) \\
&+ \sum_{j,j'=1}^{N^{USD}} H_j^{USD}(0,t)\, H_{j'}^{USD}(0,t)\, \sigma_{\gamma_j(t),b_j}^{USD,B}(t)\, \sigma_{\gamma_{j'}(t),b_{j'}}^{USD,B}(t)\, \rho\left(t, S_{\gamma_j(t),b_j}^{USD}, S_{\gamma_{j'}(t),b_{j'}}^{USD}\right) \\
&+ \sum_{j=1}^{N^{EUR}} \sum_{j'=1}^{N^{USD}} H_j^{EUR}(0,t)\, H_{j'}^{USD}(0,t)\, \sigma_{\gamma_j(t),b_j}^{EUR,B}(t)\, \sigma_{\gamma_{j'}(t),b_{j'}}^{USD,B}(t)\, \rho\left(t, S_{\gamma_j(t),b_j}^{EUR}, S_{\gamma_{j'}(t),b_{j'}}^{USD}\right) \\
&+ H^{USDEUR}(0,t)\, \sigma^{USDEUR}(t) \sum_{j=1}^{N^{EUR}} H_j^{EUR}(0,t)\, \sigma_{\gamma_j(t),b_j}^{EUR,B}(t)\, \rho\left(t, X_t^{USDEUR}, S_{\gamma_j(t),b_j}^{EUR}\right) \\
&+ H^{USDEUR}(0,t)\, \sigma^{USDEUR}(t) \sum_{j=1}^{N^{USD}} H_j^{USD}(0,t)\, \sigma_{\gamma_j(t),b_j}^{USD,B}(t)\, \rho\left(t, X_t^{USDEUR}, S_{\gamma_j(t),b_j}^{USD}\right) \\
&+ \left(H^{USDEUR}(0,t)\, \sigma^{USDEUR}(t)\right)^2
\end{aligned}$$

with σ^{USDEUR} being the normal volatility for USD/EUR FX rate.

Market data

The EUR and USD zero-coupon spot rates are reported in Table 5.6.1. Tables 5.6.2 give the ATM swaption normal volatilities for respectively the EUR and USD floating rates, with expiry date t and tenor b.

Term	Rate	Term	Rate
3*m*	-0.50%	**9***y*	0.96%
6*m*	-0.49%	**10***y*	1.11%
9*m*	-0.48%	**11***y*	1.24%
1*y*	-0.46%	**12***y*	1.35%
2*y*	-0.37%	**13***y*	1.45%
3*y*	-0.21%	**14***y*	1.54%
4*y*	-0.02%	**15***y*	1.62%
5*y*	0.19%	**20***y*	1.91%
6*y*	0.40%	**25***y*	2.08%
7*y*	0.60%	**30***y*	2.20%
8*y*	0.79%		

(a) EUR spot rates

Term	Rate	Term	Rate
3*m*	0.86%	**9***y*	2.30%
6*m*	1.00%	**10***y*	2.36%
9*m*	1.06%	**11***y*	2.40%
1*y*	1.11%	**12***y*	2.43%
2*y*	1.32%	**13***y*	2.47%
3*y*	1.51%	**14***y*	2.51%
4*y*	1.70%	**15***y*	2.55%
5*y*	1.88%	**20***y*	2.73%
6*y*	2.03%	**25***y*	2.87%
7*y*	2.17%	**30***y*	3.00%
8*y*	2.23%		

(b) USD spot rates

Table 5.6.1: Zero-coupon spot rates, continuously compounded, Act/360

$a\downarrow/b\rightarrow$	**1**y	**2**y	**3**y	**4**y	**5**y	**10**y	**20**y
1m	45	46	52	59	66	103	117
3m	50	48	53	61	70	104	114
6m	49	52	56	65	75	105	115
1y	51	56	65	74	83	106	113
2y	71	77	84	90	95	105	107
3y	93	95	99	100	102	105	105
4y	101	103	106	104	104	105	102
5y	106	106	109	106	106	105	101
10y	101	101	100	97	96	93	88

(a) EUR volatilities

$a\downarrow/b\rightarrow$	**1**y	**2**y	**3**y	**4**y	**5**y	**10**y	**20**y
1m	75	76	82	89	96	133	147
3m	80	78	83	91	100	134	144
6m	79	82	86	95	105	135	145
1y	81	86	95	104	113	136	143
2y	101	107	114	120	125	135	137
3y	123	125	129	130	132	135	135
4y	131	133	136	134	134	135	132
5y	136	136	139	136	136	135	131
10y	131	131	130	127	126	123	118

(b) USD volatilities

Table 5.6.2: Market ATM normal Swaption volatilities in bps with expiry date a and tenor b

The correlation function $\rho\left(t, S_{\gamma_j(t),b_j}, S_{\gamma_{j'}(t),b_{j'}}\right)$ refers the terminal correlation at time t between swap rates of tenors $b_j-\gamma_j(t)$ and $b_j-\gamma_{j'}(t)$. These correlations are usually deduced from market quotes of CMS spread options [5]. In the sequel, these correlations are considered constant w.r.t to the expiry t. We use the correlation surfaces given in Table 5.6.3 with a flat interpolation/extrapolation.

$\tau\downarrow/\tau'\rightarrow$	**1**y	**3**y	**5**y	**10**y	**15**y	**20**y
1y	100%	87%	76%	53%	38%	26%
3y	87%	100%	87%	61%	43%	30%
5y	76%	87%	100%	70%	50%	35%
10y	53%	61%	70%	100%	70%	50%
15y	38%	43%	50%	70%	100%	70%
20y	26%	30%	35%	50%	70%	100%

(a) EUR CMS correlations

$\tau\downarrow/\tau'\rightarrow$	**1**y	**3**y	**5**y	**10**y	**15**y	**20**y
1y	100%	82%	67%	41%	25%	15%
3y	82%	100%	82%	50%	30%	18%
5y	67%	82%	100%	61%	37%	22%
10y	41%	50%	61%	100%	61%	37%
15y	25%	30%	37%	61%	100%	61%
20y	15%	18%	22%	37%	61%	100%

(b) USD CMS correlations

Table 5.6.3: Market CMS correlations, with tenors τ and τ'

[5] Standard CMS spread options are either caps or floors. There products are provided (to a certain extent of liquidity) for EUR, USD and JPY currencies

On the other side, we consider a constant value for the correlation $\rho\left(t, S^{EUR}_{\gamma_j(t),b_j}, S^{USD}_{\gamma_{j'}(t),b_{j'}}\right) = 20\%$ between EUR and USD swap rates. The EUR/USD ATM Black volatilities are given in Table 5.6.5, extrapolated flat beyond the last option expiry. The IR/FX hybrid correlations $\rho\left(t, X^{USDEUR}_t, S^{EUR}_{\gamma_j(t),b_j}\right)$ and $\rho\left(t, X^{USDEUR}_t, S^{USD}_{\gamma_j(t),b_j}\right)$ are respectively fixed to 10% and -10%.

$\rho\left(t, S^{EUR}_{\gamma_j(t),b_j}, S^{USD}_{\gamma_{j'}(t),b_{j'}}\right)$	$\rho\left(t, X^{USDEUR}_t, S^{EUR}_{\gamma_j(t),b_j}\right)$	$\rho\left(t, X^{USDEUR}_t, S^{USD}_{\gamma_j(t),b_j}\right)$
20%	10%	-10%

Table 5.6.4: Hybrid cross-currency IR/FX correlations

The spot EUR/USD value is $X_0^{EURUSD} = 1.1$. The forward curve of the EUR/USD rate is computed using X_0^{EURUSD} and the discount curves of Table 5.6.1.

Expiry	**1**w	**1**m	**3**m	**6**m	**9**m	**1**y
ATM Vol	11.85%	7.95%	7.85%	7.75%	7.75%	7.85%

Table 5.6.5: EUR/USD ATM volatilities

Hazard rate data & models

The volatility function ν is considered to be of the form $\nu(t, y_t) = \bar{\nu}(y_t)^{\beta}$, allowing to have a CEV type dynamics of the credit intensities. $\beta = 1$ leads to a mean-reverting log-normal model (model denoted $(\mathscr{LN})$), while with $\beta = \frac{1}{2}$ we have a Cox-Ingersoll-Ross dynamics (we denote this $(\mathscr{C})$). We focus on two different sets for the credit model parameters that we name hereafter *Mid* and *High* risk settings. In particular, we consider a flat CDS curve of 200bps for the mid risk regime, and 800bps for the high risk regime. The model parameters are similar to those used in Chapter 4, they are given in Table 5.6.6.

	CDS Spread	y_0	κ	ψ	$\bar{\nu}(\mathscr{LN}, \beta = 1)$	$\bar{\nu}\left(\mathscr{C}, \beta = \frac{1}{2}\right)$	$\rho = \rho^{\lambda,X}$
Mid risk	200 bps	1%	1	2%	50%	20%	30%
High risk	800bps	3%	1.6	8%	100%	50%	90%

Table 5.6.6: Intensity model spread parameters

We now present the prices of CVA adjustments. The considered payoffs depend on the at-the-money strikes, with semi-annual payment in the EUR market. The final maturity is $T = 10y$. The Loss-Given-Default is fixed at 60% for both parties.

5.6.1 CVA for a standalone IR swap

We start with a single IR payer swap. The results are reported in Figure 5.6.1 and Table 5.6.7.

5.6.2 CVA for a mono-currency IR swaps portfolio

We consider $N = 10$ IR swaps. The value of each portfolio is divided by the number of contracts N in order to preserve homogeneity with the single swap case. We address different situations by varying the starting and ending dates.

Portfolio of swaps with co-terminal payment dates (Π_1)

In this case, the end date $\left(T_{b_j}\right)_{j=1\ldots N}$ is common, while the start dates $\left(T_{a_j}\right)_j$ are increasing. The results are reported in Figure 5.6.2 and Table 5.6.8.

Portfolio of swaps with co-starting dates (Π_2)

In this case, the start date $\left(T_{a_j}\right)_j$ is the same, while the end dates $\left(T_{b_j}\right)_j$ are decreasing.The results are reported in Figure 5.6.3 and Table 5.6.9.

Portfolio of swaps with first positive, then negative flows (Π_3)

This case is a combination of the two previous cases with 5 co-terminal payer swaps ending at T_{b_1}, and 3 co-starting receiver swap starting at T_{b_t}. The results are reported in Figure 5.6.4 and Table 5.6.10.

5.6.3 CVA for a bi-currency IR swaps portfolio

Portfolio of swaps with co-starting dates (Π_4)

This case is a combination of 5 co-starting payer swaps denominated in EUR and 5 others denominated in USD. The volatility of the exchange rate USDEUR and its correlation with the intensity process have an additional impact on the *Wrong-Way risk* exposures. The results are reported in Figure 5.6.5 and Table 5.6.11.

Portfolio of swaps with co-starting dates, long EUR and short USD (Π_5)

This case is a combination of 5 co-starting payer swaps denominated in EUR and 5 receiver swaps denominated in USD. The structure has a limited IR delta resulting in a small *WWR* impact.The results are reported in Figure 5.6.6 and Table 5.6.12.

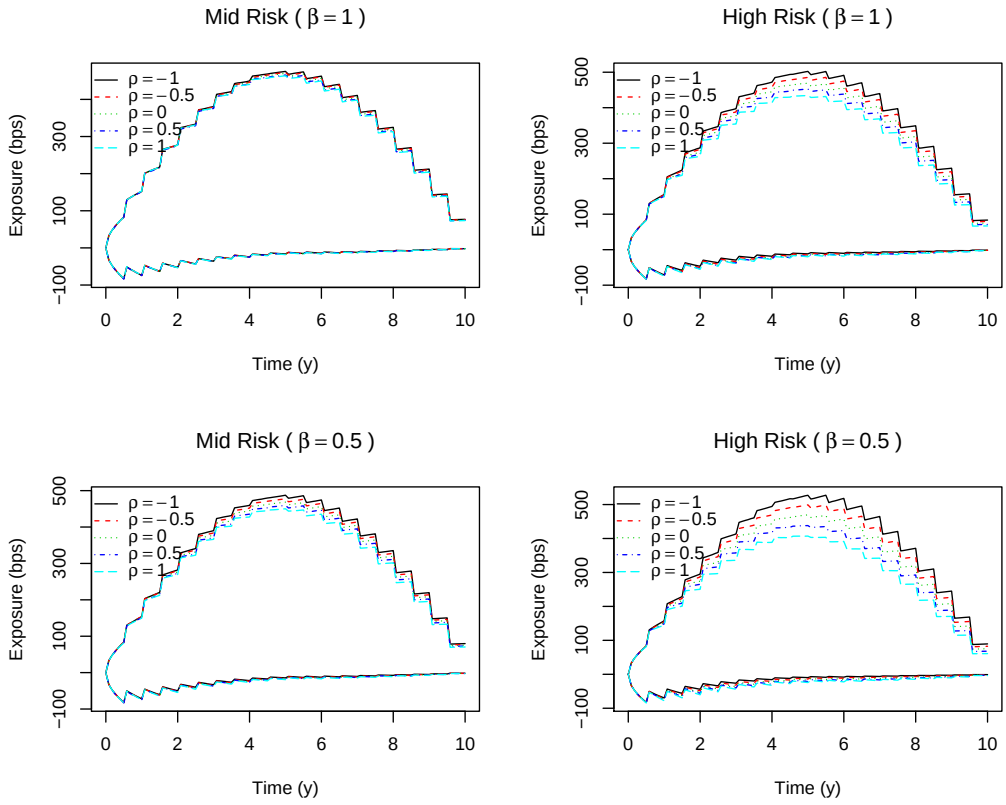


Figure 5.6.1: Expected positive/negative exposures of the ATM single swap

$\beta = 1$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	54.0	0.5	142.5	5.7	-4.3	0.1	-11.8	1.8
$\rho = -50\%$	53.6	0.2	138.9	2.2	-4.3	0.1	-12.8	0.8
$\rho = 0\%$	53.4	0.0	134.5	0.0	-4.3	0.0	-13.5	0.0
$\rho = 50\%$	53.0	-0.5	130.4	-6.3	-4.4	0.0	-13.9	-0.3
$\rho = 100\%$	52.8	-0.6	125.9	-10.8	-4.4	-0.1	-14.3	-0.7

$\beta = \frac{1}{2}$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	55.3	1.0	149.6	10.9	-4.0	0.4	-11.1	2.9
$\rho = -50\%$	54.3	0.1	142.7	4.0	-4.2	0.2	-12.4	1.5
$\rho = 0\%$	53.4	0.0	134.5	0.0	-4.3	0.0	-13.5	0.0
$\rho = 50\%$	52.3	-2.0	127.5	-11.2	-4.4	0.0	-14.7	-0.7
$\rho = 100\%$	51.3	-3.0	119.3	-19.4	-4.5	-0.2	-15.8	-1.8

Table 5.6.7: CVA and DVA of the ATM single swap

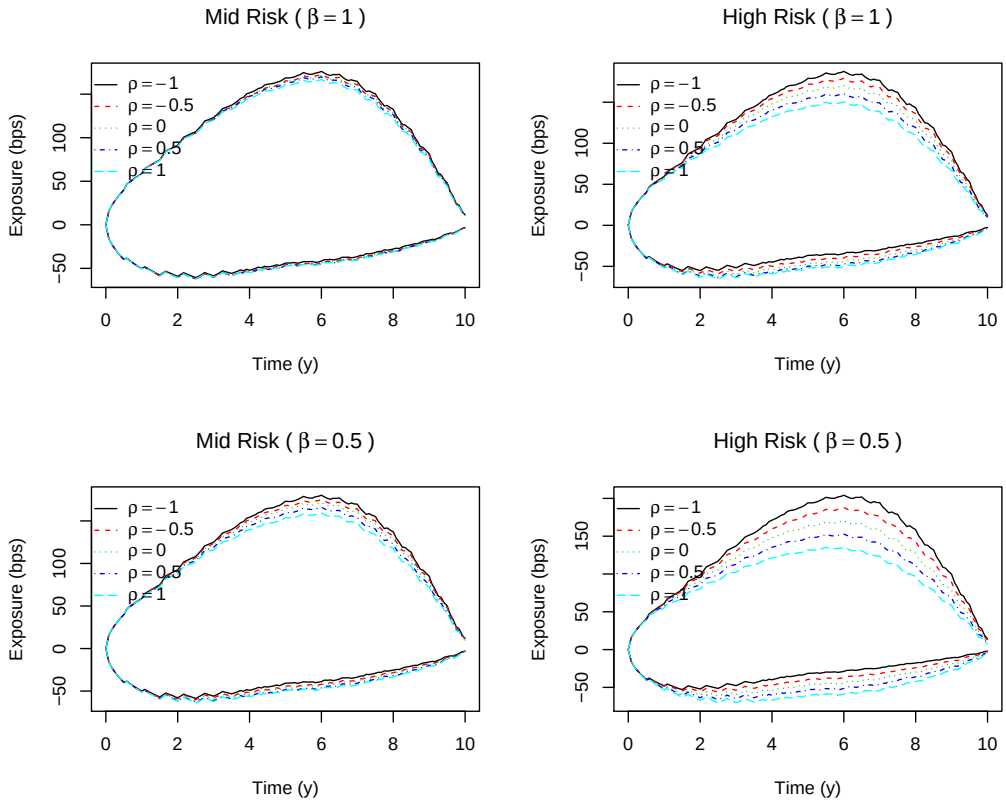


Figure 5.6.2: Expected positive/negative exposure of Portfolio Π_1

$\beta = 1$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	19.6	0.4	51.2	3.1	-6.9	0.3	-17.0	2.9
$\rho = -50\%$	19.3	0.0	49.5	1.4	-7.2	0.0	-18.4	1.5
$\rho = 0\%$	19.2	0.0	47.4	0.0	-7.2	0.0	-19.7	0.0
$\rho = 50\%$	19.0	-0.2	45.1	-3.0	-7.3	-0.1	-20.6	-0.7
$\rho = 100\%$	18.7	-0.5	42.7	-5.4	-7.3	-0.1	-21.3	-1.4

$\beta = \frac{1}{2}$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	20.0	0.6	55.2	6.3	-6.6	0.7	-15.6	4.8
$\rho = -50\%$	19.6	0.1	51.6	2.7	-7.0	0.3	-17.7	2.7
$\rho = 0\%$	19.2	0.0	47.4	0.0	-7.2	0.0	-19.7	0.0
$\rho = 50\%$	18.7	-0.8	43.7	-5.2	-7.4	-0.1	-21.6	-1.3
$\rho = 100\%$	18.0	-1.5	39.4	-9.5	-7.6	-0.3	-23.6	-3.3

Table 5.6.8: CVA and DVA of Portfolio Π_1

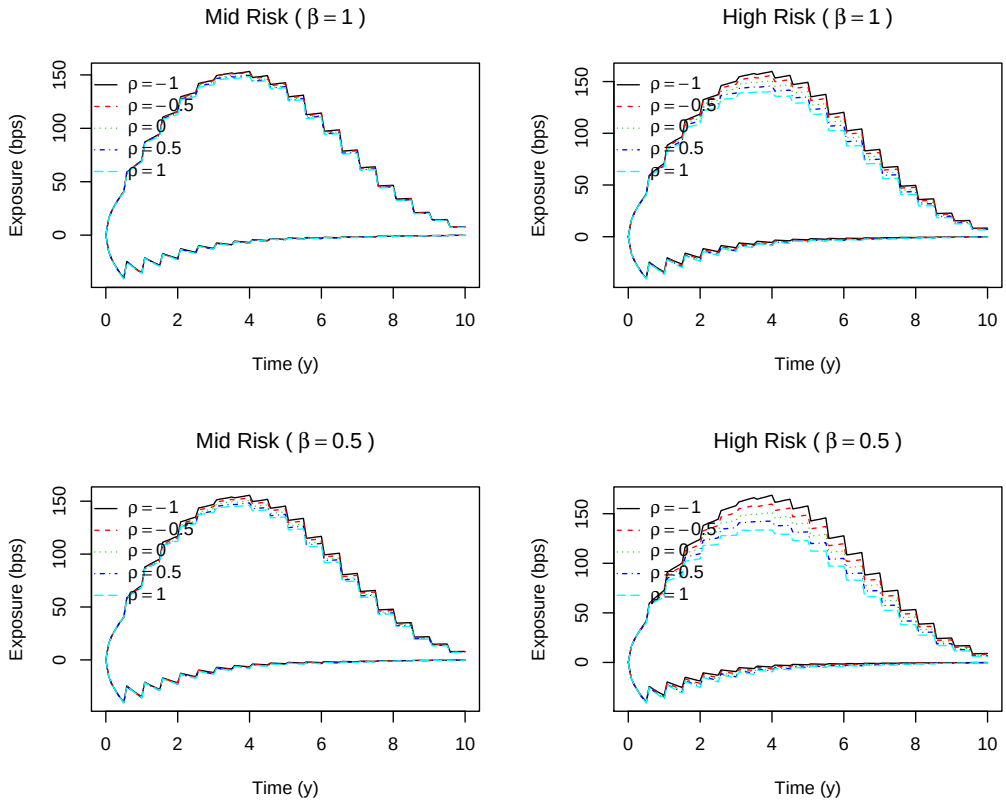


Figure 5.6.3: Expected positive/negative exposures of Portfolio Π_2

$\beta = 1$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	15.5	0.2	43.7	1.5	-1.6	0.0	-4.8	0.5
$\rho = -50\%$	15.4	0.1	42.7	0.5	-1.6	0.0	-5.1	0.3
$\rho = 0\%$	15.3	0.0	41.6	0.0	-1.6	0.0	-5.3	0.0
$\rho = 50\%$	15.2	-0.1	40.4	-1.8	-1.6	0.0	-5.5	-0.1
$\rho = 100\%$	15.0	-0.3	39.0	-3.2	-1.6	0.0	-5.6	-0.3

$\beta = \frac{1}{2}$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	15.7	0.2	46.0	2.9	-1.5	0.1	-4.5	0.9
$\rho = -50\%$	15.5	0.0	43.8	0.7	-1.6	0.1	-5.0	0.5
$\rho = 0\%$	15.3	0.0	41.6	0.0	-1.6	0.0	-5.3	0.0
$\rho = 50\%$	15.0	-0.5	39.8	-3.4	-1.6	0.0	-5.7	-0.2
$\rho = 100\%$	14.8	-0.7	37.6	-5.5	-1.7	0.0	-5.9	-0.5

Table 5.6.9: CVA and DVA of Portfolio Π_2

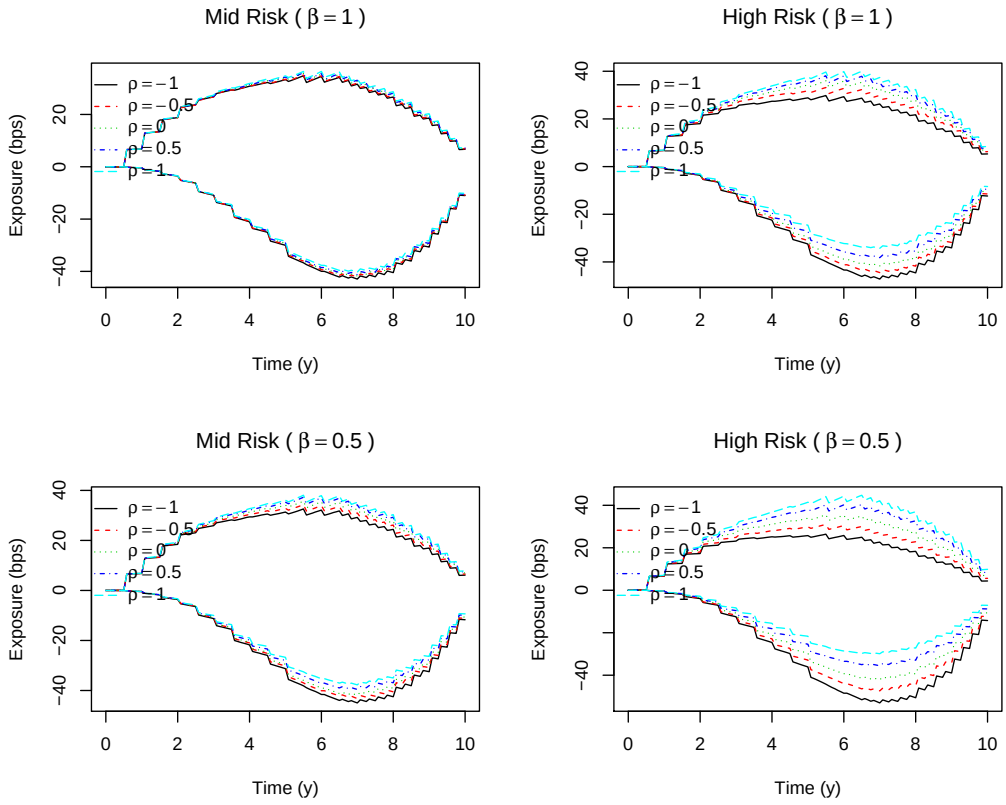


Figure 5.6.4: Expected positive/negative exposures of Portfolio Π_3

$\beta = 1$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	3.9	-0.1	8.4	-1.5	-3.7	-0.1	-8.6	-0.8
$\rho = -50\%$	3.9	-0.1	9.2	-0.7	-3.6	0.0	-8.2	-0.4
$\rho = 0\%$	4.0	0.0	9.8	0.0	-3.6	0.0	-7.7	0.0
$\rho = 50\%$	4.0	0.0	10.3	0.3	-3.5	0.1	-7.1	0.7
$\rho = 100\%$	4.1	0.1	10.7	0.8	-3.5	0.1	-6.4	1.4

$\beta = \frac{1}{2}$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	3.7	-0.4	7.7	-2.3	-3.8	-0.2	-9.6	-1.7
$\rho = -50\%$	3.8	-0.2	8.7	-1.3	-3.7	0.0	-8.8	-0.8
$\rho = 0\%$	4.0	0.0	9.8	0.0	-3.6	0.0	-7.7	0.0
$\rho = 50\%$	4.1	0.1	10.8	0.8	-3.4	0.2	-6.7	1.2
$\rho = 100\%$	4.2	0.2	11.7	1.7	-3.2	0.4	-5.7	2.2

Table 5.6.10: CVA and DVA of Portfolio Π_3

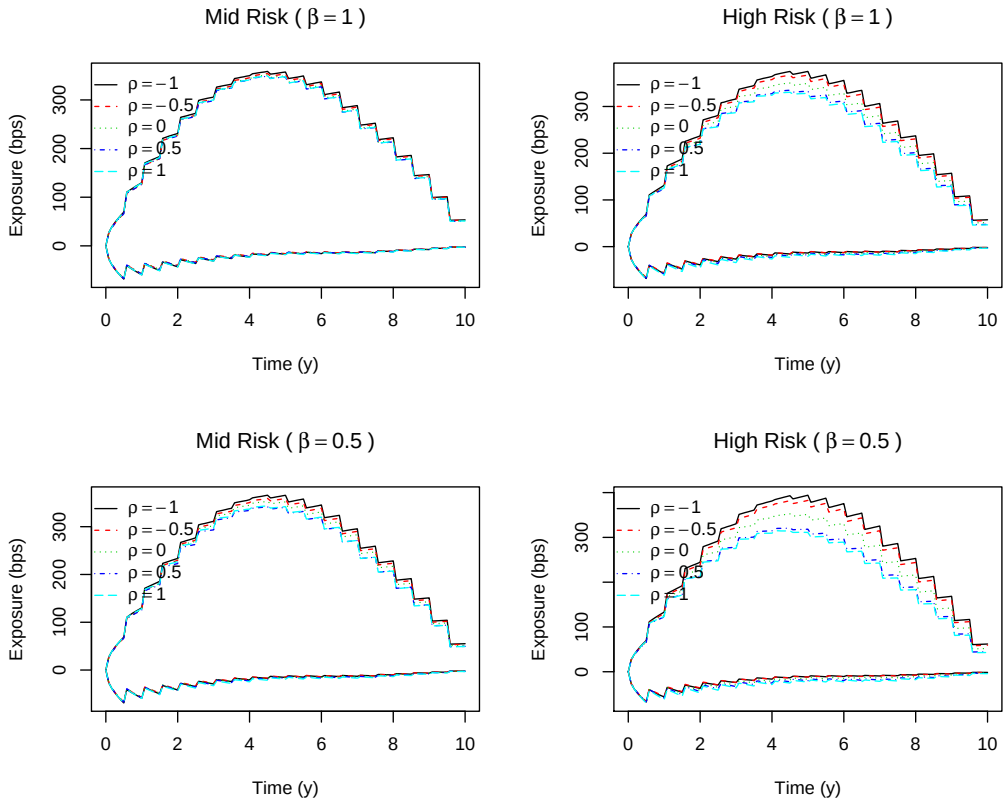


Figure 5.6.5: Expected positive/negative exposures of Portfolio Π_4

$\beta = 1$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	40.7	0.6	108.4	4.5	-3.9	-0.1	-10.6	1.0
$\rho = -50\%$	40.2	0.1	106.1	2.2	-3.7	0.1	-10.4	1.3
$\rho = 0\%$	39.8	0.0	101.8	0.0	-3.7	0.0	-11.3	0.0
$\rho = 50\%$	39.5	-0.5	97.6	-6.3	-3.7	0.1	-11.8	-0.2
$\rho = 100\%$	39.7	-0.4	96.8	-7.1	-4.0	-0.2	-12.8	-1.2

$\beta = \frac{1}{2}$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	41.4	0.6	113.6	7.7	-3.6	0.2	-10.0	1.8
$\rho = -50\%$	40.7	0.0	110.6	4.7	-3.5	0.3	-9.8	2.0
$\rho = 0\%$	39.8	0.0	101.8	0.0	-3.7	0.0	-11.3	0.0
$\rho = 50\%$	38.8	-1.9	94.3	-11.6	-3.9	0.0	-12.4	-0.6
$\rho = 100\%$	39.0	-1.7	92.9	-13.0	-4.1	-0.3	-13.5	-1.8

Table 5.6.11: CVA and DVA of Portfolio Π_4

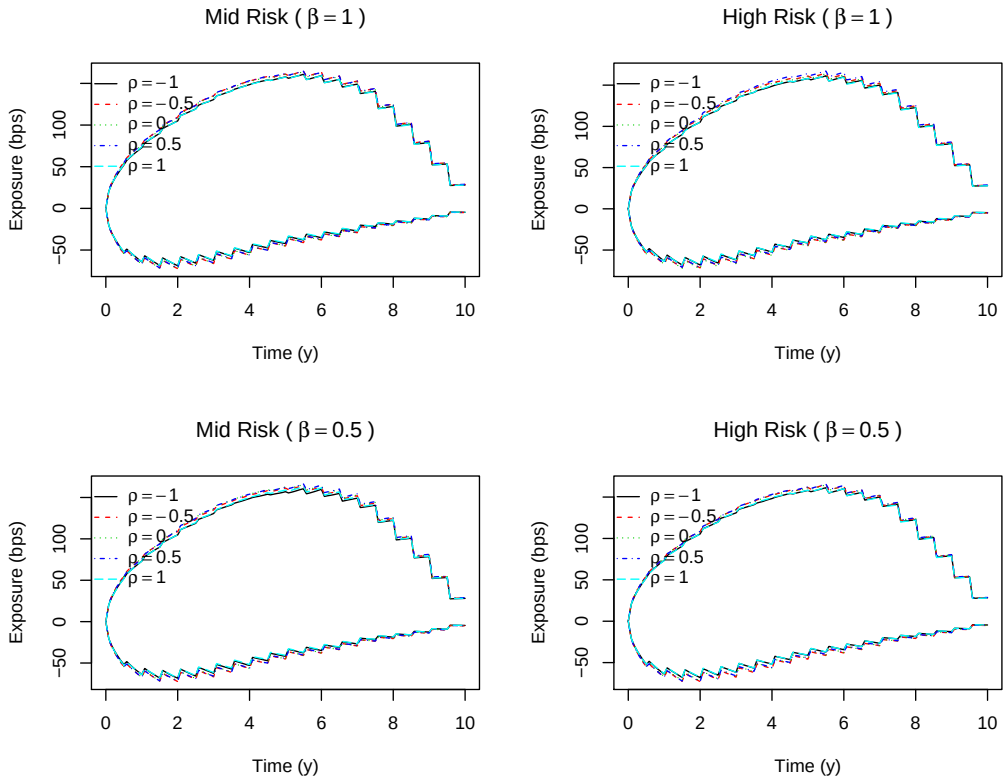


Figure 5.6.6: Expected positive/negative exposures of Portfolio Π_5

$\beta = 1$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	19.2	-0.3	48.8	-1.6	-7.1	-0.3	-20.7	-0.9
$\rho = -50\%$	19.4	-0.1	50.3	-0.3	-6.9	-0.1	-20.0	-0.3
$\rho = 0\%$	19.6	0.0	50.5	0.0	-6.8	0.0	-19.8	0.0
$\rho = 50\%$	19.8	0.2	50.7	0.3	-6.7	0.1	-19.5	0.3
$\rho = 100\%$	19.9	0.4	52.1	1.6	-6.5	0.3	-18.7	1.0

$\beta = \frac{1}{2}$	***Mid Risk***		***High Risk***		***Mid Risk***		***High Risk***	
	CVA	$CVA - CVA^0$	CVA	$CVA - CVA^0$	DVA	$DVA - DVA^0$	DVA	$DVA - DVA^0$
$\rho = -100\%$	19.2	-0.7	49.1	-2.0	-7.2	-0.3	-21.6	-1.4
$\rho = -50\%$	19.7	-0.2	50.2	-0.9	-7.0	-0.1	-20.4	-0.2
$\rho = 0\%$	19.9	0.0	51.1	0.0	-6.9	0.0	-20.2	0.0
$\rho = 50\%$	19.9	-0.1	52.0	0.9	-6.8	0.1	-20.1	0.1
$\rho = 100\%$	19.3	-0.6	53.1	2.0	-6.5	0.4	-18.7	1.5

Table 5.6.12: CVA and DVA of Portfolio Π_5

5.7 Conclusion

In the present work, pricing CVA including correlation risk between credit and the exposure factors is achieved in a parsimonious and tractable way. Our approach consists of deriving correction terms to adjust the standard CVA without *WWR*. Our expansion approach starts from the *no WWR* CVA, to which we add a sum of analytical corrections based on the exposure Greeks. Our second and third order corrections involve explicitly model parameters such as credit spread volatility and the *WWR* correlations. This tractability is of special interest for deriving CVA sensitivities w.r.t. to these parameters. We expressed CVA in a standardized scheme for XVA adjustments, by introducing the *WWR-adjusted* exposure. The later quantity can used to compute other XVA adjustments, embedding the credit *WWR*. Our approach is not restricted to credit *Wrong way risk.* We extended it to the funding *Wrong-way risk,* i.e. the correlation risk between the exposure and the stochastic funding spreads.

In order to illustrate our results, we established a toy framework based on the underlying trades sensitivities. Assuming a Gaussian distribution of the mark-to-market, the setting allows to derive closed formulas for the expected positive and negative exposures and their Greeks without any computational cost.

Appendix B

B.1 Option prices and Greeks in the Bachelier model

In a Bachelier model, the dynamics of a stock (S_t) is given by:

$$dS_t = \sigma dW_t$$

with initial value S_0 at time 0. The prices of a Call and a Put options (resp. C^B and P^B) with a strike K and maturity T is given by

$$\begin{aligned} C^B(T,S,\sigma,K) &= \sigma\sqrt{T}\,(dN(d)+n(d)) \\ P^B(T,S,\sigma,K) &= \sigma\sqrt{T}\,(-dN(d)+n(d)) \end{aligned}$$

with $d = \frac{S-K}{\sigma\sqrt{T}}$, $n(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ and $N(x) = \int_{-\infty}^{x} n(u)\,du$. The Greeks w.r.t. S are given analytically:

$$\begin{aligned} \Delta_C^B(T,S,\sigma,K) &= N(d) \\ \Delta_C^B(T,S,\sigma,K) &= -N(-d) \end{aligned}$$

and

$$\begin{aligned} \Gamma_C^B(T,S,\sigma,K) &= \frac{1}{\sigma\sqrt{T}} n(d) \\ \Gamma_P^B(T,S,\sigma,K) &= \Gamma_C^B(T,S,\sigma,K) \end{aligned}$$

The option Speed measures the rate of change in Gamma w.r.t. S:

$$\begin{aligned} \mathscr{S}_C^B(S,\sigma;T,K) &= -\frac{dn(d)}{\sigma^2 T} \\ \mathscr{S}_P^B(S,\sigma;T,K) &= \mathscr{S}_C^B(S,\sigma;T) \end{aligned}$$

B.2 Expected exposure expression conditional on default

Our approach of *WWR* is based on pricing with stochastic default intensity and correlation with risk factors driving the derivative's exposure. Default intensity volatility and correlation with other risk factors can be calibrated to market prices or estimated historically.

[Gre10] and [CdJHRS10] address another approach for *WWR* that consists of explicitly specifying a Gaussian correlation structure between the default time τ and the future market-to-market. This results in tractable expressions for the expected positive and negative exposures conditional on the default. However, we highlight a noticeable issue that is related to the absence of *observability* of the correlation parameter.

In a nutshell, we assume that the value of the contract at time t in the future is given by

$$V(t) = \mu t + \sigma\sqrt{t}Y$$

where Y is a Gaussian random variable. Like the exposure, we assume that the default time is driven by a random Gaussian variable Z such that the counterparty default if $Z \leq N^{-1}(PD(t))$ where N^{-1} is the inverse of the standard normal distribution function, and $DP(t)$ is the default probability.

$$\tau = \inf\left\{t > 0, Z \leq N^{-1}(DP(t))\right\}$$

Since PD is a increasing function, we deduce that $\tau = DP^{-1}(N(Z))$ in distribution. Under a constant hazard rate h, one has $PD(t) = 1 - e^{-ht}$ with τ and $\frac{-\ln(N(Z))}{h}$ having the same distribution. Finally, the variables Y and Z are linked via Gaussian copula with parameter ρ, i.e.

$$Y = \rho Z + \sqrt{1-\rho^2}Z^{\perp}$$

The expected exposure conditional to the default is given by

$$EPE(t|\tau) = \mathbb{E}\left[(V(\tau))|Z = N^{-1}(PD(t))\right] = C^B\left(t, \mu'(t,\tau), \sigma'(t), 0\right)$$

with $\mu'(t,\tau) = \mu t + \rho\sigma\sqrt{t}N^{-1}(DP(\tau))$ and $\sigma'(t) = \sigma\sqrt{t}\sqrt{1-\rho^2}$. The WWR-adjusted positive exposure is then given by

$$\begin{aligned} WWREPE(t) &= \frac{\int_0^t EPE(t|\tau = s)\, dDP(s)}{DP(t)} \\ &= \frac{h}{1-e^{-ht}}\int_0^t C^B\left(t, \mu'(t,s), \sigma'(t), 0\right) e^{-hs} ds \end{aligned}.$$

Chapter 6

Expansion formulas for bilateral CVA with *WWR and FTD*

This chapter extends the pricing formulas developed previously for the unilateral credit value adjustments to the bilateral context. We denote by the bilateral Wrong-way risk the mutual impact of correlations between the default events of each counterparty and the exposure. The resulting WWR adjustment is the sum of unilateral *WWR* corrections, to which is added a third term stemming from the correlation between default intensity processes.

Our results represent also an extension to the primary approach developed in Chapter 4 devoted to the pricing of credit-contingent derivatives. The derivation methodology tackles the case where cash-flows are contingent to the default events of two reference entities.

Keywords: Bilateral credit valuation adjustments, *Wrong-Way* risk*, First-to-Default* risk, credit spreads correlation, asymptotic expansion

6.1 Introduction

CVA has traditionally been a charge that is calculated in favor of the counterparty with the stronger credit worthiness. Historically, corporates were not credibly able to request CVA charges to their banking counterparties, since the probability that major banks bankrupt was utterly under-estimated.

Since the credit crisis of 2007 and the unanticipated default of Lehman Brothers, a practice that has become relevant is to consider the bilateral nature of counterparty risk. The definition of BCVA follows directly UCVA's with the assumption that both counterparties can default. In particular, the exposure at default of one party is calculated conditionally upon the survival of the other party. A simplified formula for bilateral risk that is often used in the industry, is based on subtracting the unilateral CVA and DVA from the point of view of the party who is doing the calculation. The reason why the simplified formula is popular is that it allows one to compute a bilateral CVA adjustment by combining unilateral CVAs. This way, one does not need to implement a bilateral CVA system. However, the default dependence between the two parties involved in the deal is consequently neglected.

In the present work, we allow for a correlation structure between the default times and the

portfolio's risk factors. The correlation between default times is integrated through a statistical dependence between hazard processes. We derive pricing formulas for CVA valuation in presence of the bilateral Wrong-Way risk. In particular, we obtain an explicit decomposition of the resulting adjustment into marginal Wrong-way risk corrections, to which is added a correction term due to defaults correlation.

Modeling the *FTD* risk by correlating the intensity processes of defaults is a restrictive approach. [Mor11] shows that correlation between continuous diffusion processes for the intensity generates a limited range of correlation between the events of default. The impact of the *first-to-default* time is analyzed in [BBM11]. The authors propose an explicit dependence structure, by adopting a bivariate exponential distribution for the inter-occurrence periods, the exponential variates that trigger the defaults. We discuss the use of a copula for dependence with regard to the technical requirement of our expansion technique.

Outline of the chapter

The chapter is organized as follows. In Section 6.2, a succinct introduction to bilateral counterparty risk is given. We examine the copula approach for modeling default dependence and we illustrate the impact of correlation on the joint default probability of interest in BCA. We introduce in Section 6.3 our approach for pricing bilateral counterparty risk while Section 6.4 is dedicated to the error analysis.

6.2 Bilateral credit valuation adjustments

The definition of BCVA follows directly UCVA's with the assumption that both counterparties can default. In particular, CVA's positive exposure is accounted when the default of C occurs before the default time of I and the expiry time of the portfolio. Similarly, DVA's negative exposure is subject to the the occurrence of the default of I before C's. This defines the *First-to-Default* nature of the bilateral counterparty risk.

6.2.1 Arbitrage-Free valuation

We refer to Proposition 12.1.1 of [BMP13] for a detailed derivation of bilateral CVA (BCVA) and DVA (BDVA) adjustments. *BCVA* and *BDVA* are given by

$$BCVA(0) = LGD_C \mathbb{E}\left[1_{\{\tau_C \leq T\}} 1_{\{\tau_C < \tau_I\}} D(t,\tau_C)\left(V(\tau_C)\right)^+\right] \tag{6.2.1}$$

$$BDVA(0) = LGD_I \mathbb{E}\left[1_{\{\tau_I \leq T\}} 1_{\{\tau_I < \tau_C\}} \left(-V(\tau_I)\right)^+\right] \tag{6.2.2}$$

where τ_I and τ_C respectively the default times of the investor and its counterparty, and LGD_I (resp. LGD_C) their deterministic loss given default. V is the net present value of the derivatives portfolio as seen from the point of view of I. $V(t) := \mathbb{E}_t[\Pi(t,T)]$ being the conditional expectation of the sum of future cash-flows. In presence of collateral agreements, positive and negative exposures account for posted collateral and non-segregated initial margins, reducing the exposure at default. We denote by $C(t)$ the amount of posted collateral at time t and expressed in the same valuation currency as Π. Without loss of generality, V is defined by

$$V(t) := \mathbb{E}_t[\Pi(t,T)] - C(t)$$

We recall the unilateral CVA and DVA adjustments

$$UCVA(0) = LGD_C \mathbb{E}\left[1_{\{\tau_C \leq T\}} D(t,\tau_C)(V(\tau_C))^+\right] \tag{6.2.3}$$
$$UDVA(0) = LGD_I \mathbb{E}\left[1_{\{\tau_I \leq T\}} D(t,\tau_I)(-V(\tau_I))^+\right] \tag{6.2.4}$$

Then, the First-to-Default adjustment is the difference between $BCVA$ and $UCVA$:

$$FTD_C(0) = BCVA(0) - UCVA(0) = -LGD_C \mathbb{E}\left[1_{\{\tau_I \leq \tau_C \leq T\}} D(t,\tau_C)(V(\tau_C))^+\right] \tag{6.2.5}$$
$$FTD_I(0) = BDVA(0) - UDVA(0) = -LGD_I \mathbb{E}\left[1_{\{\tau_C \leq \tau_I \leq T\}} D(t,\tau_I)(-V(\tau_I))^+\right] \tag{6.2.6}$$

6.2.2 The First-to-Default risk

To quantify the impact of FTD terms, let us consider, at a first glance, that default times and the exposure V are independent. This assumption allows to write the FTD formulas above as the product of an expected exposure and an expected cost. Formally

$$FTD_C(0) = -LGD_C \mathbb{E}\left[1_{\{\tau_I \leq \tau_C \leq T\}}\right] \mathbb{E}\left[D(t,\tau_C)(V(\tau_C))^+\right] \tag{6.2.7}$$
$$FTD_I(0) = -LGD_I \mathbb{E}\left[1_{\{\tau_C \leq \tau_I \leq T\}}\right] \mathbb{E}\left[D(t,\tau_I)(-V(\tau_I))^+\right] \tag{6.2.8}$$

We analyze the magnitude of FTD terms in comparison with $UCVA$ and $UDVA$:

$$\left|\frac{FTD_C(0)}{UCVA(0)}\right| = \frac{\mathbb{Q}(\tau_I \leq \tau_C \leq T)}{\mathbb{Q}(\tau_C \leq T)} \tag{6.2.9}$$
$$\left|\frac{FTD_I(0)}{UDVA(0)}\right| = \frac{\mathbb{Q}(\tau_C \leq \tau_I \leq T)}{\mathbb{Q}(\tau_I \leq T)} \tag{6.2.10}$$

[BBM11] adopts a bivariate copula approach for τ_I and τ_C . We examine the impacts of the Gaussian and Gumbel[1] copulas. We admit that the default intensities λ_I and λ_C of τ_I and τ_C are deterministic:

$$\tau_i = \inf\{t > 0 | \lambda_i t > \xi_i\}, i \in \{I,C\} \tag{6.2.11}$$

which yields that $\tau_i = \frac{\xi_i}{\lambda_i}$ in distribution . The dependence is then only driven by the copula functions for (ξ_I, ξ_C). Notice that for both the Gaussian and the Gumbel cases, the implied joint distribution of τ_I and τ_C admits a zero probability of simultaneous default $\tau_I = \tau_C$. With constant intensities λ_C and λ_I, one has

$$\mathbb{Q}(\tau_i \leq T) = 1 - e^{-\lambda_i T}, i \in \{I,C\} \tag{6.2.12}$$

On the other hand, $\mathbb{Q}(\tau_I \leq \tau_C \leq T)$ satisfies

$$\mathbb{Q}(\tau_I \leq \tau_C \leq T) = \mathbb{Q}\left(\frac{\xi_I}{\lambda_I} \leq \frac{\xi_C}{\lambda_C} \leq T\right) = \mathbb{Q}\left(\xi_C \in \left[\frac{\lambda_C}{\lambda_I}\xi_I, T\lambda_C\right]\right)$$

Let (U,V) a couple a random variables with $(0,1)-$uniform marginals. One has then:

$$\mathbb{Q}(\tau_I \leq \tau_C \leq T) = \mathbb{Q}\left(V \in \left[\exp(-T\lambda_C), e^{\frac{\lambda_C}{\lambda_I}} U\right]\right)$$

[1] The Gumbel copula is examined in [BBM11]

The joint density distribution function of (u,v) , i.e. $c(u,v)$ is given by the choice of the copula function. The latter equality reads then

$$\mathbb{Q}(\tau_I \leq \tau_C \leq T) = \int_{u\in(0;1)} \int_{v\in\left(\exp(-T\lambda_C),u\exp\left(\frac{\lambda_C}{\lambda_I}\right)\right)} c(u,v)\,dudv \tag{6.2.13}$$

Similarly, we obtain

$$\mathbb{Q}(\tau_C \leq \tau_I \leq T) = \int_{u\in(0;1)} \int_{v\in\left(\exp(-T\lambda_I),u\exp\left(\frac{\lambda_I}{\lambda_C}\right)\right)} c(u,v)\,dudv \tag{6.2.14}$$

Copula functions:

We refer to [CLV06] for copula cumulative and density functions. We consider the Gaussian and Gumbel cases that are given by

- Gaussian copula: Its joint cumulative distribution function reads

$$C_\rho^{Gauss}(u,v) = \Phi_\rho\left(\Phi^{-1}(u),\Phi^{-1}(v)\right) \tag{6.2.15}$$

  where Φ_ρ is the joint cumulative distribution function of a multivariate normal distribution, and Φ the cumulative distribution function of a standard univariate normal distribution. Its density distribution function is given by

$$c_\rho^{Gauss}(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v) - \left(\Phi^{-1}(u)\right)^2 - \left(\Phi^{-1}(v)\right)^2}{2(1-\rho^2)}\right) \tag{6.2.16}$$

  $\rho\in[-1,1]$ denotes the Pearson's correlation coefficient. It is related to Kendall's tau by

$$\tau^K = \frac{2}{\pi}\arcsin(\rho) \in [-1,1] \tag{6.2.17}$$

- Gumbel copula: Its joint cumulative distribution function is given by

$$C_\theta^{Gumbel}(u,v) = \exp\left(-\left((-\ln(u))^\theta + (-\ln(v))^\theta\right)^{\frac{1}{\theta}}\right) \tag{6.2.18}$$

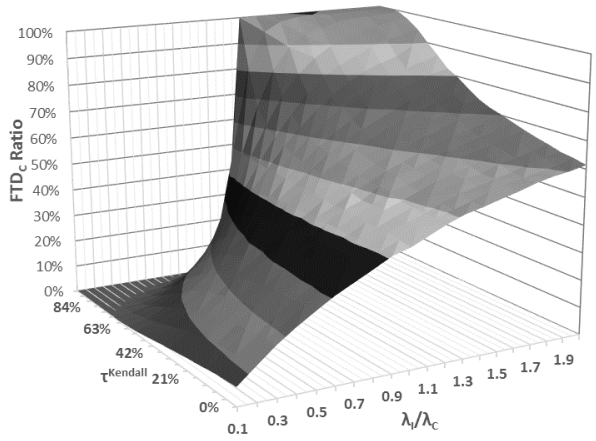
  The density distribution function is given by

$$c_\theta^{Gumbel}(u,v) = \frac{\partial^2 C_\theta^{Gumbel}(u,v)}{\partial u\partial v} \tag{6.2.19}$$

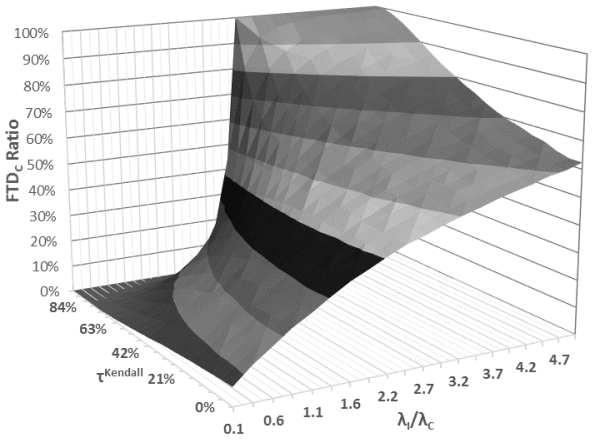
  The copula parameter $\theta\in[1,+\infty)$ is related to Kendall's tau by

$$\tau^K = 1 - \frac{1}{\theta} \in [0,1) \tag{6.2.20}$$

  where $\theta\in[1,+\infty)$.



(a) Under the Gaussian copula



(b) Under the Gumbel copula

Figure 6.2.1: $\left|\frac{FTD_C(0)}{UCVA(0)}\right|$ ratio w.r.t $\frac{\lambda_I}{\lambda_C}$ and Kendall's τ

For numerical analysis, we take the default values $\lambda^C = 0.1$ and $\lambda^I = c \times \lambda^C$ with different values of constant $c \in [0,2]$ We analyze the behavior of (6.2.9) and (6.2.10) in terms of the maturity T and the Kendall's tau under the Gaussian ((6.2.17)) and the Gumbel ((6.2.20)) copulas. The results are reported in Figures 6.2.1

In the independent case, it is well known that $\tau_I \wedge \tau_C$ is exponentially distributed with parameter $\lambda_I + \lambda_C$. It is easy then to compute $\mathbb{Q}(\tau_I \leq \tau_C \leq T)$:

$$\mathbb{Q}(\tau_I \leq \tau_C \leq T) = \mathbb{Q}(\tau_I \leq \tau_C)\mathbb{Q}(\tau_I \wedge \tau_C \leq T) = \frac{\lambda_I}{\lambda_I + \lambda_C}\left(1 - e^{-(\lambda_I+\lambda_C)T}\right)$$

The opposite situation corresponds to the comonotonic default case. We assume that $\tau_I = \frac{\xi}{\lambda_I}$ and $\tau_C = \frac{\xi}{\lambda_C}$ with ξ a standard exponential random variable. It follows that $\tau_I = \frac{\lambda_C}{\lambda_I}\tau_C$. One has then

$$\mathbb{Q}(\tau_I \leq \tau_C \leq T) = \begin{cases} 0 & \text{if } \lambda_C > \lambda_I \\ \mathbb{Q}(\tau_C \leq T) & \text{otherwise} \end{cases}$$

In particular, when $\lambda_C > \lambda_I$, $UCVA$ is an exact calculation of $BCVA$. When $\lambda_C \leq \lambda_I$, $BCVA = 0$ and the error corresponds to $UCVA$. Finally, $\tau^{Kendall} \in (0,1)$ gives intermediate values between independence and comonotonicity cases. The maximum difference between the Gaussian and Gumbel copulas is reached when $\tau^{Kendall} \in (0,1)$ and it is revealed to be not significant in our example.

6.3 *WWR* expansion formulas for bilateral CVA

We follow the intensity-based framework that we proposed in Chapters 4 and 5. For each counterparty $k \in \{I,C\}$, the default time τ_k is given by a Poisson jump process with a stochastic

intensity $\lambda^{\varepsilon,k}$. Formally, for $k \in \{I,C\}$:

$$\begin{cases} \Lambda^{\varepsilon,k}(t) & = \int_0^t \lambda_s^{\varepsilon,k} ds \\ \lambda_s^{\varepsilon,k} & = y_t^{\varepsilon,k} + \psi^{\varepsilon,k}(t,\beta) \\ dy_t^{\varepsilon,k} & = \kappa_k \left(\theta_k - y_t^{\varepsilon,k}\right) dt + \varepsilon \nu_k \left(t, y_t^{\varepsilon,k}\right) dW_t^k \\ \tau_k & = \left(\Lambda^{\varepsilon,k}\right)^{-1}(\xi_k) \end{cases} \tag{6.3.1}$$

ε is an interpolation parameter that allows to switch from a deterministic intensity model ($\varepsilon = 0$) to a stochastic one ($\varepsilon = 0$). ψ^{ε} is a deterministic shift function allowing to calibrate initial term structure of default probability implied from market instruments $t \mapsto \mathbb{Q}(\tau_k > t)$:

$$\exp\left(-\int_0^t \psi^{\varepsilon,k}(s)\, ds\right) = \frac{\mathbb{Q}(\tau_k > t)}{\mathbb{E}\left[\exp\left(-\int_0^t y_s^{\varepsilon,k} ds\right)\right]} \tag{6.3.2}$$

Each ξ_k is exponential unit-mean that is independent of any other random variable. The two variables $(\xi_k)_{k\in\{I,C\}}$ are assumed to be disconnected. A dependence structure for (ξ_I, ξ_C) can be imposed via a bivariate copula function as discussed earlier. Instead, we allow for $y_t^{\varepsilon,I}$ and $y_t^{\varepsilon,C}$ to introduce a correlation between intensity dynamics. We set

$$d\left\langle W^B, W^C \right\rangle_t = \rho^{\lambda^I,\lambda^C} dt \tag{6.3.3}$$

where $\rho^{\lambda^I,\lambda^C}$ is the correlation parameter.

We follow all notations and technical lemmas presented previously. We consider a time discretization $\mathscr{T}_M = \{0 = T_0, T_1, \ldots, T_M\}$ such that the default time τ_i, $i \in \{I,C\}$ is observed on the intervals $]T_{i-1}, T_i]$. *BCVA* and *BDVA* are approximated then by

$$\begin{aligned} BCVA^{\varepsilon}(0) &= LGD_C \sum_{i=1}^{M} D(0,T_i)\, \mathbb{E}\left[1_{\tau_I > T_i}\left(1_{T_{i-1}<\tau_C} - 1_{T_i<\tau_C}\right)(V(T_i))^+\right] \\ &= LGD_C \sum_{i=1}^{M} D(0,T_i)\, \mathbb{E}\left[e^{-\int_0^{T_i}\lambda_s^{\varepsilon,I} ds}\left(e^{-\int_0^{T_{i-1}}\lambda_s^{\varepsilon,C} ds} - e^{-\int_0^{T_i}\lambda_s^{\varepsilon,C} ds}\right)(V(T_i))^+\right] \end{aligned} \tag{6.3.4}$$

and

$$\begin{aligned} BDVA^{\varepsilon}(0) &= LGD_I \sum_{i=1}^{M} D(0,T_i)\, \mathbb{E}\left[1_{\tau_C > T_i}\left(1_{T_{i-1}<\tau_I} - 1_{T_i<\tau_I}\right)(-V(T_i))^+\right] \\ &= LGD_I \sum_{i=1}^{M} D(0,T_i)\, \mathbb{E}\left[e^{-\int_0^{T_i}\lambda_s^{\varepsilon,C} ds}\left(e^{-\int_0^{T_{i-1}}\lambda_s^{\varepsilon,I} ds} - e^{-\int_0^{T_i}\lambda_s^{\varepsilon,I} ds}\right)(-V(T_i))^+\right] \end{aligned} \tag{6.3.5}$$

V refers to the mark-to-market of the underlying portfolio at default, including collateral. As in Chapters 4 and 5, we admit that V is a deterministic function of time t and a stochastic state variable X_t

$$V(t) = V(t, X_t) \tag{6.3.6}$$

where X is a $\mathbb{R}^n-$valued diffusion process which dynamics are specified by the stochastic differential equation

$$dX_t = (\Phi(t) + \Theta(t) X_t)\, dt + \Sigma(t)\, dB_t, \qquad X_0 \in \mathbb{R}^n \tag{6.3.7}$$

with $\Phi : [0,T] \to \mathbb{R}^n$, $\Theta : [0,T] \to \mathbb{R}^{n\times n}$, $\Sigma : [0,T] \to \mathbb{R}^{n\times d}$ and $(B_t)_{t\geq 0}$ a $\mathbb{R}^d-$Brownian motion such that for $i \in [\![1,d]\!]$

$$\begin{cases} d\left\langle [B]_i, [B]_j \right\rangle dt & = \rho_{i,j}^{X,X} dt \\ d\left\langle [B]_i, W^I \right\rangle dt & = \rho_i^{X,\lambda^I} dt \\ d\left\langle [B]_i, W^C \right\rangle dt & = \rho_i^{X,\lambda^C} dt \end{cases} \tag{6.3.8}$$

We define the modified functions $\tilde{\kappa}, \tilde{\nu}$ and $\tilde{\Sigma}$ by

$$\tilde{\Sigma}(t,T) = e^{\int_t^T \Theta(\omega) d\omega} \Sigma(t) \tag{6.3.9}$$

$$\tilde{\nu}_k(t) = e^{\kappa t} \nu_k(t) \tag{6.3.10}$$

$$\tilde{\kappa}_k(t,T) = \frac{e^{-\kappa_k t} - e^{-\kappa_k T}}{\kappa_k} \tag{6.3.11}$$

Finally, we introduce the volatility upper-bounds

$$\begin{cases} M_{k,0} & = \max_{0\leq i\leq 3} \left(\left\| \nu_k^{(i)} \right\|_\infty \right) \\ M_{k,1} & = \max_{1\leq i\leq 3} \left(\left\| \nu_k^{(i)} \right\|_\infty \right) \end{cases}$$

We present now our pricing formulas in the context bilateral CVA. Similar formulas can be directly derived for *BDVA*. Let us define

$$u_{V^+}^\varepsilon : (s,t) \mapsto \frac{\mathbb{E}\left[e^{-\int_0^t \lambda_\omega^{\varepsilon,I} d\omega} e^{-\int_0^s \lambda_\omega^{\varepsilon,C} d\omega} \left(V(t, X_t) \right)^+ \right]}{\mathbb{Q}(\tau_I > t, \tau_C > s)} = \frac{\mathbb{E}\left[e^{-\int_0^t y_\omega^{\varepsilon,I} d\omega} e^{-\int_0^s y_\omega^{\varepsilon,C} d\omega} \left(V(t, X_t) \right)^+ \right]}{\mathbb{E}\left[e^{-\int_0^t y_\omega^{\varepsilon,I} d\omega} e^{-\int_0^s y_\omega^{\varepsilon,C} d\omega} \right]} \tag{6.3.12}$$

We also define the bilateral *WWR-adjusted* expected positive exposure by

$$BWWREPE^\varepsilon(T_{i-1}, T_i) = \frac{\mathbb{Q}(\tau_I > T_i, \tau_C > T_{i-1})\, u_{V^+}^\varepsilon(T_{i-1}, T_i) - \mathbb{Q}(\tau_I > T_i, \tau_C > T_i)\, u_{V^+}^\varepsilon(T_i, T_i)}{\mathbb{Q}(\tau_I > T_i)\, \mathbb{Q}(\tau_C \in]T_{i-1}, T_i])} \tag{6.3.13}$$

This allows us to write (6.3.4) in a standard representation form of BCVA

$$BCVA^\varepsilon(0) = LGD_C \sum_{i=1}^M D(0,T_i)\, \mathbb{Q}(\tau_I \geq T_i)\, \mathbb{Q}(\tau_C \in]T_{i-1}, T_i])\, BWWREPE^\varepsilon(T_{i-1}, T_i) \tag{6.3.14}$$

In addition, *the WWR-adjusted* expected exposure can be used to compute further XVA adjustments.

Approximation formulas for the price function of derivatives contingent to the default one reference entity are provided at the second and third orders in Chapter 4. They are not sufficient with regard to the definition of $u_{V^+}^\varepsilon$ since two reference entities are involved in the current setting.

6.3.1 Pricing a doubly-defaultable contingent claim

In the following, we address the technical derivation of the expansion formula at the second order for $u^{\varepsilon}_{V^+}$. We define

$$\tilde{u}^{\varepsilon}_{V^+} : (s,t) \mapsto \mathbb{E}\left[e^{-\int_0^t y^{\varepsilon,I}_{\omega} d\omega} e^{-\int_0^s y^{\varepsilon,C}_{\omega} d\omega} \left(V(t,X_t)\right)^+\right] \tag{6.3.15}$$

as the numerator function of $u^{\varepsilon}_{V^+}$ in (6.3.12). First, a Taylor expansion of the process $\left(y^{\varepsilon,k}_t\right)_t$, $k \in \{I,C\}$ w.r.t. ε is given by (4.3.12):

$$y^{\varepsilon,k}_t = y^{\varepsilon,i}_t|_{\varepsilon=0} + \varepsilon y^{\varepsilon,k}_{1,t}|_{\varepsilon=0} + \frac{\varepsilon^2}{2} y^{\varepsilon,k}_{2,t}|_{\varepsilon=0} + R^{y^{\varepsilon,k}}_{2,t}$$

where $R^{y^{\varepsilon,k}}_{2,t}$ is the remainder term. Secondly, we use the Taylor series for the exponential function $x \mapsto e^x$. We retain only terms with powers up to 2 in ε:

$$\begin{aligned}
\tilde{u}^{\varepsilon}_{V^+}(S,T) &= e^{-\int_0^T y^{0,I}_{\omega} d\omega - \int_0^S y^{0,C}_{\omega} d\omega} \mathbb{E}\left[\left(V(T,X_T)\right)^+\right] \\
&- e^{-\int_0^T y^{0,I}_{\omega} d\omega - \int_0^S y^{0,C}_{\omega} d\omega} \mathbb{E}\left[\left(\varepsilon \int_0^T y^{0,I}_{1,\omega} d\omega + \frac{\varepsilon^2}{2} \int_0^T y^{0,I}_{2,\omega} d\omega - \frac{\varepsilon^2}{2}\left(\int_0^T y^{0,I}_{1,\omega} d\omega\right)^2\right)\left(V(T,X_T)\right)^+\right] \\
&- e^{-\int_0^T y^{0,I}_{\omega} d\omega - \int_0^S y^{0,C}_{\omega} d\omega} \mathbb{E}\left[\left(\varepsilon \int_0^S y^{0,C}_{1,\omega} d\omega + \frac{\varepsilon^2}{2} \int_0^S y^{0,C}_{2,\omega} d\omega - \frac{1}{2}\left(\int_0^S y^{0,C}_{1,\omega} d\omega\right)^2\right)\left(V(T,X_T)\right)^+\right] \\
&+ e^{-\int_0^T y^{0,I}_{\omega} d\omega - \int_0^S y^{0,C}_{\omega} d\omega} \mathbb{E}\left[\left(\varepsilon^2 \int_0^T y^{0,I}_{1,\omega} d\omega \int_0^S y^{0,C}_{1,\omega} d\omega\right)\left(V(T,X_T)\right)^+\right] \\
&+ \text{Error}^{\varepsilon}_2
\end{aligned} \tag{6.3.16}$$

The first term of the expansion correspond to the proxy value

$$\tilde{u}^{\varepsilon=0}_{V^+}(S,T) = e^{-\int_0^T y^{0,I}_{\omega} d\omega - \int_0^S y^{0,C}_{\omega} d\omega} \mathbb{E}\left[\left(V(T,X_T)\right)^+\right] \tag{6.3.17}$$

The two next terms have already been developed in Chapter 5. The last term requires additional calculations that we present in sequel. We firstly define the function

$$\bar{u}^{\varepsilon}_{V^+} : (s,t) \mapsto \mathbb{E}\left[e^{-\int_0^t y^{\varepsilon,I}_{\omega} d\omega} e^{-\int_0^s y^{\varepsilon,C}_{\omega} d\omega}\right] \tag{6.3.18}$$

corresponding to the denominator of $u^{\varepsilon}_{V^+}$ in (6.3.12). A second order approximation of $\bar{u}^{\varepsilon}_{V^+}$ is given in the following proposition.

Proposition 6.3.1. *Suppose that the intensity state processes* $\left(y^{\varepsilon,k}_t\right)_{t\geq 0, k\in\{I,C\}}$ *fulfill the regularity conditions of Theorem 4.3.9, one has*

$$\mathbb{E}\left[e^{-\int_0^T y^{\varepsilon,I}_{\omega} d\omega} e^{-\int_0^S y^{\varepsilon,C}_{\omega} d\omega}\right] = \left(1 + C^I_{0,1}(T) + C^C_{0,1}(S) + C_{0,2}(S,T)\right) e^{-\int_0^T y^{0,I}_{\omega} d\omega - \int_0^S y^{0,C}_{\omega} d\omega} + Error^{\varepsilon}_2 \tag{6.3.19}$$

where

$$C_{0,1}^{k}(S) = \frac{1}{2}\int_0^S \left(\tilde{\kappa}_k(t,S)\,\tilde{\nu}_k(t)\right)^2 dt \tag{6.3.20}$$

$$C_{0,2}^{I,C}(S,T) = \rho^{\lambda^I,\lambda^C}\int_0^{T\wedge S} \tilde{\kappa}_C(t,S)\,\tilde{\kappa}_I(t,T)\,\tilde{\nu}_C(t)\,\tilde{\nu}_I(t)\,dt \tag{6.3.21}$$

for $k \in \{I,C\}$. In addition, an estimate of the error term is given in :

$$|Error| \leq \mathfrak{C}(S,T)\max_{k\in\{I,C\}} T^2 M_{k,0}\left(M_{k,1} + TM_{k,0} + T^4 M_{k,0}^3\right)$$

where $\mathfrak{C}(S,T)$ is a positive constant that depends on S and T.

Proof. From Theorem 4.3.9, one has for $k \in \{I,C\}$

$$\mathbb{E}\left[\int_0^S y_{1,\omega}^{0,k}d\omega + \frac{1}{2}\int_0^S y_{2,\omega}^{0,k}d\omega - \frac{1}{2}\left(\int_0^S y_{1,\omega}^{0,k}d\omega\right)^2\right] = -\mathbb{E}\left[\frac{1}{2}\left(\int_0^S y_{1,\omega}^{0,k}d\omega\right)^2\right] = -C_{0,1}^{k}(S)$$

In addition,

$$\int_0^S y_{1,\omega}^{0,k}d\omega = \int_0^T 1_{\omega<S} y_{1,\omega}^{0,k}d\omega = \frac{1}{\kappa_k}\int_0^T 1_{\omega<S}\tilde{\kappa}_k(\omega,S)\,\tilde{\nu}_k(\omega)\,dW_\omega^k$$

One has then

$$\begin{aligned}\int_0^T y_{1,\omega}^{0,I}d\omega \int_0^S y_{1,\omega}^{0,C}d\omega &= \rho^{\lambda^I,\lambda^C}\int_0^{T\wedge S}\tilde{\kappa}_I(\omega,T)\,\tilde{\kappa}_C(\omega,S)\,\tilde{\nu}_I(\omega)\,\tilde{\nu}_C(\omega)\,d\omega\\ &+\int_0^T \tilde{\kappa}_I(\omega,T)\,\tilde{\nu}_I(\omega)\left(\int_0^{\omega\wedge S}\tilde{\kappa}_C(\upsilon,S)\,\tilde{\nu}_C(\upsilon)\,dW_\upsilon^C\right)dW_\omega^I\\ &+\int_0^S\left(1-e^{-\kappa_C(S-\omega)}\right)\nu_C(\omega)\left(\int_0^\omega \tilde{\kappa}_I(\upsilon,T)\,\tilde{\nu}_I(\upsilon)\,dW_\upsilon^I\right)dW_\omega^C\end{aligned}$$

which leads to

$$\mathbb{E}\left[\int_0^T y_{1,\omega}^{0,I}d\omega\int_0^S y_{1,\omega}^{0,I}d\omega\right] = \rho^{\lambda^I,\lambda^C}\int_0^{T\wedge S}\tilde{\kappa}_I(\omega,T)\,\tilde{\kappa}_C(\omega,S)\,\tilde{\nu}_I(\omega)\,\tilde{\nu}_C(\omega)\,d\omega = C_{0,2}(S,T)$$

□

The correction term $C_{0,2}$ depends explicitly on the the correlation parameter $\rho^{\lambda^I,\lambda^C}$ between intensity processes of I and C. $C_{0,1}^I$ and $C_{0,1}^C$ are convexity terms due to credit spreads volatilities. An immediate consequence of the above proposition is the following approximation for the joint survival probability.

$$\begin{aligned}\mathbb{Q}(\tau_I > T, \tau_C > S) &= \mathbb{E}\left[e^{-\int_0^T \lambda_\omega^{1,I}d\omega}e^{-\int_0^S \lambda_\omega^{1,C}d\omega}\right]\\ &= \mathbb{E}\left[e^{-\int_0^T y_\omega^{1,I}d\omega}e^{-\int_0^S y_\omega^{1,C}d\omega}\right]e^{-\int_0^T \psi_\omega^{1,I}d\omega-\int_0^S \psi_\omega^{1,C}d\omega}\\ &= \frac{\mathbb{E}\left[e^{-\int_0^T y_\omega^{1,I}d\omega}e^{-\int_0^S y_\omega^{1,C}d\omega}\right]}{\mathbb{E}\left[e^{-\int_0^T y_\omega^{1,I}d\omega}\right]\mathbb{E}\left[e^{-\int_0^S y_\omega^{1,C}d\omega}\right]}\mathbb{Q}(\tau_I > T)\,\mathbb{Q}(\tau_C > S)\\ &\approx \frac{1+C_{0,1}^I(T)+C_{0,1}^C(S)+C_{0,2}^{I,C}(S,T)}{\left(1+C_{0,1}^I(T)\right)\left(1+C_{0,1}^C(S)\right)}\mathbb{Q}(\tau_I > T)\,\mathbb{Q}(\tau_C > S)\end{aligned} \tag{6.3.22}$$

where we used Corollary 4.3.11.

By regrouping approximations of $\tilde{u}^{\varepsilon}_{V^+}$ ((6.3.16)) and $\bar{u}^{\varepsilon}_{V^+}$ ((6.3.19)) , the second-order expansion formula of $u^{\varepsilon}_{V^+}$ is given in the following theorem.

Theorem 6.3.2 (***Second order approximation price formula***)**.** *Under the regularity conditions of Theorem 4.3.9 for the valuation function V, one has for $k \in \{I,C\}$*

$$\begin{aligned} u^{\varepsilon=1}_{V^+}(S,T) = \mathbb{E}\left[(V(T,X_T))^+\right] - \frac{C^I_{1,1}(T,T)+C^C_{1,1}(S,T)}{1+C^I_{0,1}(T)+C^C_{0,1}(S)+C^{I,C}_{0,2}(S,T)}.Greek^{(1)}_{V^+}(T,X_T) \\ - \frac{C^I_{2,1}(T,T)+C^C_{2,1}(S,T)+C^{I,C}_{2,3}(S,T)}{1+C^I_{0,1}(T)+C^C_{0,1}(S)+C^{I,C}_{0,2}(S,T)}.Greek^{(2)}_{V^+}(T,X_T)+Error^{\varepsilon=1}_2 \end{aligned} \tag{6.3.23}$$

where $C^k_{0,1}$ and $C^{I,C}_{0,2}$ are given in Proposition 6.3.1,

$$C^k_{1,1}(S,T) = \int_0^S \tilde{\kappa}_k(t,S)\,\tilde{\nu}_k(t)\,\tilde{\Sigma}(t,T)\,\rho^{X,\lambda^k}dt \tag{6.3.24}$$

$$C_{2,1}(S,T) = \left(c^k_{2,1}-c^k_{2,2}\right)(S,T) \tag{6.3.25}$$

$$C^{I,C}_{2,3}(S,T) = \left(c^{I,C}_{2,3}+c^{I,C}_{2,4}\right)(S,T) \tag{6.3.26}$$

$$c^k_{2,1}(S,T) = \int_0^S \left(\int_t^S \tilde{\kappa}_k(s,S)\,\nu^{(1)}_k(s)\left[\tilde{\Sigma}(s,T)\,\rho^{X,\lambda^k}\right]ds\right)\tilde{\nu}_k(t)\left[\tilde{\Sigma}(t,T)\,\rho^{X,\lambda^k}\right]dt \tag{6.3.27}$$

$$c^k_{2,2}(S,T) = \int_0^S \left(\int_t^S \tilde{\kappa}_k(s,S)\,\tilde{\nu}_k(s)\left[\tilde{\Sigma}(s,T)\,\rho^{X,\lambda^k}\right]ds\right)\tilde{\kappa}_k(t,S)\,\tilde{\nu}_k(t)\left[\tilde{\Sigma}(t,T)\,\rho^{X,\lambda^k}\right]dt \tag{6.3.28}$$

$$c^{I,C}_{2,3}(S,T) = \int_0^S \left(\int_t^T \tilde{\kappa}_I(s,T)\,\tilde{\nu}_I(s)\left[\tilde{\Sigma}(s,T)\,\rho^{X,\lambda^I}\right]ds\right)\tilde{\kappa}_C(t,S)\,\tilde{\nu}_C(t)\left[\tilde{\Sigma}(t,T)\,\rho^{X,\lambda^C}\right]dt \tag{6.3.29}$$

$$c^{I,C}_{2,4}(S,T) = \int_0^S \left(\int_t^S \tilde{\kappa}_C(s,S)\,\tilde{\nu}_C(s)\left[\tilde{\Sigma}(s,T)\,\rho^{X,\lambda^C}\right]ds\right)\tilde{\kappa}_I(t,T)\,\tilde{\nu}_I(t)\left[\tilde{\Sigma}(t,T)\,\rho^{X,\lambda^I}\right]dt \tag{6.3.30}$$

In addition, the error term is bounded by

$$|Error| \le \mathfrak{C}(S,T)\,\|V(T,X_T)\|_2 \max_{k\in\{I,C\}} T^2 M_{k,0}\left(M_{k,1}+TM_{k,0}+T^4M^3_{k,0}\right)$$

for some positive constant $\mathfrak{C}(S,T)$ that depends on S and T.

Proof. From Theorem 4.3.9, one has for $k \in \{I,C\}$

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^S y^{0,k}_{1,\omega}d\omega+\frac{1}{2}\int_0^S y^{0,k}_{2,\omega}d\omega-\frac{1}{2}\left(\int_0^S y^{0,k}_{1,\omega}d\omega\right)^2\right)(V(T,X_T))^+\right] = -C^k_{0,1}(S,T)\,\mathbb{E}\left[(V(T,X_T))^+\right] \\ +c^k_{1,1}(S,T).Greek^{(1)}_{V^+}(T,X_T)+\left(c^k_{2,1}-c^k_{2,2}\right)(S,T).Greek^{(2)}_{V^+}(T,X_T) \end{aligned}$$

In particular, using Lemma 4.6.2, one has from the Itô development of $\int_0^T y_{1,\omega}^{0,I} d\omega \int_0^S y_{1,\omega}^{0,C} d\omega$

$$\mathbb{E}\left[\int_0^T \tilde{\kappa}_I(t,T)\,\tilde{v}_I(t)\left(\int_0^t 1_{s<S}\tilde{\kappa}_C(s,S)\,\tilde{v}_C(s)\,dW_s^C\right)dW_t^I\,(V(T,X_T))^+\right] = c_{2,3}^{I,C}(S,T)\,.Greek_{V^+}^{(2)}(T,X_T)$$

and

$$\mathbb{E}\left[\int_0^S \tilde{\kappa}_C(t,S)\,\tilde{v}_C(t)\left(\int_0^t \tilde{\kappa}_I(s,T)\,\tilde{v}_I(s)\,dW_s^I\right)dW_t^C\,(V(T,X_T))^+\right] = c_{2,4}^{I,C}(S,T)\,.Greek_{V^+}^{(2)}(T,X_T)$$

Finally, based on Proposition 6.3.1, one has

$$\begin{aligned}
&\left|u_{V^+}^{\varepsilon=1}(S,T) - \frac{\mathbb{E}\left[e^{-\int_0^T y_\omega^{1,I}d\omega}e^{-\int_0^S y_\omega^{1,C}d\omega}\,(V(T,X_T))^+\right]}{e^{-\int_0^T y_\omega^{0,I}d\omega}e^{-\int_0^S y_\omega^{0,C}d\omega}\left(1+C_{0,1}^I(T)+C_{0,1}^C(S)+C_{0,2}^{I,C}(S,T)\right)}\right| \\
&\leq \mathfrak{C}(S,T)\left|\frac{\mathbb{E}\left[e^{-\int_0^T y_\omega^{1,I}d\omega}e^{-\int_0^S y_\omega^{1,C}d\omega}\right]}{e^{-\int_0^T y_\omega^{0,I}d\omega}e^{-\int_0^S y_\omega^{0,C}d\omega}} - \left(1+C_{0,1}^I(T)+C_{0,1}^C(S)+C_{0,2}^{I,C}(S,T)\right)\right| \\
&\leq \mathfrak{C}(S,T)\max_{k\in\{I,C\}} T^2 M_{k,0}\left(M_{k,1}+TM_{k,0}+T^4M_{k,0}^3\right)\|V(T,X_T)\|_2
\end{aligned}$$

for some positive constant $\mathfrak{C}(S,T)$ that depends on S and T. $\mathbb{E}\left[e^{-\int_0^T y_\omega^{\varepsilon,I}d\omega}e^{-\int_0^S y_\omega^{\varepsilon,C}d\omega}\right]$ is then approximated by

$$e^{-\int_0^T y_\omega^{0,I}d\omega}e^{-\int_0^S y_\omega^{0,C}d\omega}\left(1+C_{0,1}^I(T)+C_{0,1}^C(S)+C_{0,2}^{I,C}(S,T)\right)$$

and injected in the previous formulas. Equation (6.3.12) is obtained by applying (6.3.23). Finally, the upper bound estimate of the error term is given by Theorem 6.4.3. □

6.3.2 Expansion formula an bilateral *Wrong-way* risk

We deduce the following second order approximation for the WWR-adjusted exposure of (6.3.13): $\forall i \in [\![1,\ldots,M]\!]$,

$$\begin{aligned}
&WWREPE^{\varepsilon}(T_{i-1},T_i) \\
&= \mathbb{E}\left[(V(T_{T_i},X_{T_i}))^+\right] - \left(\tilde{C}_{1,1}^I(T_{i-1},T_i)+\tilde{C}_{1,1}^C(T_{i-1},T_i)\right).Greek_{V^+}^{(1)}(T_i,X_{T_i}) \\
&- \left(\tilde{C}_{2,1}^I(T_i,T_i)+\tilde{C}_{2,1}^C(T_{i-1},T_i)+\tilde{C}_{2,3}^{I,C}(T_{i-1},T_i)\right).Greek_{V^+}^{(2)}(T_i,X_{T_i}) + \text{Error}^{\varepsilon=1}
\end{aligned} \tag{6.3.31}$$

where

$$\tilde{C}_{i,j}(T_{i-1},T_i) = \frac{\mathbb{Q}(\tau_I>T_i,\tau_C>T_{i-1})\,C_{i,j}(T_{i-1},T_i) - \mathbb{Q}(\tau_I>T_i,\tau_C>T_i)\,C_{i,j}(T_i,T_i)}{\mathbb{Q}(\tau_I>T_i)\,\mathbb{Q}(\tau_C\in]T_{i-1},T_i])\left(1+C_{0,1}^I(T_i)+C_{0,1}^C(T_{i-1})+C_{0,2}^{I,C}(T_{i-1},T_i)\right)} \tag{6.3.32}$$

An approximation of the bilateral CVA is finally obtained by injecting (6.3.31) in .(6.3.14). Thanks to a symmetry argument, we derive analog formula for WWR-adjusted negative exposure:

$$\begin{aligned} &WWRENE^{\varepsilon}(T_{i-1},T_i) \\ &= \mathbb{E}\left[(-V(T_{T_i},X_{T_i}))^+\right] - \left(\bar{C}^C_{1,1}(T_{i-1},T_i)+\bar{C}^I_{1,1}(T_{i-1},T_i)\right).Greek^{(1)}_{(-V)^+}(T_i,X_{T_i}) \\ &- \left(\bar{C}^C_{2,1}(T_i,T_i)+\bar{C}^I_{2,1}(T_{i-1},T_i)+\bar{C}^{C,I}_{2,3}(T_{i-1},T_i)\right).Greek^{(2)}_{(-V)^+}(T_i,X_{T_i}) + \text{Error}^{\varepsilon=1} \end{aligned} \tag{6.3.33}$$

where

$$\bar{C}_{i,j}(T_{i-1},T_i) = \frac{\mathbb{Q}(\tau_C>T_i,\tau_I>T_{i-1})C_{i,j}(T_{i-1},T_i) - \mathbb{Q}(\tau_C>T_i,\tau_I>T_i)C_{i,j}(T_i,T_i)}{\mathbb{Q}(\tau_C>T_i)\mathbb{Q}(\tau_I\in]T_{i-1},T_i])\left(1+C^C_{0,1}(T_i)+C^I_{0,1}(T_{i-1})+C^{C,I}_{0,2}(T_{i-1},T_i)\right)} \tag{6.3.34}$$

6.4 Error Analysis

The error analysis is based on the explicit representation of remainder terms within the Taylor expansions. In the current case of bilateral defaults, we use the multivariate Faà Di Bruno formula with integral remainder given by Leipnik and Pearce in [LP07] (Theorem 3.1):

Lemma 6.4.1 (Di Bruno formula with two variables)**.** *Let f be an $\mathbb{R}^2$-valued bivariate function. Suppose that $\left(\frac{\partial}{\partial z_1}\right)^{q_1}\left(\frac{\partial}{\partial z_1}\right)^{q_2} f(z_1,z_2)$ exists and is jointly continuous in z_1 and z_2 for a given orders $q_1,q_2\in\mathbb{N}$. Then, for $0\le s_1\le q_1$ and $0\le s_2\le q_2$, we have*

$$f(z_1+h_1,z_2+h_2) = \sum_{m_1=0}^{s_1}\sum_{m_2=0}^{s_2}\frac{h_1^{m_1}h_2^{m_2}}{m_1!m_2!}D^{(m_1,m_2)}_{z_1,z_2}f(z_1,z_2)+R_{s_1,s_2} \tag{6.4.1}$$

where

$$\begin{aligned} R_{s_1,s_2} &= \sum_{m_1=0}^{s_1}\frac{h_1^{m_1}}{m_1!}\int_{z_2}^{z_2+h_2}\frac{(z_2+h_2-y_2)^{s_2}}{s_2!}D^{(m_1,s_2+1)}_{z_1,y_2}f(z_1,y_2)\,dy_2 \\ &+ \sum_{m_2=0}^{s_2}\frac{h_2^{m_2}}{m_2!}\int_{z_1}^{z_1+h_1}\frac{(z_1+h_1-y_1)^{s_1}}{s_1!}D^{(s_1+1,m_2)}_{y_1,z_2}f(y_1,z_2)\,dy_1 \\ &+ \int_{z_1}^{z_1+h_1}\int_{z_2}^{z_2+h_2}\frac{(z_1+h_1-y_1)^{s_1}}{s_1!}\frac{(z_2+h_2-y_2)^{s_2}}{s_2!}D^{(s_1+1,s_2+1)}_{y_1,y_2}f(y_1,y_2)\,dy_1dy_2 \end{aligned} \tag{6.4.2}$$

$D^{(m_1,m_2)}_{z_1,z_2}f(z_1,z_2)$ denotes the differential operator

$$D^{(m_1,m_2)}_{z_1,z_2}f(z_1,z_2) = \frac{\partial^{m_1+m_2}}{\partial z_1^{m_1}\partial z_2^{m_2}}f(z_1,z_2)$$

We introduce the following upper-bound operator.

Notation 6.4.2. For the upper bounds, we use the unique notation $A \leq_c B$ to assert that $A \leq cB$ where c is a positive constant depending on the modeling parameters and other pricing constants. The constant c remains bounded when the model parameters go to 0, and it is uniform w.r.t. the parameter $\varepsilon \in [0,1]$. In particular, c depends on the time parameters S and T. We denote by $\leq_{c(S,T)}$ the comparison operator where this dependence is made explicit.

Theorem 6.4.3. *The error term of Proposition 6.4.3 satisfies the following estimate*

$$|Error| \leq_{c(S,T)} \max_{k\in\{I,C\}} T^2 M_{k,0}\left(M_{k,1} + TM_{k,0} + T^4 M_{k,0}^3\right) \|V(T,X_T)\|_2 \tag{6.4.3}$$

Proof. We proceed firstly by applying Di Bruno Lemma 6.4.1 with $f(z_1,z_2) = e^{-z_1-z_2}$,$s_1 = s_2 = 2$, $z_1 = \int_0^T y_\omega^{0,I} d\omega$, $z_2 = \int_0^S y_\omega^{0,I} d\omega$, $h_1 = \varepsilon \int_0^T y_{1,\omega}^{0,I} d\omega + \frac{\varepsilon^2}{2}\int_0^T y_{2,\omega}^{0,I} d\omega + R_{2,T}^{\left(\int_0^t y_\omega^{\varepsilon,I} d\omega\right)_t}$ and $h_2 = \varepsilon \int_0^S y_{1,\omega}^{0,C} d\omega + \frac{\varepsilon^2}{2}\int_0^S y_{2,\omega}^{0,C} d\omega + R_{2,S}^{\left(\int_0^t y_\omega^{\varepsilon,C} d\omega\right)_t}$. The error term reads then

$$\begin{aligned}
&Error \\
&= -f(z_1,z_2)\,\mathbb{E}\left[R_{2,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega} - \frac{1}{2}\left(2\varepsilon\left(\int_0^T y_{1,\omega}^I d\omega\right) R_{1,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega} + \left(R_{1,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^2\right)\right] \|V(T,X_T)\|_2 \\
&- f(z_1,z_2)\,\mathbb{E}\left[R_{2,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega} - \frac{1}{2}\left(2\varepsilon\left(\int_0^S y_{1,\omega}^C d\omega\right) R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega} + \left(R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^2\right)\right] \|V(T,X_T)\|_2 \\
&+ f(z_1,z_2)\,\mathbb{E}\left[\varepsilon\left(\int_0^S y_{1,\omega}^C d\omega\right) R_{1,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega} + \varepsilon\left(\int_0^T y_{1,\omega}^I d\omega\right) R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right] \|V(T,X_T)\|_2 \\
&- \frac{1}{2} f(z_1,z_2)\,\mathbb{E}\left[R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\left(R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^2 + \left(R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^2 R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right] \|V(T,X_T)\|_2 \\
&+ R_{2,2}\,\|V(T,X_T)\|_2
\end{aligned}$$

where we have used the fact that $D_{z_1,z_2}^{(m_1,m_2)} f(z_1,z_2) = (-1)^{m_1+m_2}(z_1,z_2)$; $R_{2,2}$ being the integral remainder of Lemma 6.4.1:

$$\begin{aligned}
&R_{s_1,s_2} \\
&= \frac{1}{2}\mathbb{E}\left[\left(R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^2 \sum_{m_1=0}^{2} (-1)^{m_1+1} \frac{\left(R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^{m_1}}{m_1!} \int_0^\varepsilon (1-\xi)^2 f\left(z_1, \int_0^S \xi y_\omega^{\xi,C} + (1-\xi) y_\omega^C d\omega\right) d\xi\right] \\
&+ \frac{1}{2}\mathbb{E}\left[\left(R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^2 \sum_{m_2=0}^{2} (-1)^{m_2+1} \frac{\left(R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^{m_2}}{m_2!} \int_0^\varepsilon (1-\xi)^2 f\left(\int_0^T \xi y_\omega^{\xi,I} + (1-\xi) y_\omega^I d\omega, z_2\right) d\xi\right] \\
&+ \frac{1}{4}\mathbb{E}\left[\begin{array}{c} \left(R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^2 \left(R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^2 \times \\ \int_0^\varepsilon \int_0^\varepsilon (1-\xi_1)^2 (1-\xi_2)^2 f\left(\int_0^T \xi_1 y_\omega^{\xi_1,I} + (1-\xi_1) y_\omega^I d\omega, \int_0^S \xi_2 y_\omega^{\xi_2,C} + (1-\xi_2) y_\omega^C d\omega\right) d\xi_1 d\xi_2 \end{array}\right]
\end{aligned}$$

In a second step, observe for $k,k' \in \{I,C\}$, $x,x' \in \{S,T\}$ and $\xi,\xi' \in [0,1]$ that

$$f\left(\int_0^x \xi y_\omega^{\xi,k} + (1-\xi)\, y_\omega^k d\omega, \int_0^{x'} \xi' y_\omega^{\xi',k'} + \left(1-\xi'\right) y_\omega^{k'} d\omega\right)$$

is bounded by a constant that depends on S and T. In addition, if $x,y \in \mathbb{R}$ has finite $p-$norm, $p \geq 1$, then $|x+y|^p \leq 2^{p-1}\left(|f|^p + |g|^p\right)$, it follows that

$$\begin{aligned}
&|Error|^2 \\
&\leq_{c(S,T)} \left(\left\|R_{2,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_2^2 + \left\|\left(\int_0^T y_{1,\omega}^I d\omega\right) R_{1,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_2^2 + \left\|R_{1,T}^{\left(\int_0^t y_\omega^{\varepsilon,I} d\omega\right)_t}\right\|_4^4 + \left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_4^4\right) \|V(T,X_T)\|_2^2 \\
&+ \left(\left\|R_{2,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_2^2 + \left\|\left(\int_0^S y_{1,\omega}^C d\omega\right) R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_2^2 + \left\|R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4 + \left\|R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4\right) \|V(T,X_T)\|_2^2 \\
&+ \left(\left\|\left(\int_0^S y_{1,\omega}^C d\omega\right) R_{1,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_2^2 + \left\|\left(\int_0^T y_{1,\omega}^I d\omega\right) R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_2^2\right) \|V(T,X_T)\|_2^2 \\
&+ \left(\left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\left(R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^2\right\|_2^2 + \left\|\left(R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^2 R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_2^2\right) \|V(T,X_T)\|_2^2 \\
&+ \left(\left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega} R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4\right) \|V(T,X_T)\|_2^2
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\left|R_{2,2}\right|^2 \leq_{c(S,T)} &\left\|R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4 + \left\|\left(R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right)^2 R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_2^2 \\
&+ \left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_4^4 + \left\|\left(R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right)^2 R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_2^2 + \left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega} R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4
\end{aligned}$$

We now use the inequality $\mathbb{E}|XY| \leq \frac{1}{2}\left(\mathbb{E}\left[|X|^2\right] + \mathbb{E}\left[|Y|^2\right]\right)$:

$$\begin{aligned}
|Error|^2 \leq_{c(S,T)} &\left(\left\|R_{2,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_2^2 + \left\|\int_0^T y_{1,\omega}^I d\omega\right\|_4^4 + \left\|R_{1,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_4^4 + \left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4\right) \|V(T,X_T)\|_2^2 \\
&+ \left(\left\|R_{2,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_2^2 + \left\|\int_0^S y_{1,\omega}^C d\omega\right\|_4^4 + \left\|R_{1,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4 + \left\|R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_4^4\right) \|V(T,X_T)\|_2^2 \\
&+ \left(\left\|R_{0,T}^{\int_0 y_\omega^{\varepsilon,I} d\omega}\right\|_8^8 + \left\|R_{0,S}^{\int_0 y_\omega^{\varepsilon,C} d\omega}\right\|_8^8\right) \|V(T,X_T)\|_2^2
\end{aligned}$$

The last step consists of applying L_p upper-bound estimates of $\int_0^x y_{1,\omega}^k d\omega$ and $R_{i,x}^{\left(\int_0^t y_\omega^{\varepsilon,k} d\omega\right)_t}$ for $k \in \{I,C\}$, $x \in \{S,T\}$ and $i \in \{0,1,2\}$. From Proposition 4.6.8 and Corollary 4.6.10, one has

$$
\begin{aligned}
|Error|^2 \leq_{c(S,T)} & \left(T^4 M_{I,0}^2 M_{I,1}^2 + T^6 M_{I,0}^4 + T^8 M_{I,0}^4 M_{I,1}^4\right) \|V(T,X_T)\|_2^2 \\
& + \left(S^4 M_{C,0}^2 M_{C,1}^2 + S^6 M_{C,0}^4 + S^8 M_{C,0}^4 M_{C,1}^4\right) \|V(T,X_T)\|_2^2 \\
& + \left(T^{12} M_{I,0}^8 + S^{12} M_{I,0}^8\right) \|V(T,X_T)\|_2^2
\end{aligned}
$$

□

Remark 6.4.4. An alternative derivation of the error bound can be performed by expressing cross hazard rate terms ($R_{0,T}^{\left(\int_0^t y_\omega^{\varepsilon,I} d\omega\right)_t} R_{0,S}^{\left(\int_0^t y_\omega^{\varepsilon,C} d\omega\right)_t}$, $R_{0,T}^{\left(\int_0^t y_\omega^{\varepsilon,I} d\omega\right)_t} \int_0^S y_\omega^{\varepsilon,C} d\omega, \int_0^T y_\omega^{\varepsilon,I} d\omega R_{0,S}^{\left(\int_0^t y_\omega^{\varepsilon,C} d\omega\right)_t}$) in terms of Wiener integrals. In particular, this allows to highlight the dependence of the error bound on the correlation $\rho^{\lambda_I,\lambda_C}$.

Chapter 7

Specific WWR : The case of rating triggers

We present in this chapter a practical method for the valuation of xVA adjustments in presence of rating migration events specified in the collateral agreement. Rating migration events integrates dependence of margin thresholds and independent amounts such that more collateral is requested when the credit quality of one of the parties is worsening. The major feature of the approach is the construction of a term structure of rating transition matrices that reproduces market-implied default probabilities. In addition, the calibration procedure is constrained to ensure the generation of admissible stochastic matrices while the fitting error is almost nil.

Accounting for rating triggers can be assimilated to the valuation of *Right-way risk* exposure. We show the impact of risk-neutralizing rating migration matrices for pricing the unilateral and bilateral xVA adjustments, and we give elements for comparison with historical transition models.

Keywords: xVA valuation, Rating triggers, risk neutral rating transitions, Rating *right-way risk*

7.1 Context

Rating triggers are mechanisms for controlling the bank's exposure to counterparty risk. By making the collateral threshold based on the counterparty's rating, the bank is insured to reduce its exposure when the counterparty is downgraded. Rating triggers can also include early termination events, giving the bank the right to terminate and close-out the underlying deal. Because a default is often preceded, in most cases, by credit downgrades, rating triggers can be thought as right-way risk mechanism, where the counterparty exposure is eliminated if its credit quality worsens. However, the possibility that the counterparty defaults without triggering the threshold remains, leading to ineluctable counterparty risk.

During the financial crisis, the role of rating triggers contract in causing the vulnerability of financial institutions were pointed out. The case of AIG is probably one of the most illustrative examples of how funding and liquidity costs due to collateral posting has led the insurance giant to the verge of collapse ([RMN15]). The liquidity problems of AIG essentially stemmed from

the requirement to post additional 20$ billion collateral due to its rating downgrade. Rating triggers have also been highlighted as ineffective due to the slow reaction of credit ratings ([Cam13]). However, rating triggers remain a traditional option embedded within collateral agreements. Different institutions are entitled to follow these rating clauses, for instance when dealing with supranational banks or sovereign debt agencies.

The integration of rating migration events in pricing xVA adjustments requires risk neutral transition probability matrices. A transition probability matrix (TPM) must be a valid Markov matrix, capable of accurately reproducing market-implied term structures of default probabilities. The valuation of counterparty risk under rating migration events has been addressed in Zhou in [Zho13a, Zho13b]. While [Zho13b] tackles the risk -neutralization of rating migration matrices, [Zho13a] presents the calculation of CVA adjustments in presence of rating-based termination events.

Our approach is different from Zhou's. First, we focus on CVA valuation in the case of rating-based collateral (threshold, independent amount), without including termination. Our major contribution consists of proposing an alternative modeling and calibration approaches. We model rating migration for each counterparty and we perform calibration w.r.t. the counterparty's market default probabilities. We add appropriate constraints in order to ensure that validity of the migration matrices, and to guarantee consistency of default probabilities that result from both calibrations.

We highlight that there are often inconsistencies between bond agency ratings and their market prices. This results in numerical instabilities and significant fitting errors while performing a joint calibration of a single migration model to a synthetic (rating-based) market probabilities. Our procedure allows to overcome this issue. We define the *rating-adjusted* expected exposure in both the unilateral and bilateral xVA frameworks. We give a convenient formulation of it, in terms of the classical exposure and conditional migration probabilities to default.

Outline of the chapter

The chapter is organized as follows: In Section 7.2, we address the derivation of *rating-adjusted* exposure in presence of unilateral or bilateral migrations. Modeling and calibration approaches are detailed in Section 7.3. Section 7.4 is dedicated to numerical experiments. Our conclusions are given in Section 7.5.

Notations

- I ,C, institution and counterparty subscripts
- $\mathbb{P}$ resp.$\mathbb{Q}$ the real resp. risk-neutral probability measure, $P(t,T)$ resp. $Q(t,T)$ the real and risk-neutral transition matrix from t to T.
- $\mathscr{P}_\delta$ the $\delta-$year historical transition probability matrix
- Λ_δ is the generator matrix of $\mathscr{P}_\delta$.
- $MtM(t)$ the mark-to-market of the portfolio at time t
- $\Pi_k = \{1 \dots K\}$ the set of attainable ratings for counterparty k

- $\mathscr{R}_k(t)$ the rating of counterparty k at time t. $\mathscr{R}_k(t) = K$ denotes default.
- r_k the rating of l at inception
- $\Pi_k^\star = \{1 \dots K-1\}$ the rating set of k excluding default
- $DP_k(t)$ the market default probability over time t associated to counterparty k
- $H_k(i)$, $IA_k(i)$ resp. the margin threshold and independent amount for the counterparty k at rating i
- τ_k the default time of counterparty k
- LGD_k the loss given default of counterparty k
- $(T_i)_{i=1\dots N}$ the default time grid
- $\tau = \tau_I \wedge \tau_C$
- $[c]_K$: vector of K constant elements c
- $Diag(v)$: a zero square matrix with vector's v elements on the diagonal
- I_K the identity matrix of size K

7.2 Rating Migrations within valuation adjustments

We consider collateral agreements with a specific rating dependence imposed for margin thresholds and independent amounts. Such agreements are designed to reduce the exposure by increasing collateral requirements when one counterparty is downgraded. From the point of view of I, we denote by $PE(t, H_C, IA_C)$ the positive exposure that depends on threshold $H_C$on the *Independent Amount* A_C. Its expectation is denoted by

$$EPE(t, H_C, IA_C) = \mathbb{E}[PE(t, H_C, IA_C)] \tag{7.2.1}$$

Similarly, $NE(t, H_B, IA_B)$ denotes the counterparty's exposure in case of I default and its expectation. ENE corresponds to its expectation.

We assume that there is no correlation between exposure and the default or ratings realizations. The following framework can be easily extended in case where such a correlation risk is recognized. In the sequel, we allow for $H_k = H_k(\mathscr{R}_k(t))$ and $IA_k = IA_k(\mathscr{R}_k(t))$ to depend on the rating $\mathscr{R}_k(t)$ at time t for $k \in \{I, C\}$.

7.2.1 The unilateral framework

In the unilateral framework, the CVA with rating-dependent CSA reads

$$UCVA_R(0) = LGD_C \mathbb{E}\left[D(0, \tau_C) 1_{\tau_C \leq T} PE\left(\tau_C, H_c\left(\mathscr{R}_C\left(\tau_C^-\right)\right), IA_C\left(\mathscr{R}_C\left(\tau_C^-\right)\right)\right)\right] \tag{7.2.2}$$

where $\mathscr{R}_C\left(\tau_C^-\right)$ is the rating of C immediately before default. With a time grid $\mathscr{T} = (T_i)_{i=1\ldots N}$ for the default events, we assume that the margin is determined at T_{i-1} for a default at T_i. Using the Euler discretization, one has

$$\begin{aligned}
&UCVA_R(0)\\
&= LGD_C \sum_{i=1}^{N} D(0,T_i) \sum_{j\in\Pi_C} EPE\left(T_i, H_C(j), IA_C(j)\right) \mathbb{Q}\left(\tau_C \in]T_{i-1},T_i], \mathscr{R}_C(T_{i-1}) = j\right)\\
&= LGD_C \sum_{i=1}^{N} D(0,T_i) \sum_{j\in\Pi_C^\star} EPE\left(T_i, H_C(j), IA_C(j)\right) \mathbb{Q}\left(\tau_C \leq T_i, \mathscr{R}_C(T_{i-1}) = j\right)\\
&= LGD_C \sum_{i=1}^{N} D(0,T_i) \sum_{j\in\Pi_C^\star} EPE\left(T_i, H_C(j), IA_C(j)\right) \mathbb{Q}\left(\tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j\right) \mathbb{Q}\left(\mathscr{R}_C(T_{i-1}) = j\right)
\end{aligned} \tag{7.2.3}$$

In particular,

- $EPE\left(T_i, H_C(j), IA_C(j)\right)$ is the exposure given the rating j at T_{i-1} and the occurrence of the default in $\tau_C \in]T_{i-1}, T_i]$.
- $\mathbb{Q}\left(\tau_C \leq T_i, \mathscr{R}_C(T_{i-1}) = j\right)$ is the joint probability of default before T_i and rating realizations at time T_{i-1} in $\Pi_C^\star$. It is more appropriate to express it in terms of conditional probability

  $$\mathbb{Q}\left(\tau_C \leq T_i, \mathscr{R}_C(T_{i-1}) = j\right) = \mathbb{Q}\left(\tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j\right) \mathbb{Q}\left(\mathscr{R}_C(T_{i-1}) = j\right) \tag{7.2.4}$$

  where $\mathbb{Q}\left(\mathscr{R}_C(T_{i-1}) = j\right)$ is the cumulative rating migration to j at time T_{i-1} $\mathbb{Q}\left(\tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j\right)$ the default probability before T_i conditionally the rating j at time T_{i-1}.

We introduce the *Rating-adjusted Positive Exposure*, EPE_R^U for short, such that the generic xVA formulation is preserved

$$XVA_R(0) = \sum_{i=1}^{N} D(0,T_i)\, Cost^{XVA}(T_{i-1},T_i)\, EPE_R^U(T_{i-1},T_i) \tag{7.2.5}$$

In particular,

$$UCVA_R(0) = LGD_C \sum_{i=1}^{N} D(0,T_i)\, \mathbb{Q}\left(\tau_C \in]T_{i-1},T_i]\right) EPE_R^U(T_{i-1},T_i) \tag{7.2.6}$$

For this end, EPE_R^U satisfies the definition

$$EPE_R^U(T_{i-1},T_i) = \sum_{j\in\Pi_C^\star} EPE\left(T_i, H_C(j), IA_C(j)\right) \frac{\mathbb{Q}\left(\tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j\right) \mathbb{Q}\left(\mathscr{R}_C(T_{i-1}) = j\right)}{\mathbb{Q}\left(\tau_C \in]T_{i-1},T_i]\right)} \tag{7.2.7}$$

This contrasts with classical expected exposure $EPE(T_i)$ where $H_C(j)$ and $IA_C(j)$ are constants. We finally emphasize that EPE_R^U is a weighted sum of the expected exposures

$$EPE_R^U(T_{i-1},T_i) = \sum_{j\in\Pi_C^\star} EPE(T_i,H_C(j),IA_C(j))\,\omega_{C,j}^U(T_{i-1},T_i) \tag{7.2.8}$$

where

$$\omega_{C,j}^U(T_{i-1},T_i) = \frac{\mathbb{Q}(\tau_C \le T_i|\mathscr{R}_C(T_{i-1})=j)\,\mathbb{Q}(\mathscr{R}_C(T_{i-1})=j)}{\mathbb{Q}(\tau_C \in]T_{i-1},T_i])} \in [0,1] \tag{7.2.9}$$

One has

$$\sum_{j\in\Pi_C^\star} \omega_{C,j}^U(T_{i-1},T_i) = 1 \tag{7.2.10}$$

since $\sum_{j\in\Pi_C^\star}\mathbb{Q}(\tau_C \le T_i|\mathscr{R}_C(T_{i-1})=j)\,\mathbb{Q}(\mathscr{R}_C(T_{i-1})=j) = \mathbb{Q}(\tau_C\in]T_{i-1},T_i])$.

7.2.2 The bilateral framework

The bilateral framework is derived similarly. The bilateral CVA is defined by

$$BCVA_R(0) = LGD_C\mathbb{E}\left[D(0,\tau_C)\,1_{\tau\le T}1_{\tau=\tau_C}PE\left(\tau_C,H_C\left(\mathscr{R}_C\left(\tau_C^-\right)\right),IA_C\left(\mathscr{R}_C\left(\tau_C^-\right)\right)\right)\right] \tag{7.2.11}$$

The discretized expression reads

$$BCVA_R = LGD_C\sum_{i=1}^N D(0,T_i)\sum_{j\in\Pi_C^\star} EPE(T_i,H_C(j),IA_C(j))\,\mathbb{Q}(\tau_I>T_i\tau_C\le T_i,\mathscr{R}_C(T_{i-1})=j) \tag{7.2.12}$$

A symmetric formula can be easily derived for BDVA.

We neglect the loss or gain to the bank when both parties default in the same period $]T_{i-1},T_i]$. In fact, the probability of simultaneous default is expected to be very small. In addition, the expected exposures in case of simultaneous defaults would wipe out when taking the aggregated adjustment $BCVA_R - DVA_R$. As for the unilateral case, we exhibit rating transition probabilities in $BCVA_R$ expression

$$\mathbb{Q}(\tau_I>T_i\tau_C\le T_i,\mathscr{R}_C(T_{i-1})=j) = \mathbb{Q}(\tau_I>T_i\tau_C\le T_i|\mathscr{R}_C(T_{i-1})=j)\,\mathbb{Q}\left(\mathscr{R}_C\left(T_{j-1}\right)=j\right) \tag{7.2.13}$$

where $\mathbb{Q}(\tau_I>T_k\tau_C\le T_k|\mathscr{R}_C(T_{k-1})=j)$ is a joint conditional rating transition probability. It is calculated using single name conditional rating migration to default, as well as an explicit dependence structure (copula). We give a detailed description of this procedure in the sequel.

The bilateral *Rating-adjusted exposure* EPE_R^B is defined as

$$EPE_R^B(T_{i-1},T_i) = \sum_{j\in\Pi_C^\star} EPE(T_i,H_C(j),IA_C(j))\,\frac{\mathbb{Q}(\tau_I>T_i\tau_C\le T_i|\mathscr{R}_C(T_{i-1})=j)\,\mathbb{Q}(\mathscr{R}_C(T_{i-1})=j)}{\mathbb{Q}(\tau_I>T_k,\tau_C\in]T_{i-1},T_i])} \tag{7.2.14}$$

and leads to preserve the classical formulation for bilateral CVA

$$BCVA_R(0) = LGD_C \sum_{i=1}^{N} D(0,T_i)\,\mathbb{Q}(\tau_C \in]T_{i-1},T_i])\,\mathbb{Q}(\tau_I > T_i)\,EPE_R^B(T_{i-1},T_i) \tag{7.2.15}$$

or any other xVA adjustment

$$XVA_R(0) = \sum_{i=1}^{N} D(0,T_i)\,Cost^{XVA}(T_{i-1},T_i)\,EPE_R^B(T_{i-1},T_i) \tag{7.2.16}$$

Finally, it is more convenient to write

$$EPE_R^B(T_{i-1},T_i) = LGD_C \sum_{j\in\Pi_C^\star} EPE(T_i, H_C(j), IA_C(j))\,\omega_{C,j}^B(T_{i-1},T_i) \tag{7.2.17}$$

such that

$$\omega_{C,j}^B(T_{i-1},T_i) = \frac{\mathbb{Q}(\tau_I > T_i \tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j)\,\mathbb{Q}(\mathscr{R}_C(T_{i-1}) = j)}{\mathbb{Q}(\tau_I > T_k, \tau_C \in]T_{i-1},T_i])} \tag{7.2.18}$$

Remark 7.2.1. The unilateral or bilateral adjusted exposures in 7.2.5 and 7.3.24 are related to the nature of rating migration dynamics. XVA costs can be either unilateral or bilateral, according to the dependence on default events.

7.2.2.1 Computing joint rating transitions

Following [Zho13b], we consider that the credit quality of a counterparty $k \in \{I,C\}$ is measured by a random variable X_k such that

$$X_k = \sqrt{\rho_{\mathscr{R}}} Z + \sqrt{1-\rho_{\mathscr{R}}}\,\varepsilon_k \tag{7.2.19}$$

where Z and ε_K are the systemic and idiosyncratic risk factors modeled by two independent Gaussian variables. $\rho_{\mathscr{R}}$ is the correlation parameter that measures the co-movement of credit indices X_I and X_C. X_k can be interpreted as the asset value of the firm $k \in \{I,C\}$.

For the conditional bi-variate transition probabilities at play, the only relevant state is the default. Let θ_j^k be the marginal default threshold for party k with $j = \mathscr{R}_k(T_{i-1})$

$$\theta_j^k(T_{i-1},T_i) = \mathscr{N}^{-1}\left(Q_k(T_{i-1},T_i)_{j,K}\right) \tag{7.2.20}$$

The conditional joint transition probability where C defaults and I survives is then obtained by

$$\begin{aligned}
&\mathbb{Q}(\tau_I > T_i \tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j) \\
&= Q_C(T_{i-1},T_i)_{i,K} - \mathbb{Q}(\tau_I \leq T_i \tau_C \leq T_i | \mathscr{R}_C(T_{i-1}) = j) \\
&= Q_C(T_{i-1},T_i)_{i,K} - \sum_{l\in\Pi_I} \mathbb{Q}(\tau_I \leq T_i \tau_C \leq T_i | \mathscr{R}_I(T_{i-1}) = l, \mathscr{R}_C(T_{k-1}) = j)\,\mathbb{Q}(\mathscr{R}_I(T_{i-1}) = l) \\
&= Q_C(T_{i-1},T_i)_{i,K} - \sum_{l\in\Pi_I} \mathscr{N}\left(\theta_l^I(T_{i-1},T_i), \theta_j^C(T_{i-1},T_i); \rho_{\mathscr{R}}\right) Q_I(0,T_{i-1})_{r_I,l}
\end{aligned} \tag{7.2.21}$$

7.2.2.2 Estimation of correlation $\rho_{\mathscr{R}}$

The correlation parameter $\rho_{\mathscr{R}}$ measures the co-movement of the credit rating of two counterparties. Hence, it is an non-observable parameter. A common approach is to estimate $\rho_{\mathscr{R}}$ as the pairwise asset returns correlation. The simplest estimation method is to use the Equity return correlation as a proxy. A more advanced approach is to infer each firm's asset value from its equity and asset structures.

We emphasize that the significance of $\rho_{\mathscr{R}}$ differs from the pairwise default correlation ρ_τ. ρ_R is much higher than ρ_τ. In addition, we have an explicit relationship between both correlations: The correlation between default events occurring before time T_i reads

$$\rho_\tau(T_i) = \frac{\mathbb{E}[1_{\tau_I \leq T_i} 1_{\tau_C \leq T_i}] - \mathbb{E}[1_{\tau_I \leq T_i}]\mathbb{E}[1_{\tau_C \leq T_i}]}{\sqrt{\mathbb{V}[1_{\tau_C \leq T_i}]}\sqrt{\mathbb{V}[1_{\tau_C \leq T_i}]}} \tag{7.2.22}$$

Given default probabilities $PD_I(T_i)$ and $PD_C(T_i)$, the joint default probability is the migration probability to default from 0 to time T_i and depends on $\rho_{\mathscr{R}}$

$$PD_{I,C}(0,T_i;\rho_{\mathscr{R}}) = \mathscr{N}\left(\mathscr{N}^{-1}(PD_I(T_i)), \mathscr{N}^{-1}(PD_C(T_i)); \rho_R\right) \tag{7.2.23}$$

one has from the above equation

$$PD_{I,C}(0,T_i;\rho_{\mathscr{R}}) = PD_I(T_i)\,PD_C(T_i) + \rho_\tau(T_i) \prod_{k=I,C} \sqrt{PD_k(T_i)(1-PD_k(T_i))} \tag{7.2.24}$$

In particular, taking $\rho_\tau = 0$ leads to $\rho_{\mathscr{R}} = 0$.

7.2.3 Proxy formulas for rating-adjusted exposures

Expressions (7.2.7) and (7.2.14) giving the *RWR* exposure involve conditional rating migration to default. If we assume that the default of a counterparty $\tau_C \leq T_i$ is independent of its current rating $R(T_{i-1}) = j \in \Pi_C^\star$, then

$$\omega_{C,j}^U(T_{i-1},T_i) = \omega_{C,j}^B(T_{i-1},T_i) = \mathbb{Q}(\mathscr{R}_C(T_{i-1}) = j)$$

leading

$$\begin{aligned} EPE_R^B(T_{i-1},T_i) &= RWRPE_R^U(T_{i-1},T_i) \\ &= LGD_C \sum_{j \in \Pi_C^\star} EPE(T_i, H_C(j), IA_C(j))\,\mathbb{Q}(\mathscr{R}_C(T_{i-1}) = j) \end{aligned}$$

The adjusted exposure corresponds then to the sum of EPEs weighted by the terminal transition probabilities. In spite of calculation convenience, the independence assumption between rating and default occurrence is counterintuitive and inappropriate.

7.3 Modeling risk-neutral rating transition matrix

The calculation of the rating-adjusted exposures (7.2.7) and (7.2.14) requires risk-neutral probabilities for both the conditional and terminal rating migrations.

In the following, we recall Zhou's methodology before presenting our alternative modeling. A common approach, described in [Zho10], is based on the historical transition matrix to have an internal linkage between ratings, as well as a parametric form for the one-period conditional migration matrix $Q(T_{i-1},T_i)$. The underlying parameters can be assimilated to risk premiums that are calibrated to fit generic CDS spreads derived for all the ratings. From a numerical perspective, the approach poses the problem of constructing generic CDS spreads, while significant discrepancies can be shown for two entities with the same rating. In addition, the joint calibration for all ratings can be proven hard to achieve using real market data. In fact, for high-rank ratings, historical transition matrices embed very slow migration to default when compared to market default probability. To reduce the gap, this results in huge risk premiums that prevent from calibrating correctly low-rank ratings, and leads to numerical instabilities. This issue also reflects inconsistencies between historical transition matrices and market-implied default probabilities.

Our approach proceeds by associating a risk-neutral rating transition model to each of the two counterparties involved within the contract. Two calibrations to market-implied default probabilities are performed separately. The obtained transition matrices are specific to each one of the counterparties. In order to achieve calibration, we impose a parametric form for the conditional default probabilities w.r.t each rating. We propose an adaptive calibration algorithm that allows to preserve market probabilities for the two counterparties in both fitted models.

The common modeling approach is based on a parametrization of the conditional one-period matrix $Q(T_{i-1},T_i)$. The cumulative matrix is computed following the iterative relationship

$$Q(0,T_i) = Q(0,T_{i-1}) \times Q(T_{i-1},T_i) \tag{7.3.1}$$

In particular, we admit that $Q(T_{i-1},T_i) = F(\mu_{i-1},q_{i-1})$ is an explicit function of risk premiums μ_i and q_i. $\mu_0 = q_0 = [0]_K$.

A successful risk-neutral transformation must generate a small calibration error w.r.t reference default probabilities and the resultant transition matrix series must satisfy the following conditions

Condition 7.3.1 (Admissible transition matrix series)**.** Admissible transition matrix series $(Q(T_{i-1},T_i))_{i\geq 1}$ must satisfy:

- $\forall i \geq 1$, $Q(T_{i-1},T_i)$ is a Markov matrix, i.e

$$\forall j,j' \in [\![1\dots K]\!],\ \begin{cases} [Q(T_{i-1},T_i)]_{j,j'} & \in [0,1] \\ \sum_{j'=1}^{K} [Q(T_{i-1},T_i)]_{j,j'} & = 1 \end{cases}$$

- $\forall i \geq 1$, $Q(0,T_i) = \prod_{j=1}^{i} Q\left(T_{j-1},T_j\right)$.
- $Q(0,0) = I_K$.

7.3.1 Historical rating transition matrices

A historical transition matrix is used as a basic linkage between ratings. Such a matrix is provided by Moody's or Standard & Poor's, and corresponds to a cumulative transition probabilities over different maturities (namely 1 years or 5 years). We recall that a cumulative transition

matrix P over a period $(0,T_i)$ can be obtained recursively (Chapman-Kolmogorov equation)

$$P(0,T_i) = \prod_{k=0}^{n} P(T_{i-1},T_i) = P(0,T_{i-1}) \times P(T_{i-1},T_i) \tag{7.3.2}$$

where the $P(T_{i-1},T_i)$ refer to the one-period transition matrices from T_{i-1} to T_i.

In contrast with market default probabilities, historical rating transition probabilities are generally zero for highly rated counterparties (AAA, AA). In order to avoid numerical instabilities, it is convenient to initially adjust these probabilities before going through calibration. Following [Zho13a], the easiest way is to diagonally adjust $\mathscr{P}_\delta$ using an arbitrary minimum $\left(\bar{p}_j\right)_{1\leq j\leq K}$ for default probabilities (floor): $\forall j, j' \in [\![1,K]\!]$

$$\left(\mathscr{P}_\delta\right)_{j,j'} = \begin{cases} p_{j,j'} & , j' \neq j,K \\ \bar{p}_j & , j' = K \\ p_{jj} + p_{jK} - q_j & , j' = j \end{cases} \tag{7.3.3}$$

Given an empirical Markov matrix $\mathscr{P}_\delta$, the major assumption in the following is to express $\mathscr{P}_\delta$ as the exponential of a generator matrix Λ, i.e. $\mathscr{P}_\delta = e^{\delta\Lambda}$. This essentially allows to build a continuous-time Markov process for rating transitions. The problem of factorizing $\mathscr{P}_\delta$ as a product of two (or more) Markov transition matrices is not completely straightforward and it still raises issues of existence and uniqueness (see [KS01] for counterexamples). A simplifying assumption is to assume that $\mathscr{P}_\delta$ is embeddable:

Definition 7.3.2. A matrix A is embeddable if there is a matrix G such that $A = e^G$ and

$$\begin{cases} \sum_{j=1}^{K} G_{i,j} = 0 & \text{for } i = 1 \dots K \\ G_{i,j} \geq 0 & \text{for } i,j = 1 \dots K,\ i \neq j \end{cases} \tag{7.3.4}$$

The matrix G involved need not be unique. e^{tG} is Markov for all $t \geq 0$.

A feasible candidate of Λ may be the matrix logarithm of $\mathscr{P}_\delta$. The work in [Cul66] gives necessary and sufficient conditions on the existence and uniqueness of the matrix logarithm. The resulting solution may not be a valid generator according to Definition 7.3.2 as negative off-diagonal values may arise. [KS01] propose two simple regularization algorithms reported in Algorithm 7.1.

We use the notations exp *(or e)* and ln for the exponential and logarithm of real numbers or matrices. Matrix exponential can reliably be computed using Taylor or Padé approximation. For the logarithm, we use the method of "inverse scaling and squaring" by Kenney and Laub. We refer to [Bjö15] for definitions and computation algorithms. We now set

$$\hat{\mathscr{P}}_\delta = e^{\delta\hat{\Lambda}} \tag{7.3.8}$$

and we assume that after regularization, the error $\left\|\mathscr{P}_\delta - \hat{\mathscr{P}}_\delta\right\|$ is sufficiently small.

Assumption 7.3.3. *We admit that $\mathscr{P}_\delta$ is reasonably approximated by*

$$\mathscr{P}_\delta \approx \hat{\mathscr{P}}_\delta = e^{\delta\hat{\Lambda}} \tag{7.3.9}$$

$\hat{\mathscr{P}}_\delta$ is an embeddable matrix and Condition 7.3.1 holds.

Algorithme 7.1 Regularization algorithms of the generator Λ

1. Set
$$\hat{\Lambda}_{i,j} = \begin{cases} 0 & \text{if } (i \neq j) \text{ and } \Lambda_{i,j} < 0 \\ \Lambda_{i,j} & \text{otherwise} \end{cases}, i,j = 1,2,\ldots,K \tag{7.3.5}$$

2.

a) **Diagonal adjustment**: Set the diagonal elements to the negative sum of the non-diagonal elements
$$\hat{\Lambda}_{i,i} = -\sum_{j=1, i\neq j}^{K} \hat{\Lambda}_{i,j},\ i = 1,2,\ldots,K \tag{7.3.6}$$

b) **Weighted adjustment**: Adjust the non-zero elements according to their relative magnitudes
$$\hat{\Lambda}_{i,j} = \hat{\Lambda}_{i,j} - \left|\hat{\Lambda}_{i,j}\right| \frac{\sum_{j=1}^{n} \hat{\Lambda}_{i,j}}{\sum_{j=1}^{n} \left|\hat{\Lambda}_{i,j}\right|},\ i,j = 1,2,\ldots,K \tag{7.3.7}$$

7.3.2 Zhou's approach for risk-neutralizing transition matrices

Given rating-based term structure of market-implied default probabilities, the calibration procedure generally consists in finding $Q(T_{i-1},T_i)$ such that the K^{th} column of $Q(0,T_i)$ equals default probabilities given for each rating

$$\left[Q(0,T_{i-1})\,Q(T_{i-1},T_i)\right]_{,K} = \left[PD(T_i);1\right]^t \tag{7.3.10}$$

More formally, we write

$$Q(0,T_{i-1}) = \begin{pmatrix} \bar{Q}(0,T_{i-1}) & \alpha(0,T_{i-1}) \\ 0 & 1 \end{pmatrix} \tag{7.3.11}$$

and

$$Q(T_{i-1},T_i) = \begin{pmatrix} \bar{Q}(T_{i-1},T_i) & \beta(T_{i-1},T_i) \\ 0 & 1 \end{pmatrix} \tag{7.3.12}$$

where $\bar{Q}$ refers to $(K-1)\times(K-1)$ block of matrix Q. $\alpha_{T_{i-1}}$ and $\beta_{T_{i-1}}$ are respectively the $(K-1)$ first elements of the K^{th} column of $Q(0,T_{i-1})$ and $Q(T_{i-1},T_i)$. The general scheme of the calibration algorithm is given in Algorithm 7.2.

Numerical experiments show that the constraint (7.3.15) are necessary in order to obtain an admissible transition matrix. It also seems reasonable and corresponds to economical intuitions. The last two steps can be considered as a predictor-corrector method where 2. is the predictor for μ_{i-1} and 3. is the corrector for q_{i-1}.

The discrete Jarrow-Lando-Turnbull (JLT) model

This discrete model is defined by

$$Q(T_{i-1},T_i) \triangleq I_K + \frac{\Delta T_i}{\delta} Diag\left([\mu_{i-1},1]\right) \hat{\mathscr{P}}_\delta \tag{7.3.16}$$

Algorithme 7.2 Zhou's calibration algorithm

1. Solve $\beta(T_{i-1}, T_i)$ such that

$$[Q(0, T_{i-1})\, Q(T_{i-1}, T_i)]_{,K} = [DP(T_i); 1]^t \tag{7.3.13}$$

   i.e.

$$\bar{Q}(0, T_{i-1})\, \beta(T_{i-1}, T_i) = DP(T_i) - \alpha(0, T_{i-1}) \tag{7.3.14}$$

   under the constraints

$$0 < \beta^1 < \beta^2 < \cdots < \beta^{K-1} < 1 \tag{7.3.15}$$

2. Fix $q_{i-1} = [0.5]_K$ and solve $\mu_{i-1} > 0$, such that the last column $Q(T_{i-1}, T_i) = F(\mu_{i-1}, q_{i-1})_{,K}$ equals $[\beta(T_{i-1}, T_i); 1]$.

3. Solve q_{i-1} such that $[Q(0, T_{i-1})\, F(\mu_{i-1}, q_{i-1})] = [PD(T_{i-1}); 1]$ subject to $q_{i-1} \in]0, 1[$.

where the exponential matrix is given by a first order Taylor approximation. The generator matrix is then approximated by

$$\Lambda \approx \frac{1}{\delta} \hat{\mathscr{P}}_\delta - I \tag{7.3.17}$$

With the discrete JLT, solving μ_{i-1} is performed through explicit calculations. However, numerical experiments show that this model generates inadmissible transition matrices, mainly due to the above approximations.

The continuous JLT model

Negative transition probabilities can be avoiding by considering the continuous version of the JLT model where

$$Q(T_{i-1}, T_i) = \exp\left(\Delta T_i Diag([\mu_{i-1}, 1])\, \hat{\Lambda}\right) \tag{7.3.18}$$

The calibration can be performed by solving $[\mu_{i-1}, 1]$ such that

$$Q(0, T_{i-1})\, [Q_{1,K}(T_{i-1}, T_i), \ldots, Q_{K-1,K}(T_{i-1}, T_i), 1]^t = [PD(T_i); 1]^t \tag{7.3.19}$$

using a minimization routine. However, this straightforward calibration is shown to produce invalid forward transition matrices. The constrained calibration procedure described previously is found to be more successful but results in significant calibration error.

The mixture model

The model is defined by

$$\begin{aligned} Q(T_{i-1}, T_i) = (I_K - Diag([q_{i-1}]_K)) \exp\left(\Delta T_i Diag([g(\mu_{i-1}), 1])\, \hat{\Lambda}^{UG}\right) \\ + Diag([q_{i-1}]_K) \exp\left(\Delta T_i Diag([\mu_{i-1}, 1])\, \hat{\Lambda}^{DG}\right) \end{aligned} \tag{7.3.20}$$

where $\hat{\Lambda}^{UG}$ and $\hat{\Lambda}^{DG}$ are the upper and lower triangular matrices of Λ , allowing for upgrade and downgrade moves. g being a decreasing function that relates upgrade risk premiums to downgrade risk premiums. From a numerical point of view, the calibration of this model is found to be very time consuming. In addition, we encounter the premium squeeze effect described earlier.

7.3.3 Calibration algorithms

7.3.3.1 Reducing the initial transition matrix

The expected exposure is constant when the rating of one counterparty is located between two contract triggers. Hence, transition probabilities from this rating and toward it are not needed to perform the pricing. In order to speed-up the calibration, we reduce the size of the base transition matrix to the rating subsets that are defined with respect to the collateral specifications.

7.3.3.2 The modeling approach

In order to resolve the premium squeeze problem, the calibration procedure is performed to fit the default probabilities of one party. This results in two risk neutral transition matrices for counterparties I and C. We add then the counterparty subscript $k \in \{I,C\}$ to Q, α and β in (7.3.11) and (7.3.12), as well as the risk premium variables.

Using the default probabilities of only one rating (the party's), we consider a parametric form for β that links the conditional default probability of the reference counterparty with the conditional probabilities of the other ratings.

$$\beta_k(T_{i-1},T_i) = \tilde{\beta}_k(T_{i-1},T_i) \times \left[(\gamma_k)^{j-r_k}\right]^t_{j\in[\![1,K-1]\!]} \tag{7.3.21}$$

where the conditional default probabilities of rating j is obtained from the conditional default probability of the party $\tilde{\beta}_k$, as well as a power function that involves the distance between j and the party's rating. γ_k is a constant while $\tilde{\beta}_k(T_{i-1},T_i)$ is determined in order to recover $PD_k(T_i)$ from (7.3.14). The conditional default probabilities induced by (7.3.21) are decreasing w.r.t. ratings, satisfying the optimization constraint (7.3.15). Our numerical tests show that the continuous JLT model

$$Q(T_{i-1},T_i) = \exp\left(\Delta T_i Diag([\mu_{i-1},1])\hat{\Lambda}\right) \tag{7.3.22}$$

is sufficient to have an admissible transition model with a small calibration error. We consider this model for the conditional transition matrix.

Remark 7.3.4. One has

$$\left[Diag([\mu_{i-1},1])\hat{\Lambda}\right]_{l,m} = \mu_{i-1,l}\hat{\Lambda}_{l,m}$$

In particular, $Diag([\mu_{i-1},1])\hat{\Lambda}$ is an embedding matrix satisfying Definition 7.3.2.

7.3.3.3 The calibration procedures

For a fixed γ_k, the calibration procedure is defined as follow Algorithm 7.3.

Algorithme 7.3 Counterparty-based calibration algorithm

For $k \in \{I, C\}$

1. Solve $\tilde{\beta}_k(T_{i-1}, T_i) \in [0,1]$ such that

$$\left[Q_k(0, T_{i-1})\, Q_k(T_{i-1}, T_i)\right]_{r_k, K} = DP_k(T_i) \tag{7.3.23}$$

i.e.

$$\tilde{\beta}_k(T_{i-1}, T_i) = \frac{DP_k(T_i) - \left[Q_k(0, T_{i-1})\right]_{r_k, K}}{\sum_{j=1}^{K-1} \left[\bar{Q}_k(0, T_{i-1})\right]_{r_k, j} (\gamma_k)^{j-r_k}} \tag{7.3.24}$$

2. Solve $\mu_{k,i-1} > 0$, such that the last column of $Q_k(T_{i-1}, T_i) = \exp\left(\Delta T_i Diag\left(\left[\mu_{k,i-1}, 1\right]\right)\Lambda\right)$ equals $[\beta_k(T_{i-1}, T_i); 1]$. $\beta_k(T_{i-1}, T_i)$ being given by 7.3.21.

$\tilde{\beta}_k$ are calculated explicitly, while the optimization problem for μ_{i-1} is efficiently solved using Levenberg-Marquardt algorithm.

The calibration is performed twice, for I and C. The calibrated model for I (resp. for C) returns a term structure of default probabilities for all the ratings, and it is clear to see that (7.3.21) has a significant role in shaping those probabilities. A desired feature consists of having the same default probabilities for all ratings, within I and C calibrations. The reason behind is to guarantee the stability of the model at downgrade events, in the way that default probabilities for the new grade of I are consists with the calibrated model before the downgrade event, so re-calibration would not be needed[1]. The model should avoid important discontinuities during rating events.

We denote by $\bar{k}$ the counterparty of $k \in \{I, C\}$. In addition to (7.3.23) and (7.3.24), a minimal requirement consists of having

$$\left[Q_k(0, T_i)\right]_{r_{\bar{k}}, K} = DP_{\bar{k}}(T_i) \tag{7.3.25}$$

leading to at each time T_i:

$$\sum_{j=1}^{K-1} \left[\bar{Q}_k(0, T_{i-1})\right]_{r_{\bar{k}}, j} (\gamma_k)^{j-r_k} = \frac{DP_{\bar{k}}(T_i) - \left[Q_k(0, T_{i-1})\right]_{r_{\bar{k}}, K}}{\tilde{\beta}_k(T_{i-1}, T_i)} \tag{7.3.26}$$

Using (7.3.24), one has

$$\frac{\sum_{j=1}^{K-1} \left[\bar{Q}_k(0, T_{i-1})\right]_{r_{\bar{k}}, j} (\gamma_k)^{j-r_k}}{\sum_{j=1}^{K-1} \left[\bar{Q}_k(0, T_{i-1})\right]_{r_k, j} (\gamma_k)^{j-r_k}} = \frac{DP_{\bar{k}}(T_i) - \left[Q_k(0, T_{i-1})\right]_{r_{\bar{k}}, K}}{DP_k(T_i) - \left[Q_k(0, T_{i-1})\right]_{r_k, K}} \tag{7.3.27}$$

The calculation of γ_k is performed at each time step, before executing Algorithm 7.2. The calibration algorithm is reported in Algorithm 7.4.

[1] At least, not frequently needed, unless significant spread moves.

Algorithme 7.4 Adaptive calibration algorithm

For $k \in \{I, C\}$

1. Minimize

$$\gamma_k = \underset{\gamma \geq 1}{\arg\min} \left| \sum_{j=1}^{K-1} \xi_j \left(k, \bar{k}\right) (\gamma)^{j-r_k} \right| \tag{7.3.28}$$

with

$$\begin{aligned} \xi_j \left(k, \bar{k}\right) &= \delta_k \left[\bar{Q}_k \left(0, T_{i-1}\right)\right]_{r_{\bar{k}}, j} - \bar{\delta}_k \left[\bar{Q}_k \left(0, T_{i-1}\right)\right]_{r_k, j} \\ \delta_k &= DP_k \left(T_i\right) - \left[Q_k \left(0, T_{i-1}\right)\right]_{r_k, K} \\ \bar{\delta}_k &= DP_{\bar{k}} \left(T_i\right) - \left[Q_k \left(0, T_{i-1}\right)\right]_{r_{\bar{k}}, K} \end{aligned}$$

2. Solve $\tilde{\beta}_k \left(T_{i-1}, T_i\right) \in [0,1]$ such that

$$\tilde{\beta}_k \left(T_{i-1}, T_i\right) = \frac{DP_k \left(T_i\right) - \left[Q_k \left(0, T_{i-1}\right)\right]_{r_k, K}}{\sum_{j=1}^{K-1} \left[\bar{Q}_k \left(0, T_{i-1}\right)\right]_{r_k, j} \left(\gamma_k\right)^{j-r_k}} \tag{7.3.29}$$

3. Solve $\mu_{k,i-1} > 0$, such that the last column of $Q_k \left(T_{i-1}, T_i\right) = \exp\left(\Delta T_i Diag\left(\left[\mu_{k,i-1}, 1\right]\right) \Lambda\right)$ equals $\left[\beta_k \left(T_{i-1}, T_i\right); 1\right]$. $\beta_k \left(T_{i-1}, T_i\right)$ being given by (7.3.21).

7.4 Numerical experiments

We apply the adaptive calibration algorithm to Moody's 2014's average one-year historical transition matrix that we report in Table 7.5.1 of Appendix 7.5. The *WR* [2] state is withdrawn and the associated migration rates are distributed equally among ratings in order to preserve the validity of the historical transition matrix.

For testing purposes, we consider in Table 7.4.1 constant CDS spreads for each rating. Default probabilities are deduced consequently, taking a market *loss given default* rate of 60%:

$$DP_k \left(T_i\right) = 1 - \exp\left(-\frac{Spread_k}{0.6} T_i\right) \ , k \in [\![1, K]\!] \tag{7.4.1}$$

$\Pi\left(\Pi_I = \Pi_C\right)$	Aaa	Aa1	Aa2	Aa3	A1	A2	A3	Baa1	Baa2	Baa3	Ba1
Spread (bp)	25	50	75	100	125	150	175	200	250	300	350

$\Pi\left(\Pi_I = \Pi_C\right)$	Ba2	Ba3	B1	B2	B3	Caa1	Caa2	Caa3	Ca	C	Default
Spread (bp)	400	450	500	550	600	700	800	900	1000	1100	∞

Table 7.4.1: CDS spreads (in bps) w.r.t. Moody's rating

[2] Rating withdrawn for reasons including: debt maturity, calls, puts, conversions, etc., or business reasons (e.g. change in the size of a debt issue), or the issuer defaults

7.4.1 Calibration error and validation

We show the performance of the calibration procedure by taking all possible combinations of the couple of initial ratings $(r_I, r_C) = (\mathscr{R}_m(0), \mathscr{R}_n(0))$, $m,n \in [\![1,K]\!]$. We report in Table 7.4.2 the maximum absolute error for $(r_I, r_C) \in [\![1,K]\!]^2$. In order to assess the validity of conditional and terminal transitions matrices obtained from calibration, we also report

- The minimum and maximum of transition probabilities. Numerically, we prove that they are very close to respectively 0 and 1.
- The minimum and maximum of the sum of row probabilities. We show that they are very close to 1.

	$\min_{k,i,l}$	$\max_{k,i,l}$
$(\lvert DP_k(T_i) - Q_k \rvert)_i$	$1.73E-18$	$1.43E-17$
$(Q_k(T_i, T_{i+1}))_i$	-6.45×10^{-4}	1.0000
$(dQ_k(T_i, T_{i+1}))_i$	-9.22×10^{-5}	1.0000
$\left(\sum_{c\geq 1} [Q_k(T_i, T_{i+1})]_{l,c}\right)_{l,i}$	1.0000	0.9999
$\left(\sum_{c\geq 1} [dQ_k(T_i, T_{i+1})]_{l,c}\right)_{l,i}$	0.9999	1.0000

Table 7.4.2: Calibration Error

In addition to a small fitting error, we show in Figure 7.4.1 that the adaptive algorithm guaranties allows to preserve the default probabilities of I and C in both calibrated models. Namely, the error between $DP_C(T_i)$ (resp. $DP_I(T_i)$) and $[Q_I(0,T_i)]_{r_C,K}$ (resp. $[Q_C(0,T_i)]_{r_I,K}$) is very small.

7.4.2 Calibrated model properties

Following [Zho13b], we define respectively direction $(D(T_i))_i$ and speed $(S(T_i))_i$ of the rating migration model, by

$$D(T_i) = \frac{1}{K-1} \sum_{m=1}^{K-1} \left[\sum_{n<m} [Q(0,T_i)]_{n,m} - \sum_{n>m} [Q(0,T_i)]_{n,m} \right] \tag{7.4.2}$$

$$S(T_i) = \frac{1}{(K-1)^2} \sum_{m=1}^{K-1} \sum_{n=1}^{K-1} |i-j| \, [Q(0,T_i)]_{n,m} \tag{7.4.3}$$

Namely $D(T_i)$ is average probabilities of upgrades minus the downgrades. We report in Figure 7.4.2a $(D(T_i))_i$ with $r_C = B2$ and varying $r_I \in \Pi_I$ for $(r_I, r_C) = (Aaa, A2)$ and $(r_I, r_C) =$

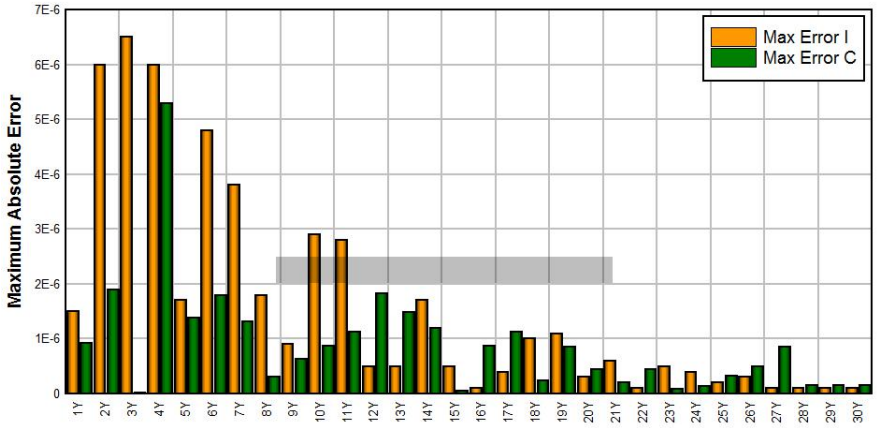
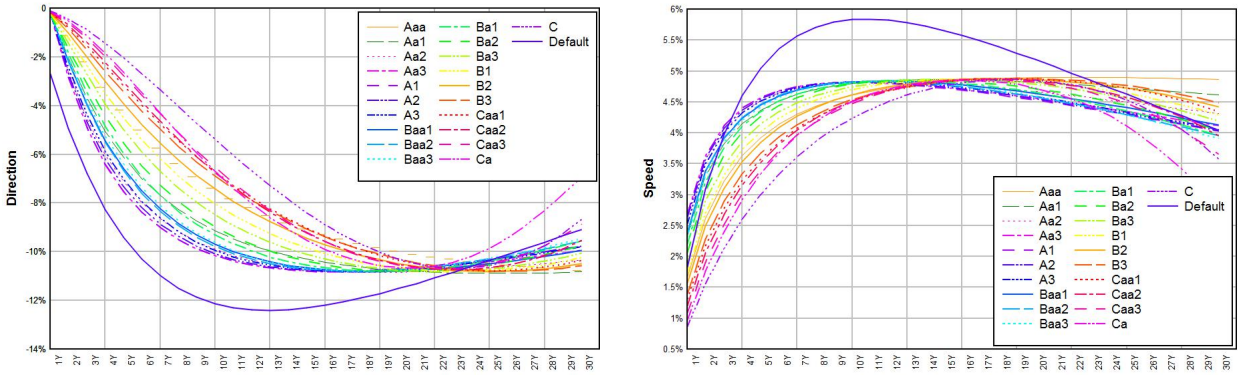


Figure 7.4.1: Maximum error $\left|DP_C(T_i) - [Q_I(0,T_i)]_{r_C,K}\right|$ (Error C) and $\left|DP_I(T_i) - [Q_C(0,T_i)]_{r_I,K}\right|$ (Error I)

$(Baa1, B2)$. $(D(T_i))_i$ is always negative, meaning that downgrade moves always dominate upgrades. This is a direct consequence of calibration to only market implied default probabilities, that are higher than the historical probabilities.

The rating speed $(S(T_i))_i$ is defined as the average magnitude of rating migration. The higher the speed is, the faster ratings migrate towards the default. Our results of $(S(T_i))_i$ are reported in Figure 7.4.2b. Since calibration is specific to each counterparty, the obtained speed and direction curves are specific to each rating.



(a) Direction (b) Speed

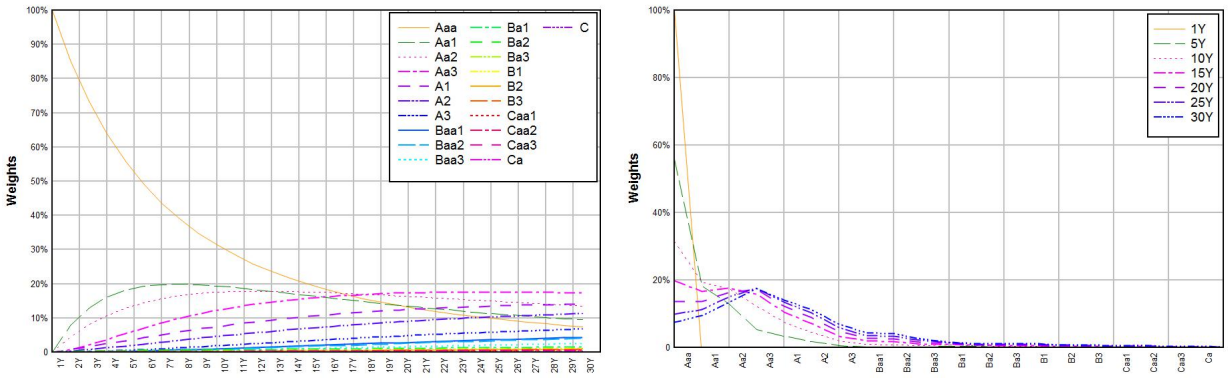
Figure 7.4.2: Transition direction and speed for $r_C = B2$ and $r_I \in \Pi_I$

7.4.3 Stylized facts of adjusted exposure with rating events

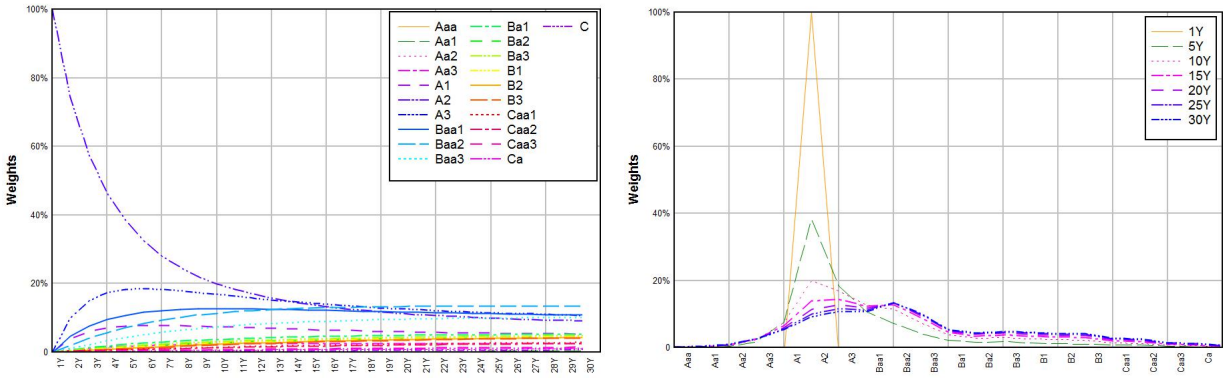
In the following, we fix $(r_I, r_C) = (A2, Aaa)$. We report in Figure 7.4.3 the unilateral weights $\left(\omega^U_{k,j}(T_{i-1},T_i)\right)_{j\in[\![1,K]\!]}$, $k \in \{I,C\}$ defined in (7.2.9). Initially, we have $\omega^U_{I,r_I}(T_0,T_1) = \omega^U_{C,r_C}(T_0,T_1) =$

100%. The exposure weights are distributed w.r.t to all ratings moving towards default when the time grows up. We show in Figure 7.4.4 the impact of the rating migration correlation $\rho_{\mathscr{R}}$ on the bilateral weights $\left(\omega^B_{k,j}(T_{i-1},T_i)\right)$ defined in (7.2.18). In particular, $\omega^B_{k,j}$ are identical to $\omega^U_{k,j}$ when $\rho_{\mathscr{R}}=-1$. The impact of $\rho_{\mathscr{R}}$ becomes significant when $\rho_{\mathscr{R}}>0.5$. Positive migration correlation decreases $\omega^B_{k,j}$ for high ratings j and increases it for low ratings. This corresponds to having an increase of the numerator and a decrease of the denominator in (7.2.18). This sensitivity of $\rho_{\mathscr{R}}$ is a consequence of negative $(D(T_i))$ highlighted previously. Positive correlation $\rho_{\mathscr{R}}$ increases then the speed of migration toward the default.

Finally, we show in Figure 7.4.4 an example of a Right-way risk exposure, expressed as a ratio of the classical exposure with no rating events. The unilateral adjusted exposure presents the minimum reduction ratio, and corresponds to the bilateral adjusted exposure with $\rho_{\mathscr{R}}=-100\%$. Positive rating $\rho_{\mathscr{R}}$ increases migration probabilities towards default, leading to more collateral requirements and hence generates a reduced exposure.



(a) $\omega^U_{C,j}(T_{i-1},T_i)$, $r_C = Aaa$



(b) $\omega^U_{I,j}(T_{i-1},T_i)$, $r_I = A2$

Figure 7.4.3: Unilateral *RWR* weights $\left(\omega^U_{k,j}(T_{i-1},T_i)\right)_{j\in[\![1,K]\!]}$, $k\in\{I,C\}$

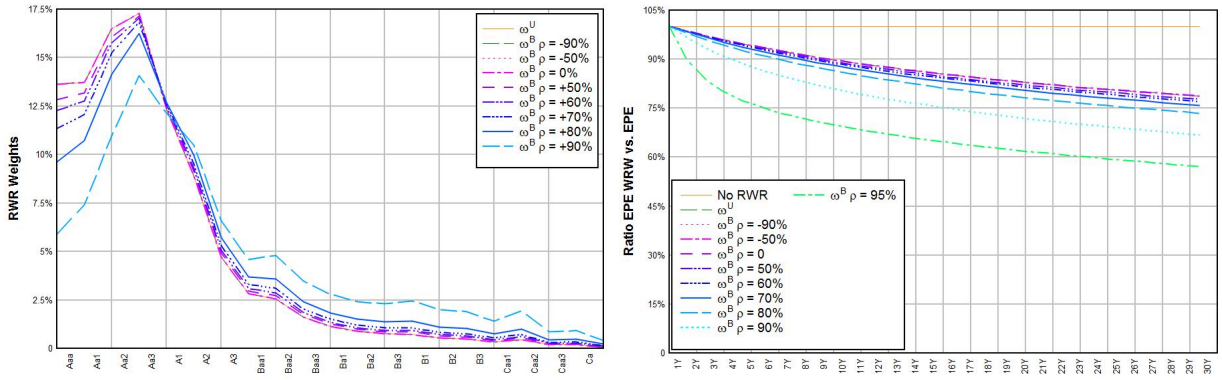


Figure 7.4.4: Impact of correlation correlation $\rho_{\mathscr{R}}$: $\omega^U_{C,j}(19Y,20Y)$ and $\omega^B_{C,j}(19Y,20Y)$ (left) and Exposure reduction ratio (right)

7.5 Conclusion

We described in this chapter a new modeling approach for rating migrations under the risk neutral measure. We abandon the rating-based representation of the historical transition models (using Moody's or S&P matrices for instance), in favor of a transition model that is specifically calibrated to the default of one reference entity. The approach allows a flexible calibration to market-implied default probabilities. Our experiments are dedicated to the valuation of xVA adjustments with collateral rating events. In particular, we highlight some stylized fact of the rating-adjusted exposure in both the unilateral and bilateral settings, and we show that collateral rating events act as a right-way risk mechanism.

Appendix C

C.1 Moody's 2014 average 1Y transition matrix

	Aaa	Aa1	Aa2	Aa3	A1	A2	A3	Baa1	Baa2	Baa3	Ba1	Ba2	Ba3	B1	B2	B3	Caa1	Caa2	Caa3	Ca	C	Default
Aaa	0.90	0.06	0.03	0.01	0.00	0.00	0.00	-	-	-	0.00	0.00	-	0.00	-	-	-	0.00	-	-	-	-
Aa1	0.02	0.79	0.09	0.07	0.02	0.01	0.00	0.00	0.00	0.00	0.00	-	0.00	-	-	-	-	-	-	-	-	-
Aa2	0.01	0.04	0.77	0.11	0.04	0.02	0.01	0.00	0.00	0.00	0.00	0.00	-	0.00	0.00	0.00	-	0.00	-	0.00	-	-
Aa3	0.00	0.01	0.04	0.79	0.09	0.04	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-	-	0.00	-	-	-	-	0.00
A1	0.00	0.00	0.01	0.05	0.79	0.09	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-	0.00	-	-	-	0.00
A2	0.00	0.00	0.00	0.01	0.05	0.80	0.09	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-	-	0.00
A3	0.00	0.00	0.00	0.00	0.02	0.07	0.78	0.08	0.03	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-	0.00
Baa1	0.00	0.00	0.00	0.00	0.00	0.02	0.07	0.78	0.08	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Baa2	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.07	0.80	0.07	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-	0.00
Baa3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09	0.77	0.05	0.03	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Ba1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.03	0.11	0.70	0.05	0.04	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.01
Ba2	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.04	0.09	0.70	0.08	0.03	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.01
Ba3	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.03	0.07	0.71	0.07	0.05	0.02	0.01	0.00	0.00	0.00	0.00	0.02
B1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.08	0.72	0.08	0.04	0.01	0.01	0.00	0.00	0.00	0.02
B2	-	-	0.00	0.00	0.00	-	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.08	0.69	0.10	0.04	0.02	0.01	0.00	0.00	0.04
B3	-	0.00	0.00	-	-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.08	0.68	0.08	0.04	0.01	0.01	0.00	0.06
Caa1	-	0.00	-	-	-	0.00	-	0.00	-	0.00	0.00	0.00	0.00	0.01	0.02	0.11	0.62	0.10	0.04	0.02	0.00	0.08
Caa2	-	-	-	-	-	0.00	-	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.04	0.10	0.54	0.06	0.04	0.01	0.18
Caa3	-	-	-	-	-	-	-	-	-	-	0.00	0.00	0.00	0.00	0.00	0.02	0.03	0.10	0.44	0.06	0.03	0.31
Ca	-	-	-	-	-	-	0.00	-	-	-	0.00	0.00	0.00	0.00	0.01	0.02	0.02	0.04	0.06	0.39	0.03	0.42
C	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	0.01	-	0.03	0.01	0.03	0.38	0.56
Default	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1.00

Table 7.5.1: Moody's 2014 average one-year rating migration rates

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