Spectraèdres tropicaux : application à la programmation semi-définie et aux jeux à paiement moyen

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Contents

CHAPTER 1

Introduction

1.1 Context of this work

Semidefinite programming (SDP) is one of the fundamental tools of convex optimization. It consists in minimizing a linear function over a spectrahedron, which is a set defined by a single linear matrix inequality of the form

$$\mathcal{S} \coloneqq \{x \in \mathbb{R}^n \colon Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succcurlyeq 0\} \,,$$

where $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{R}^{m \times m}$ is a sequence of symmetric matrices, and $\succcurlyeq$ is the Loewner order on symmetric matrices. By definition, $A \succcurlyeq B$ if $A - B$ is positive semidefinite. Because of its expressive power, SDP has found numerous applications. For instance, SDP relaxations can be used to obtain polynomial-time approximations for some NP-hard problems in combinatorial optimization, such as the Max-Cut Problem [GW95]. Another classical application of SDP in the area of combinatorial optimization is the Lovász theta function [Lov79]—this function is computable in polynomial-time by SDP and is sandwiched between the clique number and the chromatic number, which are both NP-hard to compute. We refer to [GM12, LR05] for more information about applications of SDP in combinatorial optimization. SDP is also a major tool in the area of polynomial optimization. Even though polynomial optimization problems are not convex in general, Lasserre [Las01, Las02] and Parrilo [Par03] have shown that a large class of these problems can be solved to arbitrary precision using a hierarchy of SDP relaxations. More information about the use of SDP in polynomial optimization can be found in the books [Las15, Las09b]. Thanks to its expressive power, SDP has also found applications in control theory [BEGFB94], quantum information [Wat18, FSP18, GdLL18], formal proofs [MC11, MAGW15], program verification [RVS16], experiment design [VBW98], signal processing [PE10], and other domains. Some general references for SDP and its applications include [WSV00, AL12, dK02, BPT13, BTN01, BV04].

In practice, SDPs are solved using interior point methods, which were generalized from linear programming to SDP by Alizadeh [Ali95], and to general convex programming by Nesterov and Nemirovski [NN94]. We refer to [dK02, Ren01, GM12] for more information on interior point methods. Contemporary interior point solvers can solve small and medium sized SDP programs (depending on the sparsity of the input matrices). However, a large-scale industrial optimization problem was recently solved using SDP methods in [JM18].

There are many open questions about spectrahedra and semidefinite programming. For instance, Nemirovski [Nem07] asked to classify the sets that arise as projections of spectrahedra. Helton and Nie [HN09] conjectured that every convex semialgebraic sets arises in this way. The conjecture was confirmed for several classes of sets [HN09, HV07, HN10, Las09a, GPT10, GN11, NPS08]. In particular, it is known that the conjecture is true in dimension 2 [Sch18a]. The conjecture has been recently disproved by Scheiderer, who showed that the cone of positive semidefinite forms cannot be expressed as a projection of a spectrahedron, except in some particular cases [Sch18b]. A comprehensive list of references can be found in this work.

Another open question is the generalized Lax conjecture, which asks if every hyperbolicity cone is spectrahedral. The answer is positive for several classes of hyperbolic cones [HV07, LPR05, PV13, AB18, Kum17]. However, some stronger versions of the conjecture are known to be false [Brä11, AB18, BVY14]. More information can be found in the cited works.

The geometry of spectrahedra was studied, from different perspectives, in [RG95, DI10, ORSV15, FSED18]. However, there are still many open questions in this area—for instance, the facial structure of spectrahedra and hyperbolic cones is not well understood.

In this work, we are interested in the theoretical complexity of SDP. In the Turing machine model of computation, approximate solutions for well-structured SDPs can be obtained by the ellipsoid method [GLS93, Ram93] (however, this method is considered to be inefficient in practical applications). It was only recently shown that interior point methods can achieve the same theoretical complexity bounds on the Turing machine [dKV16] (the challenging task for making a passage from the BSS model [BSS89, BCSS98] to the Turing machine was to propose an interior point method in which one can control the bit-sizes of the numbers appearing during the computation). These methods obtain only approximate solutions to SDP problems—the exact solution of a generic SDP problem is an algebraic number, and the degree of its minimal polynomial can be large [NRS10]. Furthermore, event to obtain approximate solutions, the methods mentioned above require some structural assumptions on the underlying spectrahedron. We refer to [Ram97, dKV16, LMT15] for extended discussion and to [LP18] for a class of small degenerate SDPs that are hard for the contemporary SDP solvers. The state-of-the art algorithms that are able to solve SDPs over arbitrary spectrahedra are based on critical point methods [HNSED16, Nal18, HNSED18] (the feasibility of spectrahedra can also be decided by quantifier elimination [PK97]). From the theoretical complexity perspective, Ramana [Ram97] has shown that the semidefinite feasibility problem (given the matrices $Q^{(0)}, \dots, Q^{(1)}$, decide if the associated spectrahedron is nonempty) belongs to $\mathrm{NP}_{\mathbb{R}} \cap \mathrm{coNP}_{\mathbb{R}}$, where the subscript $\mathbb{R}$ refers to the BSS model of computation. It is not known if this problem belongs to NP in the Turing machine model. A difficulty here is that all feasible points may have entries of absolute value doubly exponential in the size of the input. Also, the spectrahedron may be nonempty but contain only irrational points [Sch16]. A very concrete example of these difficulties was given by Tarasov and Vyalyi [TV08], who showed that the problem of comparing numbers represented by arithmetic circuits can be reduced to the semidefinite feasibility problem. A particular case of this task is the Sum of Square Roots Problem, whose complexity is open since (at least) 1976, see [GGJ76, Pap77, ABKPM09, EY10, JT18] for more information. Moreover, some matrix completion problems can also be expressed as semidefinite feasibility problems [Lau01, LV14]. Different certificates of (in)feasiblity, boundedness, and containment of spectrahedra are discussed in [Ram97, KS13, LP18, KTT13, The17, KPT18].

There are many equivalent definitions of a symmetric positive semidefinite matrix. For instance, one can define this notion by supposing that the matrix has nonnegative eigenvalues, nonnegative principal minors, that the associated quadratic form is nonnegative, or that this matrix admits a Cholesky decomposition. It can be shown that all of these definitions coincide not only over the field of real numbers but also over every real closed field (this follows from the completeness of the theory of real closed fields). In particular, the notion of a positive semidefinite matrix is meaningful over any such field. It follows that one can study spectrahedra and semidefinite programming in any real closed field, even if it is nonarchimedean. That is the subject of this work. The most important example for us is the field of Puiseux series. Classically, the field of Puiseux series (which can be traced back to Newton [BK12, Chapter 8.3]) is defined as the field of formal power series with rational exponents, with the additional assumption that every exponent in a given series has the same denominator. However, from the perspective of tropical geometry, it is useful to work with a larger field of series that have real exponents [Mar10]. For this reason, we consider the field of generalized Puiseux series proposed in [Mar10]. This field, by a change of variable, is isomorphic to the field of generalized Dirichlet series studied by Hardy and Riesz [HR15]. A generalized Puiseux series is a series of the form

$$\boldsymbol{x} = \sum_{i=1}^{\infty} c_{\lambda_i} t^{\lambda_i} \,,$$

where t is a formal parameter and $(\lambda_i)_{i \geqslant 1}$ is a strictly decreasing sequence of real numbers that is either finite or unbounded. If we suppose that the coefficients $c_{\lambda_i} \in \mathbb{C} \setminus \{0\}$ are complex, then the set of generalized Puiseux series forms an algebraically closed field. If we suppose that c_{λ_i} are real, then we obtain a subfield that is real closed. Moreover, it was proved by van den Dries and Speissegger [vdDS98] that if we restrict attention to the series that have real coefficients and are absolutely convergent (for t large enough), then this subset of generalized Puiseux series still forms a real closed field. This is the main field considered in this thesis, and we denote it by $\mathbb{K}$. However, we note that the particular choice of the field $\mathbb{K}$ is made mostly for the sake of simplicity and to make the exposition concrete. In the later parts of the thesis, we discuss how the results obtained for $\mathbb{K}$ can be transferred (by quantifier elimination) to other nonarchimedean real closed fields. For the sake of brevity, we omit the adjective "generalized" when we talk about Puiseux series.

As discussed above, the definition of spectrahedron over Puiseux series is the same as the definition over the reals. In other words, a spectrahedron over $\mathbb{K}$ is the set defined as

$$\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} \succcurlyeq 0\} \,,$$

where $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ is a sequence of symmetric matrices. A spectrahedron over Puiseux series gives rise to a family of spectrahedra over the reals, obtained by replacing the formal parameter t with some real, large value of the parameter:

$$\boldsymbol{\mathcal{S}}(t) \coloneqq \{x \in \mathbb{R}^n \colon \boldsymbol{Q}^{(0)}(t) + x_1 \boldsymbol{Q}^{(1)}(t) + \dots + x_n \boldsymbol{Q}^{(n)}(t) \succcurlyeq 0\} \,.$$

Quantifier elimination over real closed fields ensures that SDP over Puiseux series has the same basic properties as SDP over the reals. Moreover, it means that for large values of $t > 0$, the real spectrahedra $\boldsymbol{\mathcal{S}}(t)$ inherit the properties of the nonarchimedean spectrahedron $\boldsymbol{\mathcal{S}}$. There are numerous motivations to study these types of SDPs. Such nonarchimedean semidefinite programming problems arise when considering parametric semidefinite programming problems over the reals, or structured problems in which the entries of the matrices have different orders of magnitudes. They are also of an intrinsic interest, since, by analogy with the situation in linear programming [Meg89], shifting to the nonarchimedean case is expected to shed light on the complexity of the classical problem over the reals. In particular, polyhedra over Puiseux series encode a class of real polyhedra defined by matrices with large entries. The complexity of solving linear programming problems over such polyhedra is a particular case of Smale's ninth problem [Sma98], who asked if linear programming can be solved in strongly polynomial time in the Turing machine model and in polynomial time in the BSS model. Moreover, spectrahedra over Puiseux series may provide new classes of examples of spectrahedra with unexpected geometric properties. Our methods to study nonarchimedean spectrahedra rely on tropical geometry.

A general question, in tropical geometry, consists in providing combinatorial characterizations of nonarchimedean amoebas, i.e., images by the nonarchimedean valuation of algebraic sets over nonarchimedean, algebraically closed fields. This was the subject of the work of Bieri and Groves [BG84], extending the earlier results of Bergman [Ber71]. Kapranov's theorem [EKL06] characterizes the images of hypersurfaces using the notion of a tropical hypersurface. A generalization of this theorem to the case of algebraic varieties is known as the "fundamental theorem of tropical algebraic geometry"—see [JMM08] for a constructive proof and some historical discussion. Tropical geometry gained a lot of attention thanks to its connection with enumerative algebraic geometry [Mik05]. We refer to [RGST05, IMS09, MS15] for more information on these aspects of tropical geometry.

The study of real algebraic objects using tropical tools was initiated by Viro [Vir89, Vir08], who used combinatorial methods to construct planar curves with prescribed topology, in relation to Hilbert's sixteenth problem. This method was later used by Itenberg and Viro to disprove the Ragsdale conjecture [IV96] and it was generalized to complete intersections [Stu94, Bih02]. However, in a general setting, there is no analogue of the fundamental theorem for the tropicalization of real algebraic varieties—we refer to the works [SW05, Ale13, Vin12] for a discussion. The images by valuation of general semialgebraic sets were studied by Alessandrini [Ale13], who gave a real analogue of the Bieri–Groves theorem.

Independently of the works cited above, convexity and separation in idempotent semimodules was studied by numerous authors [Zim77, Hel88, SS92, LMS01, CGQ04, BH04, CGQS05]. The connection between the two fields was noted by Develin and Sturmfels [DS04], who introduced the name "tropical convexity." Develin and Yu [DY07] characterized tropical polyhedra in terms of valuations of polyhedra over Puiseux series. The geometry of tropical polyhedra was studied in numerous works such as [Jos05, GK07, BSS07, AGK11, AK17]. In a series of works that is closely related to this dissertation, Allamigeon, Benchimol, Gaubert, and Joswig studied the tropicalization of polytopes defined by generic matrices, developed a tropical simplex algorithm, and described the tropical analogue of central path [ABGJ15, ABGJ14, ABGJ18]. Yu [Yu15] tropicalized the SDP cone and showed that this tropicalization is described using only 2×2 minors. Akian, Gaubert, and Guterman [AGG12] proved that the problem of feasibility of tropical polyhedra is equivalent to deciding the winner of a deterministic mean payoff game and Grigoriev and Podolskii [GP15] showed the same result for the solvability problem of linear tropical prevarieties. Bodirsky and Mamino [BM16] studied the relations between tropical convexity, stochastic mean payoff games, and constrain satisfaction problems.

Two-player zero-sum repeated stochastic games were introduced by Shapley [Sha53], who studied the existence of optimal strategies in finite time and in infinite time with discounted payoffs. The idea of studying games with perfect information and long-term average payoff was proposed by Gillette [Gil57]. The fact that these games have optimal stationary strategies was proven by Liggett and Lippman [LL69]. The deterministic case of these games was also studied by Ehrenfeucht and Mycielski [EM79]. The existence of optimal stationary strategies is no longer true for the imperfect information games. Nevertheless, Bewley and Kohlberg [BK76] showed that the limiting value of the imperfect information stochastic games is well defined, and Mertens and Neyman [MN81] proved that this limit is the value of the game with a long-term average payoff. There are many other works on the theory of zero-sum repeated games. We refer to [MSZ15, LS15] for more information. Let us just note that one of the possible ways of studying these games is to use the properties of the associated Shapley operator. This approach was presented in the original paper by Shapley [Sha53] and was later extended to other classes of games [RS01, Ney03]. This approach to stochastic mean payoff games with perfect information was used in [AGG12] and we do the same in this work.

Independently, particular cases of stochastic mean payoff games with perfect information were discovered in the computer science community. Parity games were introduced by Emerson and Jutla [EJ91] and Mostowski [Mos91]. They are equivalent to modal μ-calculus model checking [EJS93, Jur98]. Furthermore, they were used by Friedmann [Fri11, DH17] to construct a subexponential lower bound for Zadeh's pivoting rule for the simplex algorithm (see also [Fea10, FHZ14] for more discussion on this subject). Deterministic mean payoff games were studied by Gurvich, Karzanov, and Khachiyan [GKK88], and simple stochastic games were introduced by Condon [Con92]. The reductions studied in [Jur98, ZP96, AM09], combined with the analysis of [Con92], show the following statements. First, parity games are polynomial-time reducible to deterministic mean payoff games, which form a subcase of stochastic mean payoff games. Second, stochastic mean payoff games are polynomial-time equivalent to simple stochastic games and to games with discounted payoffs. Third, the associated decision problem (given a state in a stochastic mean payoff games, decide if its value is nonnegative) belongs to NP $\cap$ coNP (even to UP $\cap$ coUP). On the other hand, the computational problem of calculating the value of simple stochastic games belongs to the complexity class CLS, which is a subclass of both PPAD and PLS [DP11, EY10]. Despite all of these results, there is no known polynomial time algorithm for any of these classes of games. Halman [Hal07] showed (generalizing the results of [Lud95, BSV03, BV07]) that stochastic mean payoff games can be described as an LP-type problem.[1] This implies that they can be solved in strongly subexponential time in expectation, using the randomized simplex pivoting rule of Matoušek, Sharir, and Welzl [MSW96] (see also [HZ15] for some recent improvements). It is not known if stochastic mean payoff games can be solved in subexponential time by a deterministic algorithm (this question is open even if we restrict attention to deterministic games). Therefore, various authors studied fixed-parameter complexity bounds for these games. For instance, Gimbert and Horn [GH08] showed that simple stochastic games can be solved in polynomial time when the number of random positions is fixed and Zwick and Paterson [ZP96] gave a pseudopolynomial-time algorithm for deterministic mean payoff games. We refer to [IJM12, CR17] for the improvements of these results and more discussion. Moreover, Hansen, Miltersen, and Zwick [HMZ13] gave a strongly

[1] Strictly speaking, Halman considered only a subclass of stochastic mean payoff games, but every such game can be reduced to Halman's form in strongly polynomial time using the results of [AM09].

polynomial-time algorithm for discounted games with a fixed discount factor. In contrast with the case of more general games, a deterministic subexponential-time algorithm for parity games was given by Jurdziński, Paterson, and Zwick [JPZ08]. In a more recent breakthrough, Calude et al. [CJK+17] showed that parity games can be solved in quasipolynomial time. This result has already attracted a number of follow-up works [GIJ17, JL17, FJS+17, Leh18, CDF+18]. We also refer to [Mam17, BEGM13, AGH15, ACS14, HK66, ACTDG12, vD18, Zie98, VJ00] for further discussion and information about algorithms that can be used to solved these types of games, and to [CJH04, DJL18, CDGO14, SWZ18, CD12] for related classes of games. However, let us note a few papers in this domain that are closely related to our work. Boros, Elbassioni, Gurvich, and Makino [BEGM15] used a generalization of the "pumping" algorithm, initially introduced by Gurvich, Karzanov, and Khachiyan in the deterministic setting [GKK88], to show that general stochastic mean payoff games can be solved in pseudopolynomial time if the number of random positions is fixed. (We note that this does not follow from the analogous results on simple stochastic games cited above because the reduction from general stochastic mean payoff games to simple stochastic games adds too many random positions to the game, see [BEGM13] for a discussion.) One of the results of this work deals with value iteration and its complexity bounds in the stochastic setting, as an alternative to the pumping algorithms. Value iteration for deterministic games was studied by Zwick and Paterson [ZP96]. Ibsen-Jensen and Miltersen [IJM12] showed that modified value iteration can solve simple stochastic games in polynomial time when the number of random positions is fixed, and Chatterjee and Ibsen-Jensen [CIJ14] studied the complexity of value iteration for games with imperfect information. Value iteration for 1-player games has been studied, for instance, in [Put05, ACD+17]. The relationship between deterministic mean payoff games and linear programming over Puiseux series was used by Allamigeon, Benchimol, Gaubert, and Joswig [ABGJ14] to show that any semialgebraic polynomial-time pivoting rule for simplex algorithm would lead to a polynomial-time algorithm solving deterministic mean payoff game. By similar techniques, Allamigeon, Benchimol, and Gaubert [ABG14] showed that deterministic mean payoff games are solvable in polynomial time on average.

1.2 Our contribution

Our work is divided into two parts. In the first part, we present structural results concerning the tropicalizations of semialgebraic sets. In the second part, we discuss algorithmic consequences of our approach.

In Chapter 3, we start by studying the tropicalizations of general semialgebraic sets. In the case of (complex) algebraic varieties, a fundamental result of Bieri and Groves [BG84, EKL06] states that if $(\mathscr{K}, \mathsf{val})$ is an algebraically closed valued field with value group equal to $\mathbb{R}$ and $S \subset (\mathscr{K}^*)^n$ is an algebraic variety in the torus, then $\mathsf{val}(S) \subset \mathbb{R}^n$ is a union of polyhedra. Alessandrini [Ale13] proved an analogue of this theorem for definable subsets of Hardy fields of polynomially bounded o-minimal structures. His analysis implies that if $(\mathscr{K}, \mathsf{val})$ is a real closed field with value group equal to $\mathbb{R}$ and convex valuation, and $S \subset (\mathscr{K}^*)^n$ is a semialgebraic set, then $\mathsf{val}(S)$ is a union of polyhedra. The results of Alessandrini also apply to fields with value groups smaller than $\mathbb{R}$. In Chapter 3, we give a constructive proof of this result. The proof is based on the Denef–Pas quantifier elimination [Pas89] and applies to fields with arbitrary value groups (not only the subgroups of $\mathbb{R}$). Denef–Pas quantifier elimination also gives a transfer principle that is used later to prove the tropical analogue of the Helton–Nie conjecture for arbitrary real closed valued fields.

Theorem A (Theorem 3.1)**.** *Suppose that* $(\mathscr{K}, \mathsf{val})$ *is a real closed field equipped with a convex and nontrivial valuation* $\mathsf{val}\colon \mathscr{K} \to \Gamma \cup \{-\infty\}$*. Let* $S \subset \mathscr{K}^n$ *be a semialgebraic set. Then, the set* $\mathsf{val}(S) \subset (\Gamma \cup \{-\infty\})^n$ *is semilinear and has closed strata. Conversely, any semilinear subset of* $(\Gamma \cup \{-\infty\})^n$ *that has closed strata arises as an image by valuation of a semialgebraic set.*

Given a set S, it is sometimes possible to find a simple description of $\mathsf{val}(S)$. An important result in this direction is the Kapranov theorem [EKL06, Theorem 2.1.1], which states that if $\mathscr{K}$ is the field of Puiseux series with complex coefficients and $S := \{x \in (\mathscr{K}^*)^n \colon P(x) = 0\}$ is a hypersurface given by a polynomial $P \in \mathscr{K}[X_1, \dots, X_n]$, then $\mathsf{val}(S)$ is fully described by the formal tropicalization of the polynomial P. This statement is not true for real semialgebraic states, even if they are described by one polynomial (in)equality. However, thanks to the theorem above, we can give an explicit description of the images by valuation of semialgebraic sets defined by systems of inequalities, provided that a certain regularity condition is satisfied.

Theorem B (Theorem 3.4)**.** *Let* $\mathbb{K}$ *denote the field of Puiseux series with real coefficients and let* $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{>0}$ *be a semialgebraic set defined as*

$$\boldsymbol{\mathcal{S}} := \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \boldsymbol{P}_1(\boldsymbol{x}) \,\square_1\, 0, \dots, \boldsymbol{P}_m(\boldsymbol{x}) \,\square_m\, 0\} \, ,$$

where $\boldsymbol{P}_i \in \mathbb{K}[X_1, \dots, X_n]$ *are nonzero polynomials and* $\square \in \{\geqslant, >\}^m$*. Let* $P_i := \operatorname{trop}(\boldsymbol{P}_i)$ *for all* i *and suppose that* $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ *has regular support. Then*

$$\mathsf{val}(\boldsymbol{\mathcal{S}}) = \{x \in \mathbb{R}^n \colon \forall i, P_i^+(x) \geqslant P_i^-(x)\} \, .$$

The theorem above is a generalization of a result of Develin and Yu [DY07] who proved Theorem B in the case where polynomials $\boldsymbol{P}_i$ are affine. Furthermore, we note that the condition "$\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ has regular support" is generic—it is satisfied if the coefficients of the tropical polynomials $P_1, \dots, P_m$ lay outside of some set of measure zero.

We then turn our attention to the tropicalizations of convex semialgebraic sets defined over Puiseux series. In Chapter 4 we introduce the notion of *tropical spectrahedra*, which are sets defined as images by valuation of spectrahedra over Puiseux series. We show that a class of tropical spectrahedra defined by Metzler matrices can be described explicitly using polynomial inequalities of degree two in the tropical semifield. Subsequently, we use this class to give an explicit description of generic tropical spectrahedra. It turns out that only 2×2 minors are needed to describe a generic tropical spectrahedron. This extends the result of Yu [Yu15] who tropicalized the positive semidefinite cone.

Theorem C (Theorem 4.28)**.** *Suppose that* $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ *are symmetric matrices, let* $\boldsymbol{Q}(\boldsymbol{x}) := \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)}$ *and* $\boldsymbol{\mathcal{S}} := \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ *be the associated spectrahedron. Moreover, suppose that the matrices* $\mathsf{val}(\boldsymbol{Q}^{(0)}), \dots, \mathsf{val}(\boldsymbol{Q}^{(n)}) \in \mathbb{T}^{m \times m}$ *are generic. Then, the set* $\mathsf{val}(\boldsymbol{\mathcal{S}})$ *is described by tropical polynomial inequalities that are given by a tropical variant of the* 2×2 *minors of the affine pencil* $\boldsymbol{Q}(\boldsymbol{x})$*.*

In Chapter 5, we study general convex semialgebraic sets and give multiple equivalent characterizations of their images by valuation. In particular, we show that the tropical analogue of the Helton–Nie conjecture is true.

Théorème A (Theorem 5.5)**.** *Let* $\mathcal{S} \subset \mathbb{T}^n$*. Then, the following conditions are equivalent:*

(a) $\mathcal{S}$ *is a tropicalization of a convex semialgebraic set;*

(b) $\mathcal{S}$ *is tropically convex and has closed semilinear strata;*
(c) $\mathcal{S}$ *is a projection of a tropical Metzler spectrahedron;*
(d) there exists a projected spectrahedron $\boldsymbol{S} \subset \mathbb{K}^n_{\geqslant 0}$ *such that* $\mathsf{val}(\boldsymbol{S}) = \mathcal{S}$*.*

This theorem is the first place where the connection with stochastic mean payoff games plays a role—one of the ingredients of the proof is a lemma of Zwick and Paterson [ZP96] that was originally used to prove a reduction from discounted mean payoff games to simple stochastic games. Using the transfer principle given by the Denef–Pas quantifier elimination, we can then show that the classical Helton–Nie conjecture is true "up to taking the valuation."

Theorem D (Theorem 5.2)**.** *Let* $\mathscr{K}$ *be a real closed valued field equipped with a nontrivial and convex valuation* $\mathsf{val}\colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ *and suppose that* $S \subset \mathscr{K}^n$ *is a convex semialgebraic set. Then, there exists a projected spectrahedron* $S' \subset \mathscr{K}^n$ *such that* $\mathsf{val}(S) = \mathsf{val}(S')$*.*

In the second part of the dissertation, we study the relationship between the tropicalizations of convex sets and stochastic mean payoff games. In Chapter 6 we present this class of games and analyze them using Shapley operators [Sha53], Kohlberg's theorem [Koh80], and the Collatz–Wielandt property [Nus86]. Our presentation is based on [AGG12]. We give one new result in this area—we generalize the tropical characterization of winning states of deterministic games given in [AGG12] to the case of stochastic games. By combining this result with the results of earlier chapters, we obtain the following correspondence between tropicalization of cones and stochastic mean payoff games.

Theorem E (Theorem 6.30)**.** *Let* $\mathcal{S} \subset \mathbb{T}^n$*. Then,* $\mathcal{S}$ *is a tropicalization of a closed convex semialgebraic cone if and only if there exists a stochastic mean payoff game such that its Shapley operator* $F\colon \mathbb{T}^n \to \mathbb{T}^n$ *satisfies* $\mathcal{S} = \{x \in \mathbb{T}^n \colon x \leqslant F(x)\}$*. Moreover, the support of* $\mathcal{S}$ *is given by the biggest winning dominion of this game. In particular,* $\mathcal{S}$ *is nontrivial if and only if the game has at least one winning state.*

As a consequence of the theorem above, the feasibility problem for any semialgebraic cone over Puiseux series can, in theory, be reduced to solving a mean payoff game. Nevertheless, even though our proofs are constructive, finding the appropriate game is not easy in general. Indeed, as indicated before, while deciding the winner of a stochastic mean payoff game belongs to the class NP $\cap$ coNP, there are problems of unknown complexity that can be expressed as conic semidefinite feasibility problems. However, the tropical polynomial systems mentioned in Theorem C can be converted to Shapley operators. This implies that generic nonarchimedean semidefinite feasibility problems for cones over Puiseux series can be solved by a reduction to mean payoff games. Even more, our approach can solve some semidefinite feasibility problems that do not satisfy the genericity condition. This is proven in Chapter 7 and summarized below.

Theorem F (Theorem 7.6)**.** *Suppose that* $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ *are symmetric matrices, let* $\boldsymbol{Q}(\boldsymbol{x}) \coloneqq \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)}$ *and* $\boldsymbol{S} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$*. Furthermore, suppose that the matrices* $\boldsymbol{Q}^{(k)}$ *have rational valuations,* $\mathsf{val}(\boldsymbol{Q}^{(1)}), \dots, \mathsf{val}(\boldsymbol{Q}^{(k)}) \in (\mathbb{Q} \cup \{-\infty\})^{m \times m}$*. Then, given only the signed valuations* $\mathsf{sval}(\boldsymbol{Q}^{(k)})$ *of these matrices, we can construct (in polynomial-time) a stochastic mean payoff game with the following properties. If the maximal value of the game is strictly positive, then* $\boldsymbol{S}$ *is nontrivial. If the maximal value is strictly negative, then* $\boldsymbol{S}$ *is trivial. Furthermore, if the maximal value is equal to* 0 *and the matrices* $\mathsf{val}(\boldsymbol{Q}^{(1)}), \dots, \mathsf{val}(\boldsymbol{Q}^{(n)})$ *are generic, then* $\boldsymbol{S}$ *is nontrivial. Conversely, solving stochastic mean payoff games can be reduced to the problem of checking the feasibility of spectrahedral cones* $\boldsymbol{S} \subset \mathbb{K}^n_{\geqslant 0}$ *as above.*

Along the way, we show that the problem of checking the feasibility of a tropical Metzler spectrahedron (defined only by the tropical polynomial inequalities, without the reference to Puiseux series) is equivalent to solving stochastic mean payoff games.

Theorem G (Theorem 7.4)**.** *The problem of checking the feasibility of tropical Metzler spectrahedral cones is polynomial-time equivalent to the problem of solving stochastic mean payoff games.*

In the final chapter of the thesis we study the value iteration, which is a simple algorithm that can be used to solve stochastic mean payoff games. This algorithm is based on the fact that if $F\colon \mathbb{T}^n \to \mathbb{T}^n$ is the Shapley operator of such game, then the limit $\lim_{N\to\infty} F^N(0)/N$ (where $F^N = F \circ \dots \circ F$) exists and is equal to the value vector of this game. The value iteration algorithm computes the successive values of $F^N(0)$ and deduces the properties of the limit out of this computation. As noted above, the feasibility problem of tropical spectrahedra corresponds to deciding the sign of the value. The following observation provides a condition number for this problem.

Theorem H (Theorem 8.25)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be monotone and additively homogeneous. Furthermore, suppose that the equation $f(u) = \eta + u$ has a solution $(\eta, u) \in \mathbb{R} \times \mathbb{R}^n$. Then, we have $\lim_{N\to\infty} f^N(0)/N = \eta(1, 1, \dots, 1)$. Moreover, suppose that $\eta \neq 0$ and let $R := \inf\{\|u\|_{\mathrm{H}} \in \mathbb{R}^n : f(u) = \eta + u\}$, where $\| \cdot \|_{\mathrm{H}}$ is the Hilbert seminorm. Then, for every*

$$N \geqslant \lceil 1 + \frac{R}{|\eta|} \rceil$$

the entries of $f^N(0)$ have the same sign and this sign if the same as the sign of η.

The assumptions of this theorem are fulfilled, for instance, if f is the Shapley operator of a stochastic mean payoff game with constant value. In this case η is the value of the game. However, there are other class of games for which this result may be of interest, such as the entropy games of [ACD$^+$16, AGGCG17]. When specified to tropical Metzler spectrahedra, the condition number $R/|\eta|$ has a geometric interpretation.

Proposition I (Proposition 8.16)**.** *Suppose that $F\colon \mathbb{T}^n \to \mathbb{T}^n$ is a Shapley operator associated with a tropical Metzler spectrahedral cone $\mathcal{S} \subset \mathbb{T}^n$. Furthermore, suppose that F satisfies the conditions of Theorem H and that $\eta > 0$. Then, there exists a tropical Metzler spectrahedral cone $\tilde{\mathcal{S}} \subset \mathbb{T}^{2n}$ such that $\mathcal{S}$ is the projection of $\tilde{\mathcal{S}}$ and such that $\eta/2$ is the radius of the largest ball in Hilbert seminorm that is included in $\tilde{\mathcal{S}}$. Furthermore, if $u \in \mathbb{R}^n$ is such that $F(u) = \eta + u$, then u is the projection of the center of one such ball.*

In this way, the quantity $|\eta|$ measures the width of the tropical Metzler spectrahedral cone, while the quantity R measures the distance of this cone from the origin. Intuitively speaking, this should be compared to the quantities that govern the complexity of the ellipsoid method [GLS93]. The complexity of this method depends polynomially on $\log(R/r)$, where R is the radius of a ball that contains a given convex body and r is the radius of a ball included in this body. Thus, R measures how far the body is from the origin, while r measures how large it is. The fact that value iteration depends polynomially on $R/|\eta|$ instead of $\log(R/|\eta|)$ is intuitively justified by the fact that taking valuation already corresponds to taking the logarithm of the input. The analogy with the ellipsoid is strengthened by the fact that value iteration is also based on an oracle—in order to use this method, it is enough to have an oracle that approximately evaluates f.

We also give another application of the condition number, relating the nonarchimedean and archimedean feasibility problems. As discussed earlier, a spectrahedron $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ over Puiseux series can be seen as a family of real spectrahedra $\boldsymbol{\mathcal{S}}(t) \subset \mathbb{R}^n_{\geqslant 0}$ over real numbers. By a general o-minimality argument, there exists $t_0 > 0$ such that for all $t > t_0$ the feasibility problems for $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}(t)$ coincide (i.e., $\boldsymbol{\mathcal{S}}(t)$ is nonempty if and only if $\boldsymbol{\mathcal{S}}$ is nonempty). This correspondence may potentially lead to a homotopy-type algorithm for deciding the feasibility of some spectrahedra. To create such an algorithm, it is desirable to have bounds on the quantity t_0. The next theorem shows that t_0 is not big if the associated stochastic mean payoff game is well conditioned.

Theorem J (Theorem 8.4)**.** *Suppose that symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$ create a well-formed tropical linear matrix inequality and let $\bar{\eta} \in \mathbb{R}$ denote the maximal value o the associated stochastic mean payoff game. Let $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ be the monomial lift of $Q^{(1)}, \dots, Q^{(n)}$ defied as $\boldsymbol{Q}^{(k)}_{ij} := \delta_{ij} t^{|Q^{(k)}_{ij}|}$, where $\delta_{ij} := 1$ if $i = j$ and $\delta_{ij} := -1$ otherwise. Then, for any*

$$t > (2(m-1)n)^{1/(2\bar{\eta})}$$

the real spectrahedral cone $\boldsymbol{\mathcal{S}}(t) := \{\boldsymbol{x} \in \mathbb{R}^n_{\geqslant 0} \colon x_1\boldsymbol{Q}^{(1)}(t) + \dots + x_n\boldsymbol{Q}^{(n)}(t)\}$ is nontrivial if and only if the nonarchimedean spectrahedral cone $\boldsymbol{\mathcal{S}} := \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)}\}$ is nontrivial.

The main contribution of Chapter 8 is to give explicit bounds for the condition number $R/|\eta|$ in the case when f is a Shapley operator of a stochastic mean payoff game. This is based on the following theorem that estimates the bit-size of the invariant measure of a finite Markov chain.

Theorem K (Theorem 8.44)**.** *Suppose that $P \in [0,1]^{n\times n}$ is an irreducible stochastic matrix with rational entries, let $\pi \in \,]0,1]^n$ be the stationary distribution of P, and let $M \in \mathbb{N}^*$ be the common denominator of all the entries of P. Then, the least common denominator of the entries of π is not greater that $nM^{\min\{n_r, n-1\}}$, where $n_r \leqslant n$ is the number of rows of P which are not deterministic (i.e., which have an entry in the open interval $]0,1[$). Moreover, the bound $nM^{\min\{n_r, n-1\}}$ is optimal.*

Bounds of similar nature have already appeared in the literature concerning stochastic mean payoff games [BEGM15, AGH18, Con92, ACS14]. However, the proofs in these works are based on the Hadamard inequality that leads to suboptimal results. In order to achieve the optimal bound, we replace the use of the Hadamard inequality by the combinatorial formula of Freidlin and Wentzell [FW12]. As a corollary, we can estimate the pessimistic number of iterations that is needed to solve a stochastic mean payoff game. This gives the following fixed-parameter complexity result for this class of games.

Theorem L (Theorems 8.58 and 8.68)**.** *The value iteration algorithm solves constant value games in pseudopolynomial-time and polynomial memory when the number of randomized actions is fixed. Moreover, for general games (with arbitrary value), a modification of the value iteration can find the maximal (or minimal) value of the game, and the set of states that achieve it, in pseudopolynomial-time and polynomial memory when the number of randomized actions is fixed.*

We note that the second part of the theorem relies on the characterization of dominions established in Chapter 6. The only other known algorithms that achieve the complexity given in Theorem L is the pumping algorithm studied in [BEGM15] and the variant of the ellipsoid algorithm given in [BEGM17]. However, value iteration is simpler, and our complexity bounds are better than the ones obtained for the pumping and ellipsoid algorithms. When specified to the case of tropical Metzler spectrahedra, the parametrized complexity bounds of Theorem L give bounds that depend on the sparsity of the matrices that define this tropical spectrahedron. This bounds are given in Section 8.5, where we also introduce an explicit class of tropical spectrahedra that leads to constant value games. To finish, we apply value iteration to solve random large-scale tropical semidefinite feasibility problems on this class of inputs.

1.3 Organization of the manuscript

This dissertation is organized as follows.

In Chapter 2 we discuss the preliminary notions used in the further parts of the manuscript. More precisely, in Section 2.1 we recall the basic definitions and properties related to convex polyhedra (such as relative interior or face) and polyhedral complexes. In Section 2.2 we give some fundamental facts about real closed fields, semialgebraic sets, and the notion of dimension for these sets. In Section 2.3 we define the field of Puiseux series, valuation, and tropical semifield. Section 2.4 describes the polyhedral complexes associated with (signed) tropical polynomials. In Section 2.5 we recall some definitions and properties of valued fields (such as value group, residue field, henselianity, cross-section, angular component) and state the characterization of real closed valued fields (Theorem 2.75). In Section 2.6 we discuss the basic notions of model theory of many-sorted structures (such as formula, interpretation, quantifier elimination, model completeness). This is also the place where we define the notion of semilinear sets in divisible ordered abelian groups (with or without the bottom element), and discuss the Denef–Pas quantifier elimination and its implications for real closed valued fields (Theorems 2.115 and 2.120). To finish, in Section 2.7 we recall the notion of a finite Markov chain, stochastic matrices, and discuss the general form of the ergodic theorem for Markov chains with payoffs (Theorem 2.137).

Our contribution starts in Chapter 3, where we give a constructive proof of the real analogue of the Bieri–Groves theorem (Theorem 3.1) and its application to regular tropicalizations of semialgebraic sets (Theorem 3.4). Furthermore, in Section 3.2 we discuss the tropicalizations of sets defined in fields with rational value group.

In Chapter 4 we study the fundamental objects of this dissertation—tropical spectrahedra. First, in Definition 4.13 and Proposition 4.14 we introduce a subclass of tropical spectrahedra defined by Metzler matrices. Then, we show (Theorem 4.19) that under regularity conditions, a tropicalization of a spectrahedron defined by Metlzer matrices belongs to this class. We extend this results to non-Metzler spectrahedra in Section 4.2 (Definition 4.21 and Theorem 4.28) and, in Section 4.3, we show that regularity conditions are generically satisfied. In Section 4.4 we study the tropicalization of interiors of spectrahedra.

In Chapter 5 we prove the tropical analogue of the Helton–Nie conjecture. To do so, we first introduce the basic notions of tropical convexity (Section 5.1), describe tropically convex semilinear sets as sublevel sets of monotone homogeneous operators (Proposition 5.34), and describe these operators by graphs (Lemmas 5.37 and 5.38). Next, we establish a class of graphs that encodes tropical Metzler spectrahedra (Proposition 5.40). The proof of the main theorem of this chapter is divided into a few parts. First, we establish it for real tropical cones (Proposition 5.44), then for the field of Puiseux series (Theorem 5.5), and finally generalize to arbitrary real closed valued fields (Theorem 5.2 and Section 5.6).

The second part of the dissertation starts with the introduction to stochastic mean payoff games (Chapter 6), where we discuss the Shapley operators (Definition 6.3), Kohlberg's theorem (Theorem 6.6), Collatz–Wielandt property (Corollary 6.18), and present the proof of the fact that these games have optimal policies (Theorem 6.1). Our contribution is given in Theorem 6.16, where we characterize the nonempty strata of a tropical cone defined by the Shapley operator using winning dominions of the associated game. In Section 6.3, we transfer all the results to Shapley operators associated with bipartite games, which correspond more closely to the results of Chapter 5, as shown in Lemma 6.27.

In Chapter 7 we show the equivalence between tropical semidefinite feasibility problem and stochastic mean payoff games (Theorem 7.4 and Sections 7.1 and 7.2). We also show the implications of this equivalence for the nonarchimedean semidefinite feasibility problem (Theorem 7.18). The correspondence is the most direct for a particular class of problems studied in Section 7.2.1.

In Chapter 8 we study the condition number of stochastic mean payoff games and nonarchimedean semidefinite feasibility. First, we relate the tropical feasibility problem with the archimedean one (Theorem 8.4). Then, we give a geometric interpretation of this result (Proposition 8.16). Subsequently, we move to the study of value iteration. We first study its abstract version for general operators (Theorem 8.25) and show that is works in an oracle-based model (Section 8.2.1). The main part of this chapter is to give estimates on the condition number. This is done in Section 8.3 and summarized in Theorem 8.37. The main ingredient is the estimate of stationary distribution given in Theorem 8.44. As a corollary of these results, we are able to give fixed-parameter complexity bounds for stochastic mean payoff games. These bounds, for different tasks, are collected in Section 8.4. The most general one is given in Theorem 8.68. In Section 8.5 (notably Theorem 8.72 and Proposition 8.75) we specify these bounds to the case of tropical semidefinite feasibility and present numerical results.

To finish, in Chapter 9 we discuss some open problems and possible directions of future research.

1.4 Notation

Throughout this thesis, we use the following notation:

- if V is a vector space and $v_1, \dots, v_k \in V^n$ are vectors, then we denote by $\mathrm{span}(v_1, \dots, v_k)$ the linear space spanned by $v_1, \dots, v_k$;
- we denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^n$;
- $\mathbb{N} = \{0, 1, \dots\}$ denotes the set of natural numbers with zero and we put $\mathbb{N}^* = \{1, 2, \dots\}$;
- if $n \in \mathbb{N}^*$ is a positive natural number, then we denote $[n] \coloneqq \{1, 2, \dots, n\}$;
- if X is any set, $x \in X^n$ is a vector with entries in X, and $K \subset [n]$ is a nonempty subset, then we denote by $x_K \in X^K$ the subvector of x formed by the coordinates taken from K;
- if $\lambda \in \mathbb{R}$ is a real number and $x \in \mathbb{R}^n$ is a vector, then we denote $\lambda + x \coloneqq (\lambda + x_1, \dots, \lambda + x_n)$;
- if X is a topological space, then we always equip X^n with the product topology. If $\mathcal{S} \subset X^n$ is any set, then we denote by $\mathrm{int}(\mathcal{S})$ the interior of $\mathcal{S}$, and by $\mathrm{cl}_X(\mathcal{S})$ the closure of $\mathcal{S}$ in this topology.

CHAPTER 2

Preliminaries

2.1 Polyhedra and polyhedral complexes

In this section, we recall the definition of a polyhedron, relative interior, and polyhedral complex. More information can be found in [Sch87, Zie07, Grü03].

Definition 2.1. We say that a set $\mathcal{W} \subset \mathbb{R}^n$ is a *polyhedron* if it is of the form

$$\mathcal{W} = \{x \in \mathbb{R}^n \colon \forall i = 1, \dots, p, \langle a_i, x \rangle \leqslant b_i\}, \tag{2.1}$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for all $i \in [p]$. A polyhedron $\mathcal{V} \subset \mathcal{W}$ is a *face* of $\mathcal{W}$ if there exists a set $I \subset [p]$ such that

$$\mathcal{V} = \{x \in \mathcal{W} \colon \forall i \in I, \langle a_i, x \rangle = b_i\} \, .$$

Definition 2.2. If $\mathcal{W} \subset \mathbb{R}^n$ is a polyhedron, then its *relative interior*, denoted $\mathrm{ri}(\mathcal{W})$, is the set defined as follows. We set $J \subset [p]$ as $J \coloneqq \{j \in [p] \colon \exists x \in \mathcal{W}, \langle a_j, x \rangle < b_j\}$ and we put

$$\mathrm{ri}(\mathcal{W}) \coloneqq \{x \in \mathbb{R}^n \colon \forall j \in J, \langle a_j, x \rangle < b_j, \ \forall j \notin J, \langle a_j, x \rangle = b_j\} \, .$$

The next proposition characterizes the polyhedron in terms of its interior and relative interior. We denote by $\mathrm{int}(\cdot)$ and $\mathrm{cl}_{\mathbb{R}}(\cdot)$ the interior and closure operators in the standard topology of $\mathbb{R}^n$.

Lemma 2.3. *If $\mathcal{W} \subset \mathbb{R}^n$ is a polyhedron, then $\mathrm{cl}_{\mathbb{R}}(\mathrm{ri}(\mathcal{W})) = \mathcal{W}$. In particular, if $\mathcal{W}$ is nonempty, then $\mathrm{ri}(\mathcal{W})$ is nonempty. Furthermore, if $\mathrm{int}(\mathcal{W}) \neq \emptyset$, then $\mathrm{int}(\mathcal{W}) = \mathrm{ri}(\mathcal{W})$.*

Sketch of the proof. The claim is trivial if $J = \emptyset$. Otherwise, for every $j \in J$ we take take $x^{(j)} \in \mathcal{W}$ such that $\langle a_j, x^{(j)}\rangle < b_j$. Then $x = \frac{1}{|J|}\sum_{j\in J} x^{(j)} \in \mathrm{ri}(\mathcal{W})$. Furthermore, if $y \in \mathcal{W}$, then $x_\lambda \coloneqq (1-\lambda)x + \lambda y \in \mathrm{ri}(\mathcal{W})$ for all $\lambda \in [0, 1[$. Hence $\mathrm{cl}_{\mathbb{R}}(\mathrm{ri}(\mathcal{W})) = \mathcal{W}$. To prove the second claim, note that if $x \in \mathrm{int}(\mathcal{W})$, then we have $\langle a_j, x\rangle < b_j$ for all j such that $(a_j, b_j) \neq (0, 0)$. Hence $\mathrm{int}(\mathcal{W}) \subset \mathrm{ri}(\mathcal{W})$. Conversely, every point y that satisfies $\langle a_j, y\rangle < b_j$ for all j such that $(a_j, b_j) \neq (0, 0)$ belongs to $\mathrm{int}(\mathcal{W})$ and $\mathrm{ri}(\mathcal{W}) \subset \mathrm{int}(\mathcal{W})$. □

Corollary 2.4. *If $\mathcal{W}, \mathcal{V} \in \mathbb{R}^n$ are polyhedra, then $\mathcal{W} \cup \mathcal{V} = \mathrm{cl}_{\mathbb{R}}\big(\mathrm{ri}(\mathcal{W}) \cup \mathrm{ri}(\mathcal{V})\big)$.*

Proof. We have $\mathcal{W} = \mathrm{cl}_{\mathbb{R}}(\mathrm{ri}(\mathcal{W})) \subset \mathrm{cl}_{\mathbb{R}}\big(\mathrm{ri}(\mathcal{W}) \cup \mathrm{ri}(\mathcal{V})\big)$ and analogously for $\mathcal{V}$. This proves the inclusion $\subset$. To prove the opposite inclusion, note that $\mathrm{cl}_{\mathbb{R}}\big(\mathrm{ri}(\mathcal{W}) \cup \mathrm{ri}(\mathcal{V})\big) \subset \mathrm{cl}_{\mathbb{R}}(\mathcal{W} \cup \mathcal{V}) = \mathcal{W} \cup \mathcal{V}$ because polyhedra are closed. □

Lemma 2.5. *If $\mathcal{W}$ is a polyhedron and $x \in \mathcal{W}$, then there exists a unique face $\mathcal{V}$ of $\mathcal{W}$ such that $x \in \mathrm{ri}(\mathcal{V})$.*

Sketch of the proof. Let $I \coloneqq \{i \in [p] \colon \langle a_i, x\rangle < b_i\}$ and denote $\mathcal{V} \coloneqq \{y \in \mathcal{W} \colon \forall i \notin I, \langle a_i, y\rangle = b_i\}$. The face $\mathcal{V}$ satisfies the claim. □

Definition 2.6. We say that a family $\mathcal{C} = \{\mathcal{W}_1, \dots, \mathcal{W}_m\}$ of polyhedra is a *polyhedral complex* if the following three conditions are satisfied:

- the empty set belongs to $\mathcal{C}$;
- for all $i \in [m]$, every face of $\mathcal{W}_i$ also belongs to $\mathcal{C}$;
- the relative interiors of different polyhedra belonging to $\mathcal{C}$ are pairwise disjoint.

We often refer to the polyhedra belonging to $\mathcal{C}$ as *cells of $\mathcal{C}$*.

Remark 2.7. One can replace the third condition in the definition above by "for every $i, j \in [m]$, $\mathcal{W}_i \cap \mathcal{W}_j$ is a face of $\mathcal{W}_i$ and a face of $\mathcal{W}_j$." We leave the proof of this equivalence as an exercise.

Definition 2.8. If $\mathcal{C} = \{\mathcal{W}_1, \dots \mathcal{W}_m\}$ is a polyhedral complex, then we define its *support* as the union $\bigcup_{i=1}^m \mathcal{W}_i$.

Lemma 2.9. *If $\mathcal{C}_1 = \{\mathcal{W}_1, \dots, \mathcal{W}_m\}$ and $\mathcal{C}_2 = \{\mathcal{V}_1, \dots, \mathcal{V}_\ell\}$ are two polyhedral complexes in $\mathbb{R}^n$, then $\mathcal{C} = \{\mathcal{W}_i \cap \mathcal{V}_k \colon i \in [m], k \in [\ell]\}$ is a polyhedral complex whose support is equal to the intersection of the supports of $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Sketch of the proof. It is clear that the empty set belongs to $\mathcal{C}$. Moreover, any face of the polyhedron $\mathcal{W}_i \cap \mathcal{V}_k$ is also of such form (by the definition of face and the fact that $\mathcal{C}_1, \mathcal{C}_2$ are polyhedral complexes). Finally, $\mathrm{ri}(\mathcal{W}_i \cap \mathcal{V}_k) = \mathrm{ri}(\tilde{\mathcal{W}}) \cap \mathrm{ri}(\tilde{\mathcal{V}})$, where $\tilde{\mathcal{W}}$ is a face of $\mathcal{W}_i$ and $\tilde{\mathcal{V}}$ is a face of $\mathcal{V}_k$. Hence $\mathrm{ri}(\mathcal{W}_i \cap \mathcal{V}_k) = \mathrm{ri}(\mathcal{W}_{i'}) \cap \mathrm{ri}(\mathcal{V}_{k'})$ for some $i' \in [m]$ and $k' \in [\ell]$. Therefore, by Lemma 2.3, if $\mathcal{W}_i \cap \mathcal{V}_k$ is different than $\mathcal{W}_{i''} \cap \mathcal{V}_{k''}$, then they have disjoint relative interiors. This shows that $\mathcal{C}$ is a polyhedral complex. The second claim follows from the equality $\bigcup_{i\in[m],k\in[\ell]}(\mathcal{W}_i \cap \mathcal{V}_k) = (\bigcup_{i\in[m]} \mathcal{W}_i) \cap (\bigcup_{k\in[\ell]} \mathcal{V}_k)$. □

2.2 Real closed fields

In this section we recall the basic definitions concerning real closed fields and semialgebraic sets. We refer to [BPR06] for more information.

Definition 2.10. Let $\mathscr{K} = (\mathscr{K}, \leqslant)$ be an ordered field.[1] Then, we say that $\mathscr{K}$ is *real closed* if every positive element of $\mathscr{K}$ has a square root and every univariate polynomial over $\mathscr{K}$ of odd degree has at least one root (in $\mathscr{K}$).

Example 2.11. The field $\mathbb{R}$ of real numbers is a real closed field. A second example that is important for this work is the field $\mathbb{K}$ of Puiseux series that we will introduce in Section 2.3.

A basic property of real closed fields is that the order is unique.

Lemma 2.12. *If $(\mathscr{K}, \leqslant)$ is a real closed field, then is admits a unique total order. Even more, we have $x \geqslant y$ if and only if $x - y$ is a square.*

Proof. If $\succcurlyeq$ is any total order on $\mathscr{K}$, then $x^2 \succcurlyeq 0$ for all $x \in \mathscr{K}$. Furthermore, if $y \succcurlyeq 0$ is not a square, then y is negative in the order $\leqslant$. This means that $-y \geqslant 0$ and hence $-y$ is a square because $\mathscr{K}$ is real closed. Therefore, $-y = z^2 \succcurlyeq 0$ and $y = 0 = 0^2$, which gives a contradiction. Thus, $x \succcurlyeq 0$ if and only if x is a square, which implies that $x \succcurlyeq y$ if and only if $x - y$ is a square. In particular, the order on $\mathscr{K}$ is unique. □

Real closed fields share many properties with the field of real numbers. For instance, every nonnegative element of $\mathscr{K}$ has a well-defined nth root.

Lemma 2.13. *Suppose that $x \in \mathscr{K}$ is nonnegative, $x \geqslant 0$, and let $n \geqslant 1$ be any natural number. Then, there exists a unique $y \in \mathscr{K}$ such that $y \geqslant 0$ and $y^n = x$.*

Proof. First, let us show that y is unique. If $y, z \geqslant 0$, $y^n = z^n = x$ and $y \neq z$, then we have $0 = y^n - z^n = (y - z)(y^{n-1} + y^{n-2}z + \cdots + yz^{n-2} + z^{n-1})$. Since $y, z \geqslant 0$, the second factor is equal to 0 if and only if $y = z = 0$, which gives a contradiction. Hence y is unique. To show that y exists, suppose that n is odd. Then, the polynomial $P(z) := z^n - x$ has a root y in $\mathscr{K}$. If y is negative, then $(-y)$ is positive and thus $x = y^n = -(-y)^n$ is negative, which gives a contradiction. Therefore, the claim is true for odd values of n. If n is even, then we write $n = 2^k m$, where $k \geqslant 1$ and m is odd. Let $z_0 \geqslant 0$ be the mth root of x and let $z_i \geqslant 0$ be the square root of z_{i-1} for all $i \geqslant 1$. Then $y = z_k$. □

We now recall the definition of semialgebraic sets and their basic properties.

Definition 2.14. If $\mathscr{K}$ is a real closed field, then we say that a subset $S \subset \mathscr{K}^n$ is *basic semialgebraic* if it is of the form

$$S = \{x \in \mathscr{K}^n \colon \forall i = 1, \dots p, P_i(x) > 0 \land \forall i = p+1, \dots, q, P_i(x) = 0\} ,$$

where $P_i \in \mathscr{K}[X_1, \dots, X_n]$ are polynomials. We say that S is *semialgebraic* if it is a finite union of basic semialgebraic sets.

Any real closed field $\mathscr{K}$ can be equipped with the topology induced by its order. Furthermore, we can extend this topology to $\mathscr{K}^n$ by taking the product topology. It can be checked that (as in the case of real numbers), this topology is the same as the topology induced by the *euclidean open balls* $B(x, r) \subset \mathscr{K}^n$,

$$B(x, r) := \{y \in \mathscr{K}^n \colon (y_1 - x_1)^2 + \cdots + (y_n - x_n)^2 < r^2\} ,$$

and that the addition and multiplication are continuous for this topology. The next proposition concerns the interiors and closures in this topology, denoted $\mathrm{int}(\cdot)$ and $\mathrm{cl}_{\mathscr{K}}(\cdot)$.

[1] In other words, we suppose that $\leqslant$ is a total order on $\mathscr{K}$ that fulfills the properties $x \leqslant y \implies x + z \leqslant y + z$ and $x \geqslant 0, y \geqslant 0 \implies xy \geqslant 0$ for all $x, y, z \in \mathscr{K}$.

Proposition 2.15 ([BPR06, Proposition 3.1])**.** *If $S \subset \mathscr{K}^n$ is semialgebraic, then* $\operatorname{int}(S)$ *and* $\operatorname{cl}_{\mathscr{K}}(S)$ *are semialgebraic.*

Semialgebraic sets in dimension 1 have a particularly simple structure.

Lemma 2.16. *If $S \subset \mathscr{K}$ is semialgebraic, then it is a finite union of points and open intervals.*

Sketch of the proof. If $P \in \mathscr{K}[X]$ is a polynomial, $x_1 < \cdots < x_p \in \mathscr{K}$ are its roots, and we denote $x_0 = -\infty, x_{p+1} = +\infty$, then $P(x)$ has constant sign on every interval $]x_i, x_{i+1}[$ ([BPR06, Theorem 2.11]). Hence, the claim is true form the sets $\{x \in \mathscr{K} : P(x) > 0\}$ and $\{x \in \mathscr{K} : P(x) = 0\}$. Therefore, the claim is true for basic semialgebraic sets and for semialgebraic sets. ☐

Furthermore, the class of semialgebraic sets is closed under semialgebraic transformations.

Definition 2.17. Let $S \subset \mathscr{K}^n$ be semialgebraic. We say that a function $f : S \to \mathscr{K}^m$ is *semialgebraic* if its *graph* $\{(x, y) \in S \times \mathscr{K}^m : y = f(x)\}$ is a semialgebraic set.

Proposition 2.18 ([BPR06, Proposition 2.83])**.** *If $S \subset \mathscr{K}^n$ is semialgebraic and $f : S \to \mathscr{K}^m$ is a semialgebraic function, then $f(S) \subset \mathscr{K}^m$ is a semialgebraic set.*

The next definition and proposition gathers some basic properties of the dimension of semialgebraic sets.

Definition 2.19. The *dimension* of a semialgebraic set $S \subset \mathscr{K}^n$, denoted $\dim(S)$ is the largest natural number $d \geqslant 0$ such that there exists an injective semialgebraic function $f :]0, 1[^d \to S$. (With the convention that the dimension of an empty set is equal to -1.) We say that S is *full dimensional* if $\dim(S) = n$.

Proposition 2.20 ([BPR06, Section 5.3])**.** *The dimension of a semialgebraic set $S \subset \mathscr{K}^n$ is finite and not greater than n. Furthermore, S is full dimensional if and only if it has a nonempty interior. If $S, S' \subset \mathscr{K}^n$ are semialgebraic, then* $\dim(S \cup S') = \max(\dim(S), \dim(S'))$*. Moreover, if $S \subset S'$, then* $\dim(S) \leqslant \dim(S')$*. If a function $f : S \to \mathscr{K}^m$ is semialgebraic, then* $\dim(f(S)) \leqslant \dim(S)$*. If we further suppose that f is injective, then* $\dim(f(S)) = \dim(S)$*.*

Finally, let us recall the definition of a convex set, a simplex, and characterize the dimension of convex semialgebraic sets.

Definition 2.21. We say that a set $S \subset \mathscr{K}^n$ is *convex* if for every $x, y \in S$ and every $\lambda \in \mathscr{K}$ such that $0 \leqslant \lambda \leqslant 1$ we have $\lambda x + (1 - \lambda)y \in S$. We say that $S \subset \mathscr{K}^n$ is a *(convex) cone* if for every $x, y \in S$ and every $\lambda, \mu \in \mathscr{K}$, $\lambda, \mu \geqslant 0$ we have $\lambda x + \mu y \in S$.

Example 2.22. A polyhedron $\mathcal{W} \subset \mathbb{R}^n$ is a convex semialgebraic set. Indeed, the fact that $\mathcal{W}$ is convex follows easily from the definition. Moreover, a polyhedron $\mathcal{W} \subset \mathbb{R}^n$ as in (2.1) can be described as a union of 2^p basic semialgebraic sets (for each inequality we choose whether it is satisfied as an equality or not). An euclidean open ball $B(x, r) \subset \mathbb{R}^n$ is also a convex semialgebraic set (the fact that $B(x, r)$ is convex follows from the triangle inequality of the euclidean norm).

Definition 2.23. If the vectors $u^{(1)}, \dots, u^{(m)} \in \mathscr{K}^n$ are linearly independent and $u^{(0)} \in \mathscr{K}^n$, then we define the associated *simplex* $\Delta(u^{(0)}, \dots, u^{(m)}) \subset \mathscr{K}^n$ as the set

$$\Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)}) := \{\lambda_0 u^{(0)} + \sum_{i=1}^{m} \lambda_i (u^{(0)} + u^{(i)}) : \forall i, \lambda_i \geqslant 0, \sum_{i=0}^{m} \lambda_i = 1\} .$$

Remark 2.24. We note that a simplex is a semialgebraic set. Indeed, to see that one can consider the set $S \subset \mathscr{K}^{n+m+1}$ defined as

$$S \coloneqq \Big\{(x, \lambda_0, \dots, \lambda_m) \colon \forall i, \lambda_i \geqslant 0 \land \sum_{i=0}^{m} \lambda_i = 1 \land \forall k \in [n], x_k = \lambda_0 u_k^{(0)} + \sum_{i=1}^{m} \lambda_i (u_k^{(0)} + u_k^{(i)})\Big\} .$$

Then, $\Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)})$ is the projection of S onto the first n coordinates and the claim follows from Proposition 2.18. Analogously, the affine space $u^{(0)} + \operatorname{span}(u^{(1)}, \dots, u^{(m)})$ is a semialgebraic set.

Lemma 2.25. *If the vectors $u^{(1)}, \dots, u^{(m)} \in \mathscr{K}^n$ are linearly independent and $u^{(0)} \in \mathscr{K}^n$, then the dimension of the affine space $u^{(0)} + \operatorname{span}(u^{(1)}, \dots, u^{(m)})$ is equal to m, and the same is true for the dimension of the simplex $\Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)})$.*

Sketch of the proof. Let $A \coloneqq u^{(0)} + \operatorname{span}(u^{(1)}, \dots, u^{(m)})$ and $\Delta \coloneqq \Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)})$. The function $f \colon \mathscr{K}^m \to \mathscr{K}^n$ defined as

$$f(x_1, \dots, x_m) \coloneqq u^{(0)} + x_1 u^{(1)} + \dots + x_m u^{(m)} = (1 - \sum_{i=1}^{m} x_i) u^{(0)} + \sum_{i=1}^{m} x_i (u^{(0)} + u^{(i)})$$

is an injective semialgebraic function from $\mathscr{K}^m$ to A. Hence, by Proposition 2.20, we have $\dim(A) = m$. Furthermore, note that $f(]0, 1/m[^m)$ is a subset of Δ. Since f is injective, using Proposition 2.20 again we have $m = \dim(f(]0, 1/m[^m)) \leqslant \dim(\Delta) \leqslant \dim(A) = m$. □

Corollary 2.26. *Suppose that the set $S \subset \mathscr{K}^n$ is nonempty, semialgebraic, and convex. Then, the dimension of S is equal to the largest dimension of a simplex contained in S. Equivalently, is it equal to the smallest dimension of an affine space that contains S.*

Sketch of the proof. Take the largest collection $u^{(1)}, \dots, u^{(m)} \in \mathscr{K}^n$ of linearly independent vectors such that there exists $u^{(0)} \in \mathscr{K}^n$ satisfying $\Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)}) \subset S$ (we allow $m = 0$). Let A denote the affine space $u^{(0)} + \operatorname{span}(u^{(1)}, \dots, u^{(m)})$. If there exists a point $x \in S \setminus A$, then the vectors $u^{(1)}, \dots, u^{(m)}, x - u^{(0)} \in \mathscr{K}^n$ are linearly independent. By convexity of S, this implies that the simplex $\Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)}, x - u^{(0)}) \in \mathscr{K}^n$ is contained in S, which gives a contradiction. Hence $S \subset A$. Therefore, by Lemma 2.25 we have $\dim(S) = m$. Moreover, m is the smallest dimension of an affine space that contains S. □

2.3 Puiseux series and tropical semifield

In this section, we discuss the fundamental objects of this thesis—the field of Puiseux series and the tropical semifield. We start by briefly introducing the field of Puiseux series. More information about this field can be found in Appendix A, where we discuss in detail the different fields used in tropical geometry (e.g., Puiseux series with rational exponents, generalized Puiseux series, Hahn series, both formal and convergent), prove their basic properties, and, following van den Dries and Speissegger [vdDS98], prove that all these fields are real closed (or algebraically closed in the case of complex coefficients). In this work, we use the field of convergent, generalized, real Puiseux series. In the context of tropical geometry, a variant of this field without without the convergence assumption first appeared in [Mar10]. Up to a change of variables, this field is isomorphic to the field of *generalized Dirichlet series* studied by Hardy and Riesz [HR15].

Definition 2.27. An *absolutely convergent generalized real Puiseux series* is a series of the form

$$\boldsymbol{x} = \sum_{i=1}^{\infty} c_{\lambda_i} t^{\lambda_i} \,, \tag{2.2}$$

where t is a formal parameter, $(\lambda_i)_{i\geqslant 1}$ is a strictly decreasing sequence of real numbers that is either finite or unbounded, and $c_{\lambda_i} \in \mathbb{R} \setminus \{0\}$. Furthermore, the series (2.2) is required to be absolutely convergent for t large enough. There is also a special, empty series, which is denoted by 0.

Remark 2.28. For simplicity, throughout this work we refer to absolutely convergent generalized real Puiseux series as "Puiseux series." We denote the set of Puiseux series by $\mathbb{K}$.

Definition 2.29. We denote by $\mathsf{lc}(\boldsymbol{x})$ the coefficient c_{λ_1} of the leading term in the series $\boldsymbol{x}$ as in (2.2), with the convention that $\mathsf{lc}(0) = 0$. We also endow $\mathbb{K}$ with a linear order $\leqslant$, which is defined as $\boldsymbol{y} \leqslant \boldsymbol{x}$ if $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) \geqslant 0$. Equivalently, we have $\boldsymbol{y} \leqslant \boldsymbol{x}$ if and only if $\boldsymbol{y}(t) \leqslant \boldsymbol{x}(t)$ for all sufficiently large $t > 0$. We denote by $\mathbb{K}_{\geqslant 0}$ the set of nonnegative series $\boldsymbol{x}$, i.e., the set of series satisfying $\boldsymbol{x} \geqslant 0$.

The Puiseux series can be added and multiplied in the natural way. Furthermore, given the order introduced in Definition 2.29, the ring of Puiseux series forms a real closed field. This was proven by van den Dries and Speissegger [vdDS98].

Theorem 2.30 ([vdDS98])**.** *The set of Puiseux series $\mathbb{K}$ endowed with the order $\leqslant$ forms a real closed field.*

The tropical semifield describes the algebraic structure of $\mathbb{K}$ under the valuation map.

Definition 2.31. The *valuation* of an element $\boldsymbol{x} \in \mathbb{K}$ as in (2.2) is defined as the greatest exponent λ_1 occurring in the series, $\mathsf{val}(\boldsymbol{x}) \coloneqq \lambda_1$. Equivalently, the valuation is given by

$$\mathsf{val}(\boldsymbol{x}) = \lim_{t\to+\infty} \log_t |\boldsymbol{x}(t)| \,,$$

where $\log_t(z) \coloneqq \log(z)/\log(t)$. We use the convention that $\mathsf{val}(0) = -\infty$.

Definition 2.32. The *tropical semifield* is a structure $\mathbb{T} = (\mathbb{T}, \oplus, \odot)$, where the underlying set $\mathbb{T}$ is defined as $\mathbb{T} \coloneqq \mathbb{R} \cup \{-\infty\}$, the tropical addition is defined as $x \oplus y = \max(x, y)$, and the tropical multiplication is defined as $x \odot y = x + y$.

Remark 2.33. We point out that $-\infty$ is the neutral element of the tropical addition, and 0 is the neutral element of the tropical multiplication. The word "semifield" refers to the fact that the tropical addition does not have an inverse (but all other properties of fields are satisfied by $\mathbb{T}$). Throughout this work, we use the notation $\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n$ and $a^{\odot n} = a \odot \cdots \odot a$ (n times).

Remark 2.34. We endow $\mathbb{T}$ with the standard order $\leqslant$. Since $\mathbb{T}$ is totally ordered, it has a topology defined by the order, i.e., the smallest topology such that all sets of the form $]a, b[$ and $[-\infty, b[$ for $a, b \in \mathbb{R}$ are open. We extend this topology to $\mathbb{T}^n$ by taking the product topology.

In this work, we only use a few basic properties of the tropical semifield. We refer to [But10] for more information about the algebraic properties of $\mathbb{T}$.The following observation relates the valuation over $\mathbb{K}$ to the tropical semifield $\mathbb{T}$.

Lemma 2.35. *The valuation* $\mathsf{val}\colon \mathbb{K} \to \mathbb{T}$ *satisfies the properties*

$$\mathsf{val}(\boldsymbol{x} + \boldsymbol{y}) \leqslant \max(\mathsf{val}(\boldsymbol{x}), \mathsf{val}(\boldsymbol{y})) \tag{2.3}$$

$$\mathsf{val}(\boldsymbol{x}\boldsymbol{y}) = \mathsf{val}(\boldsymbol{x}) + \mathsf{val}(\boldsymbol{y})\,. \tag{2.4}$$

Furthermore, for every $\boldsymbol{x} \geqslant \boldsymbol{y} \geqslant 0$ *we have*

$$\mathsf{val}(\boldsymbol{x}) \geqslant \mathsf{val}(\boldsymbol{y})\,. \tag{2.5}$$

Proof. The proof of the first two properties follows from the definition of addition and multiplication of the series. More precisely, the greatest exponent of the sum of two series cannot be bigger that the greatest exponent of each of them, and the greatest exponent of the product is equal to the sum of the greatest exponents of the factors. To prove the third property, note that if $\mathsf{val}(\boldsymbol{y}) > \mathsf{val}(\boldsymbol{x})$, then $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) = -\mathsf{lc}(\boldsymbol{y}) < 0$, which gives a contradiction with the fact that $\boldsymbol{x} \geqslant \boldsymbol{y}$. ☐

Remark 2.36. We point out that the equality holds in (2.3) if the leading terms of $\boldsymbol{x}$ and $\boldsymbol{y}$ do not cancel. This is the case, for instance, if $\mathsf{val}(\boldsymbol{x}) \neq \mathsf{val}(\boldsymbol{y})$ or if $\boldsymbol{x}, \boldsymbol{y} \geqslant 0$.

Remark 2.37. The properties given in Lemma 2.35 imply that the valuation is an order-preserving morphism of semifields from $\mathbb{K}_{\geqslant 0}$ to $\mathbb{T}$.

When dealing with semialgebraic sets, it is convenient to keep track not only of the valuations of the elements of $\mathbb{K}$, but also of their signs. To this end, we introduce the set of signed tropical numbers and signed valuation.

Definition 2.38. The set of *signed tropical numbers* is defined as $\mathbb{T}_{\pm} \coloneqq (\{+1, -1\} \times \mathbb{R}) \cup \{(0, -\infty)\}$. The *modulus* function $|\cdot|\colon \mathbb{T}_{\pm} \to \mathbb{T}$ is defined as the projection which forgets the first coordinate. The elements of the form $(1, a)$ of $\mathbb{T}_{\pm}$ are called *positive tropical numbers* and are denoted by $\mathbb{T}_{+}$. Similarly, the elements of the form $(-1, a)$ of $\mathbb{T}_{\pm}$ are called *negative tropical numbers* and are denoted by $\mathbb{T}_{-}$. By convention, we denote the positive tropical number $(1, a)$ by a, the negative tropical number $(-1, a)$ by $\ominus a$, and the element $(0, -\infty)$ by $-\infty$. Here, $\ominus$ is a formal symbol.

Definition 2.39. We extend the definition of tropical multiplication to $\mathbb{T}_{\pm}$ using the usual rules for signs. In other words, for $\delta_1, \delta_2 \in \{-1, 0, +1\}$ and $a, b \in \mathbb{T}$ such that $(\delta_1, a), (\delta_2, b) \in \mathbb{T}_{\pm}$ we define $(\delta_1, a) \odot (\delta_2, b) \coloneqq (\delta_1\delta_2, a + b)$. We also partially extend the tropical addition to the elements of $\mathbb{T}_{\pm}$ which have the same sign. In other words, if $\delta_1 = \delta_2$, then we define $(\delta_1, a) \oplus (\delta_1, b) \coloneqq (\delta_1, \max\{a, b\})$.

Example 2.40. We have $(\ominus 3) \odot 7 = \ominus 10$, $(\ominus 3) \odot (\ominus 7) = 10$, $3 \oplus 7 = 7$, and $(\ominus 3) \oplus (\ominus 7) = \ominus 7$, but $(\ominus 3) \oplus 7$ is not defined.

Let us note that one can extend the set $\mathbb{T}_{\pm}$ further in oder to get a semiring with a well-defined tropical addition [AGG09], or work with hyperfields [Vir10, CC11, BB16] instead of semifields to have a well-defined, but multivalued addition. However, the partial addition defined above is sufficient for this work.

Remark 2.41. We point out that the tropical semiring $\mathbb{T}$ is isomorphic to $\mathbb{T}_{+} \cup \{-\infty\}$.

Definition 2.42. We define the *sign* function $\mathsf{sign}\colon \mathbb{K} \to \{-1, 0, +1\}$ as $\mathsf{sign}(\boldsymbol{x}) = 1$ if $\boldsymbol{x} > 0$, $\mathsf{sign}(\boldsymbol{x}) = -1$ if $\boldsymbol{x} < 0$, and $\mathsf{sign}(0) = 0$. We define the *absolute value* function $|\cdot|\colon \mathbb{K} \to \mathbb{K}_{\geqslant 0}$ as $|\boldsymbol{x}| = \mathsf{sign}(\boldsymbol{x})\boldsymbol{x}$. Furthermore, we define the *signed valuation* $\mathsf{sval}\colon \mathbb{K} \to \mathbb{T}_{\pm}$ as $\mathsf{sval}(\boldsymbol{x}) = (\mathsf{sign}(\boldsymbol{x}), \mathsf{val}(\boldsymbol{x}))$.

Remark 2.43. We extend the functions $\mathsf{val}\colon \mathbb{K} \to \mathbb{T}$, $\mathsf{sval}\colon \mathbb{K} \to \mathbb{T}_\pm$, and $|\cdot|\colon \mathbb{T}_\pm \to \mathbb{T}$ to vectors and matrices by applying them coordinatewise.

2.4 Tropical polynomials

In the tropical semifield we can also define an analogue of a polynomial. In this section we discuss this notion and the polyhedral complexes that arise as solutions to tropical polynomial inequalities.

Definition 2.44. A *(signed) tropical polynomial* over the variables $X_1, \dots, X_n$ is a formal expression of the form

$$P(X) = \bigoplus_{\alpha \in \mathit{\Lambda}} a_\alpha \odot X_1^{\odot \alpha_1} \odot \dots \odot X_n^{\odot \alpha_n}\,, \tag{2.6}$$

where $\mathit{\Lambda}$ is a finite subset of $\{0, 1, 2, \dots\}^n$, and $a_\alpha \in \mathbb{T}_\pm \setminus \{-\infty\}$ for all $\alpha \in \mathit{\Lambda}$. Since the addition in $\mathbb{T}_\pm$ is only partially defined, we cannot evaluate $P(x)$ for all the points $x \in \mathbb{T}_\pm^n$. However, if $x \in \mathbb{T}_\pm^n$ is such that all of the terms $a_\alpha \odot x_1^{\odot \alpha_1} \odot \dots \odot x_n^{\odot \alpha_n}$ have the same sign, then we define $P(x)$ as the tropical sum of these terms.

Example 2.45. If $P(X) = 2 \odot X_1^{\odot 3} \odot X_2^{\odot 4} \oplus (\ominus 0 \odot X_2)$, then $P(1, \ominus 5) = 25$, $P(\ominus 1, -5) = \ominus(-5)$, whereas P is not defined for $(1, -5/3)$.

The next definition and lemma shows that a nonzero tropical polynomial divides $\mathbb{R}^n$ into cells that form a polyhedral complex. The union of the boundaries of cells of this complex is called a *tropical hypersurface*—we refer to [MS15, Section 3.1] for more information.

Definition 2.46. Given a tropical polynomial P as in (2.6), we say that P is *nonzero* if the set $\mathit{\Lambda}$ is nonempty. For every such tropical polynomial and every point $x \in \mathbb{R}^n$ we define the *set of maximizing multi-indices at x* as

$$\mathrm{Argmax}(P, x) \coloneqq \{\alpha \in \mathit{\Lambda} \colon \forall \beta \in \mathit{\Lambda},\, |a_\alpha| + \langle \alpha, x \rangle \geqslant |a_\beta| + \langle \beta, x \rangle\}\,,$$

where $\langle \cdot, \cdot \rangle$ refers to the usual scalar product.

Lemma 2.47. *If P is a nonzero tropical polynomial as in* (2.6) *and we fix a multi-index $\alpha \in \mathit{\Lambda}$, then the set*

$$\mathcal{W}_\alpha \coloneqq \mathrm{cl}_{\mathbb{R}}(\{x \in \mathbb{R}^n \colon \mathrm{Argmax}(P, x) = \alpha\})$$

is a polyhedron that is either empty or full dimensional. Furthermore, if $L \subset \mathit{\Lambda}$ is such that $\alpha \in L$, then the set

$$\mathcal{W}_L \coloneqq \mathrm{cl}_{\mathbb{R}}(\{x \in \mathbb{R}^n \colon \mathrm{Argmax}(P, x) = L\}) \tag{2.7}$$

is a (possibly empty) face of $\mathcal{W}_\alpha$. Moreover, the family $\{\mathcal{W}_L\}_{L \subset \mathit{\Lambda}}$ is a polyhedral complex whose support is equal to $\mathbb{R}^n$. (Here, $\mathrm{cl}_{\mathbb{R}}(\cdot)$ refers to the closure in $\mathbb{R}^n$ and we use the convention that $\mathcal{W}_\emptyset = \emptyset$.)

Sketch of the proof. To prove the first statement, note that a point $x \in \mathbb{R}^n$ satisfies the equality $\mathrm{Argmax}(P, x) = \alpha$ if and only if the inequality $|a_\alpha| + \langle \alpha, x \rangle > |a_\beta| + \langle \beta, x \rangle$ is true for all $\beta \in \mathit{\Lambda} \setminus \{\alpha\}$. Therefore, if $\mathcal{W}_\alpha$ is nonempty, then it contains a ball. A relative interior of a face of $\mathcal{W}_\alpha$ is obtained by fixing a subset of inequalities of the form $|a_\alpha| + \langle \alpha, x \rangle > |a_\beta| + \langle \beta, x \rangle$ and

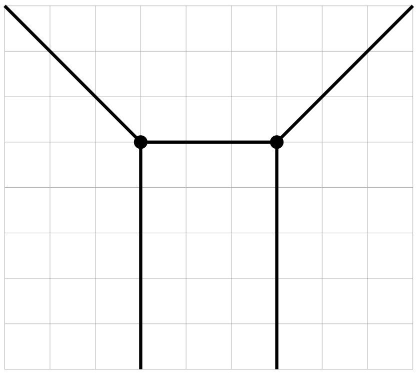

Figure 2.1: Tropical hypersurface.

turning them into equalities. In other words, $\mathcal{V}$ is a (possibly empty) face of $\mathcal{W}_\alpha$ if and only if $\mathcal{V} = \mathrm{cl}_{\mathbb{R}}(\{x \in \mathbb{R}^n \colon \mathrm{Argmax}(P, x) = L\})$ for some $L \subset \Lambda$ such that $\alpha \in L$. The polyhedra $\mathcal{W}_L$ have disjoint relative interiors, and hence the family $\{\mathcal{W}_L\}_{L \subset \Lambda}$ is a polyhedral complex. The fact that its support is equal to $\mathbb{R}^n$ follows from the definition. $\square$

Example 2.48. Take

$$P(X_1, X_2) \coloneqq (2 \odot X_1) \oplus (\ominus(-4) \odot X_1^{\odot 2})) \oplus (\ominus(-3) \odot X_1 \odot X_2) \oplus (\ominus 5)\,.$$

The tropical hypersurface associated with P is depicted in Fig. 2.1. The associated polyhedral complex consists of 11 nonempty polyhedra, 4 of which are full dimensional (one for every monomial of P).

In this work, we are interested in systems of tropical inequalities. To do so, it is convenient to make the following definitions.

Definition 2.49. Given a tropical polynomial P as in (2.6), we set $\Lambda^+ \coloneqq \{\alpha \in \Lambda \colon a_\alpha \in \mathbb{T}_+\}$ and $\Lambda^- \coloneqq \{\alpha \in \Lambda \colon a_\alpha \in \mathbb{T}_-\}$. Furthermore, we define $P^+ \coloneqq \bigoplus_{\alpha \in \Lambda^+} a_\alpha \odot X_1^{\odot \alpha_1} \odot \dots \odot X_n^{\odot \alpha_n}$ and $P^- \coloneqq \bigoplus_{\alpha \in \Lambda^-} |a_\alpha| \odot X_1^{\odot \alpha_1} \odot \dots \odot X_n^{\odot \alpha_n}$. Finally, if P is nonzero, then we set

$$\mathcal{S}^{\geqslant}(P) \coloneqq \{x \in \mathbb{R}^n \colon P^+(x) \geqslant P^-(x)\}$$

(with the convention that $P^+(x) \coloneqq -\infty$ if $\Lambda^+ = \emptyset$ and $P^-(x) \coloneqq -\infty$ if $\Lambda^- = \emptyset$).

Remark 2.50. We point out the values $P^+(x), P^-(x)$ are well defined for all $x \in \mathbb{T}^n$. Therefore, $\mathcal{S}^{\geqslant}(P)$ is also well defined.

The following definition and a lemma show the connection between polynomial inequalities over $\mathbb{K}_{\geqslant 0}$ and tropical polynomial inequalities.

Definition 2.51. To any polynomial

$$\boldsymbol{P}(X) = \sum_{\alpha \in \Lambda} \boldsymbol{a}_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n} \in \mathbb{K}[X_1, \dots, X_n] \tag{2.8}$$

over Puiseux series we associate a tropical polynomial defined as

$$\mathrm{trop}(\boldsymbol{P}) \coloneqq \bigoplus_{\alpha \in \Lambda} \mathsf{sval}(\boldsymbol{a}_\alpha) \odot X_1^{\odot \alpha_1} \odot \dots \odot X_n^{\odot \alpha_n}\,.$$

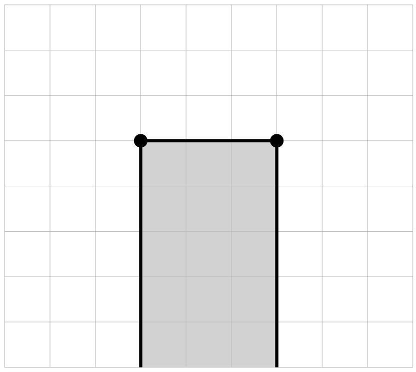

Figure 2.2: Support of the polyhedral complex $\mathcal{C}^{\geqslant}(P)$.

Lemma 2.52. *Let $\boldsymbol{P} \in \mathbb{K}[X_1, \dots, X_n]$ be a polynomial and let $P \coloneqq \operatorname{trop}(\boldsymbol{P})$. If $\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0}$ is such that $\boldsymbol{P}(\boldsymbol{x}) \geqslant 0$, then $P^+(\mathsf{val}(\boldsymbol{x})) \geqslant P^-(\mathsf{val}(\boldsymbol{x}))$.*

Proof. Define $x = \mathsf{val}(\boldsymbol{x}) \in \mathbb{T}^n_{\pm}$ and let

$$\boldsymbol{P}^+(X) = \sum_{\alpha \in \Lambda^+} \boldsymbol{a}_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}$$
$$\boldsymbol{P}^-(X) = \sum_{\alpha \in \Lambda^-} |\boldsymbol{a}_\alpha| X_1^{\alpha_1} \dots X_n^{\alpha_n} .$$

By (2.4) and Remark 2.36, we have $P^+(\mathsf{val}(\boldsymbol{x})) = \mathsf{val}(\boldsymbol{P}^+(\boldsymbol{x}))$ and $P^-(\mathsf{val}(\boldsymbol{x})) = \mathsf{val}(\boldsymbol{P}^-(\boldsymbol{x}))$. Moreover, $\boldsymbol{P}^+(\boldsymbol{x}) \geqslant \boldsymbol{P}^-(\boldsymbol{x})$ and the claim follows from (2.5). □

Next, we define the signed version of the polyhedral complex associated with a tropical polynomial and show its connections to tropical polynomial inequalities.

Definition 2.53. We say that a cell $\mathcal{W}_L \in \mathcal{C}(P)$ as in (2.7) is *positive* if there exists at least one $\alpha \in L$ such that $a_\alpha \in \mathbb{T}_+$ or if $\mathcal{W}_L$ is empty.

Lemma 2.54. *Suppose that P is a nonzero tropical polynomial and let $\mathcal{C}^{\geqslant}(P)$ be the family of positive cells of $\mathcal{C}(P)$. Then, $\mathcal{C}^{\geqslant}(P)$ is a polyhedral complex whose support is equal to $\mathcal{S}^{\geqslant}(P)$.*

Sketch of the proof. As in the proof of Lemma 2.47, if $\mathcal{W}_L$ belongs to $\mathcal{C}^{\geqslant}(P)$ and $\mathcal{V}$ is its face, then $\mathcal{V} = \mathcal{W}_{L'}$ for some $L \subset L'$. This means that all faces of $\mathcal{W}_L$ belong to $\mathcal{C}^{\geqslant}(P)$. Moreover, the polyhedra in $\mathcal{C}^{\geqslant}(P)$ have disjoint relative interiors and so $\mathcal{C}^{\geqslant}(P)$ is a polyhedral complex. To finish the proof, observe that $x \in \mathcal{S}^{\geqslant}(P)$ if and only if $\operatorname{Argmax}(P, x) \cap \Lambda^+ \neq \emptyset$. □

Example 2.55. Let $P(X_1, X_2)$ be as in Example 2.48. Then, the polyhedral complex $\mathcal{C}^{\geqslant}(P)$ consists of 6 polyhedra and its support is depicted in Fig. 2.2.

Given a system of nonzero tropical polynomials $P_1, \dots, P_m$, we also regard the refinements of complexes defined by $P_1, \dots, P_m$. More precisely, we make the following definition.

Definition 2.56. If $P_1, \dots, P_m$ are nonzero tropical polynomials, then we define $\mathcal{C}(P_1, \dots, P_m)$ and $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ as

$$\mathcal{C}(P_1, \dots, P_m) \coloneqq \{\cap_{i=1}^m \mathcal{V}_i \colon \forall i,\ \mathcal{V}_i \in \mathcal{C}(P_i)\} ,$$
$$\mathcal{C}^{\geqslant}(P_1, \dots, P_m) \coloneqq \{\cap_{i=1}^m \mathcal{V}_i \colon \forall i,\ \mathcal{V}_i \in \mathcal{C}^{\geqslant}(P_i)\} .$$

Lemma 2.57. *The families $\mathcal{C}(P_1, \dots, P_m)$ and $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ are polyhedral complexes. The support of the former is equal to $\mathbb{R}^n$ while the support of the latter coincides with*

$$\mathcal{S}^{\geqslant}(P_1, \dots, P_m) \coloneqq \{x \in \mathbb{R}^n \colon \forall i, P_i^+(x) \geqslant P_i^-(x)\} .$$

Proof. Obvious from Lemma 2.9 and the lemmas above. $\square$

Finally, in this work we often consider polyhedral complexes with regular supports. Recall that a closed set $X \subset \mathbb{R}^n$ is called *regular* if $X = \mathrm{cl}_{\mathbb{R}}(\mathrm{int}(X))$. The next lemmas show basic properties of these complexes.

Lemma 2.58. *If $\mathcal{C}$ is a polyhedral complex, then its support is regular if and only if every point of the support belongs to some full-dimensional cell of $\mathcal{C}$.*

Proof. Let $\mathcal{S}$ denote the support of $\mathcal{C}$. If $x \in \mathcal{S}$ belongs to some full-dimensional cell $\mathcal{W}$ of $\mathcal{C}$, then $x \in \mathcal{W} = \mathrm{cl}_{\mathbb{R}}(\mathrm{int}(\mathcal{W})) \subset \mathrm{cl}_{\mathbb{R}}(\mathrm{int}(\mathcal{S}))$ by Lemma 2.3. This proves the "if" direction. To prove the opposite direction, suppose that $\mathcal{S}$ is regular and take any point $y \in \mathrm{int}(\mathcal{S})$. We will show that y belongs to some full-dimensional cell of $\mathcal{C}$. Since the polyhedra that form $\mathcal{C}$ are closed (and there are finitely many of them), there exists an open ball $B(y, r) \subset \mathbb{R}^n$ such that a cell $\mathcal{W} \in \mathcal{C}$ has a nonempty intersection with $B(y, r)$ if and only if $y \in \mathcal{W}$. Let $0 < r' < r$ be such that $B(y, r') \subset \mathcal{S}$. If y does not belong to a full-dimensional cell, then the union of all cells containing y is not full dimensional and it does not contain a ball (by Proposition 2.20). In particular, $B(y, r')$ is not contained in this union, which gives a contradiction with the fact that $r' < r$. Hence, y belongs to some full-dimensional cell of $\mathcal{C}$.

To finish the proof, take any $x \in \mathcal{S}$. Then for every $k \geqslant 1$, there exists a point $y^{(k)} \in \mathrm{int}(\mathcal{S}) \cap B(x, 1/k)$. By the observation above, $y^{(k)}$ belongs to some full-dimensional cell of $\mathcal{C}$. Therefore, up to taking a subsequence, we can suppose that all $y^{(k)}$ belong to the same full-dimensional cell $\mathcal{W} \in \mathcal{C}$. Since $\mathcal{W}$ is closed, we get $x \in \mathcal{W}$. $\square$

Lemma 2.59. *Suppose that the polyhedral complex $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ has a regular support. Then this support, $\mathcal{S}^{\geqslant}(P_1, \dots, P_m)$, coincides with the closure of the set*

$$\mathcal{S}^{>}(P_1, \dots, P_m) \coloneqq \{x \in \mathbb{R}^n \colon \forall i, P_i^+(x) > P_i^-(x)\} .$$

Proof. If there is a tropical polynomial P_i such that $P = P_i^-$, then both of these sets are empty and the claim is trivially true. If there is a tropical polynomial P_i such that $P = P_i^+$, then we can remove this polynomial from the collection and this does not change the sets $\mathcal{S}^{\geqslant}(P_1, \dots, P_m)$ and $\mathcal{S}^{>}(P_1, \dots, P_m)$. Hence, we can suppose that all the tropical polynomials P_i^+, P_i^- are nonzero. Then, the set $\mathcal{S}^{\geqslant}(P_1, \dots, P_m)$ is closed since the tropical polynomial functions $P_i^{\pm}$ are continuous. We obviously have $\mathcal{S}^{>}(P_1, \dots, P_m) \subset \mathcal{S}^{\geqslant}(P_1, \dots, P_m)$. Therefore,

$$\mathrm{cl}_{\mathbb{R}}(\mathcal{S}^{>}(P_1, \dots, P_m)) \subset \mathcal{S}^{\geqslant}(P_1, \dots, P_m) .$$

Consider now $y \in \mathcal{S}^{\geqslant}(P_1, \dots, P_m)$. Since this set is regular, by Lemma 2.58, y belongs to a full-dimensional cell $\mathcal{W}$ of $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$. We have $\mathcal{W} = \cap_{1 \leqslant i \leqslant m} \mathcal{W}_i$, where $\mathcal{W}_i$ is a full-dimensional cell of $\mathcal{C}^{\geqslant}(P_i)$. This implies that $\mathcal{W}_i = \mathrm{cl}_{\mathbb{R}}(\{x \in \mathbb{R}^n \colon \mathrm{Argmax}(P_i, x) = L_i\})$ where L_i is a one element subset of $\Lambda^+(P_i)$. We conclude that $P_i^+(x) > P_i^-(x)$ holds for all $x \in \mathrm{int}(\mathcal{W}_i)$, and so, $\mathrm{int}(\mathcal{W}) \subset \mathcal{S}^{>}(P_1, \dots, P_m)$. We have $\mathcal{W} = \mathrm{cl}_{\mathbb{R}}(\mathrm{int}(\mathcal{W}))$ by Lemma 2.3 and hence $y \in \mathrm{cl}_{\mathbb{R}}(\mathcal{S}^{>}(P_1, \dots, P_m))$. $\square$

2.5 Valued fields

In this section, we recall basic definitions and properties of valued fields. We refer to [EP05, Chapter 2] for more information. We also use some basic notions from algebra, which can be found in textbooks such as [AW92, Chapters 1 and 2].

Definition 2.60. A *valued field* is a triple $(\mathscr{K}, \Gamma, \mathsf{val})$, where $\mathscr{K} = (\mathscr{K}, 0, 1, +, \cdot)$ is a field, $\Gamma = (\Gamma, 0, +, \leqslant)$ is an ordered abelian group,[2] and $\mathsf{val} \colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ is a surjective function that fulfills the following three conditions:

$$\begin{aligned} \mathsf{val}(x) = -\infty \iff x = 0\,; \\ \forall x_1, x_2 \in \mathscr{K},\ \mathsf{val}(x_1 x_2) = \mathsf{val}(x_1) + \mathsf{val}(x_2)\,; \\ \forall x_1, x_2 \in \mathscr{K},\ \mathsf{val}(x_1 + x_2) \leqslant \max(\mathsf{val}(x_1), \mathsf{val}(x_2))\,. \end{aligned} \tag{2.9}$$

The group Γ is called the *value group* of $(\mathscr{K}, \Gamma, \mathsf{val})$ and the function val is called a *valuation.* The valuation is called *trivial* if $\Gamma = \{0\}$. Otherwise, it is called *nontrivial.* The set $\mathscr{O} := \{x \in \mathscr{K} \colon \mathsf{val}(x) \leqslant 0\}$ is called the *valuation ring* of $(\mathscr{K}, \Gamma, \mathsf{val})$.

The next lemmas allow us to define the notion of the residues field of $(\mathscr{K}, \Gamma, \mathsf{val})$.

Lemma 2.61. *We have* $\mathsf{val}(1) = \mathsf{val}(-1) = 0$*.*

Proof. We have $\mathsf{val}(1) = \mathsf{val}(1 \cdot 1) = \mathsf{val}(1) + \mathsf{val}(1)$ and hence $\mathsf{val}(1) = 0$. Moreover, $0 = \mathsf{val}(1) = \mathsf{val}((-1) \cdot (-1)) = 2\,\mathsf{val}(-1)$ and therefore $\mathsf{val}(-1) = 0$. $\square$

Lemma 2.62. *The valuation ring $\mathscr{O}$ is a subring of $\mathscr{K}$ and $\mathscr{M} := \{x \in \mathscr{K} \colon \mathsf{val}(x) < 0\}$ is its maximal ideal.*

Proof. The facts that $\mathscr{O}$ is a subring of $\mathscr{K}$ and $\mathscr{M} \subset \mathscr{O}$ is its ideal are immediate consequences of (2.9). It remains to prove that $\mathscr{M}$ is maximal. To do that, note that an element $x \in \mathscr{O}$ is invertible in $\mathscr{O}$ if and only if $\mathsf{val}(x) = 0$. Indeed, if $x \neq 0$ and $x^{-1} \in \mathscr{K}$ is its inverse, then Lemma 2.61 implies that $0 = \mathsf{val}(1) = \mathsf{val}(xx^{-1}) = \mathsf{val}(x) + \mathsf{val}(x^{-1})$. In particular, $\mathscr{M}$ is exactly the set of elements of $\mathscr{O}$ that are not invertible in $\mathscr{O}$. Hence, $\mathscr{M}$ is maximal. $\square$

Definition 2.63. We say that a field $\mathscr{k}$ is a *residue field* of $(\mathscr{K}, \Gamma, \mathsf{val})$ if there exists a surjective ring homomorphism $\mathsf{res} \colon \mathscr{O} \to \mathscr{k}$ whose kernel is equal to $\mathscr{M}$. In this case, res is called a *residue map.*

Lemma 2.64. *A residue field $\mathscr{k}$ exists and is unique up to isomorphism. More precisely, if $\pi \colon \mathscr{O} \to \mathscr{O}/\mathscr{M}$ denotes the canonical projection from $\mathscr{O}$ to $\mathscr{O}/\mathscr{M}$, then $\mathscr{k}$ is isomorphic to $\mathscr{O}/\mathscr{M}$ and there exists a field isomorphism $f \colon \mathscr{O}/\mathscr{M} \to \mathscr{k}$ such that $\mathsf{res}(x) = f(\pi(x))$ for all $x \in \mathscr{O}$.*

Proof. Since the ideal $\mathscr{M}$ is maximal by Lemma 2.62, the quotient ring $\mathscr{O}/\mathscr{M}$ is a field [AW92, Theorem 2.18]. Moreover, the canonical projection $\pi \colon \mathscr{O} \to \mathscr{O}/\mathscr{M}$ is a residue map by the definition of $\mathscr{O}/\mathscr{M}$ [AW92, Section 2.2]. Finally, if $\mathscr{k}$ is any residue field and $\mathsf{res} \colon \mathscr{O} \to \mathscr{k}$ is the associated homomorphism, then the existence of f follows from the first ring isomorphism theorem [AW92, Theorem 2.6]. $\square$

[2] An abelian group $(\Gamma, \leqslant)$ is *(linearly) ordered* if $\leqslant$ is a total order on Γ such that $y_1 \leqslant y_2 \implies y_1 + y_3 \leqslant y_2 + y_3$ for all $y_1, y_2, y_3 \in \Gamma$.

Example 2.65. Let $\mathbb{K}$ be the field of Puiseux series. Then $(\mathbb{K}, \mathbb{R}, \mathsf{val})$ is a valued field. Moreover, it is easy to check that the function $\mathsf{res}\colon \mathscr{O} \to \mathbb{R}$ defined as

$$\mathsf{res}(\boldsymbol{x}) := \begin{cases} \mathsf{lc}(\boldsymbol{x}) & \text{if } \mathsf{val}(\boldsymbol{x}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

is a residue map. In particular, the residue field of Puiseux series is the field of real numbers $\mathbb{R}$.

The notions defined above are shared by all valued fields. In this work we are interested in real closed valued fields, which have more properties. The most important for this work are the notions of henselianity and the existence of an angular component.

Definition 2.66. A map $\mathsf{csec}\colon (\Gamma, +) \to (\mathscr{K}^*, \cdot)$ is called a *cross-section* if it is a group homomorphism such that $\mathsf{val} \circ \mathsf{csec}$ is the identity map. If $\mathscr{k}$ is a field, then a map $\mathsf{ac}\colon \mathscr{K} \to \mathscr{k}$ is called an *angular component* if it fulfills the following conditions:

- $\mathsf{ac}(0) = 0$;
- ac is a group homomorphism from $(\mathscr{K}^*, \cdot)$ to $(\mathscr{k}^*, \cdot)$;
- the function from $\mathscr{O}$ to $\mathscr{k}$, mapping x to $\mathsf{ac}(x)$ if $\mathsf{val}(x) = 0$, and to 0 otherwise, is a surjective ring homomorphism whose kernel is equal to $\mathscr{M}$.

Remark 2.67. We point out that Definitions 2.63 and 2.66 are not stated in the most economical way. For instance, one could simply define $\mathscr{O}/\mathscr{M}$ to be the residue field. Moreover, the last point of Definition 2.66 implies that $\mathscr{k}$ is the residue field and the mapping defined in this point is a residue map. However, the way that we state these definitions will be used in Section 2.6 to define the theory of real closed valued fields. In this context, Definition 2.63 has the advantage that the properties of res are stated as first-order formulas in the language of $(\mathscr{K}, \Gamma, \mathscr{k}, \mathsf{val}, \mathsf{res})$, while Definition 2.66 has the advantage that it does not use the notion of a residue field or a residue map.

Not every valued field admits an angular component (see [Pas90b] for a counterexample). Nevertheless, if it admits a cross-section, then it also has an angular component.

Proposition 2.68. *If* $\mathsf{csec}\colon \Gamma \to \mathscr{K}$ *is a cross-section and* $\mathscr{k}$ *is a residue field, then the map* $\mathsf{ac}\colon \mathscr{K} \to \mathscr{k}$ *defined as* $\mathsf{ac}(0) := 0$ *and* $\mathsf{ac}(x) := \mathsf{res}(\mathsf{csec}(-\mathsf{val}(x))x)$ *for* $x \neq 0$ *is an angular component.*

Proof. Note that $\mathsf{csec}(-\mathsf{val}(x))x$ is a group homomorphism from $(\mathscr{K}^*, \cdot)$ to $(\mathscr{O}^*, \cdot)$. Indeed, for all $x \in \mathscr{K}^*$ we have $\mathsf{val}(\mathsf{csec}(-\mathsf{val}(x))x) = \mathsf{val}(\mathsf{csec}(-\mathsf{val}(x))) + \mathsf{val}(x) = -\mathsf{val}(x) + \mathsf{val}(x) = 0$. Moreover, it is clear that $\mathsf{csec}(-\mathsf{val}(x))x$ is a homomorphism. Therefore, $\mathsf{res}(\mathsf{csec}(-\mathsf{val}(x))x)$ is a group homomorphism from $(\mathscr{K}^*, \cdot)$ to $(\mathscr{k}^*, \cdot)$. To finish, note that if $\mathsf{val}(x) = 0$, then $\mathsf{csec}(-\mathsf{val}(x))x = \mathsf{csec}(0)x = x$ and $\mathsf{res}(\mathsf{csec}(-\mathsf{val}(x))x) = \mathsf{res}(x)$. $\square$

Example 2.69. If $\mathbb{K}$ is the field of Puiseux series, then $\mathsf{csec}(y) = t^y$ is a cross-section, and

$$\mathsf{res}(\mathsf{csec}(-\mathsf{val}(\boldsymbol{x}))\boldsymbol{x}) = \mathsf{res}(t^{-\mathsf{val}(\boldsymbol{x})}\boldsymbol{x}) = \mathsf{lc}(\boldsymbol{x})$$

is an angular component.

Every real closed valued field has a cross-section, as shown by the following lemma.

Lemma 2.70. *Suppose that* $\mathscr{K}$ *is real closed. Then* $(\mathscr{K}, \Gamma, \mathsf{val})$ *admits a cross-section. (In particular, it has an angular component.)*

Sketch of the proof. The case where the valuation is trivial is obtained by taking a cross-section equal to 1. Therefore, we assume that the valuation is nontrivial. Observe that in this case Γ is a divisible group.[3] Indeed, if $y \in \Gamma$ is an arbitrary element, then (by surjectivity of valuation and Lemma 2.61) we can take $x \in \mathscr{K}$ such that $x > 0$ and $\mathsf{val}(x) = y$. Since $\mathscr{K}$ is real closed, Lemma 2.13 shows that for any natural $n \geqslant 1$, x has an nth root $x^{1/n} \in \mathscr{K}$, $x^{1/n} > 0$. We have $y = \mathsf{val}((x^{1/n})^n)$ and hence $y = n\,\mathsf{val}(x^{1/n})$. In other words, the equation $y = nz$ has a solution for any $y \in \Gamma$ and any $n \geqslant 1$. Moreover, this solution is unique because Γ is ordered. Indeed, if z_1, z_2 are two distinct solutions and $z_1 > z_2$, then $n(z_1 - z_2) = 0$, which is impossible by the fact that order is compatible with addition. It follows that we can regard Γ as a vector space over $\mathbb{Q}$. More precisely, for any rational number $\frac{p}{q} \in \mathbb{Q}$ and any $y \in \Gamma$, there is a unique element $z \in \Gamma$ such that $py = qz$. We denote this element $z \in \Gamma$ as $f(\frac{p}{q}, y)$. This defines a multiplication function $f\colon \mathbb{Q} \times \Gamma \to \Gamma$. One can easily check that the tuple $(\Gamma, +, f)$ forms a vector space over $\mathbb{Q}$. Let $\{y_i\}_{i\in I}$ be a basis of this space. For every i we fix $x_i \in \mathscr{K}$ such that $x_i > 0$ and $\mathsf{val}(x_i) = y_i$. Finally, for every finite subset $J \subset I$ and every $(\alpha_j) \in \mathbb{Q}^J$ we define

$$\mathsf{csec}(\sum_{j\in J} \alpha_j y_j) = \prod_{j\in J} x_j^{\alpha_j} \, .$$

It is obvious that csec is a cross-section. $\square$

We now define the class of henselian valued fields.

Definition 2.71. Let $\mathscr{O}[x]$ denote the ring of polynomials over $\mathscr{O}$. Analogously, let $\mathscr{k}[x]$ denote the ring of polynomials over a residue field $\mathscr{k}$. We extend the notion of residue map to $\mathscr{O}[x]$ in the following way. Suppose that $P \coloneqq \sum_{k=0}^{d} c_k x^k \in \mathscr{O}[x]$ is such a polynomial. Then, we put

$$\mathsf{res}(P) \coloneqq \sum_{k=0}^{d} \mathsf{res}(c_k) x^k \in \mathscr{k}[x] \, .$$

Definition 2.72. We say that a valued field if *henselian* if for every polynomial $P \in \mathscr{O}[x]$ and every $\alpha \in \mathscr{k}$ such that $\mathsf{res}(P)(\alpha) = 0$ and $(\mathsf{res}(P))'(\alpha) \neq 0$, there exists $\bar{\alpha} \in \mathscr{K}$ such that $P(\bar{\alpha}) = 0$ and $\mathsf{res}(\bar{\alpha}) = \alpha$. (Here, $(\mathsf{res}(P))' \in \mathscr{k}[x]$ denotes the formal derivative of $\mathsf{res}(P)$.)

Remark 2.73. We point out that the definition above does not depend on the particular choice of a residue field (by Lemma 2.64). Moreover, we note that there are many equivalent definitions of henselian valued fields. We refer to [EP05, Theorem 4.1.3] for a nonexhaustive list of such definitions.

To finish this section, we recall the notion of a convex valuation and the characterization of real closed valued fields.

Definition 2.74. Suppose that $\mathscr{K}$ is an ordered field with a total order $\leqslant$. We say that the valuation val is *convex with respect to* $\leqslant$ if it satisfies the following property: for every $x_1 \in \mathscr{O}$ and every $x_2 \in \mathscr{K}$ we have the implication

$$0 \leqslant x_2 \leqslant x_1 \implies x_2 \in \mathscr{O} \, .$$

Furthermore, we recall that if $\mathscr{K}$ is a real closed field, it has a unique total order (Lemma 2.12). In this case, we say that val is *convex* if it is convex with respect to this order.

[3] An abelian group Γ is *divisible* if the equation $y = nz$ has a solution for every $y \in \Gamma$ and $n \in \mathbb{N}^*$.

The following theorem gives a characterization of real closed valued fields.

Theorem 2.75 ([EP05, Theorem 4.3.7])**.** *Suppose that $\mathscr{K}$ is an ordered field with a total order $\leqslant$. Furthermore, suppose that $(\mathscr{K}, \Gamma, \mathsf{val})$ is a valued field such that its valuation is nontrivial and convex with respect to $\leqslant$. Then, $\mathscr{K}$ is real closed if and only if the following three conditions are simultaneously satisfied:*

- *$(\mathscr{K}, \Gamma, \mathsf{val})$ is henselian;*
- *Γ is divisible;*
- *$\mathscr{k}$ is real closed.*

Example 2.76. The field $\mathbb{K}$ of Puiseux series fulfills all of the properties of Theorem 2.75. In fact, the proof that $\mathbb{K}$ is real closed (see [vdDS98] or Appendix A) first shows that $\mathbb{K}$ is henselian and then deduces that it is real closed using Theorem 2.75.

2.6 Model theory

In this section, we recall basic notions of model theory. We refer to [Mar02, Chapter 1] and [TZ12, Chapter 1] for more information. Model theory studies structures that consist of a domain and distinguished constants, functions, and relations that belong to (or act upon) this domain. For example, if $\Gamma = (\Gamma, 0, +, \leqslant)$ is an ordered abelian group, then the set Γ is a domain, 0 is a constant, $+$ is a function, and $\leqslant$ is a relation. The notion of a many-sorted structure arises when we consider functions or relations acting between different domains. For example, a valued field $(\mathscr{K}, \Gamma, \mathsf{val})$ consists of two domains, $\mathscr{K}$ and $\Gamma \cup \{-\infty\}$. The valuation function $\mathsf{val}\colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ acts between the domain $\mathscr{K}$ and the domain $\Gamma \cup \{-\infty\}$. The notion of many-sorted structure is formalized in the next definition.

Definition 2.77. A *(many-sorted) structure* is a tuple

$$\mathcal{M} = \left((M_i)_{i\in I}, (\mathcal{C}_i)_{i\in I}, \mathcal{F}, \mathcal{R}\right),$$

where I is an arbitrary set, every M_i is a nonempty set (called *domain*), every set $\mathcal{C}_i$ is a collection of M_i-constants, $\mathcal{F}$ is a collection of functions, and $\mathcal{R}$ is a collection of relations. Furthermore, every function $f \in \mathcal{F}$ has its *type*. We say that f has type $(i_1, \dots, i_p, i_{p+1})$ if

$$f\colon M_{i_1} \times \cdots \times M_{i_p} \to M_{i_{p+1}}.$$

Analogously, every relation $R \in \mathcal{R}$ has its *type*. We say that a relation R has type $(i_1, \dots, i_p)$ if

$$R \subset M_{i_1} \times \cdots \times M_{i_p}.$$

Given a structure, its functions, and relations, we can build mathematical formulas that express the properties of this structure. However, for our applications, it is important to transfer properties between different structures. To do this, we construct formulas in a more abstract way. Instead of fixing a structure and building formulas using this structure, we consider formulas as strings of formal symbols. These strings can then be interpreted in a given structure. To introduce these concepts formally, we first define the notion of language.

Definition 2.78. A *language* $\mathcal{L}$ is a tuple of sets, $\mathcal{L} = ((\mathcal{C}_i)_{i\in I}, \mathcal{F}, \mathcal{R})$, where I is an arbitrary set, the set $\mathcal{C}_i$ is called the set of *constant symbols of sort i* for every $i \in I$, the set $\mathcal{F}$ is called the set of *function symbols*, and the set $\mathcal{R}$ is called the set of *relation symbols*. Each function symbol and each relation symbol is equipped with its *type*, i.e., a nonempty and finite sequence of elements of I.

Example 2.79. We emphasize the fact that the language as defined in Definition 2.78 is a collection of purely formal symbols that, for now, do not possess any interpretation. For instance, if we take take $I = \{1\}$, $\mathcal{C}_1 = \{0\}$, $\mathcal{F} = \{+\}$, $\mathcal{R} = \{\leqslant\}$, we obtain the language or totally ordered groups $\mathcal{L}_{\mathrm{og}} = (0, +, \leqslant)$, which contains three symbols. The first symbol looks like an ellipse, the second looks like a greek cross, and the third resembles an angle bracket with an additional line. In $\mathcal{L}_{\mathrm{og}}$, we suppose that $+$ has type $(1, 1, 1)$ and that $\leqslant$ has type $(1, 1)$.

As noted before, symbols can be interpreted in a given structure.

Definition 2.80. If $\mathcal{L} = ((\mathcal{C}_i)_{i\in I}, \mathcal{F}, \mathcal{R})$ is a language and $\mathcal{M} = ((M_i)_{i\in I}, (\tilde{\mathcal{C}}_i)_{i\in I}, \tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ is a structure, then we say that $\mathcal{M}$ is an $\mathcal{L}$*-structure* if there exists a tuple $\xi = ((\xi_i)_{i\in I}, \xi^{\mathcal{F}}, \xi^{\mathcal{R}})$ of bijections $\xi_i \colon \mathcal{C}_i \to \tilde{\mathcal{C}}_i$, $\xi^{\mathcal{F}} \colon \mathcal{F} \to \tilde{\mathcal{F}}$, $\xi^{\mathcal{R}} \colon \mathcal{R} \to \tilde{\mathcal{R}}$ such that

- if $f \in \mathcal{F}$ is a function symbol of type $(i_1, \dots, i_p, i_{p+1})$, then $\xi^{\mathcal{F}}(f) \in \tilde{\mathcal{F}}$ is a function of type $(i_1, \dots, i_p, i_{p+1})$;
- if $R \in \mathcal{R}$ is a relation symbol of type $(i_1, \dots, i_p)$, then $\xi^{\mathcal{R}}(R) \in \tilde{\mathcal{R}}$ is a relation of type $(i_1, \dots, i_p)$.

The tuple ξ is called *interpretation.* By abuse of notation, we use the same letters to denote the symbols in $\mathcal{L}$ and their interpretations in $\mathcal{M}$.

Example 2.81. Any ordered group Γ is an $\mathcal{L}_{\mathrm{og}}$-structure. Indeed, we can interpret the symbol "0" as the zero of Γ, the symbol "$+$" as the addition, and the symbol "$\leqslant$" as the order relation.

Given a language $\mathcal{L} = ((\mathcal{C}_i)_{i\in I}, \mathcal{F}, \mathcal{R})$, we can now define the notion of an $\mathcal{L}$-formula. There are three things that should be considered before giving the formal definition. First, we note that the symbols in $\mathcal{L}$ are not sufficient to create meaningful formulas. We have to extend $\mathcal{L}$ by adding symbols for variables (making sure that each domain has its own set of variables), logic symbols (such as quantifiers, negation, or conjunction), and delimiters (e.g., parentheses or commas). In the definitions below, we will extend $\mathcal{L}$ by putting

$$\tilde{\mathcal{L}} \coloneqq \mathcal{L} \cup \{(x_k^{(i)})_{i\in I, k\in\mathbb{N}}\} \cup \{=\} \cup \{\exists\} \cup \{\neg\} \cup \{\wedge\} \cup \{(\} \cup \{)\} \cup \{,\} .$$

As previously, one should think of the elements of $\tilde{\mathcal{L}}$ as formal symbols. Second, since we work with many-sorted structures, we have to make sure that formulas are well typed. For instance, if we work with a valued field $(\mathscr{K}, \Gamma, \mathsf{val})$, then the formula $\mathsf{val}(x) = 0$ makes sense if x is a variable associated to $\mathscr{K}$ but it is meaningless if x is a variable associated to Γ. Third, is it useful to keep track of the free and bound variables in a given formula. A variable is called "bound" if it is restricted by a quantifier and it is called "free" otherwise. We point out that we use a convention in which we disallow the variables from having both bound and free occurrences within one formula. This does not restrict the expressive power of formulas, but makes the presentation simpler. In the definitions below, Bvar denotes the set of *bound variables* of a given formula, and Fvar denotes its set of *free variables.* We define the notion of formula in three steps. Let $\mathsf{Seq}(\tilde{\mathcal{L}}) \coloneqq \bigcup_{k=1}^{\infty} \tilde{\mathcal{L}}^k$ denote the set of all finite sequences of elements of $\tilde{\mathcal{L}}$.

Definition 2.82. The set of $\mathcal{L}$*-terms* is the subset of $\mathsf{Seq}(\tilde{\mathcal{L}})$ defined by the following recurrence:

- Every constant symbol $c \in \mathcal{C}_i$ is a term. We say that $c \in \mathcal{C}_i$ has type $i \in I$, and we put $\mathsf{Bvar}(c) \coloneqq \mathsf{Fvar}(c) \coloneqq \emptyset$.
- Every variable symbol $x_k^{(i)}$ is a term. We say that $x_k^{(i)}$ has type $i \in I$, and we set $\mathsf{Bvar}(x_k^{(i)}) \coloneqq \emptyset$, $\mathsf{Fvar}(x_k^{(i)}) \coloneqq x_k^{(i)}$.
- If $f \in \mathcal{F}$ is a function symbol of type $(i_1, \dots, i_p, i_{p+1}) \in I^{p+1}$, and $\phi_1, \dots, \phi_p \in \mathsf{Seq}(\tilde{\mathcal{L}})$ are $\mathcal{L}$-terms of types $i_1, \dots, i_p \in I$ respectively, then the sequence $f(\phi_1, \dots, \phi_p) \in \mathsf{Seq}(\tilde{\mathcal{L}})$

is a term of type $i_{p+1} \in I$. We set $\mathsf{Bvar}(f(\phi_1, \dots, \phi_p)) \coloneqq \emptyset$ and $\mathsf{Fvar}(f(\phi_1, \dots, \phi_p)) \coloneqq \bigcup_{k=1}^p \mathsf{Fvar}(\phi_k)$.

Definition 2.83. The set of *atomic $\mathcal{L}$-formulas* is the subset of $\mathsf{Seq}(\tilde{\mathcal{L}})$ defined by the following recurrence:

- If $\phi, \psi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ are $\mathcal{L}$-terms of the same type, then the sequence $\phi = \psi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an atomic $\mathcal{L}$-formula. We set $\mathsf{Bvar}(\phi = \psi) \coloneqq \emptyset$ and $\mathsf{Fvar}(\phi = \psi) \coloneqq \mathsf{Fvar}(\phi) \cup \mathsf{Fvar}(\psi)$.
- If $R \in \mathcal{R}$ is a relation symbol of type $(i_1, \dots, i_p) \in I^p$, and $\phi_1, \dots, \phi_p \in \mathsf{Seq}(\tilde{\mathcal{L}})$ are $\mathcal{L}$-terms of types $i_1, \dots, i_p \in I$ respectively, then the sequence $R(\phi_1, \dots, \phi_p) \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an atomic $\mathcal{L}$-formula. We set $\mathsf{Bvar}(R(\phi_1, \dots, \phi_p)) \coloneqq \emptyset$ and $\mathsf{Fvar}(R(\phi_1, \dots, \phi_p)) \coloneqq \bigcup_{k=1}^p \mathsf{Fvar}(\phi_k)$.

Definition 2.84. The set of *$\mathcal{L}$-formulas* is the subset of $\mathsf{Seq}(\tilde{\mathcal{L}})$ defined by the following recurrence:

- Every atomic $\mathcal{L}$-formula is an $\mathcal{L}$-formula.
- If $\phi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an $\mathcal{L}$-formula, then $\neg\phi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an $\mathcal{L}$-formula. We set $\mathsf{Bvar}(\neg\phi) \coloneqq \mathsf{Bvar}(\phi)$ and $\mathsf{Fvar}(\neg\phi) \coloneqq \mathsf{Fvar}(\phi)$.
- If $\phi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an $\mathcal{L}$-formula and $x_k^{(i)} \in \mathsf{Fvar}(\phi)$ is a variable symbol, then the sequence $\exists x_k^{(i)}, \phi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an $\mathcal{L}$-formula. We set $\mathsf{Bvar}(\exists x_k^{(i)}, \phi) \coloneqq \mathsf{Bvar}(\phi) \cup \{x_k^{(i)}\}$ and $\mathsf{Fvar}(\exists x_k^{(i)}, \phi) \coloneqq \mathsf{Fvar}(\phi) \setminus \{x_k^{(i)}\}$.
- If $\phi, \psi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ are $\mathcal{L}$-formulas such that $\mathsf{Bvar}(\phi) \cap \mathsf{Fvar}(\psi) = \mathsf{Fvar}(\phi) \cap \mathsf{Bvar}(\psi) = \emptyset$, then the sequence $(\phi \wedge \psi) \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is an $\mathcal{L}$-formula. We set $\mathsf{Bvar}((\phi \wedge \psi)) \coloneqq \mathsf{Bvar}(\phi) \cup \mathsf{Bvar}(\psi)$ and $\mathsf{Fvar}((\phi \wedge \psi)) \coloneqq \mathsf{Fvar}(\phi) \cup \mathsf{Fvar}(\psi)$.

A formula without free variables is called a *sentence*.

Remark 2.85. To make sure that the definitions above are unambiguous, one should check (at each step of the construction) that formulas can be uniquely decomposed. For instance, if a formula starts with the left parenthesis symbol (, then we know (by the definition) that it is of the form $(\phi \wedge \psi)$, and it can be verified that ϕ and ψ are unique. Similarly, if a formula starts with a function symbol f and does not contain the symbol $=$, then we know (by the definition) that it is a term of the form $f(\phi_1, \dots, \phi_p)$ and it can be checked that $\phi_1, \dots, \phi_p$ are unique. We refer to [End01, Section 2.3] for the details.

Remark 2.86. We note that the definition of $\mathcal{L}$-formulas does not involve the symbols such as $\forall$, $\vee$, $\to$, $\leftrightarrow$, $\neq$, which are commonly used in mathematical formulas. However, any formula that uses these symbols can be converted to a formula that uses only the symbols from $\tilde{\mathcal{L}}$. Therefore, we will use these symbols in our examples, understanding that the string $\forall x_k^{(i)}, \psi$ is a shorthand for $\neg\exists x_k^{(i)}, \neg\psi$, the string $(\psi \vee \phi)$ is a shorthand for $\neg(\neg\psi \wedge \neg\phi)$, the string $(\psi \to \phi)$ is a shorthand for $(\neg\psi \vee (\psi \wedge \phi))$ and so on. We will also add/skip commas and parentheses, add spaces, and make other changes to the formulas in order to improve their readability. Furthermore, we use $\bigvee_{k=1}^p \phi_k$ as a shorthand for $\phi_1 \vee \dots \vee \phi_p$ and $\bigwedge_{k=1}^p \phi_k$ as a shorthand for $\phi_1 \wedge \dots \wedge \phi_p$. We also use other symbols instead of $x_k^{(i)}$ for variables (such as y, z and so on).

Suppose that $\mathcal{M} = ((M_i)_{i \in I}, (\tilde{\mathcal{C}}_i)_{i \in I}, \tilde{\mathcal{F}}, \tilde{\mathcal{R}})$ in an $\mathcal{L}$-structure and that $\phi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ is a term with a nonempty set of free variables $\mathsf{Fvar}(\phi)$. Then, we can interpret this term as a function. This is done in a natural way, by replacing every function symbol in ϕ by its interpretation in $\mathcal{M}$. Giving the formal definition is easier if we fix the values of all possible variables beforehand. More precisely, we will fix a function

$$s \colon (x_k^{(i)})_{i \in I, k \in \mathbb{N}} \to \bigcup_{i \in I} M_i$$

such that $s(x_k^{(i)}) \in M_i$ for all pairs $(i,k) \in I \times \mathbb{N}$. The value of $s(x_k^{(i)})$ is interpreted as the value associated to the variable symbol $x_k^{(i)} \in \tilde{\mathcal{L}}$.

Definition 2.87. For every $i \in I$ and every term $\phi \in \mathsf{Seq}(\tilde{\mathcal{L}})$ of type $i \in I$ we define an element $\phi(s) \in M_i$ by the following recurrence:

- If ϕ is a constant symbol $c \in \mathcal{C}_i$, then $\phi(s) \coloneqq \xi_i(c) \in M_i$.
- If ϕ is a variable symbol $x_k^{(i)}$, then $\phi(s) \coloneqq s(x_k^{(i)}) \in M_i$.
- If ϕ is of the form $f(\phi_1, \dots, \phi_p)$, where f is a function symbol of type $(i_1, \dots, i_p, i)$, then $\phi(s) \coloneqq \xi^{\mathcal{F}}(f)(\phi_1(s), \dots, \phi_p(s)) \in M_i$.

Given an $\mathcal{L}$-formula and a variable assignment s defined as above, we can now define the notion of the truth of a formula in a given structure.

Definition 2.88. If ϕ is an $\mathcal{L}$-formula, then we define a unary relation $\mathcal{M} \models \phi(s)$ by the following recurrence:

- If ϕ is of the form $\psi_1 = \psi_2$, where ψ_1, ψ_2 are $\mathcal{L}$-terms, then $\mathcal{M} \models \phi(s)$ if and only if $\psi_1(s) = \psi_2(s)$.
- If ϕ is of the form $R(\phi_1, \dots, \phi_p)$, then $\mathcal{M} \models \phi(s)$ if and only if $(\phi_1(s), \dots, \phi_p(s))$ satisfies $\xi^{\mathcal{R}}(R)$.
- If ϕ is of the form $\neg\psi$, then $\mathcal{M} \models \phi(s)$ if and only if $\mathcal{M} \not\models \psi(s)$.
- If ϕ is of the form $(\psi_1 \wedge \psi_2)$, then $\mathcal{M} \models \phi(s)$ if and only if $\mathcal{M} \models \psi_1(s)$ and $\mathcal{M} \models \psi_2(s)$.
- If ϕ is of the form $\exists x_k^{(i)}, \psi$, then $\mathcal{M} \models \phi(s)$ if and only if there exists $x \in M_i$ such that the function $\tilde{s} \colon (x_k^{(i)})_{i \in I, k \in \mathbb{N}} \to \prod_{(i,k) \in I \times \mathbb{N}} M_{i,k}$ defined as
$$\tilde{s}(x_l^{(j)}) \coloneqq \begin{cases} x & \text{if } (j,l) = (i,k) \\ s(x_l^{(j)}) & \text{otherwise} \end{cases}$$
satisfies $\mathcal{M} \models \psi(\tilde{s})$.

If $\mathcal{M} \models \phi(s)$, then we say that $\mathcal{M}$ *satisfies* $\phi(s)$ or that $\phi(s)$ *is true in* $\mathcal{M}$.

Remark 2.89. We point out that the definitions above are unambiguous by the decomposition mentioned in Remark 2.85.

Remark 2.90. One can show that the definitions above only depend on the function s restricted to the set of free variables of ϕ. In particular, if ϕ has no free variables, then these definitions do not depend on the choice of s. We often denote ϕ as $\phi(x_{k_1}^{(i_1)}, \dots, x_{k_p}^{(i_p)})$, where $x_{k_1}^{(i_1)}, \dots, x_{k_p}^{(i_p)}$ are the free variables of ϕ, $\{x_{k_1}^{(i_1)}, \dots, x_{k_p}^{(i_p)}\} = \mathsf{Fvar}(\phi)$. Then, if we fix $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p) \in M_{i_1} \times \dots \times M_{i_p}$, and ϕ is a term of type $i \in I$, the quantity $\phi(\bar{x}) \in M_i$ is well defined by putting $s(x_{k_\ell}^{(i_\ell)}) = \bar{x}_\ell$ for all ℓ. Similarly, if ϕ is a formula, then the relation $\mathcal{M} \models \phi(\bar{x})$ is well defined.

Remark 2.91. We may use a simplified notation when there is no ambiguity about the structure that we are using. For instance, if $\phi(x_1, \dots, x_n)$ is an $\mathcal{L}_{\mathrm{og}}$-formula and $(\bar{x}_1 \dots, \bar{x}_n) \in \mathbb{Q}^n$, then we may write $(\mathbb{Q}, +) \models \phi(\bar{x}_1 \dots, \bar{x}_n)$ instead of $(\mathbb{Q}, 0, +, \leqslant) \models \phi(\bar{x}_1 \dots, \bar{x}_n)$. Similarly, if Γ is an abstract ordered abelian group and $(\bar{x}_1 \dots, \bar{x}_n) \in \Gamma^n$, then we may write $\Gamma \models \phi(\bar{x}_1 \dots, \bar{x}_n)$ instead of $(\Gamma, 0, +, \leqslant) \models \phi(\bar{x}_1 \dots, \bar{x}_n)$.

Example 2.92. If we fix an ordered abelian group Γ, then the $\mathcal{L}_{\mathrm{og}}$-formula $\forall x_1, (x_1 \geqslant 0 \to \exists x_2, (x_2 \geqslant 0 \wedge x_1 = x_2 + x_2))$ is interpreted in Γ as "for every nonnegative element $x_1 \in \Gamma$, there exist a nonnegative element $x_2 \in \Gamma$ such that x_1 is equal to x_2 added to x_2." Note that this is true if we take $\Gamma = (\mathbb{Q}, +)$, but false if we take $\Gamma = (\mathbb{Z}, +)$. Similarly, the $\mathcal{L}_{\mathrm{og}}$-formula $\exists x_2,\, x_1 = x_2 + x_2$ has one free variable x_1. If we take $\Gamma = (\mathbb{Z}, +)$, then $\psi(2)$ is true, but $\psi(1)$ is false.

Throughout this work, we often use the notion of a definable set. This is the set of all assignments of variables such that a given formula is satisfied.

Definition 2.93. For any $i_1, \dots, i_p \in I$, we say that a set $S \subset M_{i_1} \times \dots \times M_{i_p}$ is *definable* if there exist $\ell \geqslant 0$, an $\mathcal{L}$-formula $\phi(x_1^{(i_1)}, \dots, x_p^{(i_p)}, x_{p+1}^{(i_{p+1})}, \dots, x_{p+\ell}^{(i_{p+\ell})})$, and a vector $\bar{y} \in M_{i_{p+1}} \times \dots \times M_{i_{p+\ell}}$ such that

$$S = \{\bar{x} \in M_{i_1} \times \dots \times M_{i_p} \colon \mathcal{M} \models \phi(\bar{x}, \bar{y})\}\,.$$

Example 2.94. If we take $(\mathbb{Z}, +)$ as an $\mathcal{L}_{\mathrm{og}}$-structure, then the set $\{(x_1, x_2) \in \mathbb{Z}^2 \colon x > 2y + 5\}$ is definable. Indeed, we take $\phi(x_1, x_2, x_3)$ to be $\neg(x_1 \leqslant x_2 + x_2 + x_3)$ and fix $\bar{x}_3 = 5$.

Example 2.95. If $\Gamma = (\Gamma, 0, + \leqslant)$ is a nontrivial ordered abelian group, then we can equip it with a topology defined by the order, and extend this topology to Γ^n (by taking the product topology). Suppose that a set $\mathscr{S} \subset \Gamma^n$ is definable in $\mathcal{L}_{\mathrm{og}}$. Then, its interior $\mathrm{int}(\mathscr{S})$ and closure $\mathrm{cl}_\Gamma(\mathscr{S})$ are also definable in $\mathcal{L}_{\mathrm{og}}$. Indeed, the closure of $\mathscr{S}$ is defined as the set of all points $(x_1, \dots, x_n)$ that satisfy the formula "for every $a_1, \dots, a_n, b_1, \dots, b_n$ such that $a_1 < x_1 < b_1, \dots, a_n < x_n < b_n$, there exist $y_1, \dots, y_n$ such that $a_1 < y_1 < b_1, \dots, a_n < y_n < b_n$ and $(y_1, \dots, y_n) \in \mathscr{S}$." Similarly, the interior of $\mathscr{S}$ is defined as the set of all $(x_1, \dots, x_n)$ that satisfy the formula "there exist $a_1, \dots, a_n, b_1, \dots, b_n$ such that $a_1 < x_1 < b_1, \dots, a_n < x_n < b_n$ and such that for every $y_1, \dots, y_n$ satisfying $a_1 < y_1 < b_1, \dots, a_n < y_n < b_n$ we have $(y_1, \dots, y_n) \in \mathscr{S}$."

We can now define the notions of a theory, logical consequence, completeness, and quantifier elimination. In our context, one can think that a theory is a set of axioms.

Definition 2.96. A *theory* Th is a set of $\mathcal{L}$-sentences. We often refer to the elements of Th as *axioms*. If $\mathcal{M}$ is an $\mathcal{L}$-structure, then we say that $\mathcal{M}$ *is a model of* Th if all sentences from Th are true in $\mathcal{M}$.

Definition 2.97. If ϕ is an $\mathcal{L}$-sentence that does not necessarily belong to Th, then we say that ϕ is a *logical consequence of* Th if ϕ is true in every model of Th. We say that $\mathcal{L}$-formulas $\phi(x_{k_1}^{(i_1)}, \dots, x_{k_p}^{(i_p)})$, $\psi(x_{k_1}^{(i_1)}, \dots, x_{k_p}^{(i_p)})$ (with the same sets of free variables) are *equivalent in* Th if the sentence $\forall x_{k_1}^{(i_1)}, \dots, \forall x_{k_p}^{(i_p)},\, \psi \leftrightarrow \phi$ is a logical consequence of Th.

Definition 2.98. We say that a theory Th is *complete* if for every $\mathcal{L}$-sentence ϕ, either ϕ or $\neg\phi$ is a logical consequence of Th.

Definition 2.99. We say that an $\mathcal{L}$-formula is *quantifier free* if it does not contain any quantifier symbol $\exists$. We say that the theory Th admits *quantifier elimination* if every $\mathcal{L}$-formula is equivalent in Th to a quantifier-free formula.

2.6.1 Theories with quantifier elimination

We now give five examples of theories that admit quantifier elimination and are complete. These examples are used in the rest of this work. The first example is the theory of nontrivial divisible ordered abelian groups.

Definition 2.100. Let $\mathcal{L}_{\mathrm{og}} = (0, +, \leqslant)$ be the language of ordered groups. The theory of nontrivial divisible ordered abelian groups, denoted $\mathrm{Th}_{\mathrm{doag}}$ consists of the $\mathcal{L}_{\mathrm{og}}$-sentences expressing the axioms of ordered abelian groups (associativity of addition, commutativity of addition, zero as the neutral element, existence of an inverse, total order axioms, compatibility of order and addition), the axiom of nontriviality $\exists y, y \neq 0$, and an infinite series of axioms defining divisibility: for every $n \geqslant 2$, $\mathrm{Th}_{\mathrm{doag}}$ contains a sentence $\forall y_1, \exists y_2, y_1 = y_2 + y_2 + \cdots + y_2$ (where the addition in taken n times).

The theory of of nontrivial divisible ordered abelian groups admits quantifier elimination.

Theorem 2.101 ([Mar02, Corollary 3.1.17])**.** *The theory* $\mathrm{Th}_{\mathrm{doag}}$ *of nontrivial divisible ordered abelian groups admits quantifier elimination and is complete.*

As a corollary, we see that if Γ is a nontrivial divisible ordered abelian group, then its definable sets have a particularly simple structure. More precisely, we make the following definition.

Definition 2.102. We say that a subset $\mathcal{S} \subset \Gamma^n$ is a *basic semilinear* set if it is of the form

$$\mathcal{S} = \{g \in \Gamma^n \colon \forall i = 1, \dots, p, f_i(g) > h^{(i)}, \forall i = p+1, \dots, q, f_i(g) = h^{(i)}\} ,$$

where $f_i \in \mathbb{Z}[X_1, \dots, X_n]$ are homogeneous linear polynomials with integer coefficients and $h^{(i)} \in \Gamma$. We say that $\mathcal{S}$ is *semilinear* if it is a finite union of basic semilinear sets.

Lemma 2.103. *A set* $\mathcal{S} \subset \Gamma^n$ *is definable in* $\mathcal{L}_{\mathrm{og}}$ *if and only if it is semilinear.*

Sketch of the proof. It is easy to see that every semilinear set is definable. To prove the converse, note that a union of two semilinear sets is semilinear. Furthermore, an intersection of two basic semilinear sets is basic semilinear. Hence, by the fact that intersection distributes over union, an intersection of two semilinear sets is semilinear. Moreover, the complement of a basic semilinear set is semilinear and hence the complement of any semilinear set is semilinear. Furthermore, one can prove by induction that if ϕ is an $\mathcal{L}_{\mathrm{og}}$-term with a nonempty set of free variables $y_1, \dots, y_n$, then there exist $m_1, \dots, m_n \in \mathbb{N}$ such that $\phi(\bar{y}) = m_1\bar{y}_1 + \cdots + m_n\bar{y}_n$ for any $\bar{y} \in \Gamma^n$. Hence, any set defined by an atomic $\mathcal{L}_{\mathrm{og}}$-formula is of the form $\{\bar{y} \in \Gamma^n \colon f(\bar{y}) \leqslant h\}$ or $\{\bar{y} \in \Gamma^n \colon f(\bar{y}) = h\}$, where $f \in \mathbb{Z}[X_1, \dots, X_n]$ is a homogeneous linear polynomial with integer coefficients and $h \in \Gamma$. In particular, these sets are semilinear. Hence, sets defined by negations, conjunctions, and disjunctions of atomic formulas are semilinear. Moreover, one can prove by induction that every quantifier-free formula can be written in the form

$$\bigvee_{k=1}^{p} \bigwedge_{\ell=1}^{q_k} \phi_{k,\ell} ,$$

where every $\phi_{k,\ell}$ is an atomic formula or its negation. Therefore, every set defined by a quantifier-free formula is semilinear. The claim follows from Theorem 2.101. ☐

Example 2.104. If $\Gamma = \mathbb{R}$, then every basic semilinear set is a relative interior of a polyhedron. Note that the converse is not true. For example, the set $\{y \in \mathbb{R}^2 \colon y_1 + y_2 > \pi\}$ is semilinear, but $\{y \in \mathbb{R}^2 \colon \sqrt{2}y_1 + y_2 > 0\}$ is not. If $\Gamma = \mathbb{Q}$, then basic semilinear sets correspond precisely to relatively open rational polyhedra.

In this work, we use divisible ordered abelian groups which arise as value groups of nonarchimedean real closed fields. Since the valuation map may evaluate to $-\infty$, we need to deal with divisible ordered abelian groups with bottom element.

Definition 2.105. We denote by $\mathcal{L}_{\mathrm{ogb}} \coloneqq (0, -\infty, +, \leqslant)$ the language of ordered groups with bottom element. The theory of nontrivial divisible ordered abelian groups with bottom element, denoted $\mathrm{Th}_{\mathrm{doagb}}$, consists of the axioms of divisible ordered abelian groups (cf. Definition 2.100),[4] the nontriviality axiom $\exists y, (y \neq 0 \land y \neq -\infty)$, and the axioms that extend the addition and order to $-\infty$, namely $\forall y,\ -\infty + y = -\infty$ and $\forall y,\ y \geqslant -\infty$.

Using Theorem 2.101, we can show that $\mathrm{Th}_{\mathrm{doagb}}$ admits quantifier elimination and is complete.

Theorem 2.106. *The theory* $\mathrm{Th}_{\mathrm{doagb}}$ *admits quantifier elimination and is complete. Moreover, any* $\mathcal{L}_{\mathrm{ogb}}$*-formula* $\phi(y_1, \dots, y_m)$ *with* $m \geqslant 1$ *is equivalent to a quantifier-free formula of the form*

$$\bigvee_{\Sigma \subset [m]} \big((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \land \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \ \land \ \psi_\Sigma \big) \,, \tag{2.10}$$

where the disjunction goes over all subsets of $[m]$*, and every* ψ_Σ *is a quantifier-free* $\mathcal{L}_{\mathrm{og}}$*-formula such that* $\mathsf{Fvar}(\psi_\Sigma) \subset \{y_\sigma\}_{\sigma \in \Sigma}$*.*

The proof of the above result is easy but technical. We have put it in Appendix B.1 for the sake of completeness. As in the previous case, quantifier elimination allows us to describe the class of definable sets. To do so, we need to observe that if $\Gamma \cup \{-\infty\}$ is a nontrivial divisible ordered abelian group with bottom element, then it has a natural stratification.

Definition 2.107. If $x \in (\Gamma \cup \{-\infty\})^n$, then we define its *support* as the set of its finite coordinates, $\{k \in [n] \colon x_k \neq -\infty\}$. Given a nonempty subset $K \subset [n]$, and a set $\mathcal{S} \subset (\Gamma \cup \{-\infty\})^n$, we define the *stratum of* $\mathcal{S}$ *associated with* K, denoted $\mathcal{S}_K \subset \Gamma^K$, as the subset of Γ^K formed by the projection of the points $x \in \mathcal{S}$ with support K. More formally, a point $y \in \Gamma^K$ belongs to $\mathcal{S}_K$ if the point $x \in (\Gamma \cup \{-\infty\})^n$ defined as $x_k = y_k$ for all $k \in K$ and $x_k = -\infty$ otherwise belongs to $\mathcal{S}$.

Lemma 2.108. *A subset* $\mathcal{S} \subset (\Gamma \cup \{-\infty\})^n$ *is definable in* $\mathcal{L}_{\mathrm{ogb}}$ *if and only if all strata of* $\mathcal{S}$ *are semilinear.*

Again, the technical proof of this claim can be found in Appendix B.1. The next theory that is used in this work is the theory of real closed fields.

Definition 2.109. The theory of real closed fields, denoted $\mathrm{Th}_{\mathrm{rcf}}$, is a theory in the language of ordered rings $\mathcal{L}_{\mathrm{or}} \coloneqq (0, 1, +, \cdot, \leqslant)$.[5] It consists of the usual axioms of ordered fields, the axiom $\forall x_1, (x_1 \geqslant 0 \to \exists x_2,\ x_1 = x_2 \cdot x_2)$ that governs the existence of square roots, and an infinite set of axioms that states the fact that every polynomial of an odd degree has a root. In other words, for every $n \geqslant 1$, $\mathrm{Th}_{\mathrm{rcf}}$ contains the axiom $\forall x_0, \dots, \forall x_{2n}, \exists y,\ y^{2n+1} + x_{2n} y^{2n} + \dots + x_1 y + x_0 = 0$.

[4]With one change required by the presence of $-\infty$: the existence of an inverse should be formulated as $\forall y, (y \neq -\infty \to \exists z, (z \neq -\infty \land y + z = 0))$. The assumption that $z \neq -\infty$ is not needed (by the nontriviality axiom) but it simplifies the presentation.

[5]We assume that $+$ and $\cdot$ have type $(1, 1, 1)$ and that $\leqslant$ has type $(1, 1)$. It it clear that any ordered filed is an $\mathcal{L}_{\mathrm{or}}$-structure with the natural interpretation of the symbols.

A classical result proved by Tarski states that the theory of real closed fields admits quantifier elimination and is complete.

Theorem 2.110 ([Mar02, Theorem 3.3.15 and Corollary 3.3.16]). *The theory* $\mathrm{Th}_{\mathrm{rcf}}$ *of real closed fields admits quantifier elimination and is complete.*

As previously, if $\mathscr{K}$ is a real closed field, then quantifier elimination allows us to characterize the class of definable sets.

Lemma 2.111. *A set* $S \subset \mathscr{K}^n$ *is definable in* $\mathcal{L}_{\mathrm{or}}$ *if and only if it is semialgebraic.*

Sketch of the proof. By induction, if $\phi(x_1, \dots x_n)$ is an $\mathcal{L}_{\mathrm{or}}$-term with a nonempty set of free variables, then there exists a polynomial $P \in \mathbb{N}[X_1, \dots, X_n]$ with natural coefficients such that $\phi(\bar{x}) = P(\bar{x})$ for all $x \in \mathscr{K}^n$. Therefore, a set definable by an atomic formula is of the form $\{\bar{x} \in \mathscr{K}^n \colon P(\bar{x}) = 0\}$ or $\{\bar{x} \in \mathscr{K}^n \colon P(\bar{x}) \geqslant 0\}$, where $P \in \mathscr{K}[X_1, \dots, X_n]$ is a polynomial with coefficients in $\mathscr{K}$. The rest of the proof is the same as the proof of Lemma 2.103 (with the word "semilinear" replaced by "semialgebraic"). □

The final two theories used in this work concern valued fields. First, we present a quantifier elimination result in henselian valued fields with angular component. This result was obtained by Pas [Pas89, Pas90a] and based on the work of Denef [Den84, Den86]. Suppose that $(\mathscr{K}, \Gamma, \mathscr{k}, \mathsf{val}, \mathsf{ac})$ is a valued field with a fixed residue field and an angular component. Furthermore, we suppose that both $\mathscr{K}$ and $\mathscr{k}$ have characteristic 0. We start by defining a three-sorted language of such fields.

Definition 2.112. We define the Denef–Pas language as $\mathcal{L}_{\mathrm{Pas}} \coloneqq (\mathcal{L}_{\mathscr{K}}, \mathcal{L}_{\Gamma}, \mathcal{L}_{\mathscr{k}}, \mathsf{val}, \mathsf{ac})$, where $\mathcal{L}_{\mathscr{K}}$ is the language of rings associated with $\mathscr{K}$, $\mathcal{L}_{\mathscr{K}} \coloneqq (0_{\mathscr{K}}, 1_{\mathscr{K}}, +_{\mathscr{K}}, \cdot_{\mathscr{K}})$, $\mathcal{L}_{\mathscr{k}}$ is the language of rings associated with $\mathscr{k}$, $\mathcal{L}_{\mathscr{k}} \coloneqq (0_{\mathscr{k}}, 1_{\mathscr{k}}, +_{\mathscr{k}}, \cdot_{\mathscr{k}})$, $\mathcal{L}_{\Gamma} \coloneqq (0_{\Gamma}, -\infty_{\Gamma}, +_{\Gamma}, \leqslant_{\Gamma})$ denotes the language of ordered groups with bottom element associated with Γ, and val, ac are two function symbols.[6]

In this way, any valued field with angular component is an $\mathcal{L}_{\mathrm{Pas}}$-structure. We now define the associated theory.

Definition 2.113. The theory of henselian valued fields with angular component in equicharacteristic 0, denoted $\mathrm{Th}_{\mathrm{Pas}}$, is a theory in the Denef–Pas language $\mathcal{L}_{\mathrm{Pas}}$. It consists of the axioms of fields (for $\mathscr{K}$ and $\mathscr{k}$), axioms of ordered abelian groups with bottom element (for $\Gamma \cup \{-\infty\}$), the axioms specifying that val is a valuation (see Definition 2.60), the axioms specifying that ac is an angular component (see Definition 2.66), and infinite sequences of axioms describing the facts that $\mathscr{K}$ and $\mathscr{k}$ have characteristic 0, and an infinite sequence of axioms describing henselianity (see Definition 2.72).

Remark 2.114. We point out that the axioms mentioned in the definition above can be written as sentences in $\mathrm{Th}_{\mathrm{Pas}}$. We leave this as an exercise. The most problematic are the axioms of henselianity. However, as noted in Remark 2.67, the definition of the residue map is already implied in the definition of angular component, because the map $\mathsf{res}(x)$ defined as $\mathsf{ac}(x)$ if $\mathsf{val}(x) = 0$ and $\mathsf{res}(x) = 0$ otherwise, is a residue map. Therefore, one can write the infinite sequence of axioms describing henselianity using only the symbols for ac and val.

[6] The types of symbols are given in a natural way: we put $I = \{1, 2, 3\}$, $\mathcal{C}_1 \coloneqq \{0_{\mathscr{K}}, 1_{\mathscr{K}}\}$, $\mathcal{C}_2 \coloneqq \{0_{\Gamma}, -\infty_{\Gamma}\}$, $\mathcal{C}_3 \coloneqq \{0_{\mathscr{k}}, 1_{\mathscr{k}}\}$. Then, val has type $(1, 2)$, ac has type $(1, 3)$, $+_{\mathscr{k}}$ has type $(3, 3, 3)$ and so on.

Pas [Pas89] has shown that $\mathrm{Th}_{\mathrm{Pas}}$ admits elimination of $\mathscr{K}$-quantifiers, i.e., the quantifiers that act on the variables associated with $\mathscr{K}$. To state this theorem, we use the following notation. If θ is a $\mathcal{L}_{\mathrm{Pas}}$-formula, then we denote it by $\theta(X,Y,Z)$, where $X=(x_1,\ldots,x_{n_1})$ (resp. $Y=(y_1,\ldots,y_{n_2})$, $Z=(z_1,\ldots,z_{n_3})$) is the sequence of free variables of ϕ associated with the sort $\mathscr{K}$ (resp. $\Gamma,\mathscr{k}$). We have the following theorem.

Theorem 2.115 ([Pas89])**.** *Any $\mathcal{L}_{\mathrm{Pas}}$-formula $\theta(X,Y,Z)$ is equivalent in $\mathrm{Th}_{\mathrm{Pas}}$ to a formula of the form*

$$\bigvee_{i=1}^{m}\Big(\phi_i\big(\mathsf{val}(f_{i1}(X)),\ldots,\mathsf{val}(f_{ik_i}(X)),Y\big)\wedge\psi_i\big(\mathsf{ac}(f_{i(k_i+1)}(X)),\ldots,\mathsf{ac}(f_{il_i}(X)),Z\big)\Big)\,, \tag{2.11}$$

where, for every $i=1,\ldots,m$, $f_{i1},\ldots,f_{il_i}\in\mathbb{Z}[X]$ are polynomials with integer coefficients, ϕ_i is an $\mathcal{L}_\Gamma$-formula, and ψ_i is an $\mathcal{L}_{\mathscr{k}}$-formula.

Remark 2.116. We point out that the original statement of this theorem given by Pas [Pas89, Theorem 4.1] is slightly weaker. We took our formulation from [CLR06, Theorem 4.2]. Nevertheless, the proof of Pas actually proves the theorem stated above.

We use Theorem 2.115 to prove a quantifier elimination result in real closed valued fields. Let $(\mathscr{K},\Gamma,\mathscr{k},\mathsf{val},\mathsf{ac})$ be valued field such that $\mathscr{K}$ is real closed, val is nontrivial and convex, and residue field and angular component are fixed. We point out that, by Theorem 2.75, $\mathscr{k}$ is real closed. As previously, we start by defining the associated language and theory.

Definition 2.117. We define the language of real closed valued fields as

$$\mathcal{L}_{\mathrm{rcvf}}\coloneqq(\mathcal{L}_{\mathscr{K}},\mathcal{L}_\Gamma,\mathcal{L}_{\mathscr{k}},\mathsf{val},\mathsf{ac})\,,$$

where $\mathcal{L}_{\mathscr{K}}$ is the language of ordered rings associated with $\mathscr{K}$, $\mathcal{L}_{\mathscr{K}}\coloneqq(0_{\mathscr{K}},1_{\mathscr{K}},+_{\mathscr{K}},\cdot_{\mathscr{K}},\leqslant_{\mathscr{K}})$, $\mathcal{L}_{\mathscr{k}}$ is the language of ordered rings associated with $\mathscr{k}$, $\mathcal{L}_{\mathscr{k}}\coloneqq(0_{\mathscr{k}},1_{\mathscr{k}},+_{\mathscr{k}},\cdot_{\mathscr{k}},\leqslant_{\mathscr{k}})$, $\mathcal{L}_\Gamma\coloneqq(0_\Gamma,-\infty_\Gamma,+_\Gamma,\leqslant_\Gamma)$ denotes the language of ordered groups with bottom element associated with Γ, and val, ac are two function symbols.[7]

Definition 2.118. The theory of real closed valued fields, denoted $\mathrm{Th}_{\mathrm{rcvf}}$, is a theory in the language $\mathcal{L}_{\mathrm{rcvf}}$. It consists of the axioms of real closed fields for $\mathscr{K}$, the axioms of ordered abelian groups with bottom element for $\Gamma\cup\{-\infty\}$, the axioms of ordered fields for $\mathscr{k}$, the axioms specifying that val is a nontrivial and convex valuation, and the axioms specifying that ac is an angular component.

Remark 2.119. We point out that the axioms of $\mathrm{Th}_{\mathrm{rcvf}}$ imply that $\mathscr{k}$ is real closed. Indeed, the axioms of ac imply that $\mathscr{k}$ is a residue field and hence it is real closed by Theorem 2.75.

Using the quantifier elimination results presented in this section, we can prove that the theory of real closed valued fields admits quantifier elimination and is complete.

Theorem 2.120. *The theory $\mathrm{Th}_{\mathrm{rcvf}}$ admits quantifier elimination and is complete. Moreover, any $\mathcal{L}_{\mathrm{rcvf}}$-formula $\theta(X,Y,Z)$ is equivalent in $\mathrm{Th}_{\mathrm{rcvf}}$ to a formula of the form*

$$\bigvee_{i=1}^{m}\Big(\phi_i\big(\mathsf{val}(f_{i1}(X)),\ldots,\mathsf{val}(f_{ik_i}(X)),Y\big)\wedge\psi_i\big(\mathsf{ac}(f_{i,k_i+1}(X)),\ldots,\mathsf{ac}(f_{il_i}(X)),Z\big)\Big)\,,$$

where, for every $i=1,\ldots,m$, $f_{i1},\ldots,f_{il_i}\in\mathbb{Z}[X]$ are polynomials with integer coefficients, ϕ_i is a quantifier-free $\mathcal{L}_\Gamma$-formula, and ψ_i is a quantifier-free $\mathcal{L}_{\mathscr{k}}$-formula.

[7] The types of symbols are given in a natural way as in Definition 2.112.

Sketch of the proof. Let $\theta(X, Y, Z)$ denote any $\mathcal{L}_{\text{rcvf}}$-formula. We recall (Lemma 2.12) that the order $\leqslant$ in any real closed field can be defined as $x_1 \leqslant x_2 \iff \exists x_3, x_2 = x_3^2 + x_1$. This enables us to eliminate the symbols $\leqslant$. More precisely, by a simple induction, $\theta(X, Y, Z)$ is equivalent in Th_{rcvf} to some $\mathcal{L}_{\text{Pas}}$-formula $\hat{\theta}(X, Y, Z)$. Moreover, by Theorem 2.75, $(\mathscr{K}, \Gamma, \mathsf{val})$ is henselian. Furthermore, since $\mathscr{K}$ and $\mathscr{k}$ are ordered, they have characteristic zero. This enables us to apply Theorem 2.115. As a result, $\hat{\theta}(X, Y, Z)$ is equivalent in Th_{rcvf} to a formula of the form (2.11). Then, we apply Theorem 2.106 and Theorem 2.110 to eliminate the quantifiers in the formulas ϕ_i and ψ_i. This shows the last part of the statement. In the case where θ is a sentence, the formulas ϕ_i and ψ_i are also sentences. The completeness results in Theorem 2.106 and Theorem 2.110 applied to each subformula ϕ_i and ψ_i allow to prove that either θ or $\neg\theta$ is a logical consequence of Th_{rcvf}. □

Remark 2.121. We note that the proof of Theorem 2.120 is constructive, in the sense that there exists an algorithm that, given a $\mathcal{L}_{\text{rcvf}}$-formula, outputs an equivalent quantifier-free formula of the form stated in Theorem 2.120. Indeed, the proof first uses the Denef–Pas quantifier elimination to eliminate quantifiers acting over $\mathscr{K}$ and the algorithm to do so is given in the proof of Pas [Pas89]. Second, we eliminate the quantifiers in the residue field $\mathscr{k}$ by using quantifier elimination in real closed fields (see [BPR06, Chapter 14] for a detailed discussion about the algorithms for this task). Third, we eliminate the quantifiers in the value group $\Gamma \cup \{-\infty\}$. An algorithm for quantifier elimination in Γ is given in [FR75] and it can be easily extended to $\Gamma \cup \{-\infty\}$ (using the proof of Theorem 2.106).

2.6.2 Model completeness

To finish this preliminary section on model theory, we recall the notion of model completeness. This requires to introduce the notion of $\mathcal{L}$-embedding. Roughly speaking, if $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structures, then a $\mathcal{L}$-embedding is an embedding of the underlying domains that preserves the interpretation of all the symbols of $\mathcal{L}$. This is formalized by the following definition.

Definition 2.122. Let $\mathcal{L} = ((\mathcal{C}_i)_{i \in I}, \mathcal{F}, \mathcal{R})$ and suppose that $\mathcal{M} = ((\tilde{M}_i)_{i \in I}, (\tilde{\mathcal{C}}_i)_{i \in I}, \tilde{\mathcal{F}}, \tilde{\mathcal{R}})$, $\mathcal{N} = ((\hat{M}_i)_{i \in I}, (\hat{\mathcal{C}}_i)_{i \in I}, \hat{\mathcal{F}}, \hat{\mathcal{R}})$ are two $\mathcal{L}$-structures with interpretations $\tilde{\xi} = ((\tilde{\xi}_i)_{i \in I}, \tilde{\xi}^{\mathcal{F}}, \tilde{\xi}^{\mathcal{R}})$ and $\hat{\xi} = ((\hat{\xi}_i)_{i \in I}, \hat{\xi}^{\mathcal{F}}, \hat{\xi}^{\mathcal{R}})$ respectively. Then, we say that a collection of functions $(\eta_i)_{i \in I}$, $\eta_i \colon \tilde{M}_i \to \hat{M}_i$ is an $\mathcal{L}$-*embedding* if it has the following properties:

- every $\eta_i \colon \tilde{M}_i \to \hat{M}_i$ is injective;
- for every $c \in \mathcal{C}_i$ we have $\eta_i(\tilde{\xi}_i(c)) = \hat{\xi}_i(c)$;
- if $f \in \mathcal{F}$ is a function symbol of $(i_1, \dots, i_p, i_{p+1})$ and $(\bar{x}_1, \dots, \bar{x}_p) \in \tilde{M}^{i_1} \times \dots \times \tilde{M}^{i_p}$, then $\eta_{i_{p+1}}(\tilde{\xi}^{\mathcal{F}}(f)(\bar{x}_1, \dots, \bar{x}_p)) = \hat{\xi}^{\mathcal{F}}(f)(\eta_{i_1}(\bar{x}_1), \dots, \eta_{i_p}(\bar{x}_p))$;
- if $R \in \mathcal{R}$ is a relation symbol of type $(i_1, \dots, i_p)$ and $(\bar{x}_1, \dots, \bar{x}_p) \in \tilde{M}^{i_1} \times \dots \times \tilde{M}^{i_p}$, then $(\bar{x}_1, \dots, \bar{x}_p)$ satisfies $\tilde{\xi}^{\mathcal{R}}(R)$ if and only if $(\eta_{i_1}(\bar{x}_1), \dots, \eta_{i_p}(\bar{x}_p))$ satisfies $\hat{\xi}^{\mathcal{R}}(R)$.

If there exists an $\mathcal{L}$-embedding from $\mathcal{M}$ to $\mathcal{N}$, then we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$. We say that the embedding $(\eta_i)_{i \in I}$ is *elementary* if for every $\mathcal{L}$-formula $\phi(x_{k_1}^{(i_1)}, \dots, x_{k_n}^{(i_n)})$ and every $(\bar{x}_{k_1}^{(i_1)}, \dots, \bar{x}_{k_n}^{(i_n)}) \in \tilde{M}_{i_1} \times \dots \times \tilde{M}_{i_n}$ we have

$$\mathcal{M} \models \phi(\bar{x}_{k_1}^{(i_1)}, \dots, \bar{x}_{k_n}^{(i_n)}) \iff \mathcal{N} \models \phi(\eta_{i_1}(\bar{x}_{k_1}^{(i_1)}), \dots, \eta_{i_n}(\bar{x}_{k_n}^{(i_n)}))\,.$$

Definition 2.123. We say that an $\mathcal{L}$-theory Th is *model complete* if every embedding between two models of Th is elementary.

Proposition 2.124 ([Mar02, Proposition 3.1.14])**.** *If an $\mathcal{L}$-theory has quantifier elimination, then it is model complete.*

Example 2.125. If we consider $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +)$ as $\mathcal{L}_{\mathrm{og}}$-structures, then $\mathbb{Z}$ is a substructure of $\mathbb{Q}$, which in turn is a substructure of $\mathbb{R}$. In both cases, the embedding is given by the identity map. The embedding from $\mathbb{Z}$ to $\mathbb{Q}$ is not elementary. For instance, the $\mathcal{L}_{\mathrm{og}}$-sentence $\forall x, \exists y, x = y + y$ is not true in $(\mathbb{Z}, +)$ but is true in $(\mathbb{Q}, +)$. The embedding from $\mathbb{Q}$ to $\mathbb{R}$ is elementary because both $\mathbb{Q}$ and $\mathbb{R}$ are models of the theory of nontrivial divisible ordered abelian groups and this theory has quantifier elimination (Theorem 2.101).

2.7 Markov chains

In this section, we recall some basic properties of Markov chains on finite spaces, referring the reader to [KS76, Chu67] for more information.

Definition 2.126. Suppose that V is a finite set. A matrix $P \in [0,1]^{V \times V}$ is called *stochastic* if the sum of entries of P in every row is equal to 1. Furthermore, suppose that $(X_0, X_1, \dots)$ is a sequence of random variables with values in V. Then, we say that this sequence is a *Markov chain* if the equality

$$P_{vw} = \mathbb{P}(X_n = w \mid X_{n-1} = v) = \mathbb{P}(X_n = w \mid X_{n-1} = v, X_{n-2} = v_{n-2}, \dots, X_0 = v_0)$$

holds whenever the right-hand side is well defined. In this case, the matrix P is also called the *transition matrix* of a Markov chain. In this context, P_{vw} denotes the probability that chain moves from state v to state w in one step. Furthermore, we say that Markov chain $(X_0, X_1, \dots)$ *starts at* $v \in V$ if $\mathbb{P}(X_0 = v) = 1$.

Remark 2.127. If P is a stochastic matrix and $v \in V$, then there exists a Markov chain such that $\mathbb{P}(X_0 = v) > 0$ for every $v \in V$ and such that its transition matrix is equal to P (see [Chu67, Part I, § 2, Theorem 1]). We will refer to any such chain a *Markov chain associated with* P.

Definition 2.128. Given a stochastic matrix $P \in [0,1]^{V \times V}$, we construct a directed graph $\vec{\mathcal{G}} := (V, E)$ by setting $E := \{(v, w) \in V^2 \colon P_{vw} > 0\}$.

Remark 2.129. We point out that, by definition, every vertex of $\vec{\mathcal{G}}$ has at least one outgoing edge.

Definition 2.130. We say that a set $C \subset V$ is *strongly connected* if for every pair $(v, w) \in C^2$ there exist a directed path in $\vec{\mathcal{G}}$ from v to w and from w to v. We say that a strongly connected set C is a a *strongly connected component of* $\vec{\mathcal{G}}$ if there is no edge going out of C (i.e., if there is no edge in $\vec{\mathcal{G}}$ that has its tail in C and head outside of C). If $(X_0, X_1, \dots)$ is a Markov chain with transition matrix P, then a strongly connected component of $\vec{\mathcal{G}}$ is also called a *recurrent class* of this Markov chain. We say that a matrix P is *irreducible* if $\vec{\mathcal{G}}$ is strongly connected.

Definition 2.131. Fix a stochastic matrix $P \in [0,1]^{V \times V}$, a state $v \in V$, and suppose that the Markov chain $(X_0, X_1, \dots)$ starts at v. We say that v is *recurrent* if the probability that the chain returns back to v is equal to 1. More formally, u is recurrent if $\mathbb{P}(\exists n \geqslant 1, X_n = v \mid X_0 = v) = 1$. Otherwise, we say that v is *transient*. A recurrent state v is *absorbing* if $\mathbb{P}(X_1 = v \mid X_0 = v) = 1$.

The next proposition classifies the states and describes the long-term behavior of a Markov chain.

Proposition 2.132 ([KS76, § 2.4]). *Every Markov chain has at least one recurrent class. Moreover, a state $v \in V$ is recurrent if and only if it belongs to a recurrent class. If a Markov chain starts at a recurrent state, then it never leaves the recurrent class of this state and visits every state of this class infinitely many times. If a Markov chain starts at a transient state, then it reaches some recurrent class in finitely many steps and stays there forever.*

We will also use the notion of a stationary distribution.

Definition 2.133. If $C \subset V$, then we say that $\pi \in [0,1]^C$ is a *stationary distribution* on the set C if $\pi_v = \sum_{w \in C} \pi_w P_{wv}$ for all $v \in C$ and $\sum_{v \in C} \pi_v = 1$. We say that $\pi \in [0,1]^n$ is a *stationary distribution* of the matrix P if it is a stationary distribution on $C = [n]$.

Proposition 2.134 ([KS76, Theorem 5.1.2]). *Every recurrent class of a Markov chain has a unique stationary distribution. In particular, every irreducible stochastic matrix has a unique stationary distribution.*

Remark 2.135. We point out that a stationary distribution of a recurrent class cannot have a zero entry. Indeed, if $\pi_v = 0$, then $\pi_w = 0$ for all $w \in C$ such that $P_{wv} > 0$. Since recurrent classes are strongly connected, this would imply that $\pi_v = 0$ for all states in C. Therefore, the system

$$\begin{cases} \pi_v = \sum_{w \in C} \pi_w P_{wv} \quad \text{for all } v \in C \\ \sum_{v \in C} \pi_v = 1 \end{cases} \tag{2.12}$$

has a unique solution in $]0,1]^n$. Moreover, it is obvious that this is a unique solution in $\mathbb{R}^n_{>0}$. Since the set of solutions of (2.12) forms an affine space, this implies that (2.12) has a unique solution in $\mathbb{R}^n$.

We now introduce Markov chains with payoffs. To this end, with every state $v \in V$ we associate a payoff $r_v \in \mathbb{R}$. This quantity is interpreted as follows: there is a controller of the chain, who receives a payoff r_v as soon as the chain leaves the state v.

Definition 2.136. A *(long-term) average payoff* of the controller is defined as

$$\forall v \in V,\, g_v = \lim_{N \to \infty} \mathbb{E} \frac{1}{N} \sum_{p=0}^{N} r_{v_p} \,,$$

where the expectation is taken over all trajectories $v_0, \dots, v_N$ starting from $v_0 = v$ in the Markov chain.

The next theorem characterizes the average payoff. Before that, let us introduce some additional notation. For any state $v \in V$, let the random variable $T_v \coloneqq \inf\{s \geqslant 1 \colon X_s = v\}$ denote the time of first return to v. By θ_v we denote the expected time of first return to v,

$$\theta_v \coloneqq \mathbb{E}(T_v | X_0 = v) \,.$$

Furthermore, let ξ_v be the expected payoff the controlled obtained before returning to v, i.e.,

$$\xi_v \coloneqq \mathbb{E}\Big(\sum_{s=0}^{T_v - 1} r_{X_s} \Big| X_0 = v \Big) \,.$$

Theorem 2.137. *If $v \in V$ is a fixed initial state, then the average payoff g_v is well defined and characterized as follows:*

(i) Suppose that v is a recurrent state belonging to the recurrent class C. Let $(\pi_w)_{w \in C}$ be the stationary distribution on C. Then $\pi_v = 1/\theta_v$. Furthermore, we have

$$g_v = \frac{\xi_v}{\theta_v} = \sum_{w \in C} r_w \pi_w \, .$$

In particular, g_v is constant for all states v belonging to C.

(ii) If v is transient and $C_1, \dots, C_p$ denote all the recurrent classes of the Markov chain, then $g_v = \sum_{s=1}^{p} g_{v_s} \psi_s$, where, for all s, ψ_s denotes the probability that the chain starting from v reaches the recurrent class C_s, and $v_s \in C_s$ is an arbitrary state of C_s.

Theorem 2.137 is well known, and can be easily derived from the analysis of Markov chains presented in the textbook of Chung [Chu67, Part I, §6, §7, and §9]. We give the details in Appendix B.2 for the sake of completeness.

Remark 2.138. If the transition matrix P is rational, then the stationary distributions $(\pi_v)_{v \in C_s}$ of recurrent classes and the absorption probabilities ψ_s are rational as well. This follows from the fact that these quantities can be computed using elementary linear algebra. Remark 2.135 shows how to compute the stationary distributions. The formula for computing the absorption probabilities is given in [KS76, Theorem 3.3.7].

Remark 2.139. We point out that given the transition matrix P, the average payoff g can be computed using the algorithm presented in [Put05, Appendix A.3 and A.4]. If the payoffs r_e are rational, then this algorithm can be implemented to run in strongly polynomial complexity using the strongly polynomial version of gaussian elimination (presented, for example, in [GLS93, Section 1.4]).

Part I

Tropical spectrahedra

CHAPTER 3

Tropicalization of semialgebraic sets

In this section we study the class of tropical semialgebraic sets, i.e., the images by valuation of semialgebraic sets over $\mathbb{K}$. As discussed in Section 1.1, Alessandrini [Ale13] has shown, among other results, that if $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is a semialgebraic set, then the real part of its image $\mathsf{val}(\boldsymbol{\mathcal{S}}) \cap \mathbb{R}^n$ has polyhedral structure, is closed, and the dimension of this set is not greater than the dimension of $\boldsymbol{\mathcal{S}}$. This can be thought of as the real analogue of the Bieri–Groves theorem [BG84, EKL06]. The proof of Alessandrini is based on o-minimal models. We give a proof that avoids the recourse to o-minimal techniques by using the Denef–Pas quantifier elimination. The main theorem of this section is the following result.

Theorem 3.1. *Let $\mathscr{K}$ be a real closed valued field equipped with a nontrivial and convex valuation $\mathsf{val} \colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ and suppose that $S \subset \mathscr{K}^n$ is a semialgebraic set. Then, every stratum of $\mathsf{val}(S) \subset (\Gamma \cup \{-\infty\})^n$ is semilinear and closed. Furthermore, we have $\mathsf{val}(\mathrm{cl}_{\mathscr{K}}(S)) = \mathrm{cl}_{\Gamma \cup \{-\infty\}}(\mathsf{val}(S))$. In particular, the image of a closed semialgebraic set is closed in $(\Gamma \cup \{-\infty\})^n$. Conversely, if a set $X \subset (\Gamma \cup \{-\infty\})^n$ has closed and semilinear strata, then there exists a semialgebraic set $S \subset \mathscr{K}^n$ such that $\mathsf{val}(S) = X$.*

Moreover, if $\mathscr{K} = \mathbb{K}$, then the dimension of every stratum of $\mathsf{val}(S)$ is not greater than the dimension of S.

Let us discuss in detail the similarities and differences between this result and the results of Alessandrini [Ale13]. The advantages of our approach are as follows. First, since we prove the theorem using the Denef–Pas quantifier elimination, our proof is constructive (see Remark 2.121). Second, the analysis of Alessandrini gives the result of Theorem 3.1 under a supplementary hypothesis that the value group Γ is a subgroup of $\mathbb{R}$, while our proof works for all possible value groups (fields with larger value groups are convenient, for instance, to do symbolic perturbations, see [ABGJ14, AK17]). Third, the Denef–Pas quantifier elimination gives us a stronger transfer principle (model completeness of real closed valued fields, Proposition 2.124 and Theorem 2.120) and we later use this principle to prove the tropical analogue of the Helton–Nie conjecture (Theorem 5.2). On the other had, the analysis of Alessandrini interprets the images by valuation of semialgebraic sets in $\mathbb{K}$ as logarithmic limit sets and this aspect is missing in our proofs. Moreover, Alessandrini's result applies not only to the class of semialgebraic sets, but also to sets definable in Hardy fields of polynomially-bounded o-minimal structures. Therefore, the two approaches overlap (for instance, they give the same result for semialgebraic sets defined over $\mathbb{K}$), but neither supersedes the other.

Before proceeding, let us discuss the different notions used in the statement of Theorem 3.1. The definition of a stratum is given in Definition 2.107. Furthermore, if $\mathscr{K} = \mathbb{K}$, then $\Gamma = \mathbb{R}$ and hence the dimension of a stratum of $\mathsf{val}(S)$ is well defined using Definition 2.19 (because every semilinear subset of $\mathbb{R}^n$ is also semialgebraic). Let us now focus on the different notions of closure that are involved in the statement of Theorem 3.1. As discussed in Section 2.2, any real closed field $\mathscr{K}$ is equipped with the topology induced by its order, and we can extend this topology to $\mathscr{K}^n$ by taking the product topology. The same is true for any divisible ordered abelian group Γ and its variant equipped with the bottom element $\Gamma \cup \{-\infty\}$. We denote by $\mathrm{cl}_{\mathscr{K}}(\cdot)$, $\mathrm{cl}_{\Gamma}(\cdot)$, and $\mathrm{cl}_{\Gamma\cup\{-\infty\}}(\cdot)$ the closure operators in the respective topologies. Theorem 3.1 uses two notions of closedness—the weaker one is the closedness of all strata, and the stronger one is the closedness in the topology of $(\Gamma\cup\{-\infty\})^n$. The following lemma and example explain the differences in these notions.

Lemma 3.2. *If a set $\mathcal{S} \subset (\Gamma \cup \{-\infty\})^n$ is closed (in the product topology of $(\Gamma \cup \{-\infty\})^n$), and $K \subset [n]$ is nonempty, then the stratum $\mathcal{S}_K \subset \Gamma^K$ of $\mathcal{S}$ is closed (in the product topology of Γ^K).*

Proof. Let $x \in \mathrm{cl}_{\Gamma}(\mathcal{S}_K)$. We want to show that $x \in \mathcal{S}_K$. To do so, consider the point $\tilde{x} \in (\Gamma \cup \{-\infty\})^n$ defined as $\tilde{x}_k \coloneqq x_k$ for $k \in K$ and $\tilde{x}_k \coloneqq -\infty$ otherwise. The fact that $x \in \mathcal{S}_K$ is equivalent to $\tilde{x} \in \mathcal{S}$. To show that $\tilde{x} \in \mathcal{S}$, take any vectors $a, b \in (\Gamma \cup \{-\infty\})^n$ such that $a_k < x_k < b_k$ for all $k \in K$ and $a_k = -\infty$, $b_k > -\infty$ for all $k \notin K$. By definition, there exists a point $(y_1, \dots, y_n) \in \mathcal{S}_K$ that satisfies $a_1 < y_1 < b_1, \dots, a_n < y_n < b_n$. Let $\tilde{y} \in (\Gamma \cup \{-\infty\})^n$ be defined as $\tilde{y}_k \coloneqq y_k$ for $k \in K$ and $\tilde{y}_k \coloneqq -\infty$ otherwise. We have $\tilde{y} \in \mathcal{S}$. Moreover, the point $\tilde{y}$ belongs to the neighborhood of $\tilde{x}$ defined by the inequalities $a_k < \tilde{y}_k < b_k$ for all $k \in K$ and $-\infty = \tilde{y}_k < b_k$ otherwise. Since the choice of a, b was arbitrary, by the closedness of $\mathcal{S}$ we obtain $\tilde{x} \in \mathcal{S}$. □

Example 3.3. The converse of the lemma above is false, and the distinction between the two notions of closedness is important for Theorem 3.1. To see this, take the set $\boldsymbol{S} \coloneqq \{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{K}^2_{\geqslant 0} \colon \boldsymbol{x}_1 < \boldsymbol{x}_2\}$. Its image by valuation is equal to $\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \leqslant x_2\} \cup \{(x_1, x_2) \in \mathbb{T}^2 \colon x_1 = -\infty, x_2 \neq -\infty\}$. This set has closed strata, but is not closed in the topology of $\mathbb{T}^2$. However, we have $\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}) = \{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{K}^2_{\geqslant 0} \colon \boldsymbol{x}_1 \leqslant \boldsymbol{x}_2\}$ and $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S})) = \{(x_1, x_2) \in \mathbb{T}^2 \colon x_1 \leqslant x_2\} = \mathsf{val}(\boldsymbol{S}) \cup \{-\infty\}$ is closed in the topology of $\mathbb{T}^2$.

As a by-product, we also get the following result, which generalizes the proposition of Develin and Yu [DY07, Proposition 2.9] on polyhedra to basic semialgebraic sets.

Theorem 3.4. *Suppose that $\boldsymbol{S} \subset \mathbb{K}^n_{>0}$ is a semialgebraic set defined as*

$$\boldsymbol{S} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \boldsymbol{P}_1(\boldsymbol{x}) \,\square_1\, 0, \dots, \boldsymbol{P}_m(\boldsymbol{x}) \,\square_m\, 0\}\,,$$

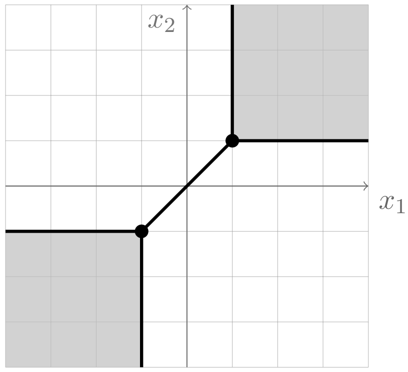

Figure 3.1: Polyhedral complex $\mathcal{C}^{\geqslant}(P)$.

where $\boldsymbol{P}_i \in \mathbb{K}[X_1, \dots, X_n]$ are nonzero polynomials and $\square \in \{\geqslant, >\}^m$. Let $P_i \coloneqq \operatorname{trop}(\boldsymbol{P}_i)$ for all i and suppose that $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ has regular support. Then

$$\mathsf{val}(\boldsymbol{\mathcal{S}}) = \{x \in \mathbb{R}^n \colon \forall i, P_i^+(x) \geqslant P_i^-(x)\}\,.$$

Example 3.5. Take $P = 0 \oplus (X_1^{\odot 2} \odot X_2^{\odot 2}) \oplus (2 \odot X_1 \odot X_2) \oplus (\ominus 2 \odot X_1^{\odot 2}) \oplus (\ominus 2 \odot X_2^{\odot 2})$. Then $\mathcal{C}^{\geqslant}(P)$ is depicted in Fig. 3.1. This support of this complex is not regular and Theorem 3.4 does not apply. Indeed, take $\boldsymbol{P}(\boldsymbol{x}_1, \boldsymbol{x}_2) = 1 + \boldsymbol{x}_1^2\boldsymbol{x}_2^2 + t^2\boldsymbol{x}_1\boldsymbol{x}_2 - t^2\boldsymbol{x}_1^2 - t^2\boldsymbol{x}_2^2$. We have $\operatorname{trop}(\boldsymbol{P}) = P$, but the set $\mathsf{val}(\{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{K}^2_{>0} \colon \boldsymbol{P}(\boldsymbol{x}_1, \boldsymbol{x}_2) \geqslant 0\})$ does not contain the open segment $](-1,-1),(1,1)[$.

The rest of this chapter is organized as follows. In Section 3.1 we give the proofs of Theorems 3.1 and 3.4. In Section 3.2 we discuss in more detail the images by valuation that are obtained if one uses the field of Puiseux series with rational exponents as the base field, instead of $\mathbb{K}$. The results of this chapter are based on the preprint [AGS16b].

3.1 Real analogue of the Bieri–Groves theorem

In this section, we prove Theorem 3.1 and Theorem 3.4. We do that in a series of lemmas. We start by proving that $\mathsf{val}(S)$ has semilinear strata.

Lemma 3.6. *If $S \subset \mathscr{K}^n$ is semialgebraic, then $\mathsf{val}(S)$ has semilinear strata.*

Proof. To prove the result, we use the quantifier elimination in real closed fields (Theorem 2.120). Let $\mathscr{k}$ denote a residue field of $(\mathscr{K}, \Gamma, \mathsf{val})$ and let $\mathsf{ac}\colon \mathscr{K} \to \mathscr{k}$ denote an angular component (its existence is proven in Lemma 2.70). The structure $\mathcal{M} = (\mathscr{K}, \Gamma \cup \{-\infty\}, \mathscr{k}, \mathsf{val}, \mathsf{ac})$ is a model of $\mathrm{Th}_{\mathrm{rcvf}}$. Let $\phi(x_1, \dots, x_{n+m})$ be an $\mathcal{L}_{\mathscr{K}}$-formula and $\bar{b} \in \mathscr{K}^m$ be a vector such that $S = \{x \in \mathscr{K}^n \colon \mathscr{K} \models \phi(x, \bar{b})\}$. Take the $\mathcal{L}_{\mathrm{rcvf}}$-formula $\theta(x_{n+1}, \dots, x_{n+m}, y_1, \dots, y_n)$ defined as

$$\exists x_1, \dots, \exists x_n, \Big(\phi(x_1, \dots, x_{n+m}) \wedge \mathsf{val}(x_1) = y_1 \wedge \dots \wedge \mathsf{val}(x_n) = y_n\Big)\,.$$

We obviously have

$$\mathsf{val}(S) = \{y \in (\Gamma \cup \{-\infty\})^n \colon \mathcal{M} \models \theta(\bar{b}, y)\}\,.$$

By Theorem 2.120, θ is equivalent to a formula of the form

$$\bigvee_{i=1}^{m} \Big(\phi_i\big(\mathsf{val}(f_{i1}(X)), \dots, \mathsf{val}(f_{ik_i}(X)), Y\big) \wedge \psi_i\big(\mathsf{ac}(f_{i(k_1+1)}(X)), \dots, \mathsf{ac}(f_{il_i}(X))\big)\Big)\,,$$

where we denote $X \coloneqq (x_{n+1}, \dots, x_{n+m})$, $Y \coloneqq (y_1, \dots, y_n)$, every ϕ_i is an $\mathcal{L}_\Gamma$-formula, every ψ_i is an $\mathcal{L}_{\not k}$-formula, and $f_{i1}, \dots, f_{il_i} \in \mathbb{Z}[X]$ are polynomials with integer coefficients. If we fix X to be equal to $\bar{b}$, then this formula is equivalent to a formula of the form

$$\bigvee_{i \in I} \phi_i(\zeta_{i1}, \dots, \zeta_{ik_i}, Y)\,,$$

where I is a subset of $[m]$ and we denote $\mathsf{val}(f_{ik}(\bar{b})) = \zeta_{ik} \in \Gamma \cup \{-\infty\}$. Hence, $\mathsf{val}(S)$ is definable in $\mathcal{L}_\Gamma$. By Lemma 2.108, $\mathsf{val}(S)$ has semilinear strata. $\square$

The next lemmas concern the closedness of $\mathsf{val}(\boldsymbol{\mathcal{S}})$. We first restrict ourselves to the case of open positive orthant of Puiseux series $\mathbb{K}^n_{>0} \coloneqq \{\boldsymbol{x} \in \mathbb{K} \colon \forall k, \boldsymbol{x}_k > 0\}$ and then we generalize the results. Let us fix a basic semialgebraic set $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{>0}$ defined as

$$\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \forall i = 1, \dots p, \boldsymbol{P}_i(\boldsymbol{x}) > 0 \land \forall i = p+1, \dots, q, \boldsymbol{Q}_i(\boldsymbol{x}) = 0\} \tag{3.1}$$

for some polynomials $\boldsymbol{P}_1, \dots, \boldsymbol{P}_p, \boldsymbol{Q}_{p+1}, \dots, \boldsymbol{Q}_q \in \mathbb{K}[X_1, \dots, X_n]$. Equivalently, we put $\boldsymbol{\mathcal{S}}$ under the form

$$\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \forall i = 1, \dots p, \boldsymbol{P}_i(\boldsymbol{x}) > 0 \land \forall i = p+1, \dots, q, \boldsymbol{P}_i(\boldsymbol{x}) \geqslant 0\}\,, \tag{3.2}$$

where we set $\boldsymbol{P}_i \coloneqq -\boldsymbol{Q}_i^2$ for all $i = p+1, \dots, q$. Denote $P_i \coloneqq \operatorname{trop}(\boldsymbol{P}_i)$ for all $i = 1, \dots, q$.

Furthermore, given a polynomial $\boldsymbol{P}$ as in (2.8), we denote by $\boldsymbol{P}^+$ the polynomial obtained by summing the terms $\boldsymbol{a}_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}$ such that $\boldsymbol{a}_\alpha > 0$. Similarly, $\boldsymbol{P}^-$ refers to the sum of the terms $-\boldsymbol{a}_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}$ satisfying $\boldsymbol{a}_\alpha < 0$. In this way, $\boldsymbol{P} = \boldsymbol{P}^+ - \boldsymbol{P}^-$. In the next lemma, we highlight a property of the full-dimensional cells of the complex $\mathcal{C}(P_1, \dots, P_q)$ whose interior is contained in $\mathsf{val}(\boldsymbol{\mathcal{S}})$.

Lemma 3.7. *Suppose that $\mathcal{W}$ is a full-dimensional cell of $\mathcal{C}(P_1, \dots, P_q)$ such that $\operatorname{int}(\mathcal{W}) \cap \mathsf{val}(\boldsymbol{\mathcal{S}}) \neq \emptyset$. Let $w \in \operatorname{int}(\mathcal{W})$, and $\boldsymbol{w} \in \mathsf{val}^{-1}(w) \cap \mathbb{K}^n_{>0}$ be an arbitrary lift. Then $\boldsymbol{w} \in \boldsymbol{\mathcal{S}}$.*

Proof. Take a point $\boldsymbol{z} \in \boldsymbol{\mathcal{S}}$ such that $z \coloneqq \mathsf{val}(\boldsymbol{z}) \in \operatorname{int}(\mathcal{W})$. For every $i = 1, \dots, q$ we have $\boldsymbol{P}_i(\boldsymbol{z}) \geqslant 0$. By Lemma 2.52 we obtain $P_i^+(z) \geqslant P_i^-(z)$. Since $\mathcal{W}$ is a full-dimensional cell of $\mathcal{C}(P_1, \dots, P_q)$, we have the equality $\operatorname{int}(\mathcal{W}) = \cap_{i=1}^q \operatorname{int}(\mathcal{W}_i)$, where, for every i, $\mathcal{W}_i$ is a full-dimensional cell of $\mathcal{C}(P_i)$. In particular, $\operatorname{Argmax}(P_i, z)$ has only one element and we have $P_i^+(z) > P_i^-(z)$. Furthermore, we have $\operatorname{Argmax}(P_i, z) = \operatorname{Argmax}(P_i, w)$ for any point $w \in \operatorname{int}(\mathcal{W})$. This implies that $P_i^+(w) > P_i^-(w)$. Therefore, if $\boldsymbol{w} \in \mathsf{val}^{-1}(w) \cap \mathbb{K}^n_{>0}$ is an arbitrary lift of w, then as in the proof of Lemma 2.52 we have $\mathsf{val}(\boldsymbol{P}_i^+(\boldsymbol{w})) = P_i^+(w) > P_i^-(w) = \mathsf{val}(\boldsymbol{P}_i^-(\boldsymbol{w}))$. Since $\boldsymbol{P}^+, \boldsymbol{P}^-$ have only positive coefficients, we have $\boldsymbol{P}_i^+(\boldsymbol{w}) > \boldsymbol{P}_i^-(\boldsymbol{w})$ by (2.5) and hence $\boldsymbol{w} \in \boldsymbol{\mathcal{S}}$. $\square$

Lemma 3.8. *Let $A \in \mathbb{Q}^{m \times n}$ be any matrix. Define a function $f_A \colon \mathbb{K}^n_{>0} \to \mathbb{K}^m_{>0}$ as*

$$\forall i \in [m],\ \big(f_A(\boldsymbol{x})\big)_i \coloneqq \boldsymbol{x}_1^{A_{i1}} \boldsymbol{x}_2^{A_{i2}} \dots \boldsymbol{x}_n^{A_{in}}\,. \tag{3.3}$$

Let $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{>0}$ be any semialgebraic set. Then $f_A(\boldsymbol{\mathcal{S}}) \subset \mathbb{K}^m_{>0}$ is semialgebraic and we have $\mathsf{val}(f_A(\boldsymbol{\mathcal{S}})) = A(\mathsf{val}(\boldsymbol{\mathcal{S}}))$.

Proof. It is easy to check that the function f_A is semialgebraic.Therefore, the first claim follows from the fact that the class of semialgebraic sets is closed under semialgebraic transformations (Proposition 2.18). The second claim follows from the identity $\mathsf{val}((f_A(\boldsymbol{x}))_i) = A_i(\mathsf{val}(\boldsymbol{x}))$. $\square$

Lemma 3.9. *Suppose that $\boldsymbol{\mathcal{S}} \subset \mathbb{K}_{>0}^n$ is a semialgebraic set. Then $\mathsf{val}(\boldsymbol{\mathcal{S}}) \subset \mathbb{R}^n$ is closed in the topology of $\mathbb{R}^n$. Furthermore, $\dim(\mathsf{val}(\boldsymbol{\mathcal{S}})) \leqslant \dim(\boldsymbol{\mathcal{S}})$.*

Proof. Throughout the proof, we will use the natural properties of dimension mentioned in Proposition 2.20. We proceed by induction over the dimension n. First, suppose that $n = 1$. The claim is obvious if $S = \emptyset$. Otherwise, by Lemma 2.16, it is a finite union of points and open intervals. Observe that the image by the valuation of an open interval in $\mathbb{K}_{>0}$ is either a closed interval in $\mathbb{R}$ or a point. Moreover, the image of a point is a point. Therefore, $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is closed. Moreover, if $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is of dimension 1, then it contains an interval. This implies that $\boldsymbol{\mathcal{S}}$ contains an interval and is of dimension 1. Therefore, the claim is true for $n = 1$.

Second, suppose that the claim holds in dimension $n-1$. Observe that it is enough to prove the claim for basic semialgebraic sets. Fix a basic semialgebraic set $\boldsymbol{\mathcal{S}} \subset \mathbb{K}_{>0}^n$ as in (3.2) and take the polyhedral complex $\mathcal{C} \coloneqq \mathcal{C}(P_1, \dots, P_q)$. Let $\tilde{\mathcal{W}}_1, \dots, \tilde{\mathcal{W}}_r$ denote the cells of $\mathcal{C}$. By Lemma 3.6, $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is a finite union of basic semilinear sets. In particular, it is a finite union of relatively open polyhedra. In other words, we have $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \bigcup_{j=1}^s \mathrm{ri}(\tilde{\mathcal{V}}_j)$, where each $\tilde{\mathcal{V}}_j$ is a polyhedron and ri denotes the relative interior. For every (i,j), let $\mathcal{W}_{ij}$ be the polyhedron defined by

$$\mathcal{W}_{ij} \coloneqq \mathrm{cl}_{\mathbb{R}}\big(\mathrm{ri}(\tilde{\mathcal{W}}_i) \cap \mathrm{ri}(\tilde{\mathcal{V}}_j)\big)\,.$$

By the definition of relative interior, we have $\mathrm{ri}(\mathcal{W}_{i,j}) = \mathrm{ri}(\tilde{\mathcal{W}}_i) \cap \mathrm{ri}(\tilde{\mathcal{V}}_j)$. Furthermore, by Lemmas 2.5 and 2.57 we have $\bigcup_{i=1}^r \mathrm{ri}(\tilde{\mathcal{W}}_i) = \mathbb{R}^n$. Therefore, we have $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \big(\bigcup_{j=1}^s \mathrm{ri}(\tilde{\mathcal{V}}_j)\big) \cap \big(\bigcup_{i=1}^r \mathrm{ri}(\tilde{\mathcal{W}}_i)\big) = \bigcup_{i,j} \mathrm{ri}(\mathcal{W}_{ij})$ and, by Corollary 2.4, $\mathrm{cl}_{\mathbb{R}}(\mathsf{val}(\boldsymbol{\mathcal{S}})) = \bigcup_{i,j} \mathcal{W}_{ij}$. We will start by proving that $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is closed. We consider an element w^* of $\mathrm{cl}_{\mathbb{R}}(\mathsf{val}(\boldsymbol{\mathcal{S}}))$. Let us look at two cases.

Case I: There is a full-dimensional polyhedron $\mathcal{W}_{ij}$ such that $w^* \in \mathcal{W}_{ij}$. In this case, the set $(-w^*) + \mathcal{W}_{ij} \subset \mathbb{R}^n$ contains an open ball,[1] and this ball contains a sequence of $n-1$ linearly independent rational vectors $u^{(1)}, \dots, u^{(n-1)} \in \mathbb{Q}^n$. Take $a \in \mathbb{Q}^n \setminus \{0\}$ such that $\mathrm{span}(u^{(1)}, \dots, u^{(n-1)}) = \{w \in \mathbb{R}^n \colon \langle w, a\rangle = 0\}$. Then, $\mathcal{H} = \{w \in \mathbb{R}^n \colon \langle a, w\rangle = \langle a, w^*\rangle\} = w^* + \mathrm{span}(u^{(1)}, \dots, u^{(n-1)})$ is a hyperplane that contains w^* and we have $w^* + u^{(1)}, \dots, w^* + u^{(n-1)} \in \mathrm{int}(\mathcal{W}_{ij})$. Therefore, as in the proof of Lemma 2.3, the sequence $(w^{(h)})_{h \geqslant 1}$ defined as $w^{(h)} \coloneqq w^* + \frac{1}{k} u^{(1)}$ satisfies $w^{(h)} \in \mathcal{H} \cap \mathrm{int}(\mathcal{W}_{ij})$ for all h and $w^{(h)} \to w^*$. Take the set $\boldsymbol{Y} \subset \mathbb{K}_{>0}^n$ defined as

$$\boldsymbol{Y} = \boldsymbol{\mathcal{S}} \cap \Big\{\boldsymbol{x} \in \mathbb{K}_{>0}^n \colon \prod_{k \in [n]} \boldsymbol{x}_k^{a_k} = t^{\langle a, w^*\rangle}\Big\}\,.$$

For every h define $\boldsymbol{w}^{(h)} \in \mathsf{val}^{-1}(w^{(h)}) \cap \mathbb{K}_{>0}^n$ as $\boldsymbol{w}_k^{(h)} = t^{w_k^{(h)}}$ for all $k \in [n]$. Note that every $w^{(h)}$ belongs to the interior of the full-dimensional polyhedron $\tilde{\mathcal{W}}_i$. Consequently, $\boldsymbol{w}^{(h)}$ belongs to $\boldsymbol{Y}$ by Lemma 3.7. Take $l \in [n]$ such that $a_l \neq 0$ and let $\boldsymbol{\pi} \colon \mathbb{K}_{>0}^n \to \mathbb{K}_{>0}^{n-1}$ denote the projection that forgets the lth coordinate. Similarly, let $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection that forgets the lth coordinate. By the induction hypothesis and Lemma 3.8, $\mathsf{val}(\boldsymbol{\pi}(\boldsymbol{Y}))$ is a closed subset of $\mathbb{R}^{n-1}$ and we have $\mathsf{val}(\boldsymbol{\pi}(\boldsymbol{Y})) = \pi(\mathsf{val}(\boldsymbol{Y}))$. The sequence $\pi(w^{(h)})$ converges to $\pi(w^*)$. Therefore, we have $\pi(w^*) \in \pi(\mathsf{val}(\boldsymbol{Y}))$. In other words, there exists a point $\boldsymbol{w}^* \in \boldsymbol{Y}$ such that $\pi(\mathsf{val}(\boldsymbol{w}^*)) = \pi(w^*)$. Moreover, we have $\mathsf{val}(\boldsymbol{w}^*) \in \mathcal{H}$ and $w^* \in \mathcal{H}$. Since $a_l \neq 0$, this implies that $\mathsf{val}(\boldsymbol{w}^*) = w^*$. Therefore $w^* \in \mathsf{val}(\boldsymbol{\mathcal{S}})$.

Case II: If w^* does not belong to any full-dimensional polyhedron $\mathcal{W}_{ij}$, then we denote by I the set of all indices (i,j) such that $\mathcal{W}_{ij}$ contains w^*. We can take $\rho > 0$ so small that the

[1] Here, by $(-w^*) + \mathcal{W}_{ij}$ we mean the translation of $\mathcal{W}_{ij}$ by vector $-w^*$.

closed Chebyshev ball $B(w^*, \rho)$ does not intersect any polyhedron $\mathcal{W}_{ij}$ with $(i,j) \notin I$. Let $w^{(1)}, w^{(2)}, \dots$ be a convergent sequence of elements of $\mathbb{R}^n$, $w^{(h)} \to w^*$ such that $w^{(h)} \in \mathsf{val}(\boldsymbol{\mathcal{S}})$ for all h. Every polyhedron $\mathcal{W}_{ij}$ such that $(i,j) \in I$ is not full dimensional. Therefore, it is contained in an affine hyperplane $\mathcal{H}_{ij}$. Let $X = \bigcup_{(i,j)\in I} \mathcal{H}_{ij}$ be the union of these hyperplanes. Observe that we have $w^* \in X$ and that $\mathsf{val}(\boldsymbol{\mathcal{S}}) \cap B(w^*, \rho) \subset X$. Let $v \in \mathbb{Q}^n$ be any rational vector such that $v \notin \big((-w^*) + X\big)$. Note that the affine line $w^* + \mathrm{span}(v)$ intersects X only in w^*. Let $A \in \mathbb{Q}^{(n-1)\times n}$ be a rational matrix such that $\ker(A) = \mathrm{span}(v)$ and take the function $f_A \colon \mathbb{K}^n_{>0} \to \mathbb{K}^{n-1}_{>0}$ defined as in (3.3). Let $\boldsymbol{U} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \forall l, \boldsymbol{x}_l \in [t^{w^*_l - \rho}, t^{w^*_l + \rho}]\}$. By Lemma 3.8, the set $f_A(\boldsymbol{\mathcal{S}} \cap \boldsymbol{U}) \subset \mathbb{K}^{n-1}_{>0}$ is semialgebraic and we have $\mathsf{val}(f_A(\boldsymbol{\mathcal{S}} \cap \boldsymbol{U})) = A(\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \boldsymbol{U}))$. Therefore, by the induction hypothesis, the set $A(\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \boldsymbol{U}))$ is closed. For every $w^{(h)}$, let $\boldsymbol{w}^{(h)} \in \boldsymbol{\mathcal{S}}$ denote any element of $\boldsymbol{\mathcal{S}}$ such that $\mathsf{val}(\boldsymbol{w}^{(h)}) = w^{(h)}$. For h large enough we have $w^{(h)} \in B(w^*, \rho/2)$ and hence $\boldsymbol{w}^{(h)} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{U}$. Moreover, the sequence $Aw^{(h)}$ converges to Aw^*. Since $A(\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \boldsymbol{U}))$ is closed, there is $\boldsymbol{w}^* \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{U}$ such that $Aw^* = A\,\mathsf{val}(\boldsymbol{w}^*)$. As $\boldsymbol{w}^* \in \boldsymbol{U}$, we have $\mathsf{val}(\boldsymbol{w}^*) \in B(w^*, \rho)$. Therefore

$$\mathsf{val}(\boldsymbol{w}^*) \in \Big((w^* + \mathrm{span}(v)) \cap B(w^*, \rho) \cap \mathsf{val}(\boldsymbol{\mathcal{S}})\Big).$$

On the other hand, we have $\mathsf{val}(\boldsymbol{\mathcal{S}}) \cap B(w^*, \rho) \subset X$ and $(w^* + \mathrm{span}(v)) \cap X = w^*$. Hence $\mathsf{val}(\boldsymbol{w}^*) = w^*$ and $w^* \in \mathsf{val}(\boldsymbol{\mathcal{S}})$.

This shows that $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is closed. It remains to prove that $\dim(\mathsf{val}(\boldsymbol{\mathcal{S}})) \leqslant \dim(\boldsymbol{\mathcal{S}})$. To do so, it is enough to prove that $\dim(\mathrm{ri}(\mathcal{W}_{ij})) \leqslant \dim(\boldsymbol{\mathcal{S}})$ for all i,j. We consider two cases. If $\dim(\mathrm{ri}(\mathcal{W}_{ij})) = n$, then $\mathrm{ri}(\mathcal{W}_{ij})$ has a nonempty interior. Therefore, there exists $w \in \mathrm{int}(\mathcal{W}_{ij}) \subset \mathrm{int}(\tilde{\mathcal{W}}_i)$, and Lemma 3.7 shows that every lift $\boldsymbol{w} \in \mathsf{val}^{-1}(w) \cap \mathbb{K}^n_{>0}$ belongs to $\boldsymbol{\mathcal{S}}$. In particular, $\boldsymbol{\mathcal{S}}$ contains the open set $\{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \forall l, \boldsymbol{x}_l \in \,]\frac{1}{2}t^{w_l}, 2t^{w_l}[\}$ and hence it is full dimensional. If $\dim(\mathrm{ri}(\mathcal{W}_{ij})) = m < n$, then, as shown in the proof of Corollary 2.26, there exist linearly independent vectors $u^{(1)}, \dots, u^{(m)} \in \mathbb{R}^n$ and $u^{(0)} \in \mathbb{R}^n$ such that $\Delta(u^{(0)}, u^{(1)}, \dots, u^{(m)}) \subset \mathrm{ri}(\mathcal{W}_{ij}) \subset u^{(0)} + \mathrm{span}(u^{(1)}, \dots, u^{(m)})$. Take any rational vector $v \in \mathbb{Q}^n$ such that $v \notin \mathrm{span}(u^{(1)}, \dots, u^{(m)})$. Let $A \in \mathbb{Q}^{(n-1)\times n}$ be a rational matrix such that $\ker(A) = \mathrm{span}(v)$ and take the function $f_A \colon \mathbb{K}^n_{>0} \to \mathbb{K}^{n-1}_{>0}$ defined as in (3.3). As previously, by Lemma 3.8, the set $f_A(\boldsymbol{\mathcal{S}}) \subset \mathbb{K}^{n-1}_{>0}$ is semialgebraic and we have $\mathsf{val}(f_A(\boldsymbol{\mathcal{S}})) = A(\mathsf{val}(\boldsymbol{\mathcal{S}}))$. By induction hypothesis we have $\dim(\boldsymbol{\mathcal{S}}) \geqslant \dim(f_A(\boldsymbol{\mathcal{S}})) \geqslant \dim(\mathsf{val}(f_A(\boldsymbol{\mathcal{S}}))) = \dim(A(\mathsf{val}(\boldsymbol{\mathcal{S}}))) = \max_{i,j} \dim(A(\mathrm{ri}(\mathcal{W}_{ij}))) \geqslant \dim(A(\mathrm{ri}(\mathcal{W}_{ij})))$. Since $\ker(A) \cap \mathrm{span}(u^{(1)}, \dots, u^{(m)}) = \emptyset$, the function $g \colon \mathrm{span}(u^{(1)}, \dots, u^{(m)}) \to \mathbb{R}^{n-1}$ defined as $g(x) \coloneqq Ax$ is injective. Hence, by Proposition 2.20, $\dim(A(\mathrm{ri}(\mathcal{W}_{ij}))) = \dim(\mathrm{ri}(\mathcal{W}_{ij}))$ and $\dim(\boldsymbol{\mathcal{S}}) \geqslant \dim(\mathrm{ri}(\mathcal{W}_{ij}))$. $\square$

The next lemma will be useful to study the images of closed semialgebraic sets.

Lemma 3.10. *Suppose that $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is a nonempty, bounded, closed, semialgebraic set. Let $K \subset [n]$ be a set of indices such that for every $\boldsymbol{a} \in \mathbb{K}_{>0}$ the set $\{\boldsymbol{x} \in \boldsymbol{\mathcal{S}} \colon \forall k \in K, \boldsymbol{x}_k \in [0, \boldsymbol{a}]\}$ is nonempty. Then, there exists a point $\boldsymbol{y} \in \boldsymbol{\mathcal{S}}$ such that $\boldsymbol{y}_k = 0$ for all $k \in K$.*

Proof. We prove that the statement holds for any real closed field $\mathscr{K}$. Fix an $\mathcal{L}_{\mathrm{or}}$-formula $\psi(x_1, \dots, x_{n+m})$. For every vector $\bar{b} \in \mathscr{K}^m$ we define the semialgebraic set $S_{\bar{b}}$ by

$$S_{\bar{b}} \coloneqq \{x \in \mathscr{K}^n \colon \mathscr{K} \models \psi(x_1, \dots, x_n, \bar{b})\}.$$

The statement "for all $(x_{n+1}, \dots, x_{n+m})$, if the set $S_{(x_{n+1},\dots,x_{n+m})}$ is nonempty, bounded, closed, and the set $\{x \in S_{(x_{n+1},\dots,x_{n+m})} \colon \forall k \in K,\, x_k \in [0, a]\}$ is nonempty for every $a > 0$, then there exists a point $y \in S_{(x_{n+1},\dots,x_{n+m})}$ such that $y_k = 0$ for all $k \in K$" is a sentence in the language of ordered rings $\mathcal{L}_{\mathrm{or}}$.[2] It is true in $\mathbb{R}$ (because bounded and closed subsets of $\mathbb{R}^n$ are compact), hence it is true in $\mathscr{K}$ by the completeness of real closed fields (Theorem 2.110). $\square$

The lemmas above lead to the main theorem of this section. Suppose that $\mathscr{K}$ is a real closed valued field equipped with a nontrivial and convex valuation $\mathsf{val}\colon \mathscr{K} \to \Gamma \cup \{-\infty\}$. We split the proof into two parts.

Lemma 3.11. *Suppose that $S \subset \mathscr{K}^n$ is a semialgebraic set. Then, every stratum of $\mathsf{val}(S) \subset (\Gamma \cup \{-\infty\})^n$ is closed. Furthermore, we have $\mathsf{val}(\mathrm{cl}_{\mathscr{K}}(S)) = \mathrm{cl}_{\Gamma\cup\{-\infty\}}(\mathsf{val}(S))$. Moreover, if $\mathscr{K} = \mathbb{K}$, then the dimension of every stratum of $\mathsf{val}(S)$ is not greater than the dimension of S.*

Proof. We first prove the result for a semialgebraic set $\boldsymbol{S} \subset \mathbb{K}^n_{\geqslant 0}$ included in the closed positive orthant of Puiseux series. Let $K \subset [n]$ be any nonempty subset and let $\boldsymbol{X}_K \subset \mathbb{K}^n$ be the set defined as

$$\boldsymbol{X}_K \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{x}_k \neq 0 \iff k \in K\}.$$

The sets $\boldsymbol{X}_K$ and $\boldsymbol{S} \cap \boldsymbol{X}_K$ are semialgebraic. Let $\boldsymbol{\pi}\colon \mathbb{K}^n \to \mathbb{K}^K$ denote the projection on the coordinates from K. Similarly, let $\pi\colon \mathbb{T}^n \to \mathbb{T}^K$ denote the projection on the coordinates from K. Observe that the stratum of $\mathsf{val}(\boldsymbol{S})$ associated with K is equal to $\pi(\mathsf{val}(\boldsymbol{S} \cap \boldsymbol{X}_K)) = \mathsf{val}(\boldsymbol{\pi}(\boldsymbol{S} \cap \boldsymbol{X}_K))$. Moreover, the set $\boldsymbol{\pi}(\boldsymbol{S} \cap \boldsymbol{X}_K)$ is included in $\mathbb{K}^K_{>0}$. Therefore, by Lemma 3.9, all strata of $\mathsf{val}(\boldsymbol{S})$ are closed and satisfy the claim about the dimension.

We will now prove the equality $\mathrm{cl}_{\mathbb{T}}(\mathsf{val}(\boldsymbol{S})) = \mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$. Observe that we trivially have $\mathrm{cl}_{\mathbb{T}}(\mathsf{val}(\boldsymbol{S})) \subset \mathrm{cl}_{\mathbb{T}}(\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S})))$. Therefore, to prove the inclusion $\mathrm{cl}_{\mathbb{T}}(\mathsf{val}(\boldsymbol{S})) \subset \mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$, it is enough to show that $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$ is closed in $\mathbb{T}^n$. Let $x \in \mathbb{T}^n$ be any point that does not belong to $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$ and let $K \subset [n]$ denote the support of x. For any $M, N > 0$ we denote $I_k(M,N) \coloneqq [-\infty, -M[$ if $k \notin K$ and $I_k(M,N) \coloneqq]x_k - \frac{1}{N}, x_k + \frac{1}{N}[$ otherwise. Similarly, we denote $\boldsymbol{I}_k(M,N) \coloneqq [0, t^{-M+1}] \subset \mathbb{K}_{\geqslant 0}$ for $k \notin K$ and $\boldsymbol{I}_k(M,N) \coloneqq [t^{x_k - \frac{2}{N}}, t^{x_k + \frac{2}{N}}] \subset \mathbb{K}_{>0}$ otherwise. We want to show that there is an open neighborhood of x that does not belong to $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$. Suppose that this is not the case. Then, for any $M, N > 0$, the set $\prod_{k=1}^n I_k(M,N)$ contains a point from $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$. Therefore, the set $\boldsymbol{S}^{(M,N)} \coloneqq \mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}) \cap \prod_{k=1}^n \boldsymbol{I}_k(M,N) \subset \mathbb{K}^n_{\geqslant 0}$ is nonempty, on top of being closed, bounded, and semialgebraic.[3] If we fix $N > 0$, then, by Lemma 3.10, there is a point $\boldsymbol{y}^{(N)} \in \boldsymbol{S}^{(1,N)}$ such that $\boldsymbol{y}^{(N)}_k = 0$ for all $k \notin K$. In other words, the set $\boldsymbol{\pi}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}) \cap \boldsymbol{X}_K)$ contains a point that belongs to $\prod_{k\in K}[t^{x_k - \frac{2}{N}}, t^{x_k + \frac{2}{N}}]$. Hence, the stratum of $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$ associated with K contains a point that belongs to $\prod_{k\in K}[x_k - \frac{2}{N}, x_k + \frac{2}{N}]$. Since this is true for all $N > 0$, and the strata of $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$ are closed, we have $x \in \mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$, which gives a contradiction.

To show the inclusion $\mathrm{cl}_{\mathbb{T}}(\mathsf{val}(\boldsymbol{S})) \supset \mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$, suppose that $x \in \mathbb{T}^n$ is a point that does not belong to $\mathrm{cl}_{\mathbb{T}}(\mathsf{val}(\boldsymbol{S}))$. We will show that x does not belong to $\mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$. As previously, let $K \subset [n]$ denote the support of x and, for $M, N > 0$, denote $I_k(M,N) = [-\infty, -M[$ if $k \notin K$ and $I_k(M,N) =]x_k - \frac{1}{N}, x_k + \frac{1}{N}[$ otherwise. By the definition of x, there exist $M, N > 0$ such that $\prod_{k=1}^n I_k(M,N)$ does not contain any point of $\mathsf{val}(\boldsymbol{S})$. Let $\boldsymbol{J}_k \coloneqq [0, t^{-M-1}[$ if $k \notin K$ and $\boldsymbol{J}_k \coloneqq]t^{x_k - \frac{1}{2N}}, t^{x_k + \frac{1}{2N}}[$ otherwise. Since $\mathsf{val}(\boldsymbol{S}) \cap \prod_{k=1}^n I_k(M,N) = \emptyset$, we have $\boldsymbol{S} \cap \prod_{k=1}^n \boldsymbol{J}_k = \emptyset$. Moreover, the set $\prod_{k=1}^n \boldsymbol{J}_k$ is open in the subspace topology of $\mathbb{K}^n_{\geqslant 0}$. Therefore, we have $\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}) \cap \prod_{k=1}^n \boldsymbol{J}_k = \emptyset$. Since any lift of x belongs to $\prod_{k=1}^n \boldsymbol{J}_k$, we have $x \notin \mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S}))$. Therefore, the claim of the lemma is true for every semialgebraic subset of $\mathbb{K}^n_{\geqslant 0}$.

[2] The fact that a set $S_{(x_{n+1},\dots,x_{n+m})}$ is closed can be expressed in $\mathcal{L}_{\mathrm{or}}$ as in Example 2.95.

[3] The set $\mathrm{cl}_{\mathbb{K}}(\boldsymbol{S})$ is semialgebraic by Proposition 2.15.

Second, suppose that $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is any semialgebraic set. Given a vector $\delta \in \{+1,-1\}^n$, we denote by $f_\delta \colon \mathbb{K}^n \to \mathbb{K}^n$ the involution which maps $\boldsymbol{x} \in \mathbb{K}^n$ to the vector with entries $\delta_k \boldsymbol{x}_k$. Since $\bigcup_\delta f_\delta(\mathbb{K}^n_{\geqslant 0}) = \mathbb{K}^n$, we have $\boldsymbol{\mathcal{S}} = \bigcup_\delta (\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))$ Moreover, note that $\mathsf{val}\big(\boldsymbol{X} \cap f_\delta(\mathbb{K}^n_{\geqslant 0})\big) = \mathsf{val}(f_\delta(\boldsymbol{X} \cap f_\delta(\mathbb{K}^n_{\geqslant 0})))$ for any $\boldsymbol{X} \subset \mathbb{K}^n$ and that the set $f_\delta(\boldsymbol{X} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))$ in contained in $\mathbb{K}^n_{\geqslant 0}$. In particular, we have the equality

$$\mathsf{val}(\boldsymbol{\mathcal{S}}) = \bigcup_\delta \mathsf{val}\Big(f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))\Big)\,.$$

Since $f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))$ is a semialgebraic set included in $\mathbb{K}^n_{\geqslant 0}$, we can apply the results of the previous paragraph to each of these sets and take the union. Therefore, we get that every stratum of $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is closed and that its dimension of not greater than the dimension of $\boldsymbol{\mathcal{S}}$ (by Proposition 2.20). The claim about closures follows from the observation that $\mathrm{cl}_\mathbb{K}(\boldsymbol{\mathcal{S}}) = \bigcup_\delta \mathrm{cl}_\mathbb{K}(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))$ and, since f_δ is a homeomorphism,

$$\begin{aligned}\mathsf{val}(\mathrm{cl}_\mathbb{K}(\boldsymbol{\mathcal{S}})) &= \bigcup_\delta \mathsf{val}\Big(f_\delta\Big(\mathrm{cl}_\mathbb{K}(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))\Big)\Big) = \bigcup_\delta \mathsf{val}\Big(\mathrm{cl}_\mathbb{K}\Big(f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))\Big)\Big) \\ &= \bigcup_\delta \mathrm{cl}_\mathbb{T}\Big(\mathsf{val}\Big(f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))\Big)\Big) = \mathrm{cl}_\mathbb{T}\Big(\bigcup_\delta \mathsf{val}\Big(f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^n_{\geqslant 0}))\Big)\Big) = \mathrm{cl}_\mathbb{T}(\mathsf{val}(\boldsymbol{\mathcal{S}}))\,.\end{aligned}$$

To prove the claim for an arbitrary field $\mathscr{K}$ we use Theorem 2.120. We fix an $\mathcal{L}_{\mathrm{or}}$-formula $\psi(x_1, \dots, x_{n+m})$. For every vector $\bar{b} \in \mathscr{K}^m$ we can look at the semialgebraic set

$$S_{\bar{b}} \coloneqq \{x \in \mathscr{K}^n \colon \mathscr{K} \models \psi(x_1, \dots, x_n, \bar{b})\}\,.$$

The statement "for all $(x_{n+1}, \dots, x_{n+m})$, the image by valuation of the set $S_{(x_{n+1}, \dots x_{n+m})}$ has closed strata" can be written as a sentence in $\mathcal{L}_{\mathrm{rcvf}}$. It is true in $\mathbb{K}$ and hence, by the completeness result of Theorem 2.120, it is also true in $\mathscr{K}$. The same is true for the statement "for all $(x_{n+1}, \dots, x_{n+m})$, the vector $(y_1, \dots, y_n)$ belongs to $\mathrm{cl}_{\Gamma \cup \{-\infty\}}(\mathsf{val}(S_{(x_{n+1}, \dots x_{n+m})}))$ if and only if it belongs to $\mathsf{val}(\mathrm{cl}_\mathscr{K}(S_{(x_{n+1}, \dots x_{n+m})}))$." □

The proof of the converse implication of Theorem 3.1 uses the following lemma.

Lemma 3.12. *Every closed semilinear subset of Γ^n is a finite union of sets of the form*

$$\{g \in \Gamma^n \colon \forall i = 1, \dots, p, f_i(g) \geqslant h^{(i)}\}\,,$$

where $f_i \in \mathbb{Z}[X_1, \dots, X_n]$ are homogeneous linear polynomials with integer coefficients and $h^{(i)} \in \Gamma$.

Proof. By definition, any semilinear subset of Γ^n is a finite union of sets of the form

$$\{g \in \Gamma^n \colon \forall i = 1, \dots, p, f_i(g) > h^{(i)}, \forall i = p+1, \dots, q, f_i(g) = h^{(i)}\}\,, \tag{3.4}$$

where $f_i \in \mathbb{Z}[X_1, \dots, X_n]$ are homogeneous linear polynomials with integer coefficients and $h^{(i)} \in \Gamma$. We will show that the closure of a set as in (3.4) is equal to

$$\{g \in \Gamma^n \colon \forall i = 1, \dots, p, f_i(g) \geqslant h^{(i)}, \forall i = p+1, \dots, q, f_i(g) = h^{(i)}\}\,. \tag{3.5}$$

Indeed, this is true for $\Gamma = \mathbb{R}$ by Lemma 2.3. Moreover, the statement "for all $(h^{(i)})_{i=1}^q$, the closure of the set given in (3.4) is equal to the set given in (3.5)" is a sentence in $\mathcal{L}_{\mathrm{og}}$. (The formula that defines the closure is given in Example 2.95.) Therefore, it is true in Γ by the completeness result of Theorem 2.106. The claim follows by taking the union. □

Lemma 3.13. *If a set $X \subset (\Gamma \cup \{-\infty\})^n$ has closed and semilinear strata, then there exists a semialgebraic set $S \subset \mathscr{K}^n$ such that $\mathsf{val}(S) = X$. Even more, we can suppose that S is included in the nonnegative orthant of $\mathscr{K}^n$.*

Proof. By Lemma 3.12, X is a finite union of sets of the form

$$\mathcal{V} = \{y \in (\Gamma \cup \{-\infty\})^n : y_K \neq -\infty, y_{[n]\setminus K} = -\infty, \forall i = 1, \dots, p, f_i(y_K) \geqslant h^{(i)}\},$$

where, for every $L \subset [n]$, y_L denotes the vector formed by the coordinates of y taken from L, $f_i \in \mathbb{Z}[X_K]$ are homogeneous linear polynomials with integer coefficient and variables indexed by K, and $h^{(i)} \in \Gamma$. Take any such polynomials f_i and a set $K \subset [n]$. denote $f_i(y_K) = \sum_{k \in K} A_{ik} y_k$ where $A_{ik} \in \mathbb{Z}$ and consider the set

$$\mathcal{W}_g := \{x \in \mathscr{K}^n : x_K > 0, x_{[n]\setminus K} = 0 \wedge \forall i \in [p], \prod_{k \in K} x_k^{A_{ik}} \geqslant g^{(i)}\},$$

where $g = (g^{(i)})_i \in \mathscr{K}^p$. Note that $\mathcal{W}_g$ is a semialgebraic set. We will show that there exists g such that $\mathsf{val}(\mathcal{W}_g) = \mathcal{V}$. First, consider the case $\mathscr{K} = \mathbb{K}$. In this case, choosing $g^{(i)} = t^{h^{(i)}}$ gives the claim. Indeed, it is obvious that in this case we have $\mathsf{val}(\mathcal{W}_g) \subset \mathcal{V}$. Furthermore, if $y \in \mathcal{V}$ and we take $x_k := t^{y_k}$ for all $k \in [n]$ (with the convention that $t^{-\infty} = 0$), then we have $x \in \mathcal{W}$. To finish the claim, observe that the statement "for all $(h^{(i)})_{i=1}^p$, there exist $(g^{(i)})_{i=1}^p$ such that $\mathsf{val}(\mathcal{W}_g) = \mathcal{V}$" is a sentence in $\mathcal{L}_{\mathrm{rcvf}}$. Therefore, this sentence is true in any $\mathscr{K}$ by the completeness result of Theorem 2.120. The claim follows by taking the union. □

This finishes the proof of Theorem 3.1 and allows us to prove Theorem 3.4.

Proof of Theorem 3.1. The claim follows from Lemmas 3.6, 3.11 and 3.13. □

Proof of Theorem 3.4. Denote $\mathcal{S}^{\geqslant} := \{x \in \mathbb{R}^n : \forall i, P_i^+(x) \geqslant P_i^-(x)\}$ and suppose that $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}$. We have $\mathsf{val}(\boldsymbol{x}) \in \mathcal{S}^{\geqslant}$ by Lemma 2.52. Therefore $\mathsf{val}(\boldsymbol{\mathcal{S}}) \subset \mathcal{S}^{\geqslant}$. On the other hand, if we take any point x such that $P_i^+(x) > P_i^-(x)$ for all i, then any lift $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}_{>0}^n$ belongs to $\boldsymbol{\mathcal{S}}$ by Lemma 3.7. Hence, we have the inclusion

$$\{x \in \mathbb{R}^n : \forall i, P_i^+(x) > P_i^-(x)\} \subset \mathsf{val}(\boldsymbol{\mathcal{S}}) \subset \{x \in \mathbb{R}^n : \forall i, P_i^+(x) \geqslant P_i^-(x)\}$$

and the claim follows from Lemma 2.59 and Theorem 3.1. □

3.2 Puiseux series with rational exponents

In this section, we study in more detail the images by valuation of semialgebraic sets defined over the field of Puiseux series with rational exponents. Let us start by defining this field.

Definition 3.14. We define the set of *Puiseux series with rational exponents*, denoted $\mathbb{R}\{\{t\}\}$, in the following way. Let $\boldsymbol{x} = \sum_{i=1}^{\infty} c_{\lambda_i} t^{\lambda_i}$ be a Puiseux series as in (2.2). Then, $\boldsymbol{x}$ belongs to $\mathbb{R}\{\{t\}\}$ if there exists $N \in \mathbb{N}^*$ such that the sequence $(\lambda_i) \geqslant 1$ consists of rational numbers with common denominator N. Moreover, the empty series 0 belongs to $\mathbb{R}\{\{t\}\}$.

One can check that $\mathbb{R}\{\{t\}\}$ is a subfield of $\mathbb{K}$. Moreover, $\mathbb{R}\{\{t\}\}$ is a real closed field, and $(\mathscr{K}, \Gamma, \Bbbk, \mathsf{val}, \mathsf{ac}) = (\mathbb{R}\{\{t\}\}, \mathbb{Q}, \mathbb{R}, \mathsf{val}, \mathsf{lc})$ is a model of the theory $\mathrm{Th}_{\mathrm{rcvf}}$ of real closed valued fields (see Appendix A for an extended discussion). Many authors studying tropical geometry use the field $\mathbb{R}\{\{t\}\}$, often without the convergence assumption (or the analogues of these fields with complex coefficients), as the base field for their investigations. This is based on historical reasons (the history of the field $\mathbb{R}\{\{t\}\}$ can be traced back to Newton, see [BK12, Chapter 8.3]), the use of $\mathbb{R}\{\{t\}\}$ use in the study of plane algebraic curves [BK12, Chapter 8.3], the simplicity of definition, the fact that the field of formal Puiseux series with rational exponents and complex coefficients is the algebraic closure of the field of Laurent series [Eis04, Corollary 13.15], and the fact that $\mathbb{R}\{\{t\}\}$ is well adapted for computations [JMM08]. However, the disadvantage of $\mathbb{R}\{\{t\}\}$ lies in the fact that it has a rational value group. In this way, the image by valuation of a set defined over $\mathbb{R}\{\{t\}\}$ consists of only rational points, which makes the analysis somewhat less natural (one often studies the closure of the image by valuation). For this reasons, some authors [Mar10, EKL06] consider more general fields with real value group. We follow this convention in our work. The next result shows that this convention is richer—the set of images under the valuation map of semialgebraic sets defined over $\mathbb{R}\{\{t\}\}$ is a subset of the analogous set of images defined over $\mathbb{K}$ (up to taking the closure). Therefore, by studying these images over $\mathbb{K}$ we work with a larger class of sets than if we restrict our attention to $\mathbb{R}\{\{t\}\}$. Furthermore, the proposition implies that the results obtained over $\mathbb{K}$ can be transferred to $\mathbb{R}\{\{t\}\}$. In this way, one can use $\mathbb{K}$ to obtain theoretical results and $\mathbb{R}\{\{t\}\}$ for computations. The same result also follows from the analysis of Alessandrini [Ale13, Theorem 4.10].

Proposition 3.15. *Let $\phi(x_1, \dots, x_n, b_1, \dots, b_m)$ be an $\mathcal{L}_{\mathrm{or}}$-formula (where $n \geqslant 1, m \geqslant 0$). Fix a vector $\boldsymbol{b} = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_m) \in \mathbb{R}\{\{t\}\}^m$ and consider the semialgebraic set $\boldsymbol{\mathcal{S}} \subset \mathbb{R}\{\{t\}\}^n$ defined as*

$$\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{R}\{\{t\}\}^n \colon \mathbb{R}\{\{t\}\} \models \phi(\boldsymbol{x}, \boldsymbol{b})\} .$$

Let $\tilde{\boldsymbol{\mathcal{S}}} \subset \mathbb{K}^n$ be the extension of $\boldsymbol{\mathcal{S}}$ to $\mathbb{K}^n$, i.e., the semialgebraic set defined as

$$\tilde{\boldsymbol{\mathcal{S}}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n \colon \mathbb{K} \models \phi(\boldsymbol{x}, \boldsymbol{b})\} .$$

Let $S \coloneqq \mathsf{val}(\boldsymbol{\mathcal{S}}) \subset (\mathbb{Q} \cup \{-\infty\})^n$ and $\tilde{S} \coloneqq \mathsf{val}(\tilde{\boldsymbol{\mathcal{S}}}) \subset \mathbb{T}^n$ be the valuations of these sets. Then, for every nonempty set $K \subset [n]$ we have the equalities

$$\mathrm{cl}_{\mathbb{R}}(S_K) = \tilde{S}_K \quad \text{and} \quad S_K = \tilde{S}_K \cap \mathbb{Q}^K .$$

Moreover, we have $-\infty \in S$ if and only if $-\infty \in \tilde{S}$.

Proof. As noted above, both $\mathscr{K}_1 \coloneqq (\mathbb{R}\{\{t\}\}, \mathbb{Q}, \mathbb{R}, \mathsf{val}, \mathsf{lc})$ and $\mathscr{K}_2 \coloneqq (\mathbb{K}, \mathbb{R}, \mathbb{R}, \mathsf{val}, \mathsf{lc})$ are models of the theory $\mathrm{Th}_{\mathrm{rcvf}}$ of real closed valued fields. Moreover, $\mathscr{K}_1$ is a substructure of $\mathscr{K}_2$ (in the language $\mathcal{L}_{\mathrm{rcvf}}$ of real closed fields), and the embedding from $\mathscr{K}_1$ to $\mathscr{K}_2$ is given by the identity maps $\mathbb{R}\{\{t\}\} \to \mathbb{K}$, $\mathbb{Q} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}$. Therefore, by the fact that $\mathrm{Th}_{\mathrm{rcvf}}$ is model complete (Theorem 2.120 and Proposition 2.124), this embedding is elementary. Fix a nonempty set $K \subset [n]$, $K = \{k_1, \dots, k_p\}$.

First, we want to show that $S_K = \tilde{S}_K \cap \mathbb{Q}^n$. To do so, let us consider the $\mathcal{L}_{\mathrm{rcvf}}$-formula $\theta(y_{k_1}, \dots, y_{k_p}, b)$ defined as

$$\exists x_1, \dots, \exists x_n,\ \big(\bigwedge_{k \notin K} x_k = 0\big) \wedge \big(\bigwedge_{i \in [p]} y_{k_i} = \mathsf{val}(x_{k_i})\big) \wedge \phi(x_1, \dots, x_n, b_1, \dots, b_m) .$$

Fix any point $w = (w_{k_1}, \dots, w_{k_p}) \in \mathbb{Q}^K$. By model completeness, the formula $\theta(w, \boldsymbol{b})$ is true over $\mathbb{R}\{\{t\}\}$ if and only if it is true over $\mathbb{K}$. In other words, $w \in S_K \iff w \in \tilde{S}_K$. Hence $S_K \cap \mathbb{Q}^K = \tilde{S}_K \cap \mathbb{Q}^K$. Since $S_K \subset \mathbb{Q}^K$, we get $S_K = \tilde{S}_K \cap \mathbb{Q}^n$. The fact that $-\infty \in S \iff -\infty \in \tilde{S}$ can be proven in the same way (consider $K = \emptyset$ in the definition of θ).

To prove the other equality, recall that $\tilde{S}_K$ is closed in $\mathbb{R}^K$ by Theorem 3.1. Since $S_K \subset \tilde{S}_K$, we have $\mathrm{cl}_{\mathbb{R}}(S_K) \subset \tilde{S}_K$. To prove the opposite inclusion, consider the $\mathcal{L}_{\mathrm{rcvf}}$-formula $\psi(\check{y}, \hat{y}, b)$ defined as

$$\exists x_1, \dots, \exists x_n,\ \big(\bigwedge_{k \notin K} x_k = 0\big) \wedge \big(\bigwedge_{i \in [p]} \check{y}_{k_i} < \mathsf{val}(x_{k_i}) < \hat{y}_{k_i}\big) \wedge \phi(x_1, \dots, x_n, b_1, \dots, b_m)\,.$$

If $w = (w_{k_1}, \dots, w_{k_p}) \in \tilde{S}_K$ and $\check{w}, \hat{w} \in \mathbb{Q}^K$ are such that $U \coloneqq \prod_{i=1}^{p} (]\check{w}_{k_i}, \hat{w}_{k_i}[)$ is an open neighborhood of w defined by rational intervals, $w \in U$, then the formula $\psi(\check{w}, \hat{w}, \boldsymbol{b})$ is true over $\mathbb{K}$. By model completeness, it is also true over $\mathbb{R}\{\{t\}\}$. In other words, there is a point $w' \in S_K$ that belongs to U. Since U was arbitrary, we have $w \in \mathrm{cl}_{\mathbb{R}}(S_K)$. $\square$

Remark 3.16. As an immediate corollary of Proposition 3.15 we get the following statement: if $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is a semialgebraic set defined by parameters belonging to $\mathbb{R}\{\{t\}\}$ and $w \in \mathsf{val}(\boldsymbol{\mathcal{S}}) \cap \mathbb{Q}^n$ is a rational point, then there exists a lift of w with rational exponents, $\boldsymbol{w} \in \mathsf{val}^{-1}(w) \cap \mathbb{R}\{\{t\}\}^n$ such that $\boldsymbol{w} \in \boldsymbol{\mathcal{S}}$. In this sense, one could potentially use the field $\mathbb{R}\{\{t\}\}$ to compute the lifts of points belonging to $\mathsf{val}(\boldsymbol{\mathcal{S}})$. However, we do not address these issues in this work—we construct some explicit lifts of points for spectrahedra (see Example 8.76), but in our case these lifts do not require to perform computations over $\mathbb{R}\{\{t\}\}$ (all computations are done in the tropical semifield).

Remark 3.17. One can also consider the field of Puiseux series with rational exponents and algebraic coefficients—this field is obtained by adding a hypothesis that all the coefficients (c_{λ_i}) are algebraic numbers. This field is probably the most suited for computations. The analysis above applies also to this field.

CHAPTER 4

Tropical spectrahedra

In this chapter, we study the tropicalization of spectrahedra, which is the main subject of this thesis. We have already introduced the definition of a spectrahedron in Section 1.1 and indicated that this definition is valid over every real closed field. Let us start by giving more details on these issues.

Definition 4.1. The *Loewner order* is defined in the following way. If $A, B \in \mathbb{R}^{m \times m}$ are real symmetric matrices, then $A \succcurlyeq B$ if $A - B$ is positive semidefinite.

Definition 4.2. If $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{R}^{m \times m}$ are real symmetric matrices, then the set

$$\mathcal{S} \coloneqq \{x \in \mathbb{R}^n \colon Q^{(0)} + x_1 Q^{(1)} + \dots + x_n Q^{(n)} \succcurlyeq 0\} \, ,$$

is called a *spectrahedron (associated with $Q^{(0)}, \dots, Q^{(n)}$)*.

Remark 4.3. It is immediate to check that spectrahedra are convex.

Before proceeding, we point out that the notion of a "positive semidefinite" matrix is meaningful in every real closed field.

Definition 4.4. A symmetric matrix $\boldsymbol{A} \in \mathbb{K}^{m \times m}$ is *positive semidefinite* if every principal minor of $\boldsymbol{A}$ is nonnegative or, equivalently, if the inequality $\boldsymbol{x}^\intercal \boldsymbol{A} \boldsymbol{x} \geqslant 0$ is true for all $\boldsymbol{x} \in \mathbb{K}^m$.

Remark 4.5. The are many equivalent definitions of positive semidefinite matrices. For instance, a real symmetric matrix is positive semidefinite if it admits a Cholesky decomposition. This is equivalent to the nonnegativity of its principal minors, its smallest eigenvalue, and the associated quadratic form (see, e.g., [Mey00, Section 7.6]). All of these properties are still equivalent for symmetric matrices defined over arbitrary real closed fields, such as Puiseux series, by the completeness of the theory of real closed fields (Theorem 2.110).

The definition of positive semidefinite matrices over $\mathbb{K}$ allows us to consider spectrahedra over $\mathbb{K}^n$ and their tropicalizations. This leads to the main definition of this chapter.

Definition 4.6. A set $\mathcal{S} \subset \mathbb{T}^n$ is said to be a *tropical spectrahedron* if there exists a spectrahedron $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ such that $\mathcal{S} = \mathsf{val}(\boldsymbol{\mathcal{S}})$. We refer to $\mathcal{S}$ as the *tropicalization* of the spectrahedron $\boldsymbol{\mathcal{S}}$, and $\boldsymbol{\mathcal{S}}$ is said to be a *lift (over the field* $\mathbb{K}$*)* of $\mathcal{S}$.

Given symmetric matrices $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ and $\boldsymbol{x} \in \mathbb{K}^n$, we denote by $\boldsymbol{Q}(\boldsymbol{x})$ the matrix pencil $\boldsymbol{Q}^{(0)} + \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)}$. By the definition of a positive semidefinite matrix, the spectrahedron $\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ can be described by a system of polynomial inequalities of the form $\det \boldsymbol{Q}_{I\times I}(\boldsymbol{x}) \geqslant 0$, where I is a nonempty subset of $[m]$, and $\det \boldsymbol{Q}_{I\times I}(\boldsymbol{x})$ corresponds to the $(I \times I)$-minor of the matrix $\boldsymbol{Q}(\boldsymbol{x})$. Following this, we obtain that the tropical spectrahedron $\mathcal{S}$ is included in the intersection of the sets $\{x \in \mathbb{T}^n \colon \operatorname{trop}(\boldsymbol{P})^+(x) \geqslant \operatorname{trop}(\boldsymbol{P})^-(x)\}$, where $\boldsymbol{P}$ is a polynomial of the form $\det \boldsymbol{Q}_{I\times I}(\boldsymbol{x})$ (Lemma 2.52). In general, this inclusion may be strict. We refer to [ABGJ15, Example 15] for an example in which $\boldsymbol{\mathcal{S}}$ is a polyhedron. Nevertheless, under the regularity assumption stated in Theorem 3.4, both sets coincide. In fact, we prove that, under similar assumptions, tropical spectrahedra have a description that is much simpler than the one provided by Theorem 3.4. This description only involves principal tropical minors of order 2.

Our results are divided into four parts. In Section 4.1 we deal with spectrahedra defined by Metzler matrices $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)}$ (i.e., matrices in which the off-diagonal entries are nonpositive). This enables us to use a lemma that is similar to Theorem 3.4 in order to give a description of tropical spectrahedra under a regularity assumption. In Section 4.2 we switch to non-Metzler matrices. In this case, tropical spectrahedra may not be regular, even under strong genericity assumptions. Nevertheless, we are able to extend our previous analysis to this case and give a description, involving only principal minors of size at most 2, of non-Metzler spectrahedra, under a regularity assumption over some associated sets. The purpose of Section 4.3 is to show that the regularity assumptions used in Sections 4.1 and 4.2 hold generically. Finally, in Section 4.4 we study the images by valuation of the interiors of spectrahedra. This chapter is based on the preprint [AGS16b].

Let us start with some introductory remarks. First, observe that in order to characterize the class of tropical spectrahedra, it is enough to restrict ourselves to tropical spectrahedral cones, as the image of a spectrahedron can be deduced from the image of its homogenized version. This is formally stated in the next lemma.

Lemma 4.7. *Let $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ be a sequence of symmetric matrices. Define*

$$\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)} \succcurlyeq 0\}$$

and

$$\boldsymbol{\mathcal{S}}^h \coloneqq \{(\boldsymbol{x}_0, \boldsymbol{x}) \in \mathbb{K}^{n+1}_{\geqslant 0} \colon \boldsymbol{x}_0\boldsymbol{Q}^{(0)} + \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)} \succcurlyeq 0\}\,.$$

Then

$$\mathsf{val}(\boldsymbol{\mathcal{S}}) = \pi(\{x \in \mathsf{val}(\boldsymbol{\mathcal{S}}^h) \colon x_0 = 0\})\,,$$

where $\pi \colon \mathbb{T}^{n+1} \to \mathbb{T}^n$ denotes the projection that forgets the first coordinate.

Proof. We start by proving the inclusion $\subset$. Take any $x \in \mathsf{val}(\boldsymbol{\mathcal{S}})$ and its lift $\boldsymbol{x} \in \boldsymbol{\mathcal{S}} \cap \mathsf{val}^{-1}(x)$. Observe that the point $(1, \boldsymbol{x})$ belongs to $\boldsymbol{\mathcal{S}}^h$. Therefore, the point $(0, x)$ belongs to $\mathsf{val}(\boldsymbol{\mathcal{S}}^h)$ and x belongs to $\pi(\{x \in \mathsf{val}(\boldsymbol{\mathcal{S}}^h) \colon x_0 = 0\})$. Conversely, let x belong to $\pi(\{x \in \mathsf{val}(\boldsymbol{\mathcal{S}}^h) \colon x_0 = 0\})$. Then $(0, x)$ belongs to $\mathsf{val}(\boldsymbol{\mathcal{S}}^h)$. In other words, there exists a lift $(\boldsymbol{z}, \boldsymbol{x}) \in \boldsymbol{\mathcal{S}}^h$ such that $\mathsf{val}(\boldsymbol{z}) = 0$ and $\mathsf{val}(\boldsymbol{x}) = x$. Take the point $(1, \boldsymbol{x}/\boldsymbol{z})$. This point also belongs to $\boldsymbol{\mathcal{S}}^h$. Moreover, $\boldsymbol{x}/\boldsymbol{z}$ belongs to $\boldsymbol{\mathcal{S}}$. Hence, the point $x = \mathsf{val}(\boldsymbol{x}/\boldsymbol{z})$ belongs to $\mathsf{val}(\boldsymbol{\mathcal{S}})$. $\square$

Our approach to the tropicalization of spectrahedra relies on the next elementary lemma. This lemma is a more precise version of a result of Yu [Yu15] who showed that the set of images by the valuation of the set of positive semidefinite matrices $\boldsymbol{A}$ over the field of Puiseux series is determined by the inequalities $\mathsf{val}(\boldsymbol{A}_{ii}) + \mathsf{val}(\boldsymbol{A}_{jj}) \geqslant 2\,\mathsf{val}(\boldsymbol{A}_{ij})$ for $i \neq j$.

Lemma 4.8. *Let $\boldsymbol{A} \in \mathbb{K}^{m \times m}$ be a symmetric matrix. Suppose that $\boldsymbol{A}$ has nonnegative entries on its diagonal and that the inequality $\boldsymbol{A}_{ii}\boldsymbol{A}_{jj} \geqslant (m-1)^2 \boldsymbol{A}_{ij}^2$ holds for all pairs (i, j) such that $i \neq j$. Then, $\boldsymbol{A}$ is positive semidefinite.*

The proof of Lemma 4.8 uses a well-known result from linear algebra.

Lemma 4.9 ([BSM03, Proposition 1.8])**.** *Suppose that a symmetric matrix $A \in \mathbb{R}^{m \times m}$ has nonnegative entries on its diagonal and is diagonally dominant, i.e., that it satisfies the inequality $A_{ii} \geqslant \sum_{j \neq i} |A_{ij}|$ for all $i \in [m]$. Then, A is positive semidefinite.*

Proof of Lemma 4.8. We will show that the lemma is true over any real closed field $\mathscr{K}$. Consider the case $\mathscr{K} = \mathbb{R}$. If $A \in \mathbb{R}^{m \times m}$ is a zero matrix, then there is nothing to show. From now on we suppose that A has at least one nonzero entry. First, let us suppose that A has positive entries on its diagonal. In this case, let $B \in \mathbb{R}^{m \times m}$ be the diagonal matrix defined as $B_{ii} := A_{ii}^{-1/2}$ for all i. Observe that A is positive semidefinite if and only if the matrix $D := BAB$ is positive semidefinite. Moreover, D has ones on its diagonal and $D_{ij} = A_{ii}^{-1/2} A_{ij} A_{jj}^{-1/2}$ for all $i \neq j$. Hence $|D_{ij}| \leqslant 1/(m-1)$ for all $i \neq j$. Therefore D is diagonally dominant and hence positive semidefinite by Lemma 4.9.

Second, if A has some zeros on its diagonal, let $I = \{i \in [m] \colon A_{ii} \neq 0\}$. Since the inequality $A_{ii}A_{jj} \geqslant (m-1)^2 A_{ij}^2$ holds, we have $A_{ij} = 0$ if either $i \notin I$ or $j \notin I$. Let A_I denote the submatrix formed by the rows and columns with indices from I. Then A is positive semidefinite if and only if A_I is positive semidefinite. Finally, A_I is positive semidefinite by the considerations from the previous paragraph.

This shows the claim for $\mathscr{K} = \mathbb{R}$. To finish the proof, observe that for every fixed m, the claim is a sentence in $\mathcal{L}_{\mathrm{or}}$. Therefore, it is true in $\mathscr{K}$ by the completeness result of Theorem 2.110. $\square$

Lemma 4.8 has two useful corollaries. The first one allows us to give an inner and an outer approximation of spectrahedra using only minors of order two. Given a spectrahedron $\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ we define two sets $\boldsymbol{\mathcal{S}}^{\mathrm{out}}, \boldsymbol{\mathcal{S}}^{\mathrm{in}} \subset \mathbb{K}^n_{\geqslant 0}$ as

$$\boldsymbol{\mathcal{S}}^{\mathrm{out}} := \Big\{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0\,, \forall i \neq j, \boldsymbol{Q}_{ii}(\boldsymbol{x})\boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant (\boldsymbol{Q}_{ij}(\boldsymbol{x}))^2\Big\}\,,$$
$$\boldsymbol{\mathcal{S}}^{\mathrm{in}} := \Big\{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0\,, \forall i \neq j, \boldsymbol{Q}_{ii}(\boldsymbol{x})\boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{x}))^2\Big\}\,.$$

Corollary 4.10. *We have $\boldsymbol{\mathcal{S}}^{\mathrm{in}} \subset \boldsymbol{\mathcal{S}} \subset \boldsymbol{\mathcal{S}}^{\mathrm{out}}$.*

Proof. Definition 4.4 gives $\boldsymbol{\mathcal{S}} \subset \boldsymbol{\mathcal{S}}^{\mathrm{out}}$, while Lemma 4.8 shows that $\boldsymbol{\mathcal{S}}^{\mathrm{in}} \subset \boldsymbol{\mathcal{S}}$. $\square$

In the sequel, in order to describe the set $\mathsf{val}(\boldsymbol{\mathcal{S}})$, we will exhibit conditions that ensure that the tropicalizations of $\boldsymbol{\mathcal{S}}^{\text{in}}$ and $\boldsymbol{\mathcal{S}}^{\text{out}}$ coincide, i.e., $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{out}}) = \mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{in}})$.

The second corollary concern the symmetric tropical matrices that have positive semidefinite lifts.

Corollary 4.11. *Let $A \in \mathbb{T}_{\pm}^{m\times m}$ be a symmetric matrix such that $A_{ii} \in \mathbb{T}_{+} \cup \{-\infty\}$ for all i and $A_{ii} \odot A_{jj} > A_{ij}^{\odot 2}$ for all $i < j$ such that $A_{ij} \neq -\infty$. Let $\boldsymbol{A} \in \mathbb{K}^{m\times m}$ be any symmetric matrix such that $\mathsf{sval}(\boldsymbol{A}) = A$. Then $\boldsymbol{A}$ fulfills the conditions of Lemma 4.8. (In particular, it is positive semidefinite.)*

Proof. Since $A_{ii} \in \mathbb{T}_{+} \cup \{-\infty\}$ for all i, we have $\boldsymbol{A}_{ii} \geqslant 0$ for all i. Moreover, if $A_{ij} = -\infty$, then $\boldsymbol{A}_{ii}\boldsymbol{A}_{jj} \geqslant 0$ and if $A_{ij} \neq -\infty$, then $\mathsf{val}(\boldsymbol{A}_{ii}\boldsymbol{A}_{jj}) = A_{ii} \odot A_{jj} > A_{ij}^{\odot 2} = \mathsf{val}((m-1)^2 \boldsymbol{A}_{ij}^2)$. Therefore $\boldsymbol{A}$ fulfills the conditions of Lemma 4.8. □

4.1 Tropical Metzler spectrahedra

In this section, we study the spectrahedra defined by Metzler matrices. First, we exhibit a class of tropical spectrahedra that arise in this way (we call this class tropical Metzler spectrahedra). Second, we show that under a genericity condition, an image of a spectrahedron defined by Metzler matrices is a tropical Metzler spectrahedron that has a natural description in terms of 2×2 tropical minors.

Definition 4.12. A square matrix $\boldsymbol{A} \in \mathbb{K}^{m\times m}$ is a *(negated) Metzler matrix* if its off-diagonal coefficients are nonpositive. Similarly, we say that a matrix $M \in \mathbb{T}_{\pm}^{m\times m}$ is a *tropical Metzler matrix* if $M_{ij} \in \mathbb{T}_{-} \cup \{-\infty\}$ for all $i \neq j$.

Let $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$ be symmetric tropical Metzler matrices. Given $i, j \in [m]$, we refer to $Q_{ij}(X)$ as the tropical polynomial:

$$Q_{ij}(X) \coloneqq Q_{ij}^{(1)} \odot X_1 \oplus \dots \oplus Q_{ij}^{(n)} \odot X_n \, .$$

Definition 4.13. If $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$ are symmetric tropical Metzler matrices, we define the *tropical Metzler spectrahedral cone* $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ described by $Q^{(1)}, \dots, Q^{(n)}$ as the set of points $x \in \mathbb{T}^n$ that fulfill the following two conditions:

- for all $i \in [m]$, $Q_{ii}^{+}(x) \geqslant Q_{ii}^{-}(x)$;
- for all $i, j \in [m]$, $i < j$, $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (Q_{ij}(x))^{\odot 2}$.

We point out that the term $Q_{ij}(x)$ ($i \neq j$) is well defined for any $x \in \mathbb{T}^n$ thanks to the Metzler property of the matrices $Q^{(k)}$. Where there is no ambiguity, we denote $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ by $\mathcal{S}$. Furthermore, if $n \geqslant 2$, then we denote by $\mathcal{S}(Q^{(1)} | Q^{(2)}, \dots, Q^{(n)})$ the set of all points $x \in \mathbb{T}^{n-1}$ such that $(0, x) \in \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$. The sets $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ and $\mathcal{S}(Q^{(1)} | Q^{(2)}, \dots, Q^{(n)})$ are called *tropical Metzler spectrahedra*.

With standard notation, the constraints defining a tropical Metzler spectrahedral cone respectively read: for all $i \in [m]$,

$$\max_{Q_{ii}^{(k)} \in \mathbb{T}_{+}} (Q_{ii}^{(k)} + x_k) \geqslant \max_{Q_{ii}^{(l)} \in \mathbb{T}_{-}} (|Q_{ii}^{(l)}| + x_l) \, , \tag{4.1}$$

and for all $i, j \in [m]$ such that $i < j$,

$$\max_{Q_{ii}^{(k)} \in \mathbb{T}_+} (Q_{ii}^{(k)} + x_k) + \max_{Q_{jj}^{(k')} \in \mathbb{T}_+} (Q_{jj}^{(k')} + x_{k'}) \geqslant 2 \max_{l \in [n]} (|Q_{ij}^{(l)}| + x_l) \,. \tag{4.2}$$

The next proposition justifies the terminology introduced in Definition 4.13, and ensures that the set $\mathcal{S}$ is indeed a tropical spectrahedron. To this end, we explicitly construct a spectrahedron $\boldsymbol{\mathcal{S}} \subset \mathbb{K}_{\geqslant 0}^n$ verifying $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$.

Proposition 4.14. *The sets $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ and $\mathcal{S}(Q^{(1)} | Q^{(2)}, \dots, Q^{(n)})$ are tropical spectrahedra.*

Proof. By Lemma 4.7, it is enough to prove the claim for $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$. Let us define the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ as follows:

- if $Q_{ij}^{(k)} \in \mathbb{T}_-$, then we set $\boldsymbol{Q}_{ij}^{(k)} \coloneqq -t^{|Q_{ij}^{(k)}|}$;
- if $Q_{ij}^{(k)} \in \mathbb{T}_+$ (which, under our assumptions, can happen only if $i = j$), then $\boldsymbol{Q}_{ij}^{(k)} \coloneqq mnt^{Q_{ij}^{(k)}}$;
- if $Q_{ij}^{(k)} = -\infty$, then $\boldsymbol{Q}_{ij}^{(k)} \coloneqq 0$.

Consider the spectrahedron $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}_{\geqslant 0}^n \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$. We claim that $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$.

We start with the inclusion $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{out}}) \subset \mathcal{S}$. Let $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{\text{out}}$. For all $i \neq j$ we have $\boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0$, $\boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant 0$, and $\boldsymbol{Q}_{ii}(\boldsymbol{x}) \boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant (\boldsymbol{Q}_{ij}(\boldsymbol{x}))^2$. Therefore, $\boldsymbol{Q}_{ii}^+(\boldsymbol{x}) \geqslant \boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0$ and $\boldsymbol{Q}_{jj}^+(\boldsymbol{x}) \geqslant \boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant 0$. This implies that $\boldsymbol{Q}_{ii}^+(\boldsymbol{x}) \boldsymbol{Q}_{jj}^+(\boldsymbol{x}) \geqslant \boldsymbol{Q}_{ii}(\boldsymbol{x}) \boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant (\boldsymbol{Q}_{ij}(\boldsymbol{x}))^2$. Moreover, we have $\mathsf{val}(\boldsymbol{Q}_{ii}^+(\boldsymbol{x})) = Q_{ii}^+(x)$, where $x = \mathsf{val}(\boldsymbol{x})$. Similarly, $\mathsf{val}(\boldsymbol{Q}_{ii}^-(\boldsymbol{x})) = Q_{ii}^-(x)$. As $Q^{(k)}$ are tropical Metzler matrices, we also have $\mathsf{val}(\boldsymbol{Q}_{ij}(\boldsymbol{x})) = |Q_{ij}(x)|$ for $i \neq j$. Since the map val is order preserving over $\mathbb{K}_{\geqslant 0}$, we deduce that $Q_{ii}^+(x) = \mathsf{val}(\boldsymbol{Q}_{ii}^+(\boldsymbol{x})) \geqslant \mathsf{val}(\boldsymbol{Q}_{ii}^-(\boldsymbol{x})) = Q_{ii}^-(x)$ for all i and $Q_{ii}^+(x) \odot Q_{jj}^+(x) = \mathsf{val}(\boldsymbol{Q}_{ii}^+(\boldsymbol{x}) \boldsymbol{Q}_{jj}^+(\boldsymbol{x})) \geqslant \mathsf{val}(\boldsymbol{Q}_{ij}(\boldsymbol{x})^2) = (Q_{ij}(x))^{\odot 2}$ for all $i < j$. Hence $x \in \mathcal{S}$.

Now, let us prove the inclusion $\mathcal{S} \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{in}})$. Take any $x \in \mathcal{S}$ and its lift $\boldsymbol{x}_k = t^{x_k}$, with the convention that $t^{-\infty} = 0$. First, as noted in the previous paragraph, we have $\mathsf{val}(\boldsymbol{Q}_{ii}^+(\boldsymbol{x})) = Q_{ii}^+(x)$ and $\mathsf{val}(\boldsymbol{Q}_{ii}^-(\boldsymbol{x})) = Q_{ii}^-(x)$. We have chosen the matrices $\boldsymbol{Q}^{(k)}$ and the point $\boldsymbol{x}$ in such a way that

$$\boldsymbol{Q}_{ii}^+(\boldsymbol{x}) = \sum_{Q_{ii}^{(k)} \in \mathbb{T}^+} mnt^{Q_{ii}^{(k)} + x_k} \geqslant mnt^{Q_{ii}^+(x)} \,. \tag{4.3}$$

Similarly, we have $\boldsymbol{Q}_{ii}^-(\boldsymbol{x}) = \sum_{Q_{ii}^{(k)} \in \mathbb{T}^-} t^{|Q_{ii}^{(k)}| + x_k} \leqslant nt^{Q_{ii}^-(x)}$. Since $Q_{ii}^+(x) \geqslant Q_{ii}^-(x)$, we deduce that $\boldsymbol{Q}_{ii}^-(\boldsymbol{x}) \leqslant \frac{1}{m} \boldsymbol{Q}_{ii}^+(\boldsymbol{x})$, and so $\boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant (1 - \frac{1}{m}) \boldsymbol{Q}_{ii}^+(\boldsymbol{x}) \geqslant 0$. Second, for all $i \neq j$ we have $0 \geqslant \boldsymbol{Q}_{ij}(\boldsymbol{x}) \geqslant -nt^{|Q_{ij}(x)|}$. Using (4.3) and the fact that $Q_{ii}^+(x) \odot Q_{jj}^+(x) \geqslant (Q_{ij}(x))^{\odot 2}$, we obtain

$$\boldsymbol{Q}_{ij}(\boldsymbol{x})^2 \leqslant n^2 t^{Q_{ii}^+(x) \odot Q_{jj}^+(x)} \leqslant \frac{1}{m^2} \boldsymbol{Q}_{ii}^+(\boldsymbol{x}) \boldsymbol{Q}_{jj}^+(\boldsymbol{x}) \,.$$

Therefore, by the previous inequalities,

$$\boldsymbol{Q}_{ii}(\boldsymbol{x}) \boldsymbol{Q}_{jj}(\boldsymbol{x}) - (m-1)^2 \boldsymbol{Q}_{ij}(\boldsymbol{x})^2 \geqslant \Big(1 - \frac{1}{m}\Big)^2 \boldsymbol{Q}_{ii}^+(\boldsymbol{x}) \boldsymbol{Q}_{jj}^+(\boldsymbol{x}) - (m-1)^2 \boldsymbol{Q}_{ij}(\boldsymbol{x})^2 \geqslant 0 \,.$$

Hence $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{\text{in}}$. Therefore, by Corollary 4.10 we have $\mathsf{val}(\boldsymbol{\mathcal{S}}) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{out}}) \subset \mathcal{S} \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{in}}) \subset \mathsf{val}(\boldsymbol{\mathcal{S}})$, which implies that $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$. ☐

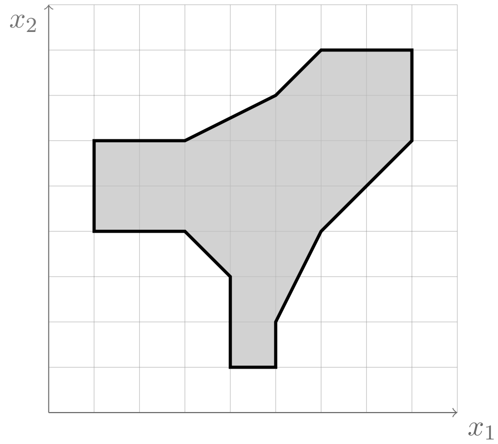

Figure 4.1: A tropical Metzler spectrahedron.

Example 4.15. If $A^{(1)}, \dots, A^{(p)}$ are matrices, then $\mathrm{tdiag}(A^{(1)}, \dots, A^{(p)})$ refers to the block diagonal matrix with blocks $A^{(s)}$ on the diagonal and all other entries equal to $-\infty$. Let $Q^{(0)}, Q^{(1)}, Q^{(2)} \in \mathbb{T}_{\pm}^{9\times 9}$ be symmetric tropical Metzler matrices defined as follows:

$$Q^{(0)} := \mathrm{tdiag}\Big(8, \ominus 1, \ominus 1, \begin{bmatrix} -\infty & \ominus 3 \\ \ominus 3 & -\infty \end{bmatrix}, \begin{bmatrix} 2 & -\infty \\ -\infty & 8 \end{bmatrix}, \begin{bmatrix} 3 & -\infty \\ -\infty & 9 \end{bmatrix}\Big),$$
$$Q^{(1)} := \mathrm{tdiag}\Big(\ominus 0, 0, -\infty, \begin{bmatrix} 0 & -\infty \\ -\infty & -2 \end{bmatrix}, \begin{bmatrix} -\infty & \ominus 0 \\ \ominus 0 & -\infty \end{bmatrix}, \begin{bmatrix} 0 & -\infty \\ -\infty & 4 \end{bmatrix}\Big),$$
$$Q^{(2)} := \mathrm{tdiag}\Big(\ominus 0, -\infty, 0, \begin{bmatrix} -1 & -\infty \\ -\infty & -1 \end{bmatrix}, \begin{bmatrix} 0 & -\infty \\ -\infty & 4 \end{bmatrix}, \begin{bmatrix} -\infty & \ominus 0 \\ \ominus 0 & -\infty \end{bmatrix}\Big).$$

The tropical Metzler spectrahedron $\mathcal{S}(Q^{(0)}|Q^{(1)}, Q^{(2)})$ is depicted in Fig. 4.1.

We now focus on the main problem of characterizing the image by the valuation of a spectrahedron defined by Metzler matrices. Our goal is to show that any spectrahedral cone $\boldsymbol{\mathcal{S}} := \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} : \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ verifying $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$ is mapped to the tropical Metzler spectrahedral cone $\mathcal{S}$, provided that some assumptions related to the genericity of the matrices $Q^{(k)}$ and the regularity of the set $\mathcal{S}$ hold. We start by giving a family of inner approximations of the set $\mathsf{val}(\boldsymbol{\mathcal{S}})$.

Definition 4.16. For any $\lambda \in \mathbb{R}$ we denote by $\mathcal{S}_\lambda$ the set of all points $x \in \mathbb{T}^n$ verifying

- for all $i \in [m]$, $Q^+_{ii}(x) \geqslant \lambda \odot Q^-_{ii}(x)$;
- for all $i, j \in [m]$, $i < j$, $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2}$.

Observe that we have $\mathcal{S}_0 = \mathcal{S}$. Furthermore, we denote $\mathcal{T} = \bigcup_{\lambda > 0} \mathcal{S}_\lambda \subset \mathbb{T}^n$. We use the notation $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ and $\mathcal{T}(Q^{(1)}, \dots, Q^{(n)})$ when we want to emphasize the dependence on $Q^{(1)}, \dots, Q^{(n)}$.

Lemma 4.17. *Let $\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} : \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ be any spectrahedral cone such that $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$. If $x \in \mathcal{S}_\lambda$ for some $\lambda > 0$ and $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}^n_{\geqslant 0}$ is any lift of x, then $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{\mathrm{in}}$. Furthermore, we have $\mathrm{cl}_{\mathbb{T}}(\mathcal{T}) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})$.*

Proof. Fix any $x \in \mathcal{S}_\lambda$ and take any lift $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}^n_{\geqslant 0}$. Let $A := \mathsf{sval}(\boldsymbol{Q}(\boldsymbol{x}))$. For any i such that Q^-_{ii} is nonzero we have $\mathsf{val}(\boldsymbol{Q}^+_{ii}(\boldsymbol{x})) = Q^+_{ii}(x) \geqslant \lambda \odot Q^-_{ii}(x) = \lambda \odot \mathsf{val}(\boldsymbol{Q}^-_{ii}(\boldsymbol{x})) > \mathsf{val}(\boldsymbol{Q}^-_{ii}(\boldsymbol{x}))$.

Therefore $A_{ii} = \mathsf{sval}(\boldsymbol{Q}_{ii}(\boldsymbol{x})) = Q_{ii}^+(x) \in \mathbb{T}_+ \cup \{-\infty\}$ for all i (even if Q_{ii}^- is zero). Furthermore, we have $A_{ij} = Q_{ij}(x)$ for any $i < j$. Therefore, for any $i < j$ such that $A_{ij} \neq -\infty$, we have $A_{ii} \odot A_{jj} = Q_{ii}^+(x) \odot Q_{jj}^+(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2} = (\lambda \odot A_{ij})^{\odot 2} > A_{ij}^{\odot 2}$. Hence, by Corollary 4.11, we have $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{\mathrm{in}}$. Moreover, this shows that $\mathcal{S}_\lambda \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})$ for all $\lambda > 0$. Hence $\mathcal{T} \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})$. Since the set $\boldsymbol{\mathcal{S}}^{\mathrm{in}}$ is a closed subset $\mathbb{K}^n_{\geqslant 0}$, $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})$ is a closed subset of $\mathbb{T}^n$ by Theorem 3.1. Therefore $\mathrm{cl}_{\mathbb{T}}(\mathcal{T}) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})$. □

We prove a result describing, under genericity assumption, the tropicalization of the spectrahedral cone restricted to the open positive orthant $\mathbb{K}^n_{>0}$.

Assumption A. We suppose that for every matrix $Q^{(k)}$ and every pair $i \neq j$ such that $Q_{ii}^{(k)}$ and $Q_{jj}^{(k)}$ belong to $\mathbb{T}_+$ the inequality $Q_{ii}^{(k)} + Q_{jj}^{(k)} \neq 2|Q_{ij}^{(k)}|$ holds.

We point out that Assumption A can be interpreted in terms of the nonsingularity of some (tropical) minors of order 2 of the matrices $Q^{(k)}$.

Remark 4.18. Note that every tropical Metzler spectrahedral cone can be described by matrices that satisfy Assumption A. Indeed, if we take matrices $Q^{(1)}, \dots, Q^{(n)}$ such that $Q_{ii}^{(k)} + Q_{jj}^{(k)} = 2|Q_{ij}^{(k)}|$, then we can replace the entry $Q_{ij}^{(k)}$ by $-\infty$ and this does not change the associated tropical Metzler spectrahedral cone.

Theorem 4.19. *Let $\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ be a spectrahedral cone described by Metzler matrices $\boldsymbol{Q}^{(1)}$, ..., $\boldsymbol{Q}^{(n)}$ such that $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$. Suppose that Assumption A holds and that the set $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is regular. Then*

$$\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n \,.$$

Proof. Let $\mathcal{T}$ be defined as in Lemma 4.17. The same arguments as in the proof of Proposition 4.14 show that $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{out}}) \subset \mathcal{S}$. Therefore $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{out}} \cap \mathbb{K}^n_{>0}) \subset \mathcal{S} \cap \mathbb{R}^n$. Then, by Corollary 4.10 and Lemma 4.17, it is enough to show that $\mathrm{cl}_{\mathbb{R}}(\mathcal{T} \cap \mathbb{R}^n) = \mathcal{S} \cap \mathbb{R}^n$. Observe that for all $\lambda \in \mathbb{R}$, the inequalities defining $\mathcal{S}_\lambda$ such that Q_{ij}^- (where $i \leqslant j$) is the zero tropical polynomial are trivially satisfied. Therefore, $\mathcal{S}_\lambda$ can be expressed as the set of points $x \in \mathbb{T}^n$ verifying:

- for all $i \in [m]$ such that Q_{ii}^- is nonzero, $Q_{ii}^+(x) \geqslant \lambda \odot Q_{ii}^-(x)$;
- for all $i, j \in [m]$, $i < j$, such that Q_{ij} (or, equivalently, Q_{ij}^-) is nonzero, $Q_{ii}^+(x) \odot Q_{jj}^+(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2}$.

This implies that $\mathcal{T} \cap \mathbb{R}^n$ is the set of points $x \in \mathbb{R}^n$ satisfying:

- for all $i \in [m]$ such that Q_{ii}^- is nonzero, $Q_{ii}^+(x) > Q_{ii}^-(x)$;
- for all $i, j \in [m]$, $i < j$, such that Q_{ij} (or, equivalently, Q_{ij}^-) is nonzero, $Q_{ii}^+(x) \odot Q_{jj}^+(x) > (Q_{ij}(x))^{\odot 2}$.

We denote by $\varXi$ the set of $(i, j) \in [m] \times [m]$ such that $i \leqslant j$ and Q_{ij}^- is nonzero. Since $\mathcal{S} \cap \mathbb{R}^n$ is supposed to be regular, we propose to use Lemma 2.59, and thus, to exhibit nonzero tropical polynomials P_{ij} such that

$$\mathcal{S} \cap \mathbb{R}^n = \{x \in \mathbb{R}^n \colon \forall (i,j) \in \varXi, P_{ij}^+(x) \geqslant P_{ij}^-(x)\} \,, \tag{4.4}$$

and

$$\mathcal{T} \cap \mathbb{R}^n = \{x \in \mathbb{R}^n \colon \forall (i,j) \in \varXi, P_{ij}^+(x) > P_{ij}^-(x)\} \,. \tag{4.5}$$

In other words, we want to express the inequalities of the form $Q_{ii}^+(x) > Q_{ii}^-(x)$ and $Q_{ii}^+(x) \odot Q_{jj}^+(x) > (Q_{ij}(x))^{\odot 2}$ as tropical polynomial inequalities in which no term appears both on the left- and on the right-hand side. The inequalities of the first kind already satisfy this condition, and it suffices to set $P_{ii} \coloneqq Q_{ii}$ for all $i \in [m]$ such that $(i,i) \in \Xi$. In contrast, we have to transform the inequalities of the second kind into equivalent contraints of the form $P^+_{ij}(x) > P^-_{ij}(x)$, where $(i,j) \in \Xi$ and $i < j$. To this end, we use Assumption A. First, observe that the functions $(Q_{ij}(x))^{\odot 2}$ and $\bigoplus_k (Q^{(k)}_{ij} \odot x_k)^{\odot 2}$ are equal. Therefore, we can replace the inequalities $Q^+_{ii}(x) \odot Q^+_{jj}(x) > (Q_{ij}(x))^{\odot 2}$ by $Q^+_{ii}(x) \odot Q^+_{jj}(x) > \bigoplus_k (Q^{(k)}_{ij} \odot x_k)^{\odot 2}$. Now, we can define a formal subtraction of these tropical expressions. More precisely, for every $(i,j) \in \Xi$ such that $i < j$, we define

$$P_{ij} \coloneqq \Bigg(\bigoplus_{\substack{k \neq l \\ Q^{(k)}_{ii}, Q^{(l)}_{jj} \in \mathbb{T}_+}} (Q^{(k)}_{ii} \odot Q^{(l)}_{jj}) \odot (X_k \odot X_l) \Bigg) \oplus \Bigg(\bigoplus_{\substack{Q^{(k)}_{ii}, Q^{(k)}_{jj} \in \mathbb{T}_+ \\ \text{or } |Q^{(k)}_{ij}| \neq -\infty}} \alpha_k \odot X_k^{\odot 2} \Bigg) ,$$

where α_k is given by:

$$\alpha_k \coloneqq \begin{cases} Q^{(k)}_{ii} \odot Q^{(k)}_{jj} & \text{if } Q^{(k)}_{ii}, Q^{(k)}_{jj} \in \mathbb{T}_+ \text{ and } Q^{(k)}_{ii} + Q^{(k)}_{jj} > 2|Q^{(k)}_{ij}| , \\ \ominus (Q^{(k)}_{ij})^{\odot 2} & \text{otherwise.} \end{cases}$$

Recall that any inequality of the form $\max(x, \alpha + y) > \max(x', \beta + y)$ is equivalent to $\max(x, \alpha + y) > x'$ if $\alpha > \beta$, and to $x > \max(x', \beta + y)$ if $\beta > \alpha$. Therefore, Assumption A ensures that $P^+_{ij}(x) > P^-_{ij}(x)$ is equivalent to $Q^+_{ii}(x) \odot Q^+_{jj}(x) > \bigoplus_k (Q^{(k)}_{ij} \odot x_k)^{\odot 2}$. The same applies to the nonstrict counterparts of these inequalities. We conclude that (4.4) and (4.5) are satisfied. $\square$

Theorem 4.20. *Let $\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ be a spectrahedral cone described by Metzler matrices $\boldsymbol{Q}^{(1)}$, …, $\boldsymbol{Q}^{(n)}$ such that $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$. Suppose that Assumption A is fulfilled and that every stratum of the set $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ is regular. Then*

$$\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) .$$

Proof. It it clear that $\mathsf{val}(\boldsymbol{\mathcal{S}})$ and $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ contain the point $-\infty$. Fix a nonempty subset $K \subset [n]$. Observe that the stratum of $\mathsf{val}(\boldsymbol{\mathcal{S}})$ associated with K is equal to $\mathsf{val}(\boldsymbol{\mathcal{S}}^{(K)} \cap \mathbb{K}^K_{>0})$, where $\boldsymbol{\mathcal{S}}^{(K)}$ is the spectrahedral cone described by $(\boldsymbol{Q}^{(k)})_{k \in K}$. Similarly, the stratum of $\mathcal{S}$ associated with K is equal to $\mathcal{S}^{(K)} \cap \mathbb{R}^K$, where $\mathcal{S}^{(K)}$ denotes the tropical Metzler spectrahedral cone described by $(Q^{(k)})_{k \in K}$. Therefore, we obtain the claim by applying Theorem 4.19 to every stratum. $\square$

4.2 Non-Metzler spectrahedra

In this section, we abandon the Metzler assumption that was imposed in the previous section. Let $Q^{(1)}$, …, $Q^{(n)} \in \mathbb{T}^{m \times m}_{\pm}$ be symmetric tropical matrices.

Definition 4.21. We introduce the set $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ (or simply $\mathcal{S}$) of points $x \in \mathbb{T}^n$ that fulfill the following two conditions:

- for all $i \in [m]$, $Q^+_{ii}(x) \geqslant Q^-_{ii}(x)$;
- for all $i, j \in [m]$, $i < j$, we have $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant (Q^+_{ij}(x) \oplus Q^-_{ij}(x))^{\odot 2}$ or $Q^+_{ij}(x) = Q^-_{ij}(x)$.

If $n \geqslant 2$, then we denote by $\mathcal{S}(Q^{(1)}|Q^{(2)},\dots,Q^{(n)})$ the set of all points $x \in \mathbb{T}^{n-1}$ such that $(0,x) \in \mathcal{S}(Q^{(1)},\dots,Q^{(n)})$. We point out that this generalizes Definition 4.13 to the case of non-Metzler matrices.

We point out that we do *not* claim that the set $\mathcal{S}$ defined above is a tropical spectrahedron. In this work we only show that this is true under some additional assumptions (which are generically fulfilled as shown in Section 4.3). Let $\boldsymbol{Q}^{(1)},\dots,\boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ be any symmetric matrices such that $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$, and $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ be the associated spectrahedral cone. The next lemma shows that the objects given in Definition 4.21 satisfy the basic inclusion $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{out}}) \subset \mathcal{S}$.

Lemma 4.22. *We have the inclusion* $\mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{out}}) \subset \mathcal{S}$.

Proof. Take a point $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{\text{out}}$ and denote $x \coloneqq \mathsf{val}(\boldsymbol{x})$. For every $i \in [m]$ we have $\boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0$ and hence $Q^+_{ii}(x) = \mathsf{val}(\boldsymbol{Q}^+_{ii}(\boldsymbol{x})) \geqslant \mathsf{val}(\boldsymbol{Q}^-_{ii}(\boldsymbol{x})) = Q^-_{ii}(x)$. Furthermore, for every $i < j$ such that $\mathsf{val}(\boldsymbol{Q}^+_{ij}(\boldsymbol{x})) \neq \mathsf{val}(\boldsymbol{Q}^-_{ij}(\boldsymbol{x}))$, we have $\mathsf{val}(\boldsymbol{Q}_{ij}(\boldsymbol{x})) = Q^+_{ij}(x) \oplus Q^-_{ij}(x)$ and hence $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant \mathsf{val}(\boldsymbol{Q}_{ii}(\boldsymbol{x})) + \mathsf{val}(\boldsymbol{Q}_{jj}(\boldsymbol{x})) \geqslant \mathsf{val}(\boldsymbol{Q}_{ij}(\boldsymbol{x})^2) = (Q^+_{ij}(x) \oplus Q^-_{ij}(x))^{\odot 2}$. On the other hand, for every $i < j$ such that $\mathsf{val}(\boldsymbol{Q}^+_{ij}(\boldsymbol{x})) = \mathsf{val}(\boldsymbol{Q}^-_{ij}(\boldsymbol{x}))$ we have $Q^+_{ii}(x) = Q^-_{jj}(x)$. In particular, $x \in \mathcal{S}$. □

To continue, we need some notation. For every subset

$$\Sigma \subset \{(i,j) \in [m]^2 \colon i < j\}$$

we denote

$$\Sigma^{\complement} \coloneqq \{(i,j) \in [m]^2 \colon i < j, (i,j) \notin \Sigma\}\,.$$

Definition 4.23. For every Σ and every $\lozenge \in \{\leqslant,\geqslant\}^{\Sigma^{\complement}}$ we define $\mathcal{S}_{\Sigma,\lozenge}(Q^{(1)},\dots,Q^{(n)})$ (or $\mathcal{S}_{\Sigma,\lozenge}$ for short) as the set of all $x \in \mathbb{T}^n$ such that

- for all $i \in [m]$, $Q^+_{ii}(x) \geqslant Q^-_{ii}(x)$;
- for all $i,j \in [m]$, $i < j$, $(i,j) \in \Sigma$, $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant (Q^+_{ij}(x) \oplus Q^-_{ij}(x))^{\odot 2}$;
- for all $i,j \in [m]$, $i < j$, $(i,j) \in \Sigma^{\complement}$, $Q^+_{ij}(x) \lozenge_{(i,j)} Q^-_{ij}(x)$.

(With the convention that if $\Sigma^{\complement} = \emptyset$, then we put $\lozenge = \emptyset$ and denote the corresponding set as $\mathcal{S}_{\Sigma,\emptyset}$.)

Lemma 4.24. *Every set $\mathcal{S}_{\Sigma,\lozenge}$ is a tropical Metzler spectrahedral cone. More precisely, it is described by the block diagonal matrices $Q^{(1)}_{\Sigma,\lozenge},\dots,Q^{(n)}_{\Sigma,\lozenge}$ of the form*

$$Q^{(k)}_{\Sigma,\lozenge} \coloneqq \begin{bmatrix} P^{(k)} & -\infty \\ -\infty & R^{(k)} \end{bmatrix}, \tag{4.6}$$

where $P^{(k)} \in \mathbb{T}^{m\times m}_{\pm}$ is the symmetric matrix defined by

$$P^{(k)}_{ij} \coloneqq \begin{cases} Q^{(k)}_{ii} & \text{if } i = j\,, \\ \ominus|Q^{(k)}_{ij}| & \text{if } (i,j) \in \Sigma\,, \\ -\infty & \text{if } (i,j) \in \Sigma^{\complement}\,. \end{cases}$$

and $R^{(k)} \in \mathbb{T}^{\Sigma^{\complement}\times\Sigma^{\complement}}_{\pm}$ is the (tropical) diagonal matrix consisting of the coefficients $Q^{(k)}_{ij}$ if $\lozenge_{(i,j)}$ is equal to $\geqslant$ and $\ominus Q^{(k)}_{ij}$ otherwise, where (i,j) ranges over the set $\Sigma^{\complement}$. Furthermore, we have the equality

$$\mathcal{S} = \bigcup_{\Sigma}\bigcap_{\lozenge} \mathcal{S}_{\Sigma,\lozenge}\,, \tag{4.7}$$

where the intersection goes over every $\lozenge \in \{\leqslant, \geqslant\}^{\Sigma^{\complement}}$ and the union goes over every $\Sigma \subset \{(i,j) \in [m]^2 \colon i < j\}$. (As previously, if $\Sigma^{\complement} = \emptyset$, then the intersection contains one element $\mathcal{S}_{\Sigma,\emptyset}$.)

Proof. Obvious from Definitions 4.13 and 4.23. □

In the sequel, we will use the following observation, which already appeared in the proof of [ABGJ15, Corollary 3.6] on the tropicalization of polyhedra. We denote by $\mathrm{conv}(X)$ the convex hull of the set $X \subset \mathbb{K}^n$.

Lemma 4.25 ([ABGJ15])**.** *Let $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(p)} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \mathbb{K}^p$. Suppose that for every sign pattern $\delta \in \{+1,-1\}^p$ there is a point $\boldsymbol{x}^{\delta} \in \mathbb{K}^n$ such that for all $s \in [p]$ we have $\delta_s(\langle \boldsymbol{a}^{(s)}, \boldsymbol{x}^{\delta}\rangle - \boldsymbol{b}_s) \geqslant 0$. Then, there exists a point $\boldsymbol{y} \in \mathrm{conv}_{\delta}\{\boldsymbol{x}^{\delta}\}$ such that for all s we have $\langle \boldsymbol{a}^{(s)}, \boldsymbol{y}\rangle = \boldsymbol{b}_s$.*

Proof. If $p = 1$, then we have two points $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ such that $\langle \boldsymbol{a}^{(1)}, \boldsymbol{x}^{(1)}\rangle \geqslant \boldsymbol{b}_1$ and $\langle \boldsymbol{a}^{(1)}, \boldsymbol{x}^{(2)}\rangle \leqslant \boldsymbol{b}_1$. Therefore, there exists $\boldsymbol{\lambda}$ such that $0 \leqslant \boldsymbol{\lambda} \leqslant 1$ and $\langle \boldsymbol{a}^{(1)}, \boldsymbol{\lambda}\boldsymbol{x}^{(1)} + (1-\boldsymbol{\lambda})\boldsymbol{x}^{(2)}\rangle = \boldsymbol{b}_1$. This completes the proof for $p = 1$.

Suppose that the claim is true for p. We will prove it for $p+1$. Take

$$\Delta^+ \coloneqq \{\delta \in \{+1,-1\}^{p+1} \colon \text{last entry of } \delta \text{ is equal to } +1\}$$

and

$$\Delta^- \coloneqq \{\delta \in \{+1,-1\}^{p+1} \colon \text{last entry of } \delta \text{ is equal to } -1\}\,.$$

By the induction hypothesis, there exists a point $\boldsymbol{x}^{(1)} \in \mathrm{conv}_{\delta\in\Delta^+}\{\boldsymbol{x}^{\delta}\}$ such that $\langle \boldsymbol{a}^{(s)}, \boldsymbol{x}^{(1)}\rangle = \boldsymbol{b}_s$ for all $s \leqslant p$. Moreover, we have $\langle \boldsymbol{a}^{(p+1)}, \boldsymbol{x}^{\delta}\rangle \geqslant \boldsymbol{b}_{p+1}$ for all $\delta \in \Delta^+$ and therefore $\langle \boldsymbol{a}^{(p+1)}, \boldsymbol{x}^{(1)}\rangle \geqslant \boldsymbol{b}_{p+1}$. Analogously, there exists a point $\boldsymbol{x}^{(2)} \in \mathrm{conv}_{\delta\in\Delta^-}\{\boldsymbol{x}^{\delta}\}$ such that $\langle \boldsymbol{a}^{(s)}, \boldsymbol{x}^{(2)}\rangle = \boldsymbol{b}_s$ for all $s \leqslant p$ and $\langle \boldsymbol{a}^{(p+1)}, \boldsymbol{x}^{(2)}\rangle \leqslant \boldsymbol{b}_{p+1}$. Therefore, there is a point $\boldsymbol{y} \in \mathrm{conv}\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\} \subset \mathrm{conv}_{\delta}\{\boldsymbol{x}^{\delta}\}$ such that $\langle \boldsymbol{a}^{(p+1)}, \boldsymbol{y}\rangle = \boldsymbol{b}_{p+1}$. Furthermore, since $\langle \boldsymbol{a}^{(s)}, \boldsymbol{x}^{(1)}\rangle = \langle \boldsymbol{a}^{(s)}, \boldsymbol{x}^{(2)}\rangle = \boldsymbol{b}_s$ for all $s \leqslant p$, we have $\langle \boldsymbol{a}^{(s)}, \boldsymbol{y}\rangle = \boldsymbol{b}_s$ for all $s \leqslant p$. □

In Lemma 4.17 we showed that the set $\mathcal{T}(Q^{(1)}, \dots, Q^{(n)})$ is included in a tropical spectrahedron $\mathsf{val}(\boldsymbol{\mathcal{S}})$ if the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ are Metzler. The following result generalizes Lemma 4.17 to the non-Metzler case.

Lemma 4.26. *For every $\{\Sigma, \lozenge\}$, we denote $\mathcal{T}_{\Sigma,\lozenge} \coloneqq \mathcal{T}(Q^{(1)}_{\Sigma,\lozenge}, \dots, Q^{(n)}_{\Sigma,\lozenge}) \subset \mathbb{T}^n$, where the matrices $Q^{(k)}_{\Sigma,\lozenge}$ are as in (4.6). Then, we have the inclusion*

$$\bigcup_{\Sigma}\bigcap_{\lozenge} \mathrm{cl}_{\mathbb{T}}(\mathcal{T}_{\Sigma,\lozenge}) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\,.$$

Proof. Fix any Σ and take $x \in \bigcap_{\lozenge} \mathrm{cl}_{\mathbb{T}}(\mathcal{T}_{\Sigma,\lozenge})$. By Lemma 4.17, for every $\lozenge \in \{\leqslant, \geqslant\}^{\Sigma^{\complement}}$ there exists a lift $\boldsymbol{x}^{\lozenge} \in \mathbb{K}^n_{\geqslant 0} \cap \mathsf{val}^{-1}(x)$ such that we have the inequalities

$$\begin{aligned} \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{x}^{\lozenge}) &\geqslant 0\,, \\ \forall (i,j) \in \Sigma, \boldsymbol{Q}_{ii}(\boldsymbol{x}^{\lozenge})\boldsymbol{Q}_{jj}(\boldsymbol{x}^{\lozenge}) &\geqslant (m-1)^2(\boldsymbol{Q}^+_{ij}(\boldsymbol{x}^{\lozenge}) + \boldsymbol{Q}^-_{ij}(\boldsymbol{x}^{\lozenge}))^2\,, \\ \forall (i,j) \in \Sigma^{\complement}, \boldsymbol{Q}_{ij}(\boldsymbol{x}^{\lozenge}) &\lozenge_{(i,j)}\, 0\,. \end{aligned}$$

Furthermore, we have $(\boldsymbol{Q}^+_{ij}(\boldsymbol{x}^{\lozenge}) + \boldsymbol{Q}^-_{ij}(\boldsymbol{x}^{\lozenge}))^2 \geqslant (\boldsymbol{Q}_{ij}(\boldsymbol{x}^{\lozenge}))^2$. Observe that the set

$$\{\boldsymbol{y} \in \mathbb{K}^n_{\geqslant 0} \colon \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{y}) \geqslant 0 \wedge \forall (i,j) \in \Sigma, \boldsymbol{Q}_{ii}(\boldsymbol{y})\boldsymbol{Q}_{jj}(\boldsymbol{y}) \geqslant (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{y}))^2\}$$

is convex. Indeed, it is a spectrahedron defined by some block diagonal matrices with blocks of size at most 2. Therefore, by Lemma 4.25, there exists a point $\boldsymbol{z} \in \operatorname{conv}_{\lozenge}\{\boldsymbol{x}^{\lozenge}\}$ such that

$$\begin{aligned} \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{z}) &\geqslant 0 \,, \\ \forall (i,j) \in \Sigma, \boldsymbol{Q}_{ii}(\boldsymbol{z})\boldsymbol{Q}_{jj}(\boldsymbol{z}) &\geqslant (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{z}))^2 \,, \\ \forall (i,j) \in \Sigma^{\complement}, \boldsymbol{Q}_{ij}(\boldsymbol{z}) &= 0 \,. \end{aligned}$$

In particular, we have $\boldsymbol{Q}_{ii}(\boldsymbol{z})\boldsymbol{Q}_{jj}(\boldsymbol{z}) \geqslant (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{z}))^2$ for all (i,j) such that $i \neq j$. Therefore $\boldsymbol{z} \in \boldsymbol{\mathcal{S}}^{\text{in}}$. Moreover, since $\boldsymbol{x}^{\lozenge} \in \mathbb{K}^n_{>0} \cap \mathsf{val}^{-1}(x)$ for all $\lozenge$, we have $\boldsymbol{z} \in \mathbb{K}^n_{>0} \cap \mathsf{val}^{-1}(x)$. ☐

Lemma 4.27. *If the matrices $Q^{(1)}, \dots, Q^{(n)}$ satisfy Assumption A, then the same is true for the matrices $Q^{(1)}_{\Sigma,\lozenge}, \dots, Q^{(n)}_{\Sigma,\lozenge}$ (where $\{\Sigma, \lozenge\}$ are arbitrary).*

Proof. This is clear from the definition of matrices $Q^{(1)}_{\Sigma,\lozenge}, \dots, Q^{(n)}_{\Sigma,\lozenge}$ given in (4.6). ☐

Theorem 4.28. *Let $\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ be a spectrahedral cone described by matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ such that $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$. Suppose that Assumption A is fulfilled and that every stratum of the set $\mathcal{S}_{\Sigma,\lozenge}(Q^{(1)}, \dots, Q^{(n)})$ is regular for every choice of $(\Sigma, \lozenge)$. Then*

$$\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \,.$$

Proof. We focus on the proof of the identity $\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$, as the generalization to all strata can be obtained analogously to the proof of Theorem 4.20. Let $(\Sigma, \lozenge)$ be fixed. Since the matrices $Q^{(1)}_{\Sigma,\lozenge}, \dots, Q^{(n)}_{\Sigma,\lozenge}$ fulfill Assumption A by Lemma 4.27, the proof of Theorem 4.19 shows the equality $\operatorname{cl}_{\mathbb{R}}(\mathcal{T}_{\Sigma,\lozenge} \cap \mathbb{R}^n) = \mathcal{S}_{\Sigma,\lozenge} \cap \mathbb{R}^n$. Therefore, by (4.7) and Lemmas 4.22 and 4.26 we have

$$(\mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{out}}) \cap \mathbb{R}^n) \subset (\mathcal{S} \cap \mathbb{R}^n) = \bigcup_{\Sigma} \bigcap_{\lozenge} (\mathcal{S}_{\Sigma,\lozenge} \cap \mathbb{R}^n) = \bigcup_{\Sigma} \bigcap_{\lozenge} \operatorname{cl}_{\mathbb{R}}(\mathcal{T}_{\Sigma,\lozenge} \cap \mathbb{R}^n) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{\text{in}} \cap \mathbb{K}^n_{>0}) \,.$$

Hence $\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) = \mathcal{S} \cap \mathbb{R}^n$. ☐

Remark 4.29. We note that if the hypotheses of Theorem 4.28 are fulfilled, then (by Lemma 4.7) the set $\mathcal{S}(Q^{(1)}|Q^{(2)}, \dots, Q^{(n)})$ is a tropicalization of the spectrahedron $\tilde{\boldsymbol{\mathcal{S}}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^{n-1}_{\geqslant 0} \colon \boldsymbol{Q}^{(1)} + \boldsymbol{x}_2\boldsymbol{Q}^{(2)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)} \succcurlyeq 0\}$, $\mathsf{val}(\tilde{\boldsymbol{\mathcal{S}}}) = \mathcal{S}(Q^{(1)}|Q^{(2)}, \dots, Q^{(n)})$.

We finish this section with some examples.

Example 4.30. Take the matrices

$$Q^{(0)} \coloneqq \begin{bmatrix} a & -\infty \\ -\infty & b \end{bmatrix}, \ Q^{(1)} \coloneqq \begin{bmatrix} -\infty & c \\ c & -\infty \end{bmatrix}, \ Q^{(2)} \coloneqq \begin{bmatrix} -\infty & \ominus d \\ \ominus d & -\infty \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$. The set $\mathcal{S}(Q^{(0)}, Q^{(1)}, Q^{(2)})$ fulfills the conditions of Theorem 4.28. Moreover, it is the set of all points $(x_0, x_1, x_2) \in \mathbb{T}^3$ such that

$$\frac{a+b}{2} + x_0 \geqslant \max\{c + x_1, d + x_2\} \quad \text{or} \quad c + x_1 = d + x_2 \,.$$

The tropical spectrahedron $\mathcal{S}(Q^{(0)}|Q^{(1)}, Q^{(2)})$ is depicted in Fig. 4.2. Note that the real part of this tropical spectrahedron is not regular for any choice of $a, b, c, d \in \mathbb{R}$.

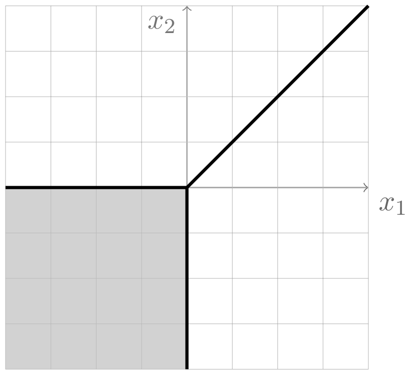

Figure 4.2: A nonregular tropical spectrahedron that fulfills the regularity conditions of Theorem 4.28.

Example 4.31. In [Yu15], Yu characterized the image by the valuation of the positive semidefinite cone. Let us show how this result can be derived from Theorem 4.28. Let $m' \coloneqq m(m+1)/2$ and let $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^{m' \times m'}$ be the cone of positive semidefinite matrices. We will compute the image by valuation of $\boldsymbol{\mathcal{S}}$ orthant-by-orthant. As in the proof of Lemma 3.11, for every $\delta \in \{+1,-1\}^{m'}$, we denote by $f_\delta \colon \mathbb{K}^{m'} \to \mathbb{K}^{m'}$ the involution which maps $\boldsymbol{x} \in \mathbb{K}^{m'}$ to the vector with entries $\delta_{ij}\boldsymbol{x}_{ij}$. We want to characterize $\mathsf{val}(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^{m'}_{\geqslant 0}))$. Note that we can restrict ourselves to the orthants such that $\delta_{ii} = 1$ for all $i \in [m]$ (because every point in $\boldsymbol{\mathcal{S}}$ fulfills the inequalities $\boldsymbol{x}_{ii} \geqslant 0$ for all $i \in [m]$). Moreover, we have $\mathsf{val}(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^{m'}_{\geqslant 0})) = \mathsf{val}\big(f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^{m'}_{\geqslant 0}))\big)$ and the set $f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^{m'}_{\geqslant 0}))$ is included in $\mathbb{K}^{m'}_{\geqslant 0}$. More precisely, the set $\boldsymbol{\mathcal{S}}_\delta \coloneqq f_\delta(\boldsymbol{\mathcal{S}} \cap f_\delta(\mathbb{K}^{m'}_{\geqslant 0}))$ is a spectrahedral cone defined as the set of all points $\boldsymbol{x} \in \mathbb{K}^{m' \times m'}_{\geqslant 0}$ such that the linear pencil

$$\boldsymbol{Q}_\delta(\boldsymbol{x}) \coloneqq \begin{bmatrix} \boldsymbol{x}_{11} & \delta_{12}\boldsymbol{x}_{12} & \dots & \delta_{1m}\boldsymbol{x}_{1m} \\ \delta_{12}\boldsymbol{x}_{12} & \boldsymbol{x}_{22} & \dots & \delta_{2m}\boldsymbol{x}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1m}\boldsymbol{x}_{1m} & \delta_{2m}\boldsymbol{x}_{2m} & \dots & \boldsymbol{x}_{mm} \end{bmatrix}$$

is positive semidefinite. Let $\boldsymbol{Q}^{(ij)} \in \mathbb{K}^{m \times m}$ denote the matrix that has δ_{ij} on the (i,j)th and (j,i)th positions and zeros otherwise (we allow $i = j$). Let $Q^{(ij)} \coloneqq \mathsf{sval}(\boldsymbol{Q}^{(ij)})$. It is clear that the matrices $(Q^{(ij)})_{i \leqslant j}$ satisfy Assumption A. Moreover, for every $\{\Sigma, \lozenge\}$, the set $\mathcal{S}_{\Sigma,\lozenge} \coloneqq \mathcal{S}_{\Sigma,\lozenge}\big((Q^{(ij)})_{i\leqslant j}\big)$ is given by

$$\mathcal{S}_{\Sigma,\lozenge} = \{x \in \mathbb{T}^{m' \times m'} \colon \forall (i,j) \in \Sigma,\, x_{ii} \odot x_{jj} \geqslant x_{ij}^{\odot 2} \wedge \forall (i,j) \in \tilde{\Sigma},\, x_{ij} = -\infty\}\,,$$

where the set $\tilde{\Sigma}$ consists of the elements of $\Sigma^{\complement}$ such that $\lozenge_{ij}$ is equal to $\leqslant$ and $\delta_{ij} = 1$ or $\lozenge_{ij}$ is equal to $\geqslant$ and $\delta_{ij} = -1$. This set has regular strata. Indeed, let $K \subset [m']$ be a nonempty set, and let $\mathcal{S}_{\Sigma,\lozenge,K}$ denote the stratum of $\mathcal{S}_{\Sigma,\lozenge}$ associated with K. If $\mathcal{S}_{\Sigma,\lozenge,K}$ is nonempty, then $\tilde{\Sigma} \cap K = \emptyset$. Furthermore, for every $(i,j) \in \Sigma \cap K$ we have $\{(i,i),(j,j)\} \subset K$. In particular, the set $\mathcal{S}_{\Sigma,\lozenge,K}$ is given by

$$\mathcal{S}_{\Sigma,\lozenge,K} = \{x \in \mathbb{R}^K \colon \forall (i,j) \in \Sigma \cap K, x_{ii} \odot x_{jj} \geqslant x_{ij}^{\odot 2}\}\,.$$

If we take any $x \in \mathcal{S}_{\Sigma,\lozenge,K}$, then for every $\varepsilon > 0$, the point $x^{(\varepsilon)}$ defined as $x_{ii}^{(\varepsilon)} \coloneqq x_{ii}^{(\varepsilon)} + \varepsilon$ for all $(i,i) \in K$ and $x_{ij}^{(\varepsilon)} = x_{ij}$ for all $(i,j) \in K$, $i < j$, belongs to the interior of $\mathcal{S}_{\Sigma,\lozenge,K}$. Hence

$x \in \mathrm{cl}_{\mathbb{R}}(\mathrm{int}(\mathcal{S}_{\Sigma,\Diamond,K}))$. Therefore, by Theorem 4.28, we obtain the equality $\mathsf{val}(\boldsymbol{\mathcal{S}}_\delta) = \mathcal{S}$, where the set $\mathcal{S} \subset \mathbb{T}^{m' \times m'}$ is given by

$$\mathcal{S} = \{x \in \mathbb{T}^{m' \times m'} \colon \forall i < j,\, x_{ii} \odot x_{jj} \geqslant x_{ij}^{\odot 2}\}\,.$$

This set does not depend on δ. Therefore, by taking the union over δ, we get $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$.

Example 4.32. In Definition 4.13 we introduced the class of tropical Metzler spectrahedral cones, and we have shown that these objects are, indeed, tropical spectrahedra. One can ask if the valuation of every spectrahedral cone is in fact a tropical Metzler spectrahedral cone. This is, however, not true. The following example shows a spectrahedral cone that fulfills the conditions of Theorem 4.28 but whose image by valuation cannot be expressed as a tropical Metzler spectrahedral cone. Take the matrices

$$\boldsymbol{Q}^{(1)} \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{Q}^{(2)} \coloneqq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \boldsymbol{Q}^{(3)} \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \boldsymbol{Q}^{(4)} \coloneqq \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The associated spectrahedron is given by $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^4_{\geqslant 0} \colon \boldsymbol{x}_1\boldsymbol{x}_2 \geqslant (\boldsymbol{x}_3 - \boldsymbol{x}_4)^2\}$ and the set $\mathcal{S} \coloneqq \mathcal{S}(\mathsf{sval}(\boldsymbol{Q}^{(1)}), \dots, \mathsf{sval}(\boldsymbol{Q}^{(4)}))$ is equal to $\mathcal{S} = \{x \in \mathbb{T}^4 \colon x_1 + x_2 \geqslant 2\max(x_3, x_4) \text{ or } x_3 = x_4\}$. It is easy to check that $\mathcal{S}$ satisfies the conditions of Theorem 4.28. One can also verify by hand that $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$. Suppose that $\mathcal{S}$ is a tropical Metzler spectrahedral cone, $\mathcal{S} = \mathcal{S}(Q^{(1)}, \dots, Q^{(4)})$ for some symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(4)} \in \mathbb{T}^{m \times m}$. As noted in Remark 4.18, we may suppose that $Q^{(1)}, \dots, Q^{(4)}$ satisfy Assumption A. Let $u = (5, -3, -2, 1) \in \mathbb{R}^4$ and observe that the half-line $\{\lambda u \colon \lambda \in \mathbb{R}_{\geqslant 0}\}$ is included in the boundary of $\mathcal{S}$. Therefore, for every λ, at least one of the nontrivial inequalities that describe $\mathcal{S}$ becomes an equality when evaluated at λu.

First, suppose that $Q_{ii}^{+}(\lambda_s u) = Q_{ii}^{-}(\lambda_s u)$ for some i and a sequence $\lambda_s > 0$, $\lambda_s \to 0$. Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined as $f(\lambda) \coloneqq Q_{ii}^{+}(\lambda u) - Q_{ii}^{-}(\lambda u)$. The function f is continuous and piecewise affine. Therefore, $f(\lambda) = 0$ on some interval $\lambda \in [0, \varepsilon]$. We can suppose that ε is so small that the functions $Q_{ii}^{+}(\lambda u)$, $Q_{ii}^{-}(\lambda u)$ are affine on $[0, \varepsilon]$. In other words, we have $Q_{ii}^{+}(\lambda u) = a + \lambda u_k$ and $Q_{ii}^{-}(\lambda u) = b + \lambda u_l$ for some $a, b \in \mathbb{R}$ and $k \neq l$. Since $f(\lambda)$ is equal to 0 on $[0, \varepsilon]$, we have $a = b$ and $u_k = u_l$, which gives a contradiction with our choice of u.

Second, suppose that $Q_{ii}^{+}(\lambda_s u) \odot Q_{jj}^{+}(\lambda_s u) = (Q_{ij}(\lambda_s u))^{\odot 2}$ for some i, j and a sequence $\lambda_s > 0$, $\lambda_s \to 0$. As previously, the function $f(\lambda) \coloneqq Q_{ii}^{+}(\lambda u) \odot Q_{jj}^{+}(\lambda u) - (Q_{ij}(\lambda u))^{\odot 2}$ is continuous and piecewise affine, there is $\varepsilon > 0$ such that $f(\lambda) = 0$ on $[0, \varepsilon]$, and all the functions $Q_{ii}^{+}(\lambda u)$, $Q_{jj}^{+}(\lambda u)$, $(Q_{ij}(\lambda u))^{\odot 2}$ are affine on $[0, \varepsilon]$, $Q_{ii}^{+}(\lambda u) = a + \lambda u_k$, $Q_{jj}^{+}(\lambda u) = b + \lambda u_l$, $(Q_{ij}(\lambda u))^{\odot 2} = 2c + 2\lambda u_p$. Since $f(\lambda)$ is equal to 0 on $[0, \varepsilon]$, we have $a + b = 2c$ and $u_k + u_l = 2u_p$. Thanks to our choice of vector u, this is possible only if $k = l = p$ or $(k, l, p) \in \{(1, 2, 4), (2, 1, 4)\}$. The first case does not occur because the matrices $Q^{(1)}, \dots, Q^{(4)}$ satisfy Assumption A (and we have $a = Q_{ii}^{(k)}, b = Q_{jj}^{(l)}, c = |Q_{ij}^{(p)}|$). Suppose that $(k, l, p) = (1, 2, 4)$ (the case $(k, l, p) = (2, 1, 4)$ is analogous). Since $Q_{jj}^{+}(\lambda u) = b + \lambda u_2$ on $[0, \varepsilon]$, the second vector of the standard basis $\epsilon_2 \in \mathbb{R}^4$ belongs to $\mathrm{Argmax}(Q_{jj}^{+}, 0)$. We claim that ϵ_2 is the only element of $\mathrm{Argmax}(Q_{jj}^{+}, 0)$. Indeed, suppose that $\epsilon_{l'} \in \mathrm{Argmax}(Q_{jj}^{+}, 0)$ for some $l' \neq 2$. Then, we have $Q_{jj}^{(l')} = b$. On the other hand, by our choice of u, we have $Q_{ij}^{(l')} + \lambda u_{l'} > b - 3\lambda = b + \lambda u_2$ for $\lambda \in]0, \varepsilon]$, which gives a contradiction with the fact that $Q_{jj}^{+}(\lambda u) = b + \lambda u_2$. Hence $\mathrm{Argmax}(Q_{jj}^{+}, 0) = \{\epsilon_2\}$. Denote $v = (0, -1, 0, 0)$. By continuity, for sufficiently small $\varepsilon' > 0$ we have $\mathrm{Argmax}(Q_{jj}^{+}, \varepsilon' v) = \{\epsilon_2\}$. Therefore $Q_{jj}^{+}(\varepsilon' v) = b - \varepsilon'$. Moreover, by the choice of v, we have $Q_{ii}^{+}(\varepsilon' v) \leqslant Q_{ii}^{+}(0) = a$ and

$Q_{ij}(\varepsilon' v) \geqslant 2c$. Therefore $(Q_{ij}(\varepsilon' v))^{\odot 2} \geqslant 2c > a + b - \varepsilon' \geqslant Q^+_{ii}(\varepsilon' v) \odot Q^+_{jj}(\varepsilon' v)$, which gives a contradiction with the fact that $\varepsilon' v \in \mathcal{S}$.

4.3 Genericity conditions

In this section we show that the requirements of Theorems 4.19 and 4.28 on the matrices $Q^{(k)}$ and the regularity of sets are fulfilled generically. In [ABGJ15] it was shown that genericity conditions for tropical polyhedra can be described by the means of tangent digraphs. We extend this characterization to tropical spectrahedra. For this purpose, we work with hypergraphs instead of graphs.

Definition 4.33. A *(directed) hypergraph* is a pair $\vec{\mathcal{G}} \coloneqq (V, E)$, where V is a finite set of *vertices* and E is a finite set of *(hyper)edges*. Every edge $e \in E$ is a pair (T_e, h_e), where $h_e \in V$ is called the *head* of the edge, and T_e is a multiset with elements taken from V. We call T_e the *multiset of tails of* e. By $|T_e|$ we denote the cardinality of T_e (counting multiplicities). We do not exclude the situation in which a head is also a tail, i.e., it is possible that $h_e \in T_e$. If $v \in V$ is a vertex of $\vec{\mathcal{G}}$, then by $\mathrm{In}(v) \subset E$ we denote the set of *incoming edges*, i.e., the set of all edges e such that $h_e = v$. By $\mathrm{Out}(v)$ we denote the multiset of *outgoing edges*, i.e., a multiset of edges e such that $v \in T_e$. We treat $\mathrm{Out}(v)$ as a multiset, with the convention that $e \in E$ appears p times in $\mathrm{Out}(v)$ if v appears p times in T_e.

Let us now define the notion of a circulation in a hypergraph.

Definition 4.34. A *circulation* in a hypergraph is a vector $\gamma = (\gamma_e)_{e \in E}$ such that $\gamma_e \geqslant 0$ for all $e \in E$, $\sum_{e \in E} \gamma_e = 1$, and for all $v \in V$ we have the equality

$$\sum_{e \in \mathrm{In}(v)} |T_e| \gamma_e = \sum_{e \in \mathrm{Out}(v)} \gamma_e \, .$$

Observe that if a hypergraph $\vec{\mathcal{G}}$ is fixed, then the set of all normalized circulations on $\vec{\mathcal{G}}$ forms a polyhedron. We say that a hypergraph *does not admit a circulation* if this polyhedron is empty.

In this section, we only consider hypergraphs such that every edge has at most two tails (counting multiplicities). Hereafter, ϵ_k denotes the kth vector of the standard basis in $\mathbb{R}^n$.

Definition 4.35. Given a sequence of tropical symmetric Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ and a point $x \in \mathbb{R}^n$, we construct a hypergraph associated with x, denoted $\vec{\mathcal{G}}_x$, as follows:

- we put $V \coloneqq [n]$;
- for every $i \in [m]$ verifying $Q^+_{ii}(x) = Q^-_{ii}(x) \neq -\infty$, and every pair $\epsilon_k \in \mathrm{Argmax}(Q^+_{ii}, x)$, $\epsilon_l \in \mathrm{Argmax}(Q^-_{ii}, x)$, the hypergraph $\vec{\mathcal{G}}_x$ contains an edge (k, l);
- for every $i < j$ such that $Q^+_{ii}(x) \odot Q^+_{jj}(x) = (Q_{ij}(x))^{\odot 2} \neq -\infty$ and every triple $\epsilon_{k_1} \in \mathrm{Argmax}(Q^+_{ii}, x)$, $\epsilon_{k_2} \in \mathrm{Argmax}(Q^+_{jj}, x)$, $\epsilon_l \in \mathrm{Argmax}(Q_{ij}, x)$, the hypergraph $\vec{\mathcal{G}}_x$ contains an edge $(\{k_1, k_2\}, l)$.

The next lemma shows that the regularity assumptions given in Theorem 4.19 are automatically satisfied if the hypergraphs constructed in Definition 4.35 do not admit a circulation. The proof uses Farkas' lemma.

Theorem 4.36 (Farkas' lemma, [Sch87, Corollary 7.1d])**.** *The polyhedron $\mathcal{W} = \{x \in \mathbb{R}^n_{\geqslant 0} \colon \forall i \in [p], \langle a_i, x\rangle = b_i\}$ is empty if and only if there exists a vector $y \in \mathbb{R}^p$ such that $\sum_{i=1}^p b_i y_i < 0$ and $\sum_{i=1}^p a_{ik} y_i \geqslant 0$ for all $k \in [n]$.*

Lemma 4.37. *Suppose that for every $x \in \mathbb{R}^n$ the hypergraph $\vec{\mathcal{G}}_x$ does not admit a circulation. Then, the matrices $Q^{(1)}, \dots, Q^{(n)}$ fulfill Assumption A and $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is regular.*

Proof. To prove the first part, suppose that we have $Q^{(k)}_{ii} + Q^{(k)}_{jj} = 2|Q^{(k)}_{ij}|$ for some $i \neq j$ and $Q^{(k)}_{ii}, Q^{(k)}_{jj} \in \mathbb{T}_+$. Take the point $x \coloneqq N\epsilon_k \in \mathbb{R}^n$. If N is large enough, then we have

$$Q^+_{ii}(x) \odot Q^+_{jj}(x) = Q^{(k)}_{ii} + Q^{(k)}_{jj} + 2N = 2|Q^{(k)}_{ij}| + 2N = (Q_{ij}(x))^{\odot 2}$$

and the hypergraph $\vec{\mathcal{G}}_x$ contains the edge $(\{k, k\}, k)$. This hypergraph admits a circulation (we put $\gamma_e \coloneqq 1$ for $e = (\{k, k\}, k)$ and $\gamma_e \coloneqq 0$ for other edges), which gives a contradiction.

We now claim that the set $\mathcal{S} \cap \mathbb{R}^n$ is regular. Let $\mathcal{T}$ be defined as in Definition 4.16. Let us show that for every $x \in \mathcal{S} \cap \mathbb{R}^n$ there exists a vector $\eta \in \mathbb{R}^n$ such that $x + \rho\eta$ belongs to $\mathcal{T}$ for $\rho > 0$ small enough. This is sufficient to prove the claim because $\mathcal{T} \cap \mathbb{R}^n$ is a subset of the interior of $\mathcal{S} \cap \mathbb{R}^n$ (as shown in the proof of Theorem 4.19). Fix a point $x \in \mathcal{S} \cap \mathbb{R}^n$. If x belongs to $\mathcal{T}$, then we can take $\eta \coloneqq 0$. Otherwise, let $\vec{\mathcal{G}}_x$ denote the hypergraph associated with x. The polytope of circulations of this hypergraph is empty. Therefore, by Farkas' lemma (Theorem 4.36), there exists a vector $\eta \in \mathbb{R}^n$ such that for every edge $e \in E$ we have

$$\sum_{v \in T_e} \eta_v > |T_e| \eta_{h_e} \, .$$

Take the vector $x^{(\rho)} \coloneqq x + \rho\eta$. Let us look at two cases.

First, suppose that there is $i \in [m]$ such that $Q^+_{ii}(x) = Q^-_{ii}(x) \neq -\infty$. Fix any k^* such that $\epsilon_{k^*} \in \operatorname{Argmax}(Q^+_{ii}, x)$ and take any l such that $\epsilon_l \in \operatorname{Argmax}(Q^-_{ii}, x)$. Then (k^*, l) is an edge in $\vec{\mathcal{G}}_x$. Therefore $\eta_{k^*} > \eta_l$. Moreover, $Q^{(k^*)}_{ii} + x_{k^*} = |Q^{(l)}_{ii}| + x_l$ and hence $Q^{(k^*)}_{ii} + x^{(\rho)}_{k^*} > |Q^{(l)}_{ii}| + x^{(\rho)}_l$. Furthermore, for every $l' \notin \operatorname{Argmax}(Q^-_{ii}, x)$ we have

$$Q^{(k^*)}_{ii} + x_{k^*} = |Q^{(l)}_{ii}| + x_l > |Q^{(l')}_{ii}| + x_{l'} \, .$$

Therefore $Q^{(k^*)}_{ii} + x^{(\rho)}_{k^*} > |Q^{(l')}_{ii}| + x^{(\rho)}_{l'}$ for ρ small enough. Since l, l' were arbitrary, for every sufficiently small ρ we have

$$Q^+_{ii}(x^{(\rho)}) \geqslant Q^{(k^*)}_{ii} + x^{(\rho)}_{k^*} > Q^-_{ii}(x^{(\rho)}) \, .$$

The second case is analogous. If there is $i < j$ such that $Q^+_{ii}(x) \odot Q^+_{jj}(x) = (Q_{ij}(x))^{\odot 2} \neq -\infty$, then we fix (k^*_1, k^*_2) such that $\epsilon_{k^*_1} \in \operatorname{Argmax}(Q^+_{ii}, x)$, $\epsilon_{k^*_2} \in \operatorname{Argmax}(Q^+_{jj}, x)$. For every $\epsilon_l \in \operatorname{Argmax}(Q_{ij}, x)$, $(\{k^*_1, k^*_2\}, l)$ is an edge in $\vec{\mathcal{G}}$. Hence $\eta_{k^*_1} + \eta_{k^*_2} > 2\eta_l$. Therefore $Q^{(k^*_1)}_{ii} + Q^{(k^*_2)}_{jj} + x^{(\rho)}_{k^*_1} + x^{(\rho)}_{k^*_2} > 2|Q^{(l)}_{ij}| + 2x^{(\rho)}_l$. As before, this implies that $Q^+_{ii}(x^{(\rho)}) \odot Q^+_{jj}(x^{(\rho)}) > (Q_{ij}(x^{(\rho)}))^{\odot 2}$ for $\rho > 0$ small enough. Since we supposed that $x \in \mathcal{S} \cap \mathbb{R}^n$, we have $x^{(\rho)} \in \mathcal{T}$ for ρ small enough. $\square$

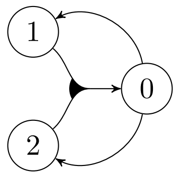

Figure 4.3: The hypergraph from Example 4.38, consisting of the edges $(0,1)$, $(0,2)$, and $(\{1,2\},0)$.

Example 4.38. Take the matrices

$$\begin{aligned} Q^{(0)} &:= \operatorname{tdiag}\Bigl(0, 0, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\Bigr), \\ Q^{(1)} &:= \operatorname{tdiag}\Bigl(\ominus 0, -\infty, 0, -\infty\Bigr), \\ Q^{(2)} &:= \operatorname{tdiag}\Bigl(-\infty, \ominus 0, -\infty, 0\Bigr). \end{aligned}$$

In this case, the set $\mathcal{S}(Q^{(0)}, Q^{(1)}, Q^{(2)}) \cap \mathbb{R}^3 = \{(\lambda,\lambda,\lambda) \colon \lambda \in \mathbb{R}\}$ is reduced to a line. In particular, it is not regular. The hypergraph associated with $(0,0,0)$ is depicted in Fig. 4.3. This hypergraph admits a circulation (we put $\gamma_e := 1/3$ for all edges).

We now want to show that the condition of Lemma 4.37 is fulfilled generically.

Lemma 4.39. *There exists a set $X \subset \mathbb{T}^d$ with $d = nm(m+1)/2$ that fulfills the following two conditions. First, every stratum of X is a finite union of hyperplanes. Second, if $Q^{(1)}, \dots, Q^{(n)}$ is any sequence of symmetric tropical Metzler matrices such that the vector with entries $|Q^{(k)}_{ij}|$ (for $i \leqslant j$) does not belong to X, then the hypergraph $\vec{\mathcal{G}}_x$ does not admit a circulation for any $x \in \mathbb{R}^n$.*

Proof. Fix a nonempty subset $D \subset [d]$, $|D| = d'$ and let $\mathbb{R}^{d'}$ be the stratum of $\mathbb{T}^d$ associated with D. Suppose that $Q^{(1)}, \dots, Q^{(n)}$ are tropical Metzler matrices, that the support of the vector $|Q^{(k)}_{ij}|$ is equal to D, and that $x \in \mathbb{R}^n$ is such that $\vec{\mathcal{G}}_x = \vec{\mathcal{G}}$ admits a circulation. Fix any such circulation γ. For every edge $e = (k,l)$ of $\vec{\mathcal{G}}$ we can fix $i_e \in [m]$ such that $Q^{(k)}_{i_e i_e} + x_k = |Q^{(l)}_{i_e i_e}| + x_l \neq -\infty$. Similarly, for every edge $e = (\{k_1,k_2\}, l)$ of $\vec{\mathcal{G}}$ we can fix $i_e < j_e$ such that $Q^{(k_1)}_{i_e i_e} + Q^{(k_2)}_{j_e j_e} + x_{k_1} + x_{k_2} = 2|Q^{(l)}_{i_e j_e}| + 2x_l \neq -\infty$. We take the sum of these equalities weighted by γ. This gives the equality

$$\begin{aligned} &\sum_{k \in [n]} \sum_{e \in \mathrm{Out}(k)} \gamma_e Q^{(k)}_{i_e i_e} + \sum_{k \in [n]} \sum_{e \in \mathrm{Out}(k)} \gamma_e x_k \\ &= \sum_{l \in [n]} \sum_{e \in \mathrm{In}_1(l)} \gamma_e |Q^{(l)}_{i_e i_e}| + \sum_{l \in [n]} \sum_{e \in \mathrm{In}_2(l)} 2\gamma_e |Q^{(l)}_{i_e j_e}| + \sum_{l \in [n]} \sum_{e \in \mathrm{In}(l)} |T_e| \gamma_e x_l \,, \end{aligned}$$

where $\mathrm{In}_1(l)$ denotes the set of incoming edges with tails of cardinality 1 and $\mathrm{In}_2(l)$ denotes the set of incoming edges with tails of cardinality 2. Since γ is a circulation, this expression simplifies to

$$\sum_{k \in [n]} \sum_{e \in \mathrm{Out}(k)} \gamma_e Q^{(k)}_{i_e i_e} = \sum_{l \in [n]} \sum_{e \in \mathrm{In}_1(l)} \gamma_e |Q^{(l)}_{i_e i_e}| + \sum_{l \in [n]} \sum_{e \in \mathrm{In}_2(l)} 2\gamma_e |Q^{(l)}_{i_e j_e}| \,.$$

Consider the set $\mathcal{H}$ of all $z \in \mathbb{R}^{d'}$ such that

$$\sum_{k\in[n]} \sum_{e\in\mathrm{Out}(k)} \gamma_e z^{(k)}_{i_e i_e} = \sum_{l\in[n]} \sum_{e\in\mathrm{In}_1(l)} \gamma_e z^{(l)}_{i_e i_e} + \sum_{l\in[n]} \sum_{e\in\mathrm{In}_2(l)} 2\gamma_e z^{(l)}_{i_e j_e} \, . \tag{4.8}$$

This set is a hyperplane. Indeed, suppose that the equality above is trivial (i.e., that it reduces to $0 = 0$). Take any edge e such that $\gamma_e \neq 0$ and any vertex $k \in T_e$. Then, the coefficient $z^{(k)}_{i_e i_e}$ appears on the left-hand side. Moreover, we have $\mathsf{sign}(Q^{(k)}_{i_e i_e}) = 1$. On the other hand, for every coefficient $z^{(l)}_{j_e j_e}$ that appears on the right-hand side we have $\mathsf{sign}(Q^{(l)}_{j_e j_e}) = -1$. This gives a contradiction.

Therefore, we can construct the stratum of X associated with D (denoted X_D) as follows: we take all possible hypergraphs that can arise in our construction (since n is fixed, we have finitely many of them). Out of them, we choose those hypergraphs that admit a circulation. For every such hypergraph we pick exactly one circulation γ. After that, for every possible choice of functions $e \to i_e$, $e \to (i_e, j_e)$,[1] we take a set $\mathcal{H}$ defined as in (4.8). If $\mathcal{H}$ is equal to $\mathbb{R}^{d'}$, then we ignore it. Otherwise, $\mathcal{H}$ is a hyperplane. We take X_D to be the union of all hyperplanes obtained in this way. ☐

The proof of Lemma 4.39 can be easily adapted to give a genericity condition both for Metzler and non-Metzler spectrahedra.

Theorem 4.40. *There exists a set $X \subset \mathbb{T}^d$ with $d = nm(m+1)/2$ that fulfills the following two conditions. First, every stratum of X is a finite union of hyperplanes. Second, if $Q^{(1)}, \dots, Q^{(n)}$ is any sequence of symmetric tropical matrices such that the vector with entries $|Q^{(k)}_{ij}|$ (for $i \leqslant j$) does not belong to X, then the matrices $Q^{(1)}, \dots, Q^{(n)}$ fulfill Assumption A and for all $(\Sigma, \Diamond)$, every stratum of $\mathcal{S}_{\Sigma,\Diamond}(Q^{(1)}, \dots, Q^{(n)})$ is regular.*

Proof. As previously, we fix a nonempty set $D \subset [d]$, $|D| = d'$, and we will present a construction of the stratum of X associated with D, denoted X_D. Take symmetric matrices $(Q^{(k)}) \in \mathbb{T}^{m\times m}$ such that the sequence $(|Q^{(k)}_{ij}|) \in \mathbb{T}^d$ has support equal to D. Take any nonempty subset $K \subset [n]$ and let $\mathcal{S}^{(K)}$ denote the set $\mathcal{S}((Q^{(k)})_{k\in K})$. Fix a pair $(\Sigma, \Diamond)$ and take the tropical Metzler spectrahedron $\mathcal{S}^{(K)}_{\Sigma,\Diamond}$. Take any $x \in \mathbb{R}^K$ and a graph $\vec{\mathcal{G}}_x$ associated with $\mathcal{S}^{(K)}_{\Sigma,\Diamond}$ (note that this graph has vertices enumerated by numbers from K). Suppose that this graph admits a circulation γ. As previously, for every edge $e = (\{k_1, k_2\}, l)$ of $\vec{\mathcal{G}}$ we can take $(i_e, j_e) \in \Sigma$ such that $Q^{(k_1)}_{i_e i_e} + Q^{(k_2)}_{j_e j_e} + x_{k_1} + x_{k_2} = 2|Q^{(l)}_{i_e j_e}| + 2x_l$. For every edge $e = (k, l)$ we have two possibilities: either there exists $i_e \in [m]$ such that $Q^{(k)}_{i_e i_e} + x_k = |Q^{(l)}_{i_e i_e}| + x_l$ or there exist $(i_e, j_e) \in \Sigma^{\complement}$ such that $|Q^{(k)}_{i_e j_e}| + x_k = |Q^{(l)}_{i_e j_e}| + x_l$. As before, we take the sum of these equalities weighted by γ. This gives the identity

$$\sum_{k\in[n]} \sum_{e\in\mathrm{Out}(k)} \gamma_e |Q^{(k)}_{i_e j_e}| = \sum_{l\in[n]} \sum_{e\in\mathrm{In}_1(l)} \gamma_e |Q^{(l)}_{i_e j_e}| + \sum_{l\in[n]} \sum_{e\in\mathrm{In}_2(l)} 2\gamma_e |Q^{(l)}_{i_e j_e}| \, .$$

As previously, the set of all $z \in \mathbb{R}^{d'}$ that fulfills this equality is a hyperplane. Indeed, any coefficient $z^{(k)}_{i_e j_e}$ which appears on the left-hand side does not appear on the right-hand side (note that here we use the fact that $\Sigma \cap \Sigma^{\complement} = \emptyset$). As before, we take all possible hypergraphs

[1] Note that the dependence on D lies here, as the choice of D restricts the amount of possible functions $e \to i_e$ $e \to (i_e, j_e)$.

(where "all possible" takes into account the fact that K can vary), one circulation for each hypergraph, all possible functions $e \to (i_e, j_e)$ (the amount of such functions depends on D), and all hyperplanes that can arise in this way. The union of these hyperplanes constitutes X_D. We deduce the result from Lemma 4.37. $\square$

4.4 Valuation of interior and regions of strict feasibility

To finish this chapter, we consider the problem of characterizing the image by valuation of the interior of a spectrahedron. As previously, let $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ be a sequence of symmetric matrices, denote $\boldsymbol{Q}(\boldsymbol{x}) \coloneqq \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)}$ and let $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ be the associated spectrahedral cone. We want to characterize the set $\mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}})\big)$. It turns out that the analysis done in the previous section extends to this problem if the matrices $\boldsymbol{Q}^{(k)}$ are Metzler, but does not solve the case of non-Metzler matrices. To partially handle this case, we study the image by valuation of strictly feasible points of $\boldsymbol{\mathcal{S}}$, and show that our techniques extend naturally to this setting. Let $Q^{(k)} \coloneqq \mathsf{sval}(\boldsymbol{Q}^{(k)})$ for all $k \in [n]$.

First, we suppose that the matrices $Q^{(k)}$ are Metzler. The following lemma extends the claim of Lemma 4.17.

Lemma 4.41. *Suppose that the matrices $Q^{(k)}$ are Metzler and let $\mathcal{T}$ be as in Definition 4.16. Then $\mathrm{cl}_{\mathbb{R}}(\mathcal{T} \cap \mathbb{R}^n) \subset \mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\big)$.*

Proof. If $x \in \mathcal{T} \cap \mathbb{R}^n$, then the proof of Lemma 4.17 shows that $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{\mathrm{in}}$ for any $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}^n_{>0}$. In particular, the ball $]t^{x_1}, 2t^{x_1}[\times \dots \times]t^{x_n}, 2t^{x_n}[$ belongs to $\boldsymbol{\mathcal{S}}^{\mathrm{in}}$ and hence $x \in \mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\big)$. The set $\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})$ is semialgebraic by Proposition 2.15 and hence $\mathrm{cl}_{\mathbb{R}}(\mathcal{T} \cap \mathbb{R}^n) \subset \mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\big)$ by Theorem 3.1. $\square$

As a corollary, we characterize $\mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\big)$ for matrices that satisfy the conditions of Theorem 4.20.

Corollary 4.42. *Suppose that the matrices $Q^{(k)}$ are Metzler and that they satisfy the conditions of Theorem 4.19. Then*

$$\mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\big) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n \, .$$

Proof. We trivially have $\mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}})\big) \subset \mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) \subset \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$. Furthermore, the proof of Theorem 4.19 shows that $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n = \mathrm{cl}_{\mathbb{R}}(\mathcal{T} \cap \mathbb{R}^n)$, and we have $\mathrm{cl}_{\mathbb{R}}(\mathcal{T} \cap \mathbb{R}^n) \subset \mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}})\big)$ by Lemma 4.41. $\square$

One may think that the claim of Corollary 4.42 extends to the case of non-Metzler matrices if we replace the conditions of Theorem 4.19 by the analogous conditions of Theorem 4.28. The following example shows that this is not true.

Example 4.43. Take the matrices

$$Q^{(0)} \coloneqq \begin{bmatrix} -\infty & a \\ a & -\infty \end{bmatrix}, \; Q^{(1)} \coloneqq \begin{bmatrix} -\infty & b \\ b & -\infty \end{bmatrix}, \; Q^{(2)} \coloneqq \begin{bmatrix} -\infty & \ominus c \\ \ominus c & -\infty \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$. The set $\mathcal{S}(Q^{(0)}, Q^{(1)}, Q^{(2)})$ fulfills the conditions of Theorem 4.28. Moreover, it is the set of all points $(x_0, x_1, x_2) \in \mathbb{T}^3$ such that $\max\{a + x_0, b + x_1\} = c + x_2$. Nevertheless, if we take any matrices $\boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)}, \boldsymbol{Q}^{(2)}$ such that $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$, then the associated spectrahedron is a plane,

$$\boldsymbol{\mathcal{S}} = \{(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{K}^3_{\geqslant 0} \colon \boldsymbol{Q}^{(0)}_{12}\boldsymbol{x}_0 + \boldsymbol{Q}^{(1)}_{12}\boldsymbol{x}_1 = \boldsymbol{Q}^{(2)}_{12}\boldsymbol{x}_2\} \, .$$

This shows that Corollary 4.42 does not carry over to non-Metzler matrices. Indeed, $\boldsymbol{\mathcal{S}}$ has empty interior but $\mathcal{S}(Q^{(0)}, Q^{(1)}, Q^{(2)}) \cap \mathbb{R}^n$ is nonempty.

Let us note, however, that we have the following partial extension of Lemma 4.41 to the case of non-Metzler matrices. (This extension will be useful in Section 7.4 where we relate the problem of deciding the feasibility to the problem of stochastic mean payoff games.)

Lemma 4.44. *Let $Q^{(1)}, \dots, Q^{(n)}$ be any tropical symmetric matrices. Put $\Sigma \coloneqq \{(i,j) \in [m]^2 \colon i < j\}$ and $\lozenge \coloneqq \emptyset$ and consider the set $\mathcal{T}_{\Sigma,\lozenge}$ defined as in Lemma 4.26. Then, we have $\mathrm{cl}_{\mathbb{R}}(\mathcal{T}_{\Sigma,\lozenge} \cap \mathbb{R}^n) \subset \mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}}^{\mathrm{in}})\big)$.*

Proof. By our choice of $(\Sigma, \lozenge)$ and Lemma 4.41, the set $\mathrm{cl}_{\mathbb{R}}(\mathcal{T}_{\Sigma,\lozenge} \cap \mathbb{R}^n)$ is included in the valuation of the interior of the set of points $\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0}$ defined by the inequalities

$$\begin{aligned} &\forall i, \boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0 \, , \\ &\forall i < j, \boldsymbol{Q}_{ii}(\boldsymbol{x})\boldsymbol{Q}_{jj}(\boldsymbol{x}) \geqslant (m-1)^2(\boldsymbol{Q}^{+}_{ij}(\boldsymbol{x}) + \boldsymbol{Q}^{-}_{ij}(\boldsymbol{x}))^2 \, . \end{aligned}$$

This set is included in $\boldsymbol{\mathcal{S}}^{\mathrm{in}}$ because $(\boldsymbol{Q}^{+}_{ij}(\boldsymbol{x}) + \boldsymbol{Q}^{-}_{ij}(\boldsymbol{x}))^2 \geqslant (\boldsymbol{Q}_{ij}(\boldsymbol{x}))^2$ for all $\boldsymbol{x}$. □

Remark 4.45. Let us point out why the lemma above does not extend to other choices of Σ. Indeed, if we take a different Σ, fix $x \in \cap_{\lozenge}\mathrm{cl}_{\mathbb{R}}(\mathcal{T}_{\Sigma,\lozenge} \cap \mathbb{R}^n)$, and try to repeat the proof of Lemma 4.26, then we can construct a point $\boldsymbol{z} \in \mathbb{K}^n_{>0}$ that belongs to the interior of the set

$$\{\boldsymbol{y} \in \mathbb{K}^n_{\geqslant 0} \colon \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{y}) \geqslant 0 \wedge \forall (i,j) \in \Sigma, \boldsymbol{Q}_{ii}(\boldsymbol{y})\boldsymbol{Q}_{jj}(\boldsymbol{y}) \geqslant (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{y}))^2\} \tag{4.9}$$

and satisfies $\boldsymbol{Q}_{ij}(\boldsymbol{z}) = 0$ for all $(i,j) \in \Sigma^{\complement}$. However, this does not imply that $\boldsymbol{z}$ belongs to the interior of $\boldsymbol{\mathcal{S}}^{\mathrm{in}}$, as shown by Example 4.43. In this example, if we take $\Sigma \coloneqq \emptyset$, then the set given by (4.9) is equal to $\mathbb{K}^3_{\geqslant 0}$, but $\boldsymbol{\mathcal{S}}^{\mathrm{in}}$ is a plane (and $\boldsymbol{z}$ is a point that belongs to this plane). This problem is avoided in cases where the weak inequalities that define (4.9) can be replaced strict inequalities, as discussed in the sequel.

Let us switch our attention to the problem of characterizing the valuation of strictly feasible points of a spectrahedron. To do so, let us recall that a matrix $\boldsymbol{A} \in \mathbb{K}^{m \times m}_{\geqslant 0}$ is called *positive definite* if it is positive semidefinite and invertible. By the completeness of the theory of real closed fields, this is equivalent to demanding that all principal minors of $\boldsymbol{A}$ are positive, that the inequality $\boldsymbol{x}^{\intercal}\boldsymbol{A}\boldsymbol{x} > 0$ is true for all $\boldsymbol{x} \neq 0$ and so on (see, e.g., [Mey00, Section 7.6]). Moreover, let us recall the following definition.

Definition 4.46. We say that $\boldsymbol{\mathcal{S}}$ is *strictly feasible* if there exists a point $\boldsymbol{x} \in \mathbb{K}^n_{>0}$ such that the matrix $\boldsymbol{Q}(\boldsymbol{x})$ is positive definite.

Let $\boldsymbol{\mathcal{S}}^{++} \subset \mathbb{K}^n_{>0}$ be the set of all strictly feasible points of $\boldsymbol{\mathcal{S}}$, i.e.,

$$\boldsymbol{\mathcal{S}}^{++} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \boldsymbol{Q}(\boldsymbol{x}) \text{ is positive definite}\} \, .$$

It is easy to check that $\boldsymbol{\mathcal{S}}^{++}$ is convex. Even more, if $\boldsymbol{\mathcal{S}}^{++}$ is nonempty, then it is equal to the interior of $\boldsymbol{\mathcal{S}}$.Therefore, it may seem that studying $\mathsf{val}(\boldsymbol{\mathcal{S}}^{++})$ is very similar to studying $\mathsf{val}\big(\mathrm{int}(\boldsymbol{\mathcal{S}})\big)$. However, there are spectrahedra that cannot be strictly feasible for trivial reasons. For instance, if there exists $i \in [m]$ such that $\boldsymbol{Q}^{(k)}_{ii} = 0$ for all k, then the set $\boldsymbol{\mathcal{S}}^{++}$ is trivially empty (because the matrix $\boldsymbol{Q}(\boldsymbol{x})$ has a zero entry on its diagonal). Therefore, it is natural to make the following assumption.

Assumption B. For every $i \in [m]$, there exists $k \in [n]$ such that $\boldsymbol{Q}_{ii}^{(k)} \neq 0$.

Let us point out that the matrices given in Example 4.43 do not satisfy Assumption B. In what follows, we show that the behavior of Example 4.43 cannot be reproduced by matrices that satisfy Assumption B. To start, we point out that the notion of diagonal dominance extends to the case of positive definite matrices.

Lemma 4.47. *Suppose that a symmetric matrix $A \in \mathbb{R}^{m\times m}$ is strictly diagonally dominant, i.e., satisfies the inequality $A_{ii} > \sum_{j\neq i} |A_{ij}|$ for all $i \in [m]$. Then A is positive definite.*

Proof. Matrix A is positive semidefinite by Lemma 4.9. It is nonsingular by [Mey00, Example 4.3.3]. □

As a corollary, we get the following results that can be proven as in Lemma 4.8 and Corollary 4.11.

Lemma 4.48. *Let $\boldsymbol{A} \in \mathbb{K}^{m\times m}$ be a symmetric matrix. Suppose that $\boldsymbol{A}$ has positive entries on its diagonal and that the inequality $\boldsymbol{A}_{ii}\boldsymbol{A}_{jj} > (m-1)^2\boldsymbol{A}_{ij}^2$ holds for all pairs (i,j) such that $i \neq j$. Then $\boldsymbol{A}$ is positive definite.*

Corollary 4.49. *Let $A \in \mathbb{T}_{\pm}^{m\times m}$ be a symmetric matrix such that $A_{ii} \in \mathbb{T}_{+}$ for all i and $A_{ii} \odot A_{jj} > A_{ij}^{\odot 2}$ for all $i < j$. Let $\boldsymbol{A} \in \mathbb{K}^{m\times m}$ be any symmetric matrix such that $\mathsf{sval}(\boldsymbol{A}) = A$. Then $\boldsymbol{A}$ fulfills the conditions of Lemma 4.48. (In particular, it is positive definite.)*

We will now give the analogues of Lemmas 4.17 and 4.26 for the regions of strict feasibility. As usual, we first consider the case of Metzler matrices.

Lemma 4.50. *Suppose that the matrices $Q^{(1)}, \dots, Q^{(n)}$ are Metzler and satisfy Assumption B. Let $\mathcal{T}$ be as in Definition 4.16. If $x \in \mathcal{T} \cap \mathbb{R}^n$ and $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}_{>0}^n$ is any lift, then $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{++}$. Moreover, we have $\mathrm{cl}_{\mathbb{R}}(\mathcal{T} \cap \mathbb{R}^n) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{++})$.*

Proof. The proof of Lemma 4.17 shows that if $x \in \mathcal{T} \cap \mathbb{R}^n$, and $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}_{>0}^n$ is any lift of x, then the matrix $A := \mathsf{sval}(\boldsymbol{Q}(\boldsymbol{x}))$ is such that $A_{ii} = Q_{ii}^{+}(x) \geqslant Q_{ii}^{-}(x)$ for all i. Moreover, by Assumption B (and the fact that $x \in \mathbb{R}^n$) the diagonal entries of A are finite, $A_{ii} \in \mathbb{T}_{+}$ for all i. Furthermore, the proof of Lemma 4.17 shows that $A_{ii} \odot A_{jj} > A_{ij}^{\odot 2}$ for all $i < j$. Hence, by Corollary 4.49, $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}^{++}$. Moreover, since $\boldsymbol{\mathcal{S}}^{++}$ is semialgebraic, its image by valuation $\mathsf{val}(\boldsymbol{\mathcal{S}}^{++})$ is closed in $\mathbb{R}^n$ (Theorem 3.1) and the claim follows. □

In the following lemma we abandon the assumption that the matrices are Metzler.

Lemma 4.51. *Suppose that the matrices $Q^{(1)}, \dots, Q^{(n)}$ satisfy Assumption B. Let $\mathcal{T}_{\Sigma,\lozenge}$ be as in Lemma 4.26. Then, we have the inclusion*

$$\bigcup_{\Sigma}\bigcap_{\lozenge} \mathrm{cl}_{\mathbb{R}}(\mathcal{T}_{\Sigma,\lozenge} \cap \mathbb{R}^n) \subset \mathsf{val}(\boldsymbol{\mathcal{S}}^{++}) \, .$$

Proof. Fix any Σ. By repeating the proof of Lemma 4.26 (replacing Lemma 4.17 by Lemma 4.50) we find a point $\boldsymbol{z} \in \mathsf{val}(x)^{-1} \cap \mathbb{K}^n_{>0}$ such that[2]

$$\begin{aligned}
\forall i, \boldsymbol{Q}_{ii}(\boldsymbol{z}) > 0\,,\\
\forall (i,j) \in \Sigma, \boldsymbol{Q}_{ii}(\boldsymbol{z})\boldsymbol{Q}_{jj}(\boldsymbol{z}) > (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{z}))^2\,,\\
\forall (i,j) \in \Sigma^{\complement}, \boldsymbol{Q}_{ij}(\boldsymbol{z}) = 0\,.
\end{aligned}$$

Hence $\boldsymbol{Q}_{ii}(\boldsymbol{z})\boldsymbol{Q}_{jj}(\boldsymbol{z}) > (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{z}))^2$ for all $i < j$ and $\boldsymbol{z}$ is strictly feasible by Lemma 4.48. ☐

Corollary 4.52. *Suppose that the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy Assumption B and the conditions of Theorem 4.28. Then*

$$\mathsf{val}(\boldsymbol{\mathcal{S}}^{++}) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n\,.$$

Proof. The proof of Theorem 4.28 shows that $\mathsf{val}(\boldsymbol{\mathcal{S}}^{++}) \subset \mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ and that $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n = \bigcup_\Sigma \bigcap_\Diamond \mathrm{cl}_\mathbb{R}(\mathcal{T}_{\Sigma,\Diamond} \cap \mathbb{R}^n)$. Hence, the claim follows from Lemma 4.51. ☐

[2] This proof requires to check that the set

$$\{\boldsymbol{y} \in \mathbb{K}^n_{>0} \colon \forall i, \boldsymbol{Q}_{ii}(\boldsymbol{y}) > 0 \land \forall (i,j) \in \Sigma, \boldsymbol{Q}_{ii}(\boldsymbol{y})\boldsymbol{Q}_{jj}(\boldsymbol{y}) > (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{y}))^2\}$$

is convex. Note that this is true because for every $(i,j) \in \Sigma$ the set

$$\{\boldsymbol{y} \in \mathbb{K}^n_{>0} \colon \boldsymbol{Q}_{ii}(\boldsymbol{y}) > 0, \boldsymbol{Q}_{jj}(\boldsymbol{y}) > 0, \boldsymbol{Q}_{ii}(\boldsymbol{y})\boldsymbol{Q}_{jj}(\boldsymbol{y}) > (m-1)^2(\boldsymbol{Q}_{ij}(\boldsymbol{y}))^2\}$$

is the set of strictly feasible points of a spectrahedron defined by matrices of size 2×2.

CHAPTER 5

Tropical analogue of the Helton–Nie conjecture

As discussed in Section 1.1, an important question in semidefinite optimization consists in characterizing the sets that arise as projections of spectrahedra [Nem07]. Helton and Nie [HN09] conjectured that every convex semialgebraic set is a projected spectrahedron. The conjecture has been recently disproved by Scheiderer [Sch18b], who showed that the cone of positive semidefinite forms cannot be expressed as a projection of a spectrahedron, except in some particular cases.

Theorem 5.1 ([Sch18b])**.** *The cone of positive semidefinite forms of degree $2d$ in n variables can be expressed as a projection of a spectrahedron only when $2d = 2$ or $n \leqslant 2$ or $(n, 2d) = (3, 4)$.*

In this section we study the tropicalizations of convex semialgebraic sets and we show the following theorem, which may be thought of as a “Helton–Nie conjecture for valuations.”

Theorem 5.2. *Let $\mathscr{K}$ be a real closed valued field equipped with a nontrivial and convex valuation $\mathsf{val}\colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ and suppose that $S \subset \mathscr{K}^n$ is a convex semialgebraic set. Then, there exists a projected spectrahedron $S' \subset \mathscr{K}^n$ such that $\mathsf{val}(S) = \mathsf{val}(S')$.*

As in Chapter 3, in order to prove Theorem 5.2 we first study the case of Puiseux series $\mathscr{K} = \mathbb{K}$ and then use model theory to generalize the result to other fields. Along the way, we obtain a more precise characterization of sets that arise as tropicalizations of convex semialgebraic sets over $\mathbb{K}$.

Definition 5.3. We say that a set $S \subset \mathbb{T}^n$ is a *tropicalization of a convex semialgebraic set* if there exists a convex semialgebraic set $\boldsymbol{S} \subset \mathbb{K}^n$ such that $\mathsf{val}(\boldsymbol{S}) = S$.

Definition 5.4. We say that a tropical Metzler spectrahedron $\mathcal{S} \subset \mathbb{T}^n$ is *real* if it is included in $\mathbb{R}^n$.

Our characterization of tropicalizations of convex semialgebraic sets is given in the next result.

Theorem 5.5. *Fix a set $\mathcal{S} \subset \mathbb{T}^n$. Then, the following conditions are equivalent:*
(a) $\mathcal{S}$ is a tropicalization of a convex semialgebraic set;
(b) $\mathcal{S}$ is tropically convex and has closed semilinear strata;
(c) $\mathcal{S}$ is tropically convex and every stratum of $\mathcal{S}$ is a projection of a real tropical Metzler spectrahedron;
(d) $\mathcal{S}$ is a projection of a tropical Metzler spectrahedron;
(e) there exists a projected spectrahedron $\boldsymbol{S} \subset \mathbb{K}^n_{\geqslant 0}$ such that $\mathsf{val}(\boldsymbol{S}) = \mathcal{S}$.

The rest of this chapter is organized as follows. In Section 5.1 we recall some basic notions of tropical convexity. Then, in Sections 5.2 and 5.3 we study real tropical cones and show that these objects can be described by monotone homogeneous operators and by graphs. Then, we show Theorems 5.2 and 5.5. This is done is three steps. First, we prove a simpler variant of Theorem 5.5 for real tropical cones (Section 5.4), then we prove both theorems for Puiseux series (Section 5.5), and finally we extend the result to more general fields (Section 5.6). This chapter is based on the article [AGS19].

Before starting, let us point out that the result of Scheiderer generalizes to all real closed fields. In other words, the Helton–Nie conjecture is false over every such field.

Corollary 5.6 (of [Sch18b, Corollary 4.25])**.** *Let $\mathscr{K}$ be a real closed field. Then, the cone of positive semidefinite forms of degree $2d$ in n variables over $\mathscr{K}$ can be expressed as a projection of a spectrahedron over $\mathscr{K}$ only when $2d = 2$ or $n \leqslant 2$ or $(n, 2d) = (3, 4)$.*

Proof. Fix the integers d, m, n, and p. The statement "the cone of positive semidefinite forms of degree $2d$ in n variables over $\mathscr{K}$ is the projection of a spectrahedron in $\mathscr{K}^p$ associated with matrices of size $m \times m$" is a sentence in the language of ordered rings $\mathcal{L}_{\mathrm{or}}$. Since the theory of real closed fields is complete (Theorem 2.110), this sentence is true over $\mathbb{R}$ if and only if it is true over $\mathscr{K}$. □

5.1 Tropical convexity

In this section, we recall some basic facts about convexity in the usual and tropical sense. The convex hull of a set $\boldsymbol{X} \subset \mathbb{K}^n$, denoted $\mathrm{conv}(\boldsymbol{X})$, can be defined as the smallest (inclusionwise) convex set that contains $\boldsymbol{X}$. This set is characterized by Carathéodory's theorem.

Theorem 5.7 (Carathéodory's theorem, [Sch87, Corollary 7.1j])**.** *If $\boldsymbol{X} \subset \mathbb{K}^n$, then we have the equality*

$$\mathrm{conv}(\boldsymbol{X}) = \Big\{ \sum_{k=1}^{n+1} \boldsymbol{\lambda}_k \boldsymbol{x}^{(k)} \in \mathbb{K}^n \colon \forall k,\, \boldsymbol{x}^{(k)} \in \boldsymbol{X} \,\wedge\, \forall k,\, \boldsymbol{\lambda} \geqslant 0 \,\wedge\, \sum_{k=1}^{n+1} \boldsymbol{\lambda}_k = 1 \Big\} .$$

Remark 5.8. We point out that the proof of Theorem 5.7 given in [Sch87, Corollary 7.1j] is valid over every ordered field.

As a corollary, we obtain that the class of semialgebraic sets is closed under taking convex hulls.

Lemma 5.9. *If $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is a semialgebraic set, then* $\operatorname{conv}(\boldsymbol{\mathcal{S}})$ *is also semialgebraic.*

Proof. By Theorem 5.7, $\operatorname{conv}(\boldsymbol{\mathcal{S}})$ is definable in $\mathcal{L}_{\mathrm{or}}$. Hence, it is semialgebraic by Lemma 2.111. □

Let us now move to tropical convexity, referring the reader to [CGQ04, DS04] for more information.

Definition 5.10. We say that a set $X \subset \mathbb{T}^n$ is *tropically convex* if for every $x, y \in X$ and every $\lambda, \mu \in \mathbb{T}$ such that $\lambda \oplus \mu = 0$ the point $(\lambda \odot x) \oplus (\mu \odot y)$ belongs to X. We say that $X \subset \mathbb{T}^n$ is a *tropical (convex) cone* if $(\lambda \odot x) \oplus (\mu \odot y) \in X$ for all $\lambda, \mu \in \mathbb{T}$.

Remark 5.11. The quantity $(\lambda \odot x) \oplus (\mu \odot y)$ for $\lambda \oplus \mu = 0$ corresponds to the tropical analogue of a convex combination of x and y. Indeed, the scalars λ and μ are implicitly "nonnegative" in the tropical sense, as they are greater than or equal to the tropical zero element $-\infty$. Besides, their tropical sum equals the tropical unit 0.

Example 5.12. Any tropical Metzler spectrahedral cone is a tropical cone. Indeed, if we use the notation as in Definition 4.13, then for all $x, y \in \mathbb{T}^n$, $\lambda, \mu \in \mathbb{T}$, and all $i, j \in [m]$ such that $i \leqslant j$ we have $Q_{ij}^-\big((\lambda \odot x) \oplus (\mu \odot y)\big) = (\lambda \odot Q_{ij}^-(x)) \oplus (\mu \odot Q_{ij}^-(y))$ and the same is true for Q_{ii}^+. Hence, if $x, y \in \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$, then

$$\begin{aligned} Q_{ii}^+\big((\lambda \odot x) \oplus (\mu \odot y)\big) &= (\lambda \odot Q_{ii}^+(x)) \oplus (\mu \odot Q_{ii}^+(y)) \\ &\geqslant (\lambda \odot Q_{ii}^-(x)) \oplus (\mu \odot Q_{ii}^-(y)) = Q_{ii}^-\big((\lambda \odot x) \oplus (\mu \odot y)\big) \end{aligned}$$

for all $i \in [m]$ and

$$\begin{aligned} &Q_{ii}^+\big((\lambda \odot x) \oplus (\mu \odot y)\big) \odot Q_{jj}^+\big((\lambda \odot x) \oplus (\mu \odot y)\big) \\ &\qquad = \Big((\lambda \odot Q_{ii}^+(x)) \oplus (\mu \odot Q_{ii}^+(y))\Big) \odot \Big((\lambda \odot Q_{jj}^+(x)) \oplus (\mu \odot Q_{jj}^+(y))\Big) \\ &\qquad \geqslant (\lambda^{\odot 2} \odot Q_{ii}^+(x) \odot Q_{jj}^+(x)) \oplus (\mu^{\odot 2} \odot Q_{ii}^+(y) \odot Q_{jj}^+(y)) \\ &\qquad \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2} \oplus (\mu \odot Q_{ij}(x))^{\odot 2} \\ &\qquad = \Big((\lambda \odot Q_{ij}(x)) \oplus (\mu \odot Q_{ij}(x))\Big)^{\odot 2} \\ &\qquad = \Big(Q_{ij}^-\big((\lambda \odot x) \oplus (\mu \odot y)\big)\Big)^{\odot 2} \end{aligned}$$

for all $i < j$. A more abstract proof of the same fact can be done using Lemma 5.18 below combined with Proposition 4.14.

Definition 5.13. For any set $X \subset \mathbb{T}^n$ we can define its *tropical convex hull*, denoted $\operatorname{tconv}(X)$, as the smallest (inclusionwise) tropically convex set that contains X.

Remark 5.14. The tropical convex hull is well defined because the intersection of any number of tropically convex sets is tropically convex.

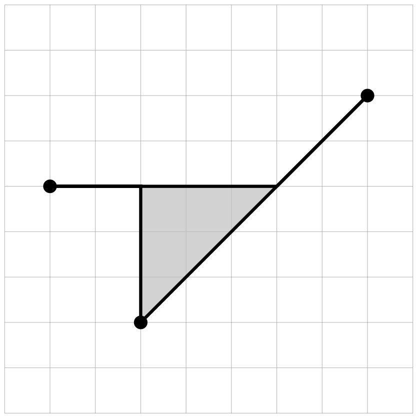

Figure 5.1: Tropical convex hull of three points.

Carathéodory's theorem is still true in the tropical setting:

Theorem 5.15 ([Hel88], [BH04], [DS04])**.** *If $X \subset \mathbb{T}^n$, then we have the equality*

$$\operatorname{tconv}(X) = \Big\{ \bigoplus_{k=1}^{n+1} (\lambda_k \odot x^{(k)}) \colon \forall k,\, x^{(k)} \in X \,\wedge\, \bigoplus_{k=1}^{n+1} \lambda_k = 0 \Big\} .$$

Sketch of the proof. For every $p \geqslant 1$ we denote

$$Y_p = \Big\{ \bigoplus_{k=1}^{p} (\lambda_k \odot x^{(k)}) \colon \forall k,\, x^{(k)} \in X \,\wedge\, \bigoplus_{k=1}^{p} \lambda_k = 0 \Big\} .$$

An easy induction shows that $Y_p \subset \operatorname{tconv}(X)$ for all $p \geqslant 0$. Moreover, the set $\bigcup_{p=1}^{\infty} Y_p$ is tropically convex and hence $\operatorname{tconv}(X) = \bigcup_{p=1}^{\infty} Y_p$. Suppose that $y \in Y_p$ for some $p \geqslant 1$. For every coordinate $l \in [n]$ we can find $k(l) \in [p]$ such that $y_l = \lambda_{k(l)} \odot x_l^{k(l)}$. Moreover, there exists $k^* \in [p]$ such that $\lambda_{k^*} = 0$. Hence $y = (\bigoplus_{l=1}^{p} (\lambda_{k(l)} \odot x^{k(l)})) \oplus x^{(k^*)}$, which implies that $y \in Y_{n+1}$. □

Example 5.16. Figure 5.1 depicts a tropical convex hull of three points: $(1, 5)$, $(3, 2)$, and $(8, 7)$. Note that the point $(4, 4)$ can be written as $(4, 4) = ((-4) \odot (8, 7)) \oplus ((-1) \odot (1, 5)) \oplus (3, 2)$.

We also need the following lemma.

Lemma 5.17. *Suppose that sets $X, Y \subset \mathbb{T}^n$ are tropically convex. Then, we have the equality*

$$\operatorname{tconv}(X \cup Y) = \{(\lambda \odot x) \oplus (\mu \odot y) \in \mathbb{T}^n \colon x \in X, y \in Y, \lambda \oplus \mu = 0\} .$$

Sketch of the proof. The inclusion $\supset$ follows immediately from the definition of tropical convex hull. The other inclusion holds because the set on the right-hand side contains X and Y and is tropically convex. □

A relation between the convexity in $\mathbb{K}$ and the tropical convexity is shown in the next two lemmas.

Lemma 5.18. *If $\boldsymbol{X} \subset \mathbb{K}^n$ is a convex set, then $\mathsf{val}(\boldsymbol{X})$ is tropically convex.*

Proof. Let $x, y \in \mathsf{val}(\boldsymbol{X})$ and take any $\lambda, \mu \in \mathbb{T}$ such that $\lambda \oplus \mu = 0$. Without loss of generality, suppose that $\lambda = 0$. Take any points $\boldsymbol{x} \in \boldsymbol{X} \cap \mathsf{val}^{-1}(x)$ and $\boldsymbol{y} \in \boldsymbol{X} \cap \mathsf{val}^{-1}(y)$. Let us look at two cases. If $\mu < 0$, then for any real positive constant c, we have $1 - ct^{\mu} > 0$, and so the point $\boldsymbol{z} = (1 - ct^{\mu})\boldsymbol{x} + ct^{\mu}\boldsymbol{y}$ belongs to $\boldsymbol{X}$. Hence, if we choose c such that $c \neq -\mathsf{lc}(\boldsymbol{x}_k)/\mathsf{lc}(\boldsymbol{y}_k)$ for all $k \in [n]$ satisfying $\boldsymbol{y}_k \neq 0$, then $\mathsf{val}(\boldsymbol{z}) = (\lambda \odot x) \oplus (\mu \odot y)$. If $\mu = 0$, then we take a real constant $c \in \,]0, 1[$ such that for all $k \in [n]$ satisfying $\boldsymbol{y}_k \neq 0$ we have $c/(1-c) \neq -\mathsf{lc}(\boldsymbol{x}_k)/\mathsf{lc}(\boldsymbol{y}_k)$. Then, the point $\boldsymbol{z} = (1-c)\boldsymbol{x} + c\boldsymbol{y}$ belongs to $\boldsymbol{X}$ and we have $\mathsf{val}(\boldsymbol{z}) = (\lambda \odot x) \oplus (\mu \odot y)$. ☐

The next lemma shows that a tighter relation holds for sets included in the nonnegative orthant of $\mathbb{K}$.

Lemma 5.19. *If $\boldsymbol{X} \subset \mathbb{K}^n_{\geqslant 0}$ is any set, then we have $\mathsf{val}(\mathrm{conv}(\boldsymbol{X})) = \mathrm{tconv}(\mathsf{val}(\boldsymbol{X}))$.*

Proof. We start by proving the inclusion $\subset$. Take a point $\boldsymbol{y} \in \mathrm{conv}(\boldsymbol{X})$. By Theorem 5.7, there exist $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{n+1} \geqslant 0$ and $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n+1)} \in \boldsymbol{X}$ such that $\boldsymbol{y} = \boldsymbol{\lambda}_1\boldsymbol{x}^{(1)} + \dots + \boldsymbol{\lambda}_{n+1}\boldsymbol{x}^{(n+1)}$. Hence, using the fact that $\boldsymbol{X} \subset \mathbb{K}^n_{\geqslant 0}$, we have

$$\mathsf{val}(\boldsymbol{y}) = \big(\mathsf{val}(\boldsymbol{\lambda}_1) \odot \mathsf{val}(\boldsymbol{x}_1)\big) \oplus \dots \oplus \big(\mathsf{val}(\boldsymbol{\lambda}_{n+1}) \odot \mathsf{val}(\boldsymbol{x}_{n+1})\big) \, .$$

Furthermore, we have $\sum_{k=1}^{n+1} \boldsymbol{\lambda}_k = 1$ and hence $\bigoplus_{k=1}^{n+1} \mathsf{val}(\boldsymbol{\lambda}_k) = 0$. Therefore, $\mathsf{val}(\boldsymbol{y}) \in \mathrm{tconv}(X)$ by Theorem 5.15. Conversely, take any point $y \in \mathrm{tconv}(X)$. By Theorem 5.15, we can find $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{T}$, $\bigoplus_{k=1}^{n+1} \lambda_k = 0$ and $x^{(1)}, \dots, x^{(n+1)} \in X$ such that $y = (\lambda_1 \odot x^{(1)}) \oplus \dots (\lambda_{n+1} \odot x^{(n+1)})$. We define $\boldsymbol{\lambda}_k := t^{\lambda_k}/(\sum_{l=1}^{n+1} t^{\lambda_l})$. Observe that for all k, $\mathsf{val}(\boldsymbol{\lambda}_k) = \lambda_k$ because the term $\sum_{l=1}^{n+1} t^{\lambda_l}$ has valuation $\bigoplus_{l=1}^{n+1} \lambda_l = 0$. Moreover, we have $\boldsymbol{\lambda}_k \geqslant 0$ and $\sum_{k=1}^{n+1} \boldsymbol{\lambda}_k = 1$. Take any points $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n+1)} \in \boldsymbol{X}$ such that $\mathsf{val}(\boldsymbol{x}^{(k)}) = x^{(k)}$ for all $k \in [n+1]$. The point $\boldsymbol{y} = \boldsymbol{\lambda}_1\boldsymbol{x}_1 + \dots + \boldsymbol{\lambda}_{n+1}\boldsymbol{x}_{n+1}$ belongs to $\mathrm{conv}(\boldsymbol{X})$ and verifies $\mathsf{val}(\boldsymbol{y}) = y$. ☐

Example 5.20. The assumption that $\boldsymbol{X}$ lies in the nonnegative orthant cannot be omitted. To see this, take $\boldsymbol{X} = \{-1, 1\} \subset \mathbb{K}$. We have $\mathsf{val}(\boldsymbol{X}) = \{0\} = \mathrm{tconv}(\mathsf{val}(\boldsymbol{X}))$, but $\mathsf{val}(\mathrm{conv}(\boldsymbol{X})) = \mathsf{val}([-1, 1]) = [-\infty, 0]$.

The last two lemmas, together with Theorem 3.1, allow us to give our first characterization of tropicalizations of convex semialgebraic sets.

Proposition 5.21. *A set $\mathscr{S} \subset \mathbb{T}^n$ is a tropicalization of a convex semialgebraic set if and only if $\mathscr{S}$ is tropically convex and every stratum of $\mathscr{S}$ is a closed semilinear set.*

Proof. The "only if" part follows from Theorem 3.1 and Lemma 5.18. To prove the opposite implication, suppose that $\mathscr{S}$ is tropically convex and has closed semilinear strata. Then, Lemma 3.13 shows that there exists a semialgebraic set $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ included in the nonnegative orthant of $\mathbb{K}^n$ and such that $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathscr{S}$. The set $\mathrm{conv}(\boldsymbol{\mathcal{S}})$ is semialgebraic by Lemma 5.9 and satisfies $\mathsf{val}(\mathrm{conv}(\boldsymbol{\mathcal{S}})) = \mathrm{tconv}(\mathscr{S}) = \mathscr{S}$ by Lemma 5.19. ☐

Remark 5.22. The considerations of this section extend to the case of tropicalization of convex cones. Indeed, if $X \subset \mathbb{T}^n$ is any set, the we can define its *tropical conic hull*, $\mathrm{tcone}(X)$ as the smallest tropical cone that contains X. In this case, the tropical Carathéodory theorem states that we have the equality

$$\mathrm{tcone}(X) = \Big\{ \bigoplus_{k=1}^{n} (\lambda_k \odot x^{(k)}) \colon \forall k,\, x^{(k)} \in X, \lambda_k \in \mathbb{T} \Big\} \, .$$

The analogues of the other propositions stated above are also true for cones. If $\boldsymbol{X} \subset \mathbb{K}^n$ is a convex cone, then $\mathsf{val}(\boldsymbol{X})$ is a tropical cone. Indeed, if we take $x, y \in \mathsf{val}(\boldsymbol{X})$, $\lambda, \mu \in \mathbb{T}$, and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ such that $\mathsf{val}(\boldsymbol{x}) = x$, $\mathsf{val}(\boldsymbol{y}) = y$, then there is $c > 0$ such that $\boldsymbol{z} := t^{\lambda}\boldsymbol{x} + ct^{\mu}\boldsymbol{y} \in \boldsymbol{X}$ and $\mathsf{val}(\boldsymbol{z}) = (\lambda \odot x) \oplus (\mu \odot y)$ (we choose c such that $c \neq -\mathsf{lc}(\boldsymbol{x}_k)/\mathsf{lc}(\boldsymbol{y}_k)$ for all $k \in [n]$ satisfying $\boldsymbol{y}_k \neq 0$). Moreover, if $\boldsymbol{X} \subset \mathbb{K}^n_{\geqslant 0}$ is any set included in the nonnegative orthant and $\mathrm{cone}(\cdot)$ denotes the conic hull in $\mathbb{K}^n$, then we have $\mathsf{val}(\mathrm{cone}(\boldsymbol{X})) = \mathrm{tcone}(\mathsf{val}(\boldsymbol{X}))$. The proof of this fact is done as in Lemma 5.19, replacing the Carathéodory's theorem by its analogue for cones [Sch87, Corollary 7.1i]. In particular, by repeating the proof of Proposition 5.21, we get that a set $\mathcal{S} \subset \mathbb{T}^n$ is a tropicalization of a convex semialgebraic cone if and only if $\mathcal{S}$ is a tropical cone that has closed semilinear strata.

5.2 Real tropical cones as sublevel sets of dynamic programming operators

As a first step towards the proof of the tropical Helton–Nie conjecture, we study the case of real tropical cones. We will show that semilinear real tropical cones are characterized as sublevel sets of semilinear, monotone, homogeneous operators. The next definitions and lemmas introduce these class of cones and operators and their basic properties.

Definition 5.23. We say that a set $X \subset \mathbb{R}^n$ is a *real tropical cone* if for every $x, y \in X$ and every $\lambda, \mu \in \mathbb{R}$ we have $(\lambda \odot x) \oplus (\mu \odot y) \in X$.

Remark 5.24. We point out that a real tropical cone is nothing but the main stratum of a tropical cone as defined in Section 5.1. Indeed, if Y is a tropical cone, then $Y \cap \mathbb{R}^n$ is a real tropical cone, whereas if X is a real tropical cone, then $X \cup \{-\infty\}$ is a tropical cone.

Definition 5.25. We say that a function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is *piecewise affine* if there exists a set of full-dimensional polyhedra $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(p)} \subset \mathbb{R}^n$ satisfying $\bigcup_{s=1}^{p} \mathcal{W}^{(s)} = \mathbb{R}^n$ and such that the restriction of F to $\mathcal{W}^{(s)}$ is affine, i.e., $F_{|\mathcal{W}^{(s)}}(x) = A^{(s)}x + b^{(s)}$ for some matrix $A^{(s)} \in \mathbb{R}^{m\times n}$ and vector $b^{(s)} \in \mathbb{R}^m$. We shall say that the family $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_s$ is a *piecewise description* of the function F.

Remark 5.26. We point out that our definition implies that piecewise affine functions are continuous (because the polyhedra $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(p)}$ are closed).

The following minimax representation result was proved by Ovchinnikov [Ovc02]. In the sequel, we use the notation $F(x) = (F_1(x), \dots, F_m(x))$.

Theorem 5.27 ([Ovc02])**.** *Suppose that the function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is piecewise affine, and let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ be a piecewise description of F. Then, for every $k \in [m]$ there exists a number $2^p \geqslant M_k \geqslant 1$ and a family $\{S_{ki}\}_{i\in[M_k]}$ of subsets of $[p]$ such that for all $x \in \mathbb{R}^n$ we have*

$$F_k(x) = \min_{i\in[M_k]} \max_{s\in S_{ki}} (A_k^{(s)}x + b_k^{(s)})\,.$$

Definition 5.28. We say that a function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is *semilinear* if its graph $\{(x, y) \in \mathbb{R}^{n\times m}\colon y = F(x)\}$ is a semilinear set.

The next lemma shows that continuous semilinear functions are piecewise affine.

Lemma 5.29. *Suppose that the continuous function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is semilinear. Then, it is piecewise affine. Moreover, it admits a piecewise description $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ such that the polyhedra $\mathcal{W}^{(s)}$ are semilinear, and the matrices $A^{(s)}$ are rational.*

Proof. Since F is continuous and semilinear, the graph of F is a closed semilinear set. Therefore, by Lemma 3.12, it is a finite union of semilinear polyhedra. Let $\{(x, y)\colon Bx + Cy \geqslant d\}$, where $B \in \mathbb{Q}^{p\times n}$, $C \in \mathbb{Q}^{p\times m}$, $d \in \mathbb{R}^p$ be one of these polyhedra. If we fix $\bar{x}$, then, by the definition of a graph, the polyhedron consisting of all y such that $Cy \geqslant d - B\bar{x}$ reduces to a point $\bar{y}$. Thus, if $J \subset [p]$ denotes the maximal set such that $C_J\bar{y} = d_J - B_J\bar{x}$, then this system of affine equalities has a unique solution. Hence, there exists a set $I \subset J$, $|I| = m$ such that the matrix $C_I \in \mathbb{Q}^{m\times m}$ is invertible and satisfies $\bar{y} = C_I^{-1}(d_I - B_I\bar{x}) = C_I^{-1}d_I - C_I^{-1}B_I\bar{x}$. In other words, the graph of F is a finite union of polyhedra of the form

$$\mathcal{W} = \{(x, y)\colon Bx + Cy \geqslant d, y = C_I^{-1}d_I - C_I^{-1}B_Ix\} ,$$

where C_I is an invertible submatrix of C. As a result, if $\pi\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ denotes the projection on the first n coordinates, and $x \in \pi(\mathcal{W})$ is any point, then we have $F(x) = C_I^{-1}d_I - C_I^{-1}B_Ix$. By eliminating the polyhedra $\pi(\mathcal{W})$ that are not full dimensional,[1] we obtain a piecewise description of F satisfying the expected requirements. □

Definition 5.30. We say that a selfmap $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if $F(x) \leqslant F(y)$ as soon as $x \leqslant y$, where $\leqslant$ denotes the coordinatewise partial order over $\mathbb{R}^n$. Such a function is said to be *(additively) homogeneous* if $F(\lambda + x) = \lambda + F(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Here, if $z \in \mathbb{R}^n$, then $\lambda + z$ stands for the vector with entries $\lambda + z_k$.

The following observation is well known [CT80].

Lemma 5.31. *Every monotone homogeneous operator is nonexpansive in the supremum norm.*

Proof. Observe that $x \leqslant \|x - y\|_\infty + y$. Therefore, we get $F(x) \leqslant F(\|x - y\|_\infty + y) = \|x - y\|_\infty + F(y)$. Analogously, $F(y) \leqslant \|x - y\|_\infty + F(x)$ and $\|F(x) - F(y)\|_\infty \leqslant \|x - y\|_\infty$. □

Kolokoltsov showed that every monotone homogeneous operator F has a minimax representation as a dynamic programming operator of a zero-sum game [Kol92]. When F is semilinear, the following lemma and its corollary show that we have a finite representation of the same nature.

Lemma 5.32. *Suppose that $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is piecewise affine, monotone, homogeneous and let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ be any piecewise description of F. Then, every matrix $A^{(s)}$ is stochastic.*

Proof. Take any $x \in \operatorname{int}(\mathcal{W}_s)$. Let y be the sum of the columns of $A^{(s)}$. Since F is homogeneous, for any $\rho > 0$ small enough we have $F(\rho + x) = A^{(s)}x + b^{(s)} + \rho y = \rho + F(x)$. In other words, the sum of every line of $A^{(s)}$ is equal to 1. Let ϵ_k denote the kth vector of standard basis in $\mathbb{R}^n$. Since F is monotone, for $\rho > 0$ small enough we have $F(x+\rho\epsilon_k) = A^{(s)}x+b^{(s)}+\rho A^{(s)}\epsilon_k \geqslant F(x)$. In other words, the matrix $A^{(s)}$ has nonnegative entries in its kth column. Since k was arbitrary, $A^{(s)}$ is stochastic. □

[1] The fact that a projection of a polyhedron is also a polyhedron follows from the Fourier–Motzkin elimination, see [Sch87, Section 12.2].

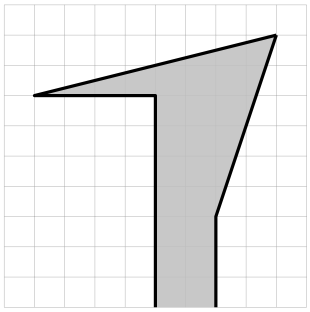

Figure 5.2: A real tropical cone from Example 5.35 (for $x_3 = 0$).

Corollary 5.33. *If $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is semilinear, monotone, and homogeneous, then it can be written in the form*

$$\forall k,\ F_k(x) = \min_{i\in[M_k]} \max_{s\in S_{ki}} (A_k^{(s)}x + b_k^{(s)})\,, \tag{5.1}$$

where $A^{(1)},\dots,A^{(p)} \in \mathbb{Q}^{n\times n}$ is a sequence of stochastic matrices, $b^{(s)} \in \mathbb{R}^n$ for all $s \in [n]$, $M_k \geqslant 1$ for all $k \in [n]$, and S_{ki} is a subset of $[p]$ for every $k \in [n]$ and $i \in [M_k]$.

Proof. Lemma 5.31 shows that F is continuous. Let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ a piecewise description of F as provided by Lemma 5.29. In particular, every matrix $A^{(s)}$ is rational. Furthermore, it is stochastic by Lemma 5.32. Therefore, the claim follows from Theorem 5.27. □

We now characterize the class of closed, semilinear, real tropical cones. In this context, "closed" means "closed in the standard topology of $\mathbb{R}^n$."

Proposition 5.34. *A set $\mathcal{S} \subset \mathbb{R}^n$ is a closed, semilinear, real tropical cone if and only if there exists a semilinear, monotone, homogeneous operator $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{S} = \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$.*

Proof. To prove the first implication, we consider two cases. If $\mathcal{S}$ is empty, then we take $F(x) = x - (1,\dots,1)$. Otherwise, we define F by $F_k(x) \coloneqq \sup\{y_k \colon y \in \mathcal{S}\,,\ y \leqslant x\}$ for all $k \in [n]$. We claim that every supremum is attained. Indeed, the set $\{y \in \mathcal{S} \colon y \leqslant x\}$ is nonempty (take an arbitrary $z \in \mathcal{S}$, and consider $y \coloneqq \lambda + z$ for $\lambda \in \mathbb{R}$ small enough), closed, and bounded by x. Hence $F_k(x) \coloneqq \max\{y_k \colon y \in \mathcal{S}\,,\ y \leqslant x\}$ for all $k \in [n]$. Observe that, since $\mathcal{S}$ is semilinear, the graph of the operator F is definable in the language $\mathcal{L}_{\mathrm{og}}$. Therefore, F is semilinear by Lemma 2.103. Besides, F is obviously monotone. It is also homogeneous because if $y \in \mathcal{S}$, then $\lambda + y = (\lambda \odot y) \oplus (\lambda \odot y) \in \mathcal{S}$ for all $\lambda \in \mathbb{R}$. Moreover, the inclusion $\mathcal{S} \subset \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$ is straightforward. To prove the inverse inclusion, let $x \in \mathbb{R}^n$ be such that $x \leqslant F(x)$ and let $y^{(k)} \in \mathcal{S}$ be a point attaining the maximum in $F_k(x)$. Then, the point $y \coloneqq y^{(1)} \oplus \dots \oplus y^{(n)}$ is an element of $\mathcal{S}$ smaller than or equal to x. Hence $F(x) = y \leqslant x$. Since we supposed that $x \leqslant F(x)$, we have $x = y \in \mathcal{S}$.

Conversely, fix a semilinear, monotone, homogeneous operator F and take the set $\mathcal{S} = \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$. This set is is definable in $\mathcal{L}_{\mathrm{og}}$ and hence semilinear. Moreover, $\mathcal{S}$ is closed because F is continuous. To prove that this a real tropical cone, fix a pair $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathcal{S}$. Since F is monotone and homogeneous, we have $F(\max\{\lambda + x, \mu + y\}) \geqslant F(\lambda + x) = \lambda + F(x) \geqslant \lambda + x$ and similarly $F(\max\{\lambda+x, \mu+y\}) \geqslant \mu + y$. Hence $\max\{\lambda+x, \mu+y\} \in \mathcal{S}$. □

Example 5.35. We illustrate our results on the following example. Take $n = 3$, $p = 2$, $M_k = 1$, $S_{k,1} = \{1, 2\}$ for all $k \in \{1, 2, 3\}$,

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 0 & 3/4 \\ 1 & 0 & 0 \end{bmatrix} \qquad b^{(1)} = \begin{bmatrix} 4 \\ 5/2 \\ -2 \end{bmatrix},$$
$$A^{(2)} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad b^{(2)} = \begin{bmatrix} 14/3 \\ 1 \\ -2 \end{bmatrix}.$$

Then, the operator $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F_1(x) = \max\Big\{x_3 + 4, \frac{1}{3}x_2 + \frac{2}{3}x_3 + \frac{14}{3}\Big\},$$
$$F_2(x) = \max\Big\{\frac{1}{4}x_1 + \frac{3}{4}x_3 + \frac{5}{2}, x_3 + 1\Big\},$$
$$F_3(x) = \max\{x_1 - 2, x_2 - 2\}.$$

The real tropical cone $\{x \in \mathbb{R}^3 \colon x \leqslant F(x)\}$ is depicted in Fig. 5.2.

5.3 Description of real tropical cones by directed graphs

We now describe how semilinear, monotone, homogeneous operators can be encoded by directed graphs. To this end we take a directed graph $\vec{\mathcal{G}} := (V, E)$, where the set of vertices is divided into Max vertices, Min vertices, and Random vertices, i.e., $V := V_{\mathrm{Min}} \uplus V_{\mathrm{Rand}} \uplus V_{\mathrm{Max}}$, where the symbol $\uplus$ denotes the disjoint union of sets. We suppose that the sets of Max vertices and Min vertices are nonempty. If $v \in V$ is a vertex of $\vec{\mathcal{G}}$, then by $\mathrm{In}(v) := \{(w, v) \colon (w, v) \in E\}$ we denote the set of its incoming edges, and by $\mathrm{Out}(v) := \{(v, w) \colon (v, w) \in E\}$ we denote the set of its outgoing edges. We suppose that the every vertex has at least one outgoing edge. If v is a Min vertex or a Max vertex and $e \in \mathrm{Out}(v)$ is its outgoing edge, then we equip this edge with a real number r_e. Furthermore, if v is a Random vertex, then we equip its set of outgoing edges with a rational probability distribution. More precisely, every edge $e \in \mathrm{Out}(v)$ is equipped with a strictly positive rational number $q_e \in \mathbb{Q}$, $q_e > 0$, and we suppose that $\sum_{e \in \mathrm{Out}(v)} q_e = 1$. We also make the following assumption.

Assumption C. (i) Every path between any two Min vertices contains at least one Max vertex;

(ii) Every path between any two Max vertices contains at least one Min vertex;

(iii) From every Random vertex, there is a path to a Min or a Max vertex.

We now construct a semilinear, monotone, homogeneous operator from such a graph. We construct a stochastic matrix $P \in [0, 1]^{V^2}$ with transition probabilities $p_{vv} := 1$ for all $v \in V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$, $p_{vw} := q_{(v,w)}$ if $v \in V_{\mathrm{Rand}}$ and $(v, w) \in \mathrm{Out}(v)$, and $p_{vw} := 0$ otherwise. In other words, if we consider a Markov chain with P as its transition matrix, then every state of $V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$ is absorbing, and a trajectory of the Markov chain visits the states of V_{Rand} by picking at random, for each vertex $v \in V_{\mathrm{Rand}}$, one edge in $\mathrm{Out}(v)$ according to the probability law given by $q_{(v,\cdot)}$, until it reaches a state of $V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$. In this way, after leaving a Min vertex, the trajectory reaches a Max vertex, and vice versa. If e is an edge and v is a Max or Min vertex, then we denote by p^e_v the conditional probability to reach the absorbing state v starting from the head of e. For the sake of simplicity, we assume that $V_{\mathrm{Min}} = [n]$ and $V_{\mathrm{Max}} = [m]$.

Definition 5.36. The *operator encoded by* $\vec{\mathcal{G}}$ is the function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\forall v \in [n],\ (F(x))_v := \min_{e \in \mathrm{Out}(v)} \Bigl(r_e + \sum_{w \in [m]} p^e_w \max_{e' \in \mathrm{Out}(w)} \bigl(r_{e'} + \sum_{u \in [n]} p^{e'}_u x_u \bigr) \Bigr) . \tag{5.2}$$

Lemma 5.37. *The operator encoded by* $\vec{\mathcal{G}}$ *is semilinear, monotone, and homogeneous.*

Proof. Let F be the operator encoded by $\vec{\mathcal{G}}$. Note that every p^e_v is rational since we have assumed that the q_e are in $\mathbb{Q}$ (see Remark 2.138). This means that the graph of F is definable in $\mathcal{L}_{\mathrm{og}}$ and hence F is semilinear (by Lemma 2.103). Moreover, F is obviously monotone. We already observed that the Max and Min vertices are absorbing states in the Markov chain constructed from $\vec{\mathcal{G}}$. Besides, Assumption C (iii) still holds in the subgraph obtained by removing the edges going out of the Max and Min vertices. As a consequence, the Max and Min vertices are the only recurrent classes in the Markov chain. Let v be a Min vertex, and $e \in \mathrm{Out}(v)$. We claim that if $u \in [n]$ is a Min state, then $p^e_u = 0$. Indeed, by Assumption C (i), any path from the head of e to u in $\vec{\mathcal{G}}$ contains a Max vertex. As a consequence, there is no path from the head of e to u in the subgraph in which we have removed the edges going out of the Max and Min vertices. We deduce that for every Min vertex v and edge $e \in \mathrm{Out}(v)$, we have $\sum_{w \in [m]} p^e_w = 1$. Analogously, we can show that for all Max vertices w and edge $e' \in \mathrm{Out}(w)$, $\sum_{u \in [n]} p^{e'}_u = 1$. We deduce that the operator F is homogeneous. □

In the following lemma, we show that any semilinear, monotone, homogeneous operator is encoded by some directed graph:

Lemma 5.38. *Let* $F\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a semilinear, monotone, homogeneous operator. Then, there exists a directed graph* $\vec{\mathcal{G}}$ *satisfying Assumption C such that* F *is encoded by* $\vec{\mathcal{G}}$.

Proof. The idea is to identify the representation (5.1) to a special case of (5.2), in which the probabilities p^e_w with $e \in \mathrm{Out}(v)$ and $v \in V_{\mathrm{Min}}$ take only the values 0 and 1. Formally, let $A^{(1)}, \dots, A^{(p)} \in \mathbb{Q}^{n \times n}$ and $b^{(1)}, \dots, b^{(p)} \in \mathbb{R}^n$ such that Corollary 5.33 holds. We build $\vec{\mathcal{G}}$ as the graph in which the set of Min vertices is $[n]$, the set of Max vertices is $\uplus_k [M_k]$, and the set of Random vertices is $\uplus_{k \in [n],\, i \in [M_k]} S_{ki}$. Let k be a Min vertex. We add an edge (k, i) for every $i \in [M_k]$, with $r_{(k,i)} := 0$. Moreover, for every $i \in [M_k]$, we add an edge (i, s) for each $s \in S_{ki}$, with $r_{(i,s)} := b^{(s)}_k$. Finally, if $i \in [M_k]$ and $s \in S_{ki}$, we add an edge (s, l) with $q_{(s,l)} := A^{(s)}_{kl}$ for every $l \in [n]$ such that $A^{(s)}_{kl} > 0$. The requirements of Assumption C are straightforwardly satisfied. □

Example 5.39. The graph presented in Fig. 5.3 encodes the operator from Example 5.35.

The following proposition characterizes the semilinear, monotone, homogeneous operators associated with tropical Metzler spectrahedral cones.

Proposition 5.40. *Suppose that the graph* $\vec{\mathcal{G}}$ *fulfills Assumption C and has the following properties:*

- *every Random vertex has exactly two outgoing edges, the heads of these edges are different, and the probability distribution associated with these edges is equal to* $(1/2, 1/2)$*;*

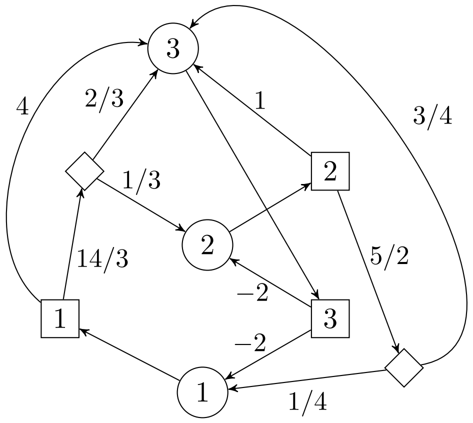

Figure 5.3: Graph that encodes the operator from Example 5.35. Min vertices are depicted by circles, Max vertices are depicted by squares, Random vertices are depicted by diamonds. We put $r_e = 0$ for every edge $e \in E$ that has no label.

- *every edge outgoing from a Random vertex has a Max vertex as its head.*

Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the semilinear monotone homogeneous operator encoded by $\vec{\mathcal{G}}$. We extend F to a function $F\colon \mathbb{T}^n \to \mathbb{T}^n$ by applying (5.2) *to all points of $\mathbb{T}^n$.*[2] *Then, the set $\{x \in \mathbb{T}^n \colon x \leqslant F(x)\}$ is a tropical Metzler spectrahedral cone.*

Proof. Consider the Markov chain introduced before Definition 5.36, take a Min vertex $v \in [n]$ and an outgoing edge $e \in \operatorname{Out}(v)$. Under the assumptions over the graph $\vec{\mathcal{G}}$, the absorbing states reachable from the head of e form a set $\{w_e, w'_e\} \subset [m]$ of cardinality at most 2 (we use the convention $w_e = w'_e$ if there is only one such absorbing state). Moreover, if $w_e \neq w'_e$, then $p^e_{w_e} = p^e_{w'_e} = 1/2$. Furthermore, observe that if $w \in [m]$ is a Max vertex and $e' \in \operatorname{Out}(w)$ is an outgoing edge, then our assumptions imply that the head of e' is a Min vertex. We denote it by $u_{e'}$. With this notation, we have:

$$(F(x))_v = \min_{e \in \operatorname{Out}(v)} \Bigl(r_e + \frac{1}{2}\bigl(\max_{e' \in \operatorname{Out}(w_e)} (r_{e'} + x_{u_{e'}}) + \max_{e' \in \operatorname{Out}(w'_e)} (r_{e'} + x_{u_{e'}})\bigr)\Bigr) \tag{5.3}$$

for all $v \in [n]$. Out of this data we construct symmetric Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$. If a Min vertex $v \in [n]$ has an outgoing edge $e \in \operatorname{Out}(v)$ such that $w_e \neq w'_e$, then we define

$$Q^{(v)}_{w_e w'_e} = Q^{(v)}_{w'_e w_e} \coloneqq \ominus \max\{-r_{e'} \colon e' \in \operatorname{Out}(v) \wedge \{w_{e'}, w'_{e'}\} = \{w_e, w'_e\}\} \,.$$

The diagonal coefficients $Q^{(v)}_{ww}$ of the matrices $Q^{(v)}$ are defined as follows:

- if $e = (v, w)$ is an edge, but (w, v) is not, then we set $Q^{(v)}_{ww} \coloneqq \ominus(-r_e)$;
- if $e = (w, v)$ is an edge, but (v, w) is not, then we set $Q^{(v)}_{ww} \coloneqq r_e$;
- if both $e = (v, w)$ and $e' = (w, v)$ are edges, then we set $Q^{(v)}_{ww}$ to $\ominus(-r_e)$ if $-r_e > r_{e'}$, and to $r_{e'}$ if $r_{e'} \geqslant -r_e$.

[2] We use the convention that $-\infty \cdot x = -\infty$ for all $x > 0$ and $-\infty \cdot 0 = 0$.

Finally, all the other entries of the matrices $Q^{(v)}$ are set to $-\infty$. Let $\mathcal{S} = \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ denote the associated tropical Metzler spectrahedral cone. We want to show that $\mathcal{S} = \{x \in \mathbb{T}^n \colon x \leqslant F(x)\}$. Note that we have the equivalence

$$\begin{aligned} &\forall v \in [n],\ x_v \leqslant (F(x))_v \iff \\ &\forall v,\ \forall e \in \operatorname{Out}(v),\ x_v - r_e \leqslant \frac{1}{2}\Big(\max_{e' \in \operatorname{Out}(w_e)} (r_{e'} + x_{u_{e'}}) + \max_{e' \in \operatorname{Out}(w'_e)} (r_{e'} + x_{u_{e'}})\Big)\,. \end{aligned}$$

The last set of inequalities is equivalent to

$$\begin{aligned} \forall \{w, w'\},\ &\max_{\{\{v,e\} \colon e \in \operatorname{Out}(v), \{w,w'\} = \{w_e, w'_e\}\}} (x_v - r_e) \\ &\leqslant \frac{1}{2}\Big(\max_{e' \in \operatorname{Out}(w)} (r_{e'} + x_{u_{e'}}) + \max_{e' \in \operatorname{Out}(w')} (r_{e'} + x_{u_{e'}})\Big)\,. \end{aligned}$$

We want to show that the last set of constraints describes $\mathcal{S}$. To do this, recall that an inequality of the form $\max(x, \alpha + y) \geqslant \max(x', \beta + y)$ is equivalent to $\max(x, \alpha + y) \geqslant x'$ if $\alpha \geqslant \beta$, and to $x \geqslant \max(x', \beta + y)$ if $\beta > \alpha$. Therefore, for every $w \in [m]$ we have the equivalence

$$\begin{aligned} \max_{\{\{v,e\} \colon e \in \operatorname{Out}(v), w = w_e\}} (x_v - r_e) \leqslant \max_{e' \in \operatorname{Out}(w)} (r_{e'} + x_{u_{e'}}) \iff \\ \max_{Q^{(v)}_{ww} \in \mathbb{T}_-} (x_v + |Q^{(v)}_{ww}|) \leqslant \max_{Q^{(u)}_{ww} \in \mathbb{T}_+} (Q^{(u)}_{ww} + x_u)\,. \end{aligned} \tag{5.4}$$

Moreover, note that if $x \in \mathbb{T}^n$ verifies (5.4), then we have the equality

$$\max_{Q^{(u)}_{ww} \in \mathbb{T}_+} (Q^{(u)}_{ww} + x_u) = \max_{e' \in \operatorname{Out}(w)} (r_{e'} + x_{u_{e'}})\,. \tag{5.5}$$

Indeed, if we have $\max_{Q^{(u)}_{ww} \in \mathbb{T}_+} (Q^{(u)}_{ww} + x_u) < r_{e'} + x_{u_{e'}}$ for some $e' \in \operatorname{Out}(w)$, then by the construction $e = (u_{e'}, w)$ is an edge and we have $(-r_e) > r_{e'}$. This implies that $Q^{(u_{e'})}_{ww} = \ominus(-r_e)$. In particular, $\max_{Q^{(v)}_{ww} \in \mathbb{T}_-} (x_v + |Q^{(v)}_{ww}|) > r_{e'} + x_{u_{e'}}$, which gives a contradiction with (5.4). Furthermore, observe that for any $w \neq w'$ we have the equality

$$\max_{\{\{v,e\} \colon e \in \operatorname{Out}(v), \{w,w'\} = \{w_e, w'_e\}\}} (x_v - r_e) = \max_{Q^{(v)}_{ww'} \in \mathbb{T}_-} (x_v + |Q^{(v)}_{ww'}|)\,. \tag{5.6}$$

Suppose that $x \in \mathcal{S}$. Then, x verifies (5.4) for all $w \in [m]$. Moreover, by (5.5) and (5.6), for any $w \neq w'$ we have

$$\begin{aligned} \max_{\{\{v,e\} \colon e \in \operatorname{Out}(v), \{w,w'\} = \{w_e, w'_e\}\}} (x_v - r_e) &= \max_{Q^{(v)}_{ww'} \in \mathbb{T}_-} (x_v + |Q^{(v)}_{ww'}|) \\ &\leqslant \frac{1}{2}\Big(\max_{Q^{(u)}_{ww} \in \mathbb{T}_+} (Q^{(u)}_{ww} + x_u) + \max_{Q^{(u)}_{w'w'} \in \mathbb{T}_+} (Q^{(u)}_{w'w'} + x_u)\Big) \\ &= \frac{1}{2}\Big(\max_{e' \in \operatorname{Out}(w)} (r_{e'} + x_{u_{e'}}) + \max_{e' \in \operatorname{Out}(w')} (r_{e'} + x_{u_{e'}})\Big)\,. \end{aligned} \tag{5.7}$$

Thus $x \leqslant F(x)$. Conversely, if $x \leqslant F(x)$, then x also verifies (5.4) for all $w \in [m]$, and the same argument as in (5.7) shows that $x \in \mathcal{S}$. □

Figure 5.4: Lowering the degree. Random vertices are depicted by diamonds.

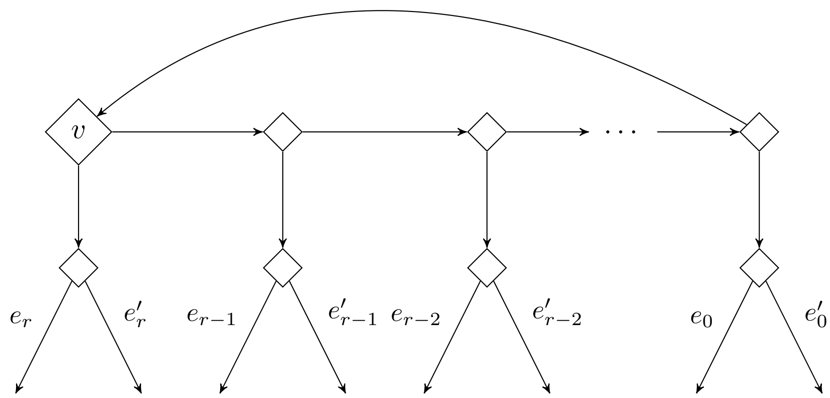

Figure 5.5: The construct of Zwick and Paterson.

5.4 Tropical Helton–Nie conjecture for real tropical cones

In this section, we want to show that every real tropical cone associated with a graph $\vec{\mathcal{G}}$ is a projection of a tropical Metzler spectrahedron. The idea of the proof is to take an arbitrary graph $\vec{\mathcal{G}}$ and transform it (by adding auxiliary states) into a graph $\vec{\mathcal{G}}'$ that fulfills the conditions of Proposition 5.40. Furthermore, our construction needs to preserve the projection. A key ingredient is the following construction, which was used by Zwick and Paterson [ZP96] to show the reduction from discounted games to simple stochastic games.

Lemma 5.41 ([ZP96])**.** *One can transform any graph $\vec{\mathcal{G}}$ as defined in Section 5.3 into a graph $\vec{\mathcal{G}}'$ such that*

- *every Random vertex of $\vec{\mathcal{G}}'$ has exactly two outgoing edges and the probability distribution associated with these edges is equal to* $(1/2, 1/2)$*;*
- *$\vec{\mathcal{G}}$ and $\vec{\mathcal{G}}'$ encode the same operator.*

Let us present the construction of Zwick and Paterson for the sake of completeness.

Proof. Fix a Random vertex v belonging to $\vec{\mathcal{G}}$. If this vertex has only one outgoing edge e, then we can delete v by joining all incoming edges $\mathrm{In}(v)$ with the head of e.

If v has at least three outgoing edges, then we enumerate the outgoing edges $\mathrm{Out}(v)$ by $\{e_1, \dots, e_d\}$, $d \geqslant 3$. Let us recall that the vertex v is equipped with a probability distribution $(q_{e_s})_{s=1}^d$. We now perform the transformation presented on Fig. 5.4. We replace the vertex v by a pair of vertices (w, u) such that all incoming edges of v are connected to w and w has two outgoing edges: one going to the head of e_1 with probability q_{e_1} and the other going to u with probability $1 - q_{e_1}$. Finally, u has $d-1$ outgoing edges, the head of the sth outgoing edge is the head of e_s (for $s \geqslant 2$), and the associated probability is equal to $q_{e_s}/(1 - q_{e_1})$. We repeat this transformation until we reach a graph in which all Random vertices have exactly two outgoing edges.

If v has exactly two outgoing edges, then we denote the heads of these edges by w and u, and the associated probability distribution by $(q, 1-q)$, where $q = a/b$, $a, b \in \mathbb{N}^*$ and $a < b$. If $q \neq 1/2$, then we take $r \geqslant 1$ such that $2^r \leqslant b < 2^{r+1}$. We write a and $b - a$ in binary, $a = \sum_{s=0}^r c_s 2^s$ and $b - a = \sum_{s=0}^r d_s 2^s$ for $c_s, d_s \in \{0, 1\}$. We now replace the outgoing edges of vertex v by the construct presented on Fig. 5.5. In this construction, every Random node has exactly two outgoing edges and the associated probability distribution is equal to $(1/2, 1/2)$. Furthermore, for any s, if $c_s = 1$, then head of e_s is w and if $c_s = 0$, then the head of e_s is v. Similarly, if $d_s = 1$, then head of e'_s is u and if $d_s = 0$, then the head of e'_s is v. Suppose that the Markov chain starts at v. Then, with probability

$$\frac{c_r}{4} + \frac{c_{r-1}}{8} + \dots + \frac{c_0}{2^{r+2}} = \frac{a}{2^{r+2}}$$

the Markov chain goes to w without coming back to v. Similarly, with probability $(b-a)/2^{r+2}$ the Markov chain moves to u without coming back to v. Therefore, the probability that the Markov chain finally reaches w is equal to

$$\frac{a}{2^{r+2}}(1 + (1 - \frac{b}{2^{r+2}}) + (1 - \frac{b}{2^{r+2}})^2 + \dots) = \frac{a}{b}$$

and the probability that it finally reaches u is equal to $(b-a)/b$. We repeat this procedure for every Random vertex of our graph.

To finish the proof, observe that the operations described above do not affect the associated semilinear, monotone, homogeneous operator because this operator is defined by absorption probabilities which remain unchanged throughout the construction. □

We now describe how to transform a graph given in Lemma 5.41 into a graph that verifies the conditions of Proposition 5.40. More precisely, we transform the graph $\vec{\mathcal{G}}$ (which has n Min vertices) into a graph $\vec{\mathcal{G}}'$ (which has n' Min vertices, where $n' \geqslant n$) in such a way that the real tropical cone $\{x \in \mathbb{R}^n : x \leqslant F(x)\}$ associated with $\vec{\mathcal{G}}$ is a projection of the real tropical cone $\{x \in \mathbb{R}^{n'} : x \leqslant F'(x)\}$ associated with $\vec{\mathcal{G}}'$. The main difficulty here is that the operators arising from tropical Metzler spectrahedra have a special structure in which there edges connect Random vertices directly to Max vertices and Max vertices to Min vertices. By comparison, the Zwick–Paterson construction (Lemma 5.41) leads to a graph with consecutive sequences of Random nodes and it does not exclude the edges that connect Max vertices to Random vertices. We shall see, however, that the latter situation can be reduced from the former one by applying, as a basic ingredient, two transformations, the validity of which is expressed in Lemmas 5.42 and 5.43.

The first transformation that we execute is presented in Fig. 5.6. It is defined as follows. Suppose that we are given a graph $\vec{\mathcal{G}}$ and we denote $V_{\mathrm{Min}} = [n]$ and $V_{\mathrm{Max}} = [m]$. Furthermore,

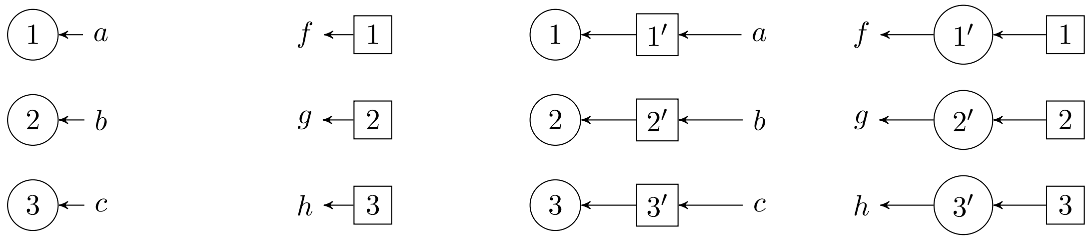

Figure 5.6: First transformation of a graph. Min vertices are presented by circles, Max vertices are presented by squares.

let $E_{\text{Max}} \subset E$ denote the set of all edges that have a Max vertex as their tail. For every Max vertex $w \in [m]$ and an outgoing edge $e \in \text{Out}(w)$, we insert a Min vertex between w and the head of e, as illustrated in Fig. 5.6. In a similar way, for every Min vertex $v \in [n]$ and incoming edge $e \in \text{In}(v)$, we insert a Max vertex between the tail of e and v. We denote the transformed graph by $\vec{\mathcal{G}}'$. Observe that this graph fulfills Assumption C. We refer to the Min vertices in $\vec{\mathcal{G}}'$ as follows: the vertices that were present in $\vec{\mathcal{G}}$ are denoted by $[n]$, whereas the added Min vertices are denoted by $e \in E_{\text{Max}}$.

Lemma 5.42. *Let $F \colon \mathbb{R}^n \to \mathbb{R}^n$ denote the operator associated with $\vec{\mathcal{G}}$ and let $\vec{\mathcal{G}}'$ denote the graph obtained from $\vec{\mathcal{G}}$ by the first transformation above. Let $F' \colon \mathbb{R}^{n+|E_{\text{Max}}|} \to \mathbb{R}^{n+|E_{\text{Max}}|}$ denote the operator associated with $\vec{\mathcal{G}}'$. Then, the real tropical cone $\{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$ is the projection of the real tropical cone $\{(x,x') \in \mathbb{R}^n \times \mathbb{R}^{|E_{\text{Max}}|} \colon (x,x') \leqslant F'(x,x')\}$.*

Proof. Denote the operator F as

$$\forall v \in [n],\ (F(x))_v = \min_{e \in \text{Out}(v)} \Bigl(r_e + \sum_{w \in [m]} p_w^e \max_{e' \in \text{Out}(w)} \bigl(r_{e'} + \sum_{u \in [n]} p_u^{e'} x_u\bigr)\Bigr) .$$

Observe that for every $v \in [n]$ we have

$$(F'(x,x'))_v = \min_{e \in \text{Out}(v)} \Bigl(r_e + \sum_{w \in [m]} p_w^e \max_{e' \in \text{Out}(w)} \bigl(r_{e'} + x_{e'}\bigr)\Bigr) .$$

Furthermore, for every $e \in E_{\text{Max}}$ we have

$$(F'(x,x'))_e = \sum_{v \in [n]} p_v^e x_v .$$

Therefore, if $x \leqslant F(x)$ and for every $e \in E_{\text{Max}}$ we set $x_e = \sum_{v \in [n]} p_v^e x_v$, then for every $v \in [n]$ we have $x_v \leqslant (F(x))_v = (F'(x,x'))_v$ and for every $e \in E_{\text{Max}}$ we have $x_e = (F'(x,x'))_e$. Conversely, if $(x,x') \leqslant F'(x,x')$, then we have $x_v \leqslant (F'(x,x'))_v \leqslant (F(x))_v$ for every $v \in [n]$. ☐

The second transformation is given as follows. As previously, we are given a graph $\vec{\mathcal{G}}$ and we denote $V_{\text{Min}} = [n]$ and $V_{\text{Max}} = [m]$. Furthermore, let $E_{\text{Max}} \subset E$ denote the set of all edges that have a Max vertex as their tail. Moreover, we suppose that $\vec{\mathcal{G}}$ is such that every edge $e \in E_{\text{Max}}$ has a Min vertex as its head. Let $e^* \in E$ be a fixed edge in $\vec{\mathcal{G}}$ that connects two Random vertices. We add a Max vertex $m+1$ and a Min vertex $n+1$ onto e^* as presented on Fig. 5.7. We denote the transformed graph by $\vec{\mathcal{G}}'$. Since every edge $e \in E_{\text{Max}}$ has a Min vertex as its head, every path that joins a Max vertex with a Min vertex has length 1. In particular, e^* does not belong to any such path. Hence, the transformed graph $\vec{\mathcal{G}}'$ fulfills Assumption C.

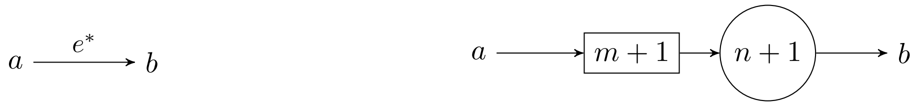

Figure 5.7: Second transformation of a graph.

Lemma 5.43. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the operator associated with $\vec{\mathcal{G}}$ and let $\vec{\mathcal{G}}'$ denote the graph obtained from $\vec{\mathcal{G}}$ by the second transformation above. Let $F'\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the operator associated with $\vec{\mathcal{G}}'$. Then, the real tropical cone $\{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$ is a projection of the real tropical cone $\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \colon (x, x_{n+1}) \leqslant F'(x, x_{n+1})\}$.*

Proof. Denote the operator F as

$$\forall v \in [n],\ (F(x))_v = \min_{e \in \mathrm{Out}(v)} \Bigl(r_e + \sum_{w \in [m]} p_w^e \max_{e' \in \mathrm{Out}(w)} (r_{e'} + x_{u_{e'}})\Bigr),$$

where $u_{e'}$ denotes the head of edge e' (by our assumptions, $u_{e'}$ is a Min vertex, $u_{e'} \in [n]$). Let us introduce the following notation. For every $e \in E$, we denote by p_{e*}^e the conditional probability that the Markov chain reaches the head of e^* from the head of e. Moreover, for every Max vertex w and every edge $e \in E$, we denote by $p_{w\dagger}^e$ the conditional probability that the Markov chain reaches w from the head of e without passing by the head of e^*. Thus, for every Max vertex w and every $e \in E$ we have $p_w^e = p_{e*}^e p_w^{e^*} + p_{w\dagger}^e$. Therefore, for any $x_{n+1} \in \mathbb{R}$ we have

$$(F'(x, x_{n+1}))_{n+1} = \sum_{w \in [m]} p_w^{e^*} \max_{e' \in \mathrm{Out}(w)} (r_{e'} + x_{u_{e'}}) = (F'(x, 0))_{n+1}$$

and

$$\begin{aligned}(F(x))_v &= \min_{e \in \mathrm{Out}(v)} \Bigl(r_e + \sum_{w \in [m]} (p_{e^*}^e p_w^{e^*} + p_{w\dagger}^e) \max_{e' \in \mathrm{Out}(w)} (r_{e'} + x_{u_{e'}})\Bigr) \\ &= \min_{e \in \mathrm{Out}(v)} \Bigl(r_e + p_{e^*}^e (F'(x,0))_{n+1} + \sum_{w \in [m]} p_{w\dagger}^e \max_{e' \in \mathrm{Out}(w)} (r_{e'} + x_{u_{e'}})\Bigr).\end{aligned}$$

Furthermore, for every $v \in [n]$ we have

$$(F'(x, x_{n+1}))_v = \min_{e \in \mathrm{Out}(v)} \Bigl(r_e + p_{e^*}^e x_{n+1} + \sum_{w \in [m]} p_{w\dagger}^e \max_{e' \in \mathrm{Out}(w)} (r_{e'} + x_{u_{e'}})\Bigr).$$

Therefore, if $x \leqslant F(x)$ and we set $x_{n+1} = (F'(x,0))_{n+1}$, then $(x, x_{n+1}) \leqslant F'(x, x_{n+1})$. Conversely, if $(x, x_{n+1}) \leqslant F'(x, x_{n+1})$, then $x_{n+1} \leqslant (F'(x,0))_{n+1}$ and hence $x_v \leqslant (F(x))_v$ for all $v \in [n]$. □

As a corollary, we may now show that the tropical analogue of the Helton–Nie conjecture is true for real tropical cones.

Proposition 5.44. *Every closed, semilinear, real tropical cone is a projection of a real tropical Metzler spectrahedron.*

Proof. Take any closed, semilinear, real tropical cone $\mathcal{S} \subset \mathbb{R}^n$. By Proposition 5.34, we can write it as $\mathcal{S} = \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$, where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a semilinear, monotone, homogeneous operator. Let $\vec{\mathcal{G}}$ denote the graph associated with F (as given by Lemma 5.38). By Lemma 5.41

we may suppose that the probabilities associated with Random vertices in $\vec{\mathcal{G}}$ are equal to 1/2. We perform the first transformation on the graph $\vec{\mathcal{G}}$ and denote the transformed graph by $\vec{\mathcal{G}}_1$. We then perform the second transformation on every edge in $\vec{\mathcal{G}}_1$ that joins two Random vertices. We denote the transformed graph by $\vec{\mathcal{G}}'$ and the associated operator as $F' \colon \mathbb{R}^{n+n'} \to \mathbb{R}^{n+n'}$. By Lemmas 5.42 and 5.43, the real tropical cone $\{x \in \mathbb{R}^n : x \leqslant F(x)\}$ is the projection of the real tropical cone $\{(x, x') \in \mathbb{R}^n \times \mathbb{R}^{n'} : (x, x') \leqslant F'(x, x')\}$. Furthermore, $\vec{\mathcal{G}}'$ fulfills the conditions of Proposition 5.40. Therefore, the set $\mathcal{S}' = \{(x, x') \in \mathbb{T}^n \times \mathbb{T}^{n'} : (x, x') \leqslant F'(x, x')\}$ is a tropical Metzler spectrahedral cone. Finally, we take the set

$$\begin{aligned}\mathcal{S}'' &= \{(x, x', y) \in \mathbb{T}^n \times \mathbb{T}^{n'} \times \mathbb{T}^{n+n'} : (x, x') \leqslant F'(x, x') \wedge (x, x') + y \geqslant 0\} \\ &= \{(x, x', y) \in \mathbb{R}^n \times \mathbb{R}^{n'} \times \mathbb{R}^{n+n'} : (x, x') \leqslant F'(x, x') \wedge (x, x') + y \geqslant 0\} .\end{aligned}$$

The set $\mathcal{S}''$ is a real tropical Metzler spectrahedron because a constraint of the form $x_v + y_v \geqslant 0$ can be encoded by adding a 2×2 block

$$\begin{bmatrix} x_v & \ominus 0 \\ \ominus 0 & y_v \end{bmatrix}$$

to the matrices that describe $\mathcal{S}'$. Moreover, $\mathcal{S}$ is a projection of $\mathcal{S}''$. Indeed, if (x, x') is a real vector such that $(x, x') \leqslant F'(x, x')$, then there exists $y \in \mathbb{R}^{n+n'}$ such that $(x, x') + y \geqslant 0$. □

Example 5.45. Take the graph from Fig. 5.3 and consider the Random vertex that has Min vertices 2 and 3 as its neighbors. Figure 5.8 presents the outcome of the procedure described in the lemmas above when applied to this vertex.

5.5 Tropical Helton–Nie conjecture for Puiseux series

We now generalize Proposition 5.44 to tropically convex sets in $\mathbb{T}^n$. In order to study this case, we use the notion of homogenization of a convex set. There are many possible homogenizations of a given set and we need to use three of them.

Definition 5.46. If $\mathcal{S}$ is a tropically convex set with only finite points (i.e., $\mathcal{S} \subset \mathbb{R}^n$), then we define its *real homogenization* as

$$\mathcal{S}^{rh} := \{(x_0, x_0 + x) \in \mathbb{R}^{n+1} : x \in \mathcal{S}\} .$$

If $\mathcal{S} \subset \mathbb{T}^n$ is a tropically convex set, then we define its *homogenization* as

$$\mathcal{S}^{h} := \{(x_0, x_0 + x) \in \mathbb{T}^{n+1} : x \in \mathcal{S}\} .$$

If $\mathcal{S}(Q^{(0)} | Q^{(1)}, \dots, Q^{(n)}) \subset \mathbb{T}^n$ is a tropical Metzler spectrahedron, then we define its *formal homogenization* $\mathcal{S}^{fh} \subset \mathbb{T}^{n+1}$ as the tropical Metzler spectrahedral cone

$$\mathcal{S}^{fh} := \mathcal{S}(Q^{(0)}, Q^{(1)}, \dots, Q^{(n)}) .$$

Lemma 5.47. *If $\mathcal{S} \subset \mathbb{T}^n$ is tropically convex, then its homogenization is a tropical cone. Moreover, if $\mathcal{S}$ is included in $\mathbb{R}^n$, then its real homogenization if a real tropical cone.*

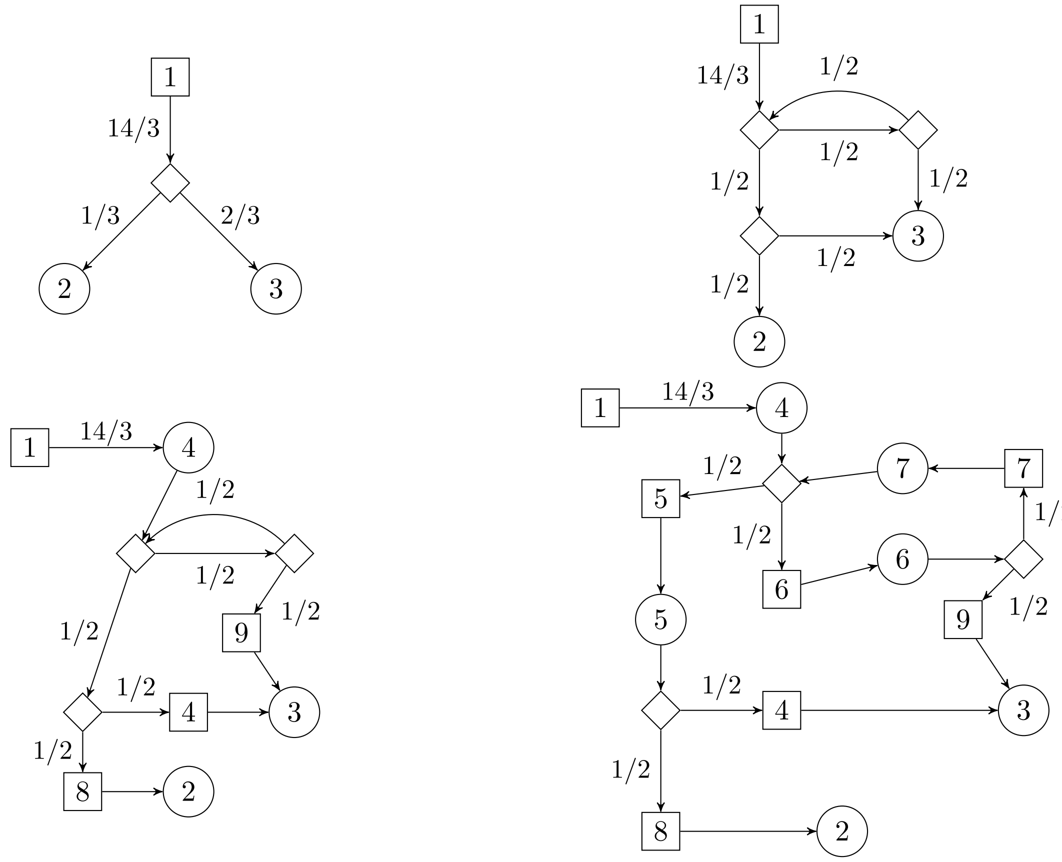

Figure 5.8: The transformation of Lemmas 5.41 to 5.43 applied to one Random vertex from the graph presented in Fig. 5.3. Top left: the initial graph. Top right: the graph after the application of Lemma 5.41. Bottom left: the graph after the application of Lemma 5.42. Bottom right: the graph after the application of Lemma 5.43.

Proof. Suppose that $\mathcal{S} \subset \mathbb{T}^n$ is tropically convex, take its homogenization $\mathcal{S}^h$, two points $(x_0, x_0 + x), (y_0, y_0 + y) \in \mathcal{S}^{rh}$ (such that $x, y \in \mathcal{S}$ and $x_0, y_0 \in \mathbb{T}$) and $\lambda, \mu \in \mathbb{T}$. Without loss of generality we may suppose that $\lambda + x_0 \geqslant \mu + y_0$. Let $z \coloneqq (\lambda \odot (x_0, x_0 + x)) \oplus (\mu \odot (y_0, y_0 + y))$. If $\mu + y_0 = -\infty$, then $z = (\lambda + x_0, \lambda + x_0 + x) \in \mathcal{S}^h$. Otherwise, we denote $\eta \coloneqq \mu + y_0 - \lambda - x_0 \leqslant 0$ and we have

$$z = \Big(\lambda + x_0, \lambda + x_0 + \big(x \oplus (\eta \odot y)\big)\Big) \in \mathcal{S}^h .$$

The proof for the real homogenization is analogous. $\square$

Example 5.48. The three notions of homogenization given in Definition 5.46 are different. Indeed, if we take the set $\mathcal{S} \coloneqq \{x \in \mathbb{T} \colon x \geqslant 0\}$, then its real homogenization if equal to $\mathcal{S}^{rh} = \{(x, y) \in \mathbb{R}^2 \colon y \geqslant x\}$, while its homogenization is equal to $\mathcal{S}^h = \mathcal{S}^{rh} \cup \{-\infty\}$. Moreover, $\mathcal{S} = \mathcal{S}(Q^{(0)} | Q^{(1)})$ is a tropical Metzler spectrahedron defined by the matrices $Q^{(0)} = \ominus 0$, $Q^{(1)} = 0$. Then, its formal homogenization is equal to $\mathcal{S}^{fh} = \{(x, y) \in \mathbb{T}^2 \colon y \geqslant x\}$.

Lemma 5.49. *Every closed, semilinear, tropically convex set in $\mathbb{R}^n$ is a projection of a real tropical Metzler spectrahedron.*

Proof. Take any closed, semilinear, tropically convex set $\mathcal{S} \subset \mathbb{R}^n$ and consider its real homogenization $\mathcal{S}^{rh}$. This set is a real tropical cone in $\mathbb{R}^{n+1}$ by Lemma 5.47. Moreover, it is closed and definable in $\mathcal{L}_{\mathrm{og}}$. This implies that it is semilinear by Lemma 2.103. By Proposition 5.44, $\mathscr{S}^{rh}$ is a projection of a real tropical Metzler spectrahedron $\mathcal{S}_1 \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n'}$. Consider the set

$$\mathcal{S}_2 = \{(x_0, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n'} : (x_0, x, y) \in \mathcal{S}_1 \wedge x_0 = 0\} .$$

The set $\mathcal{S}_2$ is a tropical Metzler spectrahedron because the constraint $x_0 = 0$ can be encoded by adding a 2×2 block

$$\begin{bmatrix} \ominus 0 \oplus x_0 & -\infty \\ -\infty & 0 \oplus (\ominus x_0) \end{bmatrix}$$

to the matrices that describe $\mathcal{S}_1$. Furthermore, $\mathscr{S}$ is its projection. Indeed, if $x \in \mathscr{S}$, then $(0, x) \in \mathscr{S}^{rh}$ and there exists $y \in \mathbb{R}^{n'}$ such that $(0, x, y) \in \mathcal{S}_1$. Conversely, if $(0, x, y) \in \mathcal{S}_1$, then $(0, x) \in \mathscr{S}^{rh}$ and hence $x \in \mathscr{S}$. $\square$

We now want to extend the result of Lemma 5.49 to tropically convex sets in $\mathbb{T}^n$. In order to do this, we proceed stratum-by-stratum. This requires to show that a tropical convex hull of finitely many projected tropical Metzler spectrahedra is a projected tropical Metzler spectrahedron. In the classical case of real spectrahedra, it is known that a convex hull of finitely many projected spectrahedra is a projected spectrahedron. This fact has a very short proof presented in [NS09]. The proof in the tropical case is exactly the same (we only change the classical notation to the tropical one). Let us present this proof for the sake of completeness.

Lemma 5.50. *A tropically convex set $\mathscr{S} \subset \mathbb{T}^n$ is a projected tropical Metzler spectrahedron if and only if its homogenization is a projected tropical Metzler spectrahedron.*

Proof. First, suppose that $\mathscr{S}^h$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_1 \subset \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'}$. Consider the set

$$\mathcal{S}_2 = \{(x_0, x, y) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'} : (x_0, x, y) \in \mathcal{S}_1 \wedge x_0 = 0\} .$$

As in the proof of Lemma 5.49, the set $\mathcal{S}_2$ is a tropical Metzler spectrahedron and $\mathscr{S}$ is its projection. Conversely, suppose that $\mathscr{S}$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_1 \subset \mathbb{T}^n \times \mathbb{T}^{n'}$. Consider its formal homogenization $\mathcal{S}_1^{fh} \subset \mathbb{T}^{1+n+n'}$ and take the set

$$\mathcal{S}_2 = \{(x_0, x, y, z) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'} \times \mathbb{T}^n : (x_0, x, y) \in \mathcal{S}_1^{fh} \wedge \forall k \in [n], x_0 + z_k \geqslant 2x_k\} .$$

The set $\mathcal{S}_2$ is a tropical Metzler spectrahedron because a constraint of the form $x_0 + z_k \geqslant 2x_k$ can be encoded by adding a 2×2 block

$$\begin{bmatrix} x_0 & \ominus x_k \\ \ominus x_k & z_k \end{bmatrix}$$

to the matrices that describe $\mathcal{S}_1$. We will show that $\mathscr{S}^h$ is a projection of $\mathcal{S}_2$. To see this, take any point $(x_0, x_0 + x) \in \mathscr{S}^h$, where $x \in \mathscr{S}$ and $x_0 \in \mathbb{T}$. If $x_0 = -\infty$, then $-\infty \in \mathscr{S}^h$ belongs to the projection of $\mathcal{S}_2$. Otherwise, take y such that $(x, y) \in \mathcal{S}_1$ and observe that we have $(x_0, x_0 + x, x_0 + y) \in \mathcal{S}_1^{fh}$. Since $x_0 \in \mathbb{R}$, if we take z_k large enough, then $(x_0, x_0 + x, x_0 + y, z) \in \mathcal{S}_2$. This shows that $\mathscr{S}^h$ is included in the projection of $\mathcal{S}_2$. Conversely, suppose that $(x_0, x, y, z) \in \mathcal{S}_2$. If $x_0 = -\infty$, then $x = -\infty$ and hence $(x_0, x) \in \mathscr{S}^h$. If $x_0 \neq -\infty$, then we have $(0, -x_0 + x, -x_0 + y, -x_0 + z) \in \mathcal{S}_2$. Hence $(0, -x_0 + x, -x_0 + y) \in \mathcal{S}_1^{fh}$, $(-x_0 + x, -x_0 + y) \in \mathcal{S}_1$, and $-x_0 + x \in \mathscr{S}$. Therefore $(x_0, x) \in \mathscr{S}^h$. $\square$

Lemma 5.51. *Suppose that $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{T}^n$ are projected tropical Metzler spectrahedra. Then* $\operatorname{tconv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ *is a projected tropical Metzler spectrahedron.*

Proof. Let $\mathcal{S} = \operatorname{tconv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ and consider

$$\mathcal{S}_1^h \oplus \mathcal{S}_2^h := \{x \in \mathbb{T}^{n+1} \colon \exists (u,w) \in \mathcal{S}_1^h \times \mathcal{S}_2^h,\, x = u \oplus w\}\,.$$

First, we will show that we have the identity $\mathcal{S}^h = \mathcal{S}_1^h \oplus \mathcal{S}_2^h$. Indeed, since $\mathcal{S}_1 \subset \mathcal{S}$, we have $\mathcal{S}_1^h \subset \mathcal{S}^h$. Similarly, $\mathcal{S}_2^h \subset \mathcal{S}^h$. Therefore, we have $\mathcal{S}_1^h \oplus \mathcal{S}_2^h \subset \mathcal{S}^h$. Conversely, take a point $z \in \mathcal{S}^h$. By Lemma 5.17, we can write z as

$$z = \Big(z_0, z_0 \odot \big((\lambda \odot x) \oplus (\mu \odot y)\big)\Big) \in \mathcal{S}^h\,,$$

where $\lambda \oplus \mu = 0$, $x \in \mathcal{S}_1$, and $y \in \mathcal{S}_2$. Then $z = \tilde{x} \oplus \tilde{y}$, where

$$\begin{aligned}\tilde{x} &:= (\lambda \odot z_0, (\lambda \odot z_0) \odot x) \in \mathcal{S}_1^h\,,\\ \tilde{y} &:= (\mu \odot z_0, (\mu \odot z_0) \odot y) \in \mathcal{S}_2^h\,.\end{aligned}$$

Hence $\mathcal{S}^h = \mathcal{S}_1^h \oplus \mathcal{S}_2^h$. Since $\mathcal{S}_1, \mathcal{S}_2$ are projected tropical Metzler spectrahedra, the same is true for $\mathcal{S}_1^h, \mathcal{S}_2^h$ by Lemma 5.50. Let $\mathcal{S}_1^h$ be a projection of $\mathcal{S}_1 \subset \mathbb{T}^{n+1+n_1}$ and $\mathcal{S}_2^h$ be a projection of $\mathcal{S}_2 \subset \mathbb{T}^{n+1+n_2}$. Then, the set $\mathcal{S}_1^h \times \mathcal{S}_2^h$ is a projection of the set $\mathcal{S}_1 \times \mathcal{S}_2$. Moreover, the set $\mathcal{S}_1 \times \mathcal{S}_2$ is a tropical Metzler spectrahedron (described by block-diagonal matrices such that the first block is given by the matrices that describe $\mathcal{S}_1$ and the second block is given by the matrices that describe $\mathcal{S}_2$). Therefore, the set

$$\mathcal{S}_3 := \{(u,u',w,w',x) \colon \mathbb{T}^{(n+1)+n_1+(n+1)+n_2+(n+1)} \colon (u,u',w,w') \in \mathcal{S}_1 \times \mathcal{S}_2, x = u \oplus w\}$$

is a tropical Metzler spectrahedron (because a constraint of the form $x_k = u_k \oplus w_k$ can be encoded by adding a 2×2 block to the matrices that describe $\mathcal{S}_1 \times \mathcal{S}_2$). Moreover, $\mathcal{S}_1^h \oplus \mathcal{S}_2^h$ is a projection of $\mathcal{S}_3$. Since $\mathcal{S}^h = \mathcal{S}_1^h \oplus \mathcal{S}_2^h$, the set $\mathcal{S}$ is a projected tropical Metzler spectrahedron by Lemma 5.50. □

We are now ready to present the proof of Theorem 5.5.

Proof of Theorem 5.5. The equivalence between Theorem 5.5 (a) and Theorem 5.5 (b) is given in Proposition 5.21. The implication from Theorem 5.5 (b) to Theorem 5.5 (c) follows from Lemma 5.49. We now prove the implication from Theorem 5.5 (c) to Theorem 5.5 (d). Let $\mathcal{S} \subset \mathbb{T}^n$ be as in Theorem 5.5 (c). If $\mathcal{S}$ is empty, then it is a tropical Metzler spectrahedron defined by a single inequality $-\infty \geqslant 0$. Otherwise, let $K \subset [n]$ be any nonempty set such that the stratum $\mathcal{S}_K \subset \mathbb{R}^K$ is nonempty. The set $\mathcal{S}_K$ is a projection of a real tropical Metzler spectrahedron $\mathcal{S}_K \subset \mathbb{R}^K \times \mathbb{R}^{n'}$. For any $x \in \mathbb{T}^n$ we denote by $x_K \in \mathbb{T}^K$ the subvector formed by the coordinates of x with indices in K. Furthermore, let $X_K \subset \mathbb{T}^n$ denote the set

$$X_K := \{x \in \mathbb{T}^n \colon x_k \neq -\infty \iff k \in K\}\,.$$

The set $\mathcal{S} \cap X_K$ is a projection of a tropical Metzler spectrahedron defined as

$$\tilde{\mathcal{S}}_K = \{(x,y) \in \mathbb{T}^n \times \mathbb{T}^{n'} \colon (x_K, y) \in \mathcal{S}_K \wedge \forall k \notin K,\, -\infty \geqslant x_k\}\,.$$

Moreover, for $K = \emptyset$, let us denote $X_\emptyset = -\infty$. Note that the intersection $\mathcal{S} \cap X_\emptyset$ is either empty or is equal to $-\infty$, and that $-\infty$ is a tropical Metzler spectrahedron (defined by the inequalities

$-\infty \geqslant x_k$ for all $k \in [n]$). Hence, we have $\mathcal{S} = \cup_{K \subset [n]} \mathcal{S} \cap X_K = \operatorname{tconv}(\cup_{K \subset [n]} \mathcal{S} \cap X_K)$. Therefore, the claim follows from Lemma 5.51. To prove the implication Theorem 5.5 (d) to Theorem 5.5 (e), let $\mathcal{S} \subset \mathbb{T}^n$ be a projection of a tropical Metzler spectrahedron $\mathcal{S} \subset \mathbb{T}^n \times \mathbb{T}^{n'}$. By Proposition 4.14, there is a spectrahedron $\boldsymbol{\mathcal{S}} \subset \mathbb{K}_{\geqslant 0}^{n+n'}$ such that $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$. Let $\boldsymbol{\pi} \colon \mathbb{K}^{n+n'} \to \mathbb{K}^n, \pi \colon \mathbb{T}^{n+n'} \to \mathbb{T}^n$ denote the projections on the first n coordinates. Then $\mathsf{val}(\boldsymbol{\pi}(\boldsymbol{\mathcal{S}})) = \pi(\mathsf{val}(\boldsymbol{\mathcal{S}})) = \pi(\mathcal{S}) = \mathcal{S}$. The implication Theorem 5.5 (e) to Theorem 5.5 (a) follows trivially from the fact that projected spectrahedra are semialgebraic (Proposition 2.18) and convex. □

5.6 Extension to general fields

In this section we partially extend Theorem 5.5 to general real closed valued fields and prove Theorem 5.2. We want to do so using a model-theoretic argument. However, this requires to control the size of the spectrahedra constructed in the proof of Theorem 5.5. As a consequence, the proof is rather technical. We start by presenting a more effective version of Proposition 5.34. This variant allows us to give a uniform bounds on the complexity of describing the pieces involved in the piecewise description of the operator F. To formalize this uniformity, we fix rational matrices $B^{(1)}, \dots, B^{(q)}$, $B^{(s)} \in \mathbb{Q}^{m_s \times n}$ and, for every real vector $\bar{c} = (c^{(1)}, \dots, c^{(q)})$, $c^{(s)} \in \mathbb{R}^{m_s}$, consider the set $\mathcal{S}_{\bar{c}} \subset \mathbb{R}^n$ defined as $\mathcal{S}_{\bar{c}} \coloneqq \bigcup_{s=1}^q \{x \in \mathbb{R}^n \colon B^{(s)}x \leqslant c^{(s)}\}$.

Lemma 5.52. *There exists a number $N \geqslant 1$ that depends on $B^{(1)}, \dots, B^{(q)}$, but not on $\bar{c}$, such that if $\mathcal{S}_{\bar{c}}$ is a nonempty real tropical cone, then the operator $F \colon \mathbb{R}^n \to \mathbb{R}^n$, $F_k(x) \coloneqq \max\{y_k \colon y \in \mathcal{S}_{\bar{c}}\,,\ y \leqslant x\}$ defined in the proof of Proposition 5.34 has a piecewise description $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s \in [p]}$ such that $p \leqslant N$, the matrices $A^{(s)}$ are rational and stochastic, and the common denominator of all the numbers $A^{(s)}_{kl}$ is not greater than N.*

Proof. Fix $\bar{c}$ such that $\mathcal{S}_{\bar{c}}$ is a nonempty real tropical cone. We will show how to construct the piecewise description of the operator F that has the desired properties. Given $x \in \mathbb{R}^n$ we denote by $Q(x) \subset [q]$ the set of all $s \in [q]$ such that the polyhedron $\{y \in \mathbb{R}^n \colon B^{(s)}y \leqslant c^{(s)}, y \leqslant x\}$ is nonempty. Since $\mathcal{S}_{\bar{c}}$ is a nonempty real tropical cone, the the set $Q(x)$ is never empty (take an arbitrary $z \in \mathcal{S}_{\bar{c}}$ and consider $\lambda + z$ for $\lambda \in \mathbb{R}$ small enough). Therefore, by the strong duality of linear programming ([Sch87, Corollary 7.1g]) we have

$$F_k(x) = \max_{s \in Q(x)} \max\{y_k \colon B^{(s)}y \leqslant c^{(s)}, y \leqslant x\} \tag{5.8}$$

$$= \max_{s \in Q(x)} \min\{z^\intercal c^{(s)} + w^\intercal x \colon (B^{(s)})^\intercal z + w = \epsilon_k, z \geqslant 0, w \geqslant 0\}\,, \tag{5.9}$$

where ϵ_k denotes the kth vector of the standard basis. The polyhedron $\mathcal{W}^{(s)}_k \coloneqq \{(z, w) \in \mathbb{R}^{m_s} \times \mathbb{R}^n \colon (B^{(s)})^\intercal z + w = \epsilon_k, z \geqslant 0, w \geqslant 0\}$ is pointed, and hence the minimum in (5.9) is attained in some vertex of this polyhedron and these vertices are rational ([Sch87, Chapter 8]). Let $V^{(s)}_k$ denote the set of vertices of $\mathcal{W}^{(s)}_k$. Therefore, we have

$$F_k(x) = \max_{s \in Q(x)} \min_{(z,w) \in V^{(s)}_k} \{z^\intercal c^{(s)} + w^\intercal x\}\,. \tag{5.10}$$

Moreover, by Farkas' lemma (Theorem 4.36), the polyhedron $\{y \in \mathbb{R}^n \colon B^{(s)}y \leqslant c^{(s)}, y \leqslant x\}$ is nonempty if and only if for all (z, w) such that $(B^{(s)})^\intercal z + w = 0$, $z \geqslant 0$, $w \geqslant 0$, we have

$z^\intercal c^{(s)} + w^\intercal x \geqslant 0$. As previously, if $U^{(s)}$ consists of precisely one rational representative of every extreme ray of the polyhedral cone $\{(z,w) \in \mathbb{R}^{m_s} \times \mathbb{R}^n : (B^{(s)})^\intercal z + w = 0, z \geqslant 0, w \geqslant 0\}$, then $\{y \in \mathbb{R}^n : B^{(s)} y \leqslant c^{(s)}, y \leqslant x\}$ is nonempty if and only if $z^\intercal c^{(s)} + w^\intercal x \geqslant 0$ for all $(z,w) \in U^{(s)}$. Let U denote the disjoint union of $U^{(1)}, \dots, U^{(q)}$. For every possible choice $\lozenge \in \{<,>,=\}^U$ we define $\tilde{\mathcal{W}}^\lozenge := \{x \in \mathbb{R}^n : \forall s \in [q], \forall (z,w) \in U^{(s)}, z^\intercal c^{(s)} + w^\intercal x \lozenge_{(s,z,w)} 0\}$. Note that $Q(x)$ is constant on each $\tilde{\mathcal{W}}^\lozenge$. By fixing the terms achieving the maximum and minimum in (5.10) we subdivide the sets $\tilde{\mathcal{W}}^\lozenge$ into smaller sets, $\tilde{\mathcal{W}}^\lozenge = \bigcup_{j=1}^{N_\lozenge} \tilde{\mathcal{W}}_j^\lozenge$ such that every $\tilde{\mathcal{W}}_j^\lozenge$ is an intersection of half-spaces (both open and closed) and F is affine on $\tilde{\mathcal{W}}_j^\lozenge$. Since F is continuous (by Lemma 5.31), we can then restrict ourselves to these sets $\tilde{\mathcal{W}}_j^\lozenge$ that are full dimensional, and this gives the piecewise description of F. To see that F has the expected property, note that the definition of $U^{(s)}$ does not depend on $\bar{c}$. Therefore, the number of sets $\tilde{\mathcal{W}}^\lozenge$ does not depend on $\bar{c}$. Furthermore, the definition of $V_k^{(s)}$ also does not depend on $\bar{c}$. Hence, the numbers $N_\lozenge$ are bounded by a quantity that does not depend on $\bar{c}$. Moreover, by (5.10), the matrices $A^{(s)}$ are of the form $A_k^{(s)} = w^\intercal$. Hence, the common denominator of the numbers $A_{kl}^{(s)}$ is bounded by a quantity that does not depend on $\bar{c}$. The fact that $A^{(s)}$ must be stochastic follows from Lemma 5.32. □

Theorem 5.2 follows from Lemma 5.52 by a careful examination of the proofs presented in Section 5.4. We split the proof into a series of lemmas. The first lemma is a uniform version of Proposition 5.44.

Lemma 5.53. *There exists a number $N \geqslant 1$ that depends on $B^{(1)}, \dots, B^{(q)}$, but not on $\bar{c}$, such that if $\mathcal{S}_{\bar{c}}$ is a real tropical cone, then it is a projection of a real tropical Metzler spectrahedron $\mathcal{S}(Q^{(0)} | Q^{(1)}, \dots, Q^{(n')})$, $Q^{(k)} \in \mathbb{T}_\pm^{m \times m}$, satisfying $n' \leqslant N$ and $m \leqslant N$.*

Proof. Suppose that $\mathcal{S}_{\bar{c}}$ is a real tropical cone. We will examine all the steps involved in the proof of Proposition 5.44. By Lemma 5.52 and the proof of Proposition 5.34, there exists $N_1 \geqslant 1$ independent of $\bar{c}$ such that we have $\mathcal{S}_{\bar{c}} = \{x \in \mathbb{R}^n : x \leqslant F(x)\}$, where the operator $F : \mathbb{R}^n \to \mathbb{R}^n$ is semilinear, monotone, homogeneous, and has a piecewise description $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s \in [p]}$ satisfying $p \leqslant N_1$. Moreover, the matrices $A^{(s)}$ are stochastic, rational, and the common denominator of all the numbers $A_{kl}^{(s)}$ is not greater than N_1. By Theorem 5.27, the operator F can be written as $\forall k$, $F_k(x) = \min_{i \in [M_k]} \max_{s \in S_{ki}} (A_k^{(s)} x + b_k^{(s)})$, where $1 \leqslant M_k \leqslant 2^{N_1}$ for all $k \geqslant 1$ and every set S_{ki} is a subset of $[p]$.

Given the operator F, the proof of Lemma 5.38 constructs a directed graph $\vec{\mathcal{G}}$ satisfying Assumption C. By the construction presented in the proof of Lemma 5.38, the vertices, edges, and probability distributions of $\vec{\mathcal{G}}$ depend on $(n, (M_k)_k, (S_{ki})_{ki}, (A^{(s)})_s)$, but not on $(b^{(s)})_{s \in [p]}$. The vector $(b^{(s)})_{s \in [p]}$ is used only to define the weights r_e associated with the edges of the graph. Subsequently, the proof of Proposition 5.44 proceeds by applying Lemmas 5.41 to 5.43 to the graph $\vec{\mathcal{G}}$ and transforming it into a graph $\vec{\mathcal{G}}'$. Let n'' denote the number of Min vertices of $\vec{\mathcal{G}}'$ and m' denote the number of Max vertices of $\vec{\mathcal{G}}'$. By the constructions given in Lemmas 5.41 to 5.43, the numbers n'', m' do not depend on the weights r_e associated with the edges of $\vec{\mathcal{G}}$. Hence, they do not depend on $(b^{(s)})_{s \in [p]}$. Given $\vec{\mathcal{G}}'$, we apply Proposition 5.40 to construct some symmetric tropical matrices $\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n'')}$ such that $\tilde{Q}^{(k)} \in \mathbb{T}_\pm^{m' \times m'}$. As a final step, the proof of Proposition 5.44 transforms these matrices into matrices $Q^{(0)}, Q(1), \dots, Q^{(n')} \in \mathbb{T}_\pm^{m \times m}$ by adding some additional variables and 2×2 blocks to the matrices $\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n'')}$. However, the number of these additional variables and blocks depends only on n and n''. As a consequence, the numbers n', m depend on $(n, (M_k)_k, (S_{ki})_{ki}, (A^{(s)})_s)$, but not on $(b^{(s)})_{s \in [p]}$.

Since the number of possible tuples $((M_k)_k, (S_{ki})_{ki}, (A^{(s)})_s)$ is finite and bounded by a quantity that depends only on n, N_1 (not on $\bar{c}$), the numbers n', m are also bounded by a quantity that does not depend on $\bar{c}$. □

The claim of the previous lemma can be extended to tropically convex sets using the proof of Lemma 5.49.

Lemma 5.54. *There exists a number $N \geqslant 1$ that depends on $B^{(1)}, \dots, B^{(q)}$, but not on $\bar{c}$, such that if $\mathcal{S}_{\bar{c}}$ is tropically convex, then it is a projection of a real tropical Metzler spectrahedron $\mathcal{S}(Q^{(0)}|Q(1), \dots, Q^{(n')})$, $Q^{(k)} \in \mathbb{T}_{\pm}^{m \times m}$, satisfying $n' \leqslant N$ and $m \leqslant N$.*

Proof. Let $e = (1, 1, \dots, 1)$. For every $s \in [q]$ we denote $\tilde{B}^{(s)} = [-B^{(s)}e|B^{(s)}] \in \mathbb{Q}^{m_s \times (n+1)}$. Then, the real homogenization of $\mathcal{S}_{\bar{c}}$ is given by

$$\mathcal{S}_{\bar{c}}^{rh} = \bigcup_{s=1}^{q} \{(x_0, x) \in \mathbb{R}^{n+1} \colon \tilde{B}^{(s)}(x_0, x)^{\intercal} \leqslant c^{(s)}\}\,. \tag{5.11}$$

Indeed, if we take any $x_0 \in \mathbb{R}$ and $x \in \mathcal{S}_{\bar{c}}$ such that $B^{(s)}x \leqslant c^{(s)}$, then $\tilde{B}^{(s)}(x_0, x_0 + x)^{\intercal} = -x_0B^{(s)}e + B^{(s)}(x_0 + x) = B^{(s)}x \leqslant c^{(s)}$. Conversely, if (x_0, x) is such that $\tilde{B}^{(s)}(x_0, x)^{\intercal} \leqslant c^{(s)}$, and we denote $y = -x_0 + x$, then $B^{(s)}y = -x_0B^{(s)}e + B^{(s)}x = \tilde{B}^{(s)}(x_0, x)^{\intercal} \leqslant c^{(s)}$ and $y \in \mathcal{S}_{\bar{c}}$. Given $\mathcal{S}_{\bar{c}}$, the proof of Lemma 5.49 considers its real homogenization $\mathcal{S}_{\bar{c}}^{rh}$, describes it as a projection of a real tropical Metzler spectrahedron $\mathcal{S}(\tilde{Q}^{(0)}|\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')})$, and adds a 2×2 block to the matrices $\tilde{Q}^{(0)}, \tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')}$. Therefore, the claim follows from Lemma 5.53 and (5.11). □

The next lemma is a more precise version of Lemma 5.51.

Lemma 5.55. *Suppose that $\mathcal{S}_1 \subset \mathbb{T}^n$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_1 = \mathcal{S}(\tilde{Q}^{(0)}|\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n_1)})$, $\tilde{Q}^{(k)} \in \mathbb{T}_{\pm}^{m_1 \times m_1}$ and that $\mathcal{S}_2 \subset \mathbb{T}^n$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_2 = \mathcal{S}(\hat{Q}^{(0)}|\hat{Q}^{(1)}, \dots, \hat{Q}^{(n_2)})$, $\hat{Q}^{(k)} \in \mathbb{T}_{\pm}^{m_2 \times m_2}$. Then, there exists a number N that depends only on (n, n_1, m_1, n_2, m_2) such that* $\operatorname{tconv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ *is a projection of a tropical Metzler spectrahedron $\mathcal{S}(Q^{(0)}|Q^{(1)}, \dots, Q^{(n')})$, $Q^{(k)} \in \mathbb{T}_{\pm}^{m' \times m'}$ satisfying $n', m' \leqslant N$.*

Proof. The proof follows from an examination of the construction given in Lemma 5.51. More precisely, the construction is as follows. The first step is to consider the homogenizations $\mathcal{S}_1^h, \mathcal{S}_2^h$ and describe them as projections of tropical Metzler spectrahedra $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2$. The matrices that describe $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2$ are constructed using Lemma 5.50, by taking the formal homogenizations $\mathcal{S}_1^{fh}, \mathcal{S}_2^{fh}$ and adding variables and 2×2 blocks to the matrices that describe these formal homogenizations. The number of these variables and blocks depends only on (n, n_1, n_2). Then, the construction presented in Lemma 5.51 considers the set $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$, which is a tropical Metzler spectrahedron described by block diagonal matrices obtained by stacking the matrices that describe $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2$. Afterwards, the construction adds some 2×2 blocks to the matrices that describe $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$ and the number of these blocks depends only on (n, n_1, n_2). This gives a tropical Metzler spectrahedron $\tilde{\mathcal{S}}_3$ such that the homogenization of $\operatorname{tconv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ is its projection. Finally, the other part of Lemma 5.50 is used: we add a 2×2 block to the matrices that describe $\tilde{\mathcal{S}}_3$ and find the matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(n')}$. Hence, the numbers n', m' depend only on (n, n_1, m_1, n_2, m_2). □

We are now ready to give a stratified version of Lemma 5.54 and prove Theorem 5.2. Suppose that $(f_{K,i})_{K \subset [n], i \in [q_K]}$ is a collection of homogeneous polynomials with integer coefficients, $f_{K,i} \in \mathbb{Z}[X_K]$, where $X_K = (x_{k_1}, \dots, x_{k_\ell})$ for $K = \{k_1, \dots, k_\ell\}$, and we use the convention that $f_{\emptyset,i}$ is the zero polynomial. Given a real vector $\bar{c} = (c^{(K,i)})_{K,i}$, $c^{(K,i)} \in \mathbb{R}$ we denote by $\mathscr{S}_{\bar{c}} \subset \mathbb{T}^n$ the set

$$\bigcup_{K \in [n]} \bigcup_{i \in [q_K]} \{x \in \mathbb{T}^n \colon \forall k \notin K, x_k = -\infty \wedge \forall k \in K, x_k \neq -\infty \wedge f_{K,i}(x_K) \leqslant c^{(K,i)}\} . \tag{5.12}$$

Lemma 5.56. *There exists a number $N \geqslant 1$ that depends on $(f_{K,i})_{K \subset [n], i \in [q_K]}$, but not on $\bar{c}$, such that if $\mathscr{S}_{\bar{c}}$ is tropically convex, then it is a projection of a tropical Metzler spectrahedron $\mathcal{S}(Q^{(0)}|Q(1), \dots, Q^{(n')})$, $Q^{(k)} \in \mathbb{T}_{\pm}^{m \times m}$, satisfying $n' \leqslant N$ and $m \leqslant N$.*

Proof. The claim follows from the previous lemmas by analyzing the proof of Theorem 5.5. Denote $\mathscr{S}_{\bar{c}} = \mathscr{S}$ and take any $K \subset [n]$. If K is nonempty, then the stratum of $\mathscr{S}$ associated with K, denoted $\mathscr{S}_K$, is tropically convex and given by

$$\mathscr{S}_K = \bigcup_{i \in [q_K]} \{x \in \mathbb{T}^K \colon f_{K,i}(x) \leqslant c^{(K,i)}\} .$$

Therefore, by Lemma 5.54, there exists N_K (independent on $\bar{c}$) and such that $\mathscr{S}_K$ is a projection of a real tropical Metzler spectrahedron $\mathcal{S}_K \subset \mathbb{R}^{n_K}$, $n_K \leqslant N_K$, described by matrices of size $m_K \leqslant N_K$. Moreover, as noted in the proof of Theorem 5.5, the set

$$\tilde{\mathscr{S}}_K = \bigcup_{i \in [q_K]} \{x \in \mathbb{T}^n \colon \forall k \notin K, x_k = -\infty \wedge \forall k \in K, x_k \neq -\infty \wedge f_{K,i}(x_K) \leqslant c^{(K,i)}\}$$

is a projection of a tropical Metzler spectrahedron $\tilde{\mathcal{S}}_K$. This spectrahedron is constructed by adding variables and 1×1 blocks to the matrices that describe $\mathcal{S}_K$, and the number of these blocks and variables does not depend on $\bar{c}$. Similarly, the set $\mathscr{S}_\emptyset$ is either empty or reduced to $-\infty$ and thus it is a tropical Metzler spectrahedron. We conclude by applying Lemma 5.55 to $\operatorname{tconv}(\bigcup \tilde{\mathscr{S}}_K)$. □

Proof of Theorem 5.2. Suppose that $S \subset \mathscr{K}^n$ is convex and semialgebraic. Fix a $\mathcal{L}_{\mathrm{or}}$-formula ψ and a vector $\bar{a} \in \mathscr{K}^n$ such that $S = \{x \in \mathscr{K}^n \colon \mathscr{K} \models \phi(x, \bar{a})\}$. Then, by Theorem 3.1, the set $\mathsf{val}(S)$ has closed semilinear strata. In particular, by Lemma 3.12, there exists a collection $(f_{K,i})_{K \subset [n], i \in [q_K]}$ and a vector $\bar{c} = (c^{(K,i)})_{K,i}$, $c^{(K,i)} \in \mathit{\Gamma}$ such that $\mathsf{val}(S)$ is of the form given in (5.12), $\mathsf{val}(S) = \mathscr{S}_{\bar{c}}$ (we note that this form is also correct for $K = \emptyset$ because $\mathit{\Gamma}$ is nontrivial). For every $\bar{b} \in \mathscr{K}^m$ we denote $S_{\bar{b}} \coloneqq \{x \in \mathscr{K}^n \colon \mathscr{K} \models \phi(x, \bar{b})\}$. If we fix $N \geqslant 1$, then the statement "for all $\bar{b}, \bar{c}$ such that $S_{\bar{b}}$ is convex and $\mathsf{val}(S_{\bar{b}}) = \mathscr{S}_{\bar{c}}$ there exist symmetric matrices $Q^{(0)}, \dots, Q^{(n')}$, $Q^{(k)} \in \mathscr{K}^{m' \times m'}$ such that $n', m' \leqslant N$, $n' \geqslant n$, and $S_{\bar{b}}$ has the same image by valuation as the projection of the spectrahedron associated with $Q^{(0)}, \dots, Q^{(n')}$" is a sentence in $\mathcal{L}_{\mathrm{rcvf}}$. This sentence is true if $\mathscr{K} = \mathbb{K}$ and N is large enough. Indeed, in this case we can apply Lemma 5.56 to $\mathscr{S}_{\bar{c}}$ to describe it as a projection of a tropical Metzler spectrahedron and lift this spectrahedron into $\mathbb{K}$ by Proposition 4.14. This spectrahedron fulfills the claim (as noted in the proof of Theorem 5.5). Therefore, if we fix N such that the sentence above in true over $\mathbb{K}$, then the completeness result of Theorem 2.120 shows that this sentence is true in $\mathscr{K}$. □

Part II

Relation to stochastic mean payoff games

CHAPTER 6

Introduction to stochastic mean payoff games

In this chapter we introduce the notion of stochastic mean payoff games and we show that these games can be analyzed using monotone homogeneous operators (called Shapley operators). Our presentation follows [AGG12]. The reference [AGG12] considered only deterministic mean payoff games, but numerous proofs of [AGG12] extend immediately to the case of stochastic mean payoff games. However, we add our two contributions. First, in Theorem 6.16 we show that the supports of tropical cones defined as sublevels sets of Shapley operators correspond to winning dominions of the underlying game (which is a nontrivial extension of the analogous result of [AGG12]). Second, in Theorem 6.30 we combine the analysis of this chapter with the previous one, showing the structural equivalence between tropicalizations of closed convex semialgebraic cones and stochastic mean payoff games. The chapter is organized as follows. In Section 6.1 we define the notion of a stochastic mean payoff game and the associated Shapley operator. We also show how Kohlberg's theorem [Koh80] can be used to prove that these games have optimal policies. In Section 6.2 we study the dominions and show a simple version of the Collatz–Wielandt property of [Nus86, AGG12]. In Section 6.3 we specify the analysis to the case of bipartite games and give the relationship with the results of the previous chapter. A simplified version of the results presented here is included in the paper [AGS18].

6.1 Optimal policies and Shapley operators

A *stochastic mean payoff game* involves two players, Max and Min, who control disjoint sets of states. The states owned by Player Max and Player Min are respectively denoted as $V_{\mathrm{Max}} \coloneqq [m]$ and $V_{\mathrm{Min}} \coloneqq [n]$. The space of states of the games is a disjoint union of V_{Max} and V_{Min}, $V \coloneqq V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$. We will use the symbols i, j to refer to states of Player Max, and k, l to states of Player Min. Every state $k \in [n]$ of Player Min is equipped with a finite and nonempty set $A^{(k)}$ (called the set of *actions*). Similarly, every state $i \in [m]$ of Player Max is equipped with a finite and nonempty set $B^{(i)}$. Every action $a \in \biguplus_k A^{(k)}$ is equipped with a probability distribution p^a_k over V, $p^a_k = (p^a_{kv})_{v \in V} \in [0,1]^V$, $\sum_{v \in V} p^a_{kv} = 1$ and a real number $r_a \in \mathbb{R}$. Likewise, every action $b \in \biguplus_i B^{(i)}$ is equipped with a probability distribution $p^b_i = (p^b_{iv})_{v \in V} \in [0,1]^V$, $\sum_{v \in V} p^b_{iv} = 1$ over V and a real number $r_b \in \mathbb{R}$. Both players alternatively move a pawn over these states by the rules described as follows. When the pawn is on a state $k \in [n]$, Player Min chooses an action $a \in A^{(k)}$. Then, Player Min moves the pawn to state $v \in V$ with probability p^a_{kv} and pays r^a to Player Max. Similarly, once the pawn is on a state $i \in [m]$, Player Max picks an action $b \in B^{(i)}$. Then, Player Max moves the pawn to the state $v \in V$ with probability p^b_{iv} and Player Min pays him r^b.

A *policy for Player Min* is a function $\sigma \colon [n] \to \biguplus_k A^{(k)}$ mapping every state $k \in [n]$ to an action $\sigma(k)$ in $A^{(k)}$. Analogously, a *policy for Player Max* is a function $\tau \colon [m] \to \biguplus_i B^{(i)}$ such that $\tau(i) \in B^{(i)}$ for all $i \in [m]$. Suppose that the game starts from a state v^*. When players play according to a couple (σ, τ) of policies, the movement of the pawn is described by a Markov chain on the space V. The average payoff of Player Max in the long-term is then defined as the average payoff of the controller in this Markov chain, see Definition 2.136. In other words, the payoff of Player Max is given by

$$g_{v^*}(\sigma, \tau) = \lim_{N \to \infty} \mathbb{E}_{\sigma,\tau}\Big(\frac{1}{N}\sum_{p=1}^{N} r_{\xi(v_p)}\Big) \,, \tag{6.1}$$

where the expectation $\mathbb{E}_{\sigma,\tau}$ is taken over all the trajectories $v_1, \dots, v_N$ starting from $v_1 = v^*$ in the Markov chain, and the function ξ is defined as $\xi(k) \coloneqq \sigma(k)$ for $k \in [n]$ and $\xi(i) \coloneqq \tau(i)$ for $i \in [m]$. This payoff is well defined by Theorem 2.137. The goal of Player Max is to find a policy that maximizes his average payoff, while Player Min aims at minimizing this quantity. The main theorem of stochastic mean payoff games ensures that this game has a well-defined optimal policies. This was shown by Liggett and Lippman [LL69].

Theorem 6.1 ([LL69])**.** *There exists a pair of policies $(\bar{\sigma}, \bar{\tau})$ and a unique vector $\chi \in \mathbb{R}^V$ such that for all initial states $v \in V$, the following two conditions are satisfied:*

- *for each policy σ of Player Min, $\chi_v \leqslant g_v(\sigma, \bar{\tau})$;*
- *for each policy τ of Player Max, $\chi_v \geqslant g_v(\bar{\sigma}, \tau)$.*

The vector χ called the value *of the game, while the policies $(\bar{\sigma}, \bar{\tau})$ are called* optimal. *We note that we have $\chi_v = g_v(\bar{\sigma}, \bar{\tau})$ for all $v \in V$.*

The first condition in Theorem 6.1 states that, by playing according to the policy $\bar{\tau}$, Player Max is certain to get an average payoff greater than or equal to the value χ_v associated with the initial state. Symmetrically, Player Min is ensured to limit her average loss to the quantity χ_v by following the policy $\bar{\sigma}$.

Example 6.2. Consider the game is depicted in Fig. 6.1. The states of Player Min are depicted by circles. The states of Player Max are depicted by squares. Moreover, all probabilities in

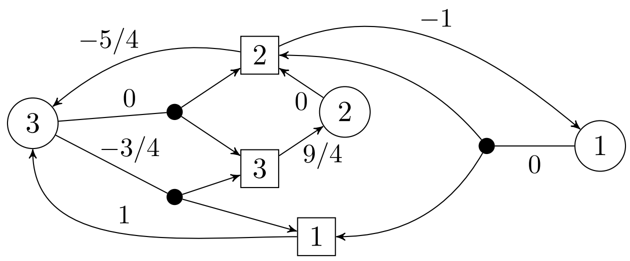

Figure 6.1: Stochastic mean payoff game considered in Example 6.2.

this example belong to $\{0, 1/2, 1\}$. More precisely, edges represent the actions of players. Edges without full dots represent deterministic actions and full dots indicate actions where a coin toss is made. The corresponding payments received by Player Max are indicated on the edges. Observe that both players in this game have only two policies: at state $\boxed{2}$ Player Max can choose the move that goes to 1 or the one that goes to 3, whereas at state 3 Player Min can choose the move $\{\boxed{1}, \boxed{3}\}$ or the move $\{\boxed{2}, \boxed{3}\}$. Both players in this example game have only two policies: at state $\boxed{2}$ Player Max can choose the action that goes to 1 or the action that goes to 3, whereas at state 3 Player Min can choose the action that goes to $\{\boxed{1}, \boxed{3}\}$ or the action that goes to $\{\boxed{2}, \boxed{3}\}$. Suppose that Player Max chooses the action that goes to 1 and that Player Min chooses the action that goes to $\{\boxed{1}, \boxed{3}\}$. The Markov chain obtained in this way has the transition matrix of form

$$P = \begin{bmatrix} 0 & Y \\ Z & 0 \end{bmatrix} ,$$

where Y describes the probabilities of transition from circle states to square states, and Z describes the probabilities of transition from square states to circle states, i.e.,

$$Y = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \end{array}, \quad Z = \begin{array}{c} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} .$$

This chain has only one recurrent class and all states belong to this class. Moreover, it is easy to verify that

$$\pi = \frac{1}{10}(2, 1, 2, 2, 2, 1) \in \mathbb{R}^{V_{\mathrm{Min}}} \times \mathbb{R}^{V_{\mathrm{Max}}}$$

is the stationary distribution of this chain. Therefore, by Theorem 2.137, the payoff of Player Max is equal to

$$g\Big(\{\boxed{1}, \boxed{3}\}, \{1\}\Big) = \frac{1}{10}(-\frac{6}{4} + 2 - 2 + \frac{9}{4}) = \frac{3}{40}$$

for every initial state. Similarly, if Player Max chooses the action that goes to 1 and Player Min chooses the action that goes to $\{\boxed{2}, \boxed{3}\}$, then the transition matrix of the induced Markov chain has the form $P' = \left[\begin{smallmatrix} 0 & Y' \\ W & 0 \end{smallmatrix}\right]$, where

$$Y' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

and Z is the same as previously. In this case the chain also has only one recurrent class and every state belongs to this class. Furthermore,

$$\pi' = \frac{1}{14}(4, 1, 2, 2, 4, 1) \in \mathbb{R}^{V_{\mathrm{Min}}} \times \mathbb{R}^{V_{\mathrm{Max}}}$$

is the stationary distribution of this chain. Hence the payoff of Player Max is equal to

$$g\Big(\{\boxed{2}, \boxed{3}\}, \{①\}\Big) = \frac{1}{14}(2 - 4 + \frac{9}{4}) = \frac{1}{56}$$

for every initial state. In both cases, the payoff of Player Max is positive. Therefore, if Player Max chooses the action that goes to ①, then he is guaranteed to obtain a nonnegative payoff. One can check, by doing the same calculations for the remaining policies, that $\Big(\{\boxed{2}, \boxed{3}\}, \{①\}\Big)$ is the unique couple of optimal policies in this game and the value of the game is equal to $\chi = \frac{1}{56}(1, 1, 1, 1, 1, 1)$.

In this section we prove Theorem 6.1 using the associated Shapley operator. This proof was given in [AGG12] in the case of deterministic mean payoff games (i.e., when all probability distributions are reduced to vectors in $\{0, 1\}^V$). However, the proof in the stochastic case is the same as the one given in [AGG12], and we reproduce it here for the sake of completeness.

Definition 6.3. Given a stochastic mean payoff game, we define its *Shapley operator* $T \colon \mathbb{T}^V \to \mathbb{T}^V$ as

$$\forall v \in V,\ (T(x))_v := \begin{cases} \min_{a \in A^{(v)}} (r_a + \sum_{w \in V} p_{vw}^a x_w) & \text{if } v \in V_{\mathrm{Min}} \\ \max_{b \in B^{(v)}} (r_b + \sum_{w \in V} p_{vw}^b x_w) & \text{if } v \in V_{\mathrm{Max}}\,, \end{cases} \tag{6.2}$$

where we use the convention that $-\infty \cdot 0 = 0$ and $-\infty \cdot x = -\infty$ for all $x > 0$.

The basic properties of Shapley operators are summarized in the next lemma.

Lemma 6.4. *The Shapley operator T is monotone and homogeneous. Furthermore, it preserves $\mathbb{R}^V$ (i.e., if $x \in \mathbb{R}^V$, then $T(x) \in \mathbb{R}^V$) and its restriction $T_{|\mathbb{R}^V} \colon \mathbb{R}^V \to \mathbb{R}^V$ is piecewise affine.*

Proof. The facts that T is monotone, homogeneous, and preserves $\mathbb{R}^V$ follow immediately from the definition. Its restriction to $\mathbb{R}^V$ is piecewise affine because it is defined as a coordinatewise minimum or maximum of affine functions. □

Our proof of Theorem 6.1 is based on the notion of invariant half-line and the following result of Kohlberg [Koh80].

Definition 6.5. Given a function $f \colon \mathbb{R}^n \to \mathbb{R}^n$, we say that a pair $(u, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ is an *invariant half-line* of f if there exists $\gamma_0 \geqslant 0$ such that the equality $f(u + \gamma\eta) = u + (\gamma + 1)\eta$ holds for all $\gamma \geqslant \gamma_0$. Given a function $f \colon \mathbb{T}^n \to \mathbb{T}^n$ that preserves $\mathbb{R}^n$, we say that $(u, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ is an invariant half-line of f if it is an invariant half-line of $f_{|\mathbb{R}^n} \colon \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 6.6 ([Koh80])**.** *Suppose that function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is piecewise affine and nonexpansive in any norm. Then, it admits and invariant half-line. Furthermore, if (u_1, η_1) and (u_2, η_2) are invariant half-lines, then $\eta_1 = \eta_2$.*

Later on, we will need a more precise formulation of Kohlberg's theorem. Therefore, we present its proof in Appendix B.3.

Corollary 6.7. *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a monotone, homogeneous, and piecewise affine function. Then, it admits and invariant half-line (u, η, γ_0). Furthermore, if $x \in \mathbb{R}^n$ is any point, then we have the equality*

$$\lim_{N\to\infty} \frac{1}{N} f^N(x) = \eta \, ,$$

where $f^N = f \circ f \circ \dots \circ f$ (N times). In particular, this limit exists.

Proof. The function f is nonexpansive in the supremum norm by Lemma 5.31. As a consequence, it admits an invariant half-line by Theorem 6.6. In particular, for every $x \in \mathbb{R}^n$ we have

$$\|f^N(x) - N\eta - u - \gamma_0\eta\|_\infty = \|f^N(x) - f^N(u + \gamma_0\eta)\|_\infty \leqslant \|x - u - \gamma_0\eta\|_\infty$$

and $\lim_{N\to\infty} \frac{1}{N} f^N(x) = \eta$. □

In order to prove Theorem 6.1, we introduce the following notation. If σ is a policy of Player Min, then we denote by T^σ the Shapley operator of a game in which Player Min uses σ (i.e., a game in which every state $k \in [n]$ controlled by Player Min is equipped with exactly one action and this action is $\sigma(k)$). Analogously, if τ is a policy of Player Max, then we denote by T^τ the Shapley operator of a game in which Player Max uses τ. Furthermore, we denote by $T^{\sigma,\tau}$ the Shapley operator of a game in which Player Min uses σ and Player Max uses τ. In an explicit way, these operators are given by

$$\forall v \in V,\ (T^\sigma(x))_v := \begin{cases} r_{\sigma(v)} + \sum_{w\in V} p_{vw}^{\sigma(v)} x_w & \text{if } v \in V_{\text{Min}} \\ \max_{b\in B^{(v)}}(r_b + \sum_{w\in V} p_{vw}^b x_w) & \text{if } v \in V_{\text{Max}} \, , \end{cases}$$

$$\forall v \in V,\ (T^\tau(x))_v := \begin{cases} \min_{a\in A^{(v)}}(r_a + \sum_{w\in V} p_{vw}^a x_w) & \text{if } v \in V_{\text{Min}} \\ r_{\tau(v)} + \sum_{w\in V} p_{vw}^{\tau(v)} x_w & \text{if } v \in V_{\text{Max}} \, , \end{cases}$$

$$\forall v \in V,\ (T^{\sigma,\tau}(x))_v := \begin{cases} r_{\sigma(v)} + \sum_{w\in V} p_{vw}^{\sigma(v)} x_w & \text{if } v \in V_{\text{Min}} \\ r_{\tau(v)} + \sum_{w\in V} p_{vw}^{\tau(v)} x_w & \text{if } v \in V_{\text{Max}} \, . \end{cases}$$

The properties of these operators are given in the next lemmas.

Lemma 6.8. *If we fix $x \in \mathbb{R}^V$, then the set $\{T^\sigma(x)\colon \sigma \text{ is a policy of Player Min}\} \subset \mathbb{R}^V$ has a well defined minimum in the partial order given by $y \leqslant z \iff \forall v \in V, y_v \leqslant z_v$. Moreover, this minimum is equal to $T(x)$. Analogously, the set $\{T^\tau(x)\colon \tau \text{ is a policy of Player Max}\} \subset \mathbb{R}^V$ has a well defined maximum in this partial order and this maximum is equal to $T(x)$.*

Proof. By definition we have $(T^\sigma(x))_v \geqslant (T(x))_v$ for all $v \in V$. Furthermore, for every state $k \in [n]$ we can take an action $a^* \in A^{(k)}$ that satisfies $r_{a^*} + \sum_{w\in V} p_{vw}^{a^*} x_w = \min_{a\in A^{(k)}}(r_a + \sum_{w\in V} p_{vw}^a x_w)$ and define $\sigma^*(k) := a^*$. The policy σ^* satisfies $(T^{\sigma^*}(x))_v = (T(x))_v$ for all $v \in V$. The other case is analogous. □

Lemma 6.9. *If Player Min uses a policy σ and Player Max uses a policy τ, then the expected total payoff of Player Max after the Nth move of the pawn is equal to $(T^{\sigma,\tau})^N(0)$.*

Proof. We prove the claim by induction. Let $g^N \in \mathbb{R}^V$ denote the expected payoff of Player Max after the Nth move of the pawn. We want to show that $g^N = (T^{\sigma,\tau})^N(0)$ for all $N \geqslant 1$. The case $N = 1$ is trivial. Moreover, observe that for all $N \geqslant 2$, g^N satisfies the recurrence

$$\forall v \in V,\ g_v^N = \begin{cases} r_{\sigma(v)} + \sum_{w\in V} p_{vw}^{\sigma(v)} g_w^{N-1} & \text{if } v \in V_{\text{Min}} \\ r_{\tau(v)} + \sum_{w\in V} p_{vw}^{\tau(v)} g_w^{N-1} & \text{if } v \in V_{\text{Max}} \, . \end{cases}$$

Moreover, for every $v \in V$ we have

$$((T^{\sigma,\tau})^N(0))_v = \Big(T((T^{\sigma,\tau})^{N-1}(0))\Big)_v = \begin{cases} r_{\sigma(v)} + \sum_{w \in V} p_{vw}^{\sigma(v)} \big((T^{\sigma,\tau})^{N-1}(0)\big)_w & \text{if } v \in V_{\mathrm{Min}} \\ r_{\tau(v)} + \sum_{w \in V} p_{vw}^{\tau(v)} \big((T^{\sigma,\tau})^{N-1}(0)\big)_w & \text{if } v \in V_{\mathrm{Max}} \,. \end{cases}$$

Hence $g^N = (T^{\sigma,\tau})^N(0)$ for all $N \geqslant 1$. ☐

Lemma 6.10. *Suppose that (u, η) is an invariant half-line of T. Then, there exists a couple of policies $(\bar{\sigma}, \bar{\tau})$ such that (u, η) is also an invariant half-line of the operators $T^{\bar{\sigma}}$ and $T^{\bar{\tau}}$.*

Proof. By Lemma 6.8, for every sufficiently large $\ell \geqslant 0$, there exists a policy σ_ℓ such that $T^{\sigma_\ell}(u + \ell\eta) = T(u + \ell\eta) = u + (\ell + 1)\eta$. Let $\bar{\sigma}$ be a policy such that $\bar{\sigma} = \sigma_\ell$ for infinitely many values of ℓ that belong to natural numbers. In this way we have $T^{\bar{\sigma}}(u + \ell\eta) = u + (\ell + 1)\eta$ for infinitely many natural numbers ℓ. Since the function $T^{\bar{\sigma}}_{|\mathbb{R}^V}$ is piecewise affine, the same is true for the function of one variable $\ell \mapsto T^{\bar{\sigma}}(u + \ell\eta)$. Hence, we have $T^{\bar{\sigma}}(u + \ell\eta) = u + (\ell + 1)\eta$ for every sufficiently large ℓ. The construction of $\bar{\tau}$ is analogous. ☐

We can now present the proof of Theorem 6.1.

Proof of Theorem 6.1. By Lemma 6.4 and Corollary 6.7, T admits an invariant half-line (u, η). Let $(\bar{\sigma}, \bar{\tau})$ be a couple of policies that satisfy the claim of Lemma 6.10. Suppose that τ is any policy of Player Max. By Lemma 6.9 and Corollary 6.7, for all $v \in V$ we have

$$g(\bar{\sigma}, \tau)_v = \lim_{N \to \infty} \frac{\Big((T^{\bar{\sigma},\tau})^N(0)\Big)_v}{N} \leqslant \lim_{N \to \infty} \frac{\Big((T^{\bar{\sigma}})^N(0)\Big)_v}{N} = \eta_v \,.$$

Analogously, if σ is any policy of Player Min, then for all $v \in V$ we have $g(\sigma, \bar{\tau})_v \geqslant \eta_v$. Hence, the policies $(\bar{\sigma}, \bar{\tau})$ are optimal. ☐

Remark 6.11. The proof above shows that if (u, η) is an invariant half-line of T, then η is the value of the game, $\chi = \eta$.

6.2 Sublevel sets, dominions, and the Collatz–Wielandt property

In this section we analyze the sublevel sets $\mathscr{S} := \{x \in \mathbb{T}^V : x \leqslant T(x)\}$, where T is a Shapley operator. Since Shapley operators are monotone and homogeneous, the set $\mathscr{S}$ is a tropical cone (this can be proven as in Proposition 5.34). Furthermore, observe that $\mathscr{S}$ always contains the point $-\infty$. We are interested in characterizing the situations in which this cone is nontrivial, i.e., contains a point different than $-\infty$. This was done for deterministic games in [AGG12, Theorem 3.2] and we extend this result to stochastic games. This requires to introduce the definition of a dominion.

Definition 6.12. We say that a set $W \subset V$ is a *dominion (for Player Max)* if the following two conditions are satisfied:

(i) Player Min cannot leave W. In other words, for every state $k \in V_{\mathrm{Min}} \cap W$ and every action $a \in A^{(k)}$ we have $\sum_{v \in W} p_{kv}^a = 1$.

(ii) Player Max can ensure that the game stays in W provided that it starts in W. In other words, for every state $i \in V_{\mathrm{Max}} \cap W$ there exists an action $b \in B^{(i)}$ such that $\sum_{v \in W} p_{iv}^b = 1$.

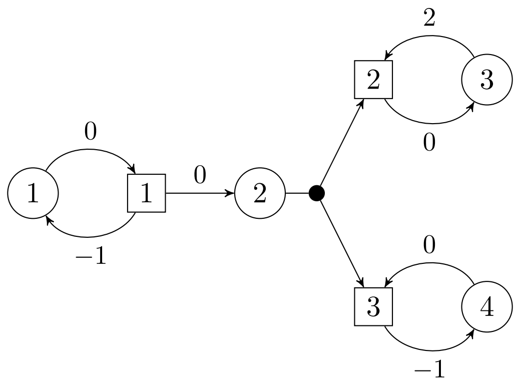

Figure 6.2: A mean payoff games illustrating the notion of dominions.

If $W \subset V$ is a dominion, then we defined a *game induced by* W in the following way. The space of states of the induced game is equal to W, the actions of Player Min are the same as her actions in the original game, and the set of actions of Player Max consist of these actions that fulfill the condition of Item (ii). The value of the induced game is called an *induced value.* If the induced value is nonnegative for every state belonging to W, then we say that this dominion is *winning (for Player Max).*

We note that the dominions are extensively used in the deterministic subexponential algorithm for parity games of [JPZ08] (however, the authors of [JPZ08] use the term "dominion" for what we call a winning dominion). Moreover, dominions are related to ergodicity conditions of stochastic games. This was studied in [BEGM15] for the class of games considered here and, in a more general setting of games with infinite action spaces, in [AGH15].

Lemma 6.13. *Let $\chi \in \mathbb{R}^V$ denote the value of a stochastic mean payoff game. If $W \subset V$ is a dominion, and $\tilde{\chi} \in \mathbb{R}^W$ denotes the value of the game induced by W, then $\tilde{\chi}_v \leqslant \chi_v$ for all $v \in W$.*

Proof. Let $\bar{\tau}$ be an optimal policy of Player Max in the induced game. Take any policy τ^* of Player Max in the original game that agrees with $\bar{\tau}$ on W, i.e., satisfies $\tau^*(v) = \bar{\tau}(v)$ for every $v \in W$. Let σ be any policy of Player Min in the original game. By the definition of dominion, if the Markov chain induced by (σ, τ^*) starts in a state belonging to W, then it never leaves W. In particular, the choice of τ^* guarantees an expected payoff of at least $\tilde{\chi}$ to Player Max, $g(\sigma, \tau^*)_v \geqslant \tilde{\chi}_v$ for every $v \in W$. Hence, by the definition of value, we have $\chi_v \geqslant \tilde{\chi}_v$ for all $v \in V$. ☐

Lemma 6.14. *Let $\chi \in \mathbb{R}^V$ denote the value of a stochastic mean payoff game, and let $W \subset V$ denote the set of all states with the maximal value, $v \in W \iff \chi_v = \max_{w \in V} \chi_w$. Then, the set W is a dominion. Moreover, the induced value of every state in this dominion is the same as in the original game, i.e., it is equal to $\max_{w \in V} \chi_w$.*

Proof. Let $T \colon \mathbb{T}^V \to \mathbb{T}^V$ denote the Shapley operator associated with the game and let (u, χ) be an invariant half-line of T (such a half-line exists by Remark 6.11). Moreover, denote $\chi^* = \max_{v \in V} \chi_v$. Suppose that $k \in W \cap V_{\mathrm{Min}}$ and let $a \in A^{(k)}$ be any action. Then, for large

$\gamma \geqslant 0$ we have

$$u_k + (\gamma+1)\chi^* = (T(u+\gamma\chi))_k \leqslant r^a + \sum_{v \in V} p_{kv}^a(u_v + \gamma\chi_v) = r^a + \sum_{v \in V} p_{kv}^a u_v + \gamma \sum_{v \in V} p_{kv}^a \chi_v \,. \quad (6.3)$$

Observe that if $\sum_{v \in W} p_{kv}^a < 1$, then $\sum_{v \in V} p_{kv}^a \chi_v < \chi^*$, which gives a contradiction with (6.3) for sufficiently large γ. Hence $\sum_{v \in W} p_{kv}^a = 1$. Similarly, let $i \in W \cap V_{\text{Max}}$ and let $\tilde{B}^{(i)} \subset B^{(i)}$ denote the set of actions of Player Max at state i such that $b \in \tilde{B}^{(i)} \iff \sum_{v \in W} p_{iv}^b = 1$. Observe that if $b \notin \tilde{B}^{(i)}$, then $\sum_{v \in V} p_{iv}^b \chi_v < \chi^*$ and the inequality

$$u_i + (\gamma+1)\chi^* = (T(u+\gamma\chi))_i \geqslant r^b + \sum_{v \in V} p_{iv}^b(u_v + \gamma\chi_v)$$

is strict for sufficiently large γ. Therefore, the set $\tilde{B}^{(i)}$ is nonempty and

$$u_i+(\gamma+1)\chi^* = (T(u+\gamma\chi))_i = \max_{b \in B^{(i)}} \bigl(r_b + \sum_{v \in V} p_{iv}^b(u_v+\gamma\chi_v)\bigr) = \max_{b \in \tilde{B}^{(i)}} \bigl(r_b + \sum_{v \in V} p_{iv}^b(u_v+\gamma\chi_v)\bigr) \quad (6.4)$$

for sufficiently large γ. Thus, W is a dominion. Moreover, if $\tilde{T} \colon \mathbb{T}^W \to \mathbb{T}^W$ denotes the Shapley operator of the induced game, then (6.4) shows that (u_W, χ_W) is an invariant half-line of $\tilde{T}$ and the claim follows from Remark 6.11. $\square$

Example 6.15. Consider the game presented in Fig. 6.2. The states ①, ②, and ③ have positive values. The minimal (inclusionwise) dominions are given by $\{①, \boxed{1}\}$, $\{③, \boxed{2}\}$, $\{④, \boxed{3}\}$, and $\{②, ③, ④, \boxed{2}, \boxed{3}\}$. Note that the value of the game induced by $\{①, \boxed{1}\}$ is negative. On the other hand, ③ has the greatest value and the game induced by $\{③, \boxed{2}\}$ has the same value as ③. The state ① has positive value, but it does not belong to any winning dominion. This highlights the difference between deterministic and stochastic mean payoff games—in the deterministic case, the set of all states with nonnegative values forms a winning dominion.

The following theorem characterizes the feasibility of the set $\mathcal{S}$.

Theorem 6.16. *Let $\lambda \in \mathbb{R}$ be a real number and consider the set $\mathcal{S}_\lambda := \{x \in \mathbb{T}^V : \lambda + x \leqslant T(x)\}$. If $K \subset V$ is a nonempty subset, then the stratum of $\mathcal{S}_\lambda$ associated with K is nonempty if and only if K is a dominion and every state of this dominion has an induced value that is not smaller than λ.*

The proof requires the following observation.

Lemma 6.17. *Suppose that $\lambda + x \leqslant T(x)$ for some $x \in \mathbb{R}^V$ and $\lambda \in \mathbb{R}$. Then, the value of every state of the game is not smaller than λ.*

Proof. Since T is monotone and homogeneous, we have $T^N(x) \geqslant N\lambda + x$ for all $N \geqslant 1$ and the claim follows from Corollary 6.7 and Remark 6.11. $\square$

Proof of Theorem 6.16. Fix a nonempty subset $K \subset V$ and let $x \in \mathcal{S}_\lambda$ be a point with support equal to K. We will show that K is a dominion. To see that, let $k \in K$ be a state controlled by Player Min. If there exists an action $a \in A^{(k)}$ such that $\sum_{v \in K} p_{kv}^a < 1$, then $\lambda + x_k \leqslant (T(x))_k \leqslant r_a + \sum_{v \in V} p_{kv}^a x_v = -\infty$, which gives a contradiction. Similarly, if $i \in K$ is a state controlled by Player Max and we have $\sum_{v \in W} p_{iv}^b < 1$ for every action $b \in B^{(i)}$, then $\lambda + x_i \leqslant (T(x))_i = -\infty$, which gives a contradiction. Hence K is a dominion. Furthermore, let $\tilde{T} \colon \mathbb{T}^K \to \mathbb{T}^K$ denote the Shapley operator of the induced game. Observe that $(\tilde{T}(x_K))_v = (T(x))_v \geqslant \lambda + x_v$ for all $v \in K$. Therefore, by Lemma 6.17, the induced value of every state in K is not smaller than λ.

To prove the opposite implication, suppose that $K \subset V$ is a dominion such that every state of this dominion has an induced value that is not smaller than λ. Let $\tilde{T} \colon \mathbb{T}^K \to \mathbb{T}^K$ be the Shapley operator of the induced game, and let (u, η) be an invariant half-line of $\tilde{T}$. Take any $\gamma \geqslant 0$ such that $\tilde{T}(u + \gamma\eta) = u + (\gamma + 1)\eta \geqslant \lambda + u + \gamma\eta$ and consider the point $x := u + \gamma\eta$. Let $\tilde{x} \in \mathbb{T}^V$ be defined as $\tilde{x}_v := x_v$ if $v \in K$ and $\tilde{x}_v := -\infty$ otherwise. Observe that for every $v \in K$ we have $\lambda + \tilde{x}_v = \lambda + x_v \leqslant (\tilde{T}(x))_v = (T(\tilde{x}))_v$. Moreover, for every $v \notin K$ we have $\lambda + \tilde{x}_v = -\infty \leqslant (T(\tilde{x}))_v$. □

As a corollary, we obtain the following property.

Corollary 6.18 (Collatz–Wielandt property)**.** *Let T be a Shapley operator of a stochastic mean payoff game and $\chi \in \mathbb{R}^V$ be its value. Then, we have the equalities*

$$\begin{aligned}
\max_{v \in V} \chi_v &= \max\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{T}^V, x \neq -\infty, \lambda + x \leqslant T(x)\}\,, \\
\min_{v \in V} \chi_v &= \max\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{R}^V, \lambda + x \leqslant T(x)\}\,.
\end{aligned}$$

Proof. Note that V is trivially a dominion. Therefore, by Theorem 6.16, the set $\{x \in \mathbb{R}^V : \lambda + x \leqslant T(x)\}$ is nonempty if and only if $\lambda \leqslant \min_{v \in V} \chi_v$. This shows the second equality. To show the first equality, observe that $\max_{v \in V} \chi_v \geqslant \sup\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{T}^V, x \neq -\infty, \lambda + x \leqslant T(x)\}$. Indeed, if $x \in \mathbb{T}^V \setminus \{-\infty\}$ is such that $\lambda + x \leqslant T(x)$, then Theorem 6.16 shows that the support of x is a dominion and the induced value of every state of this dominion is not smaller than λ. Since the induced value is not greater than the value in the original game (by Lemma 6.13), this means that every state in the support of x has a value not smaller than λ. Conversely, let $W \subset V$ denote the set of all vertices with the maximal value and let $\lambda^* = \max_{v \in V} \chi_v$. By Lemma 6.14 and Theorem 6.16, there exists a point $x \in \mathbb{T}^V$ with support W such that $\lambda^* + x \leqslant T(x)$. This shows the first equality. □

Remark 6.19. The results of this section can be easily dualized by considering the dominions of Player Min instead of Player Max and by replacing the inequalities $\lambda + x \leqslant T(x)$ with $\lambda + x \geqslant T(x)$.[1] In particular, the Collatz–Wielandt property gives the equalities

$$\begin{aligned}
\max_{v \in V} \chi_v &= \min\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{R}^V, \lambda + x \geqslant T(x)\}\,, \\
\min_{v \in V} \chi_v &= \min\{\lambda \in \mathbb{R} \colon \exists x \in (\mathbb{R} \cup \{+\infty\})^V, x \neq +\infty, \lambda + x \geqslant T(x)\}\,.
\end{aligned}$$

6.3 Bipartite games, their operators, and graphs

We now specify the notions introduced in the previous section to the case of bipartite games and their operators. This is useful because bipartite games correspond to the operators considered in Chapter 5.

Definition 6.20. We say that a stochastic mean payoff game is *bipartite* if none of the players controls the pawn for two moves in a row. More formally, a game is bipartite if $\sum_{i \in V_{\mathrm{Max}}} p_{ki}^a = 1$

[1] One should also consider the Shapley operator as a function $T \colon (\mathbb{R} \cup \{+\infty\})^V \to (\mathbb{R} \cup \{+\infty\})^V$.

for every state $k \in V_{\mathrm{Min}}$ and every action $a \in A^{(k)}$ and $\sum_{k \in V_{\mathrm{Min}}} p^b_{ik} = 1$ for every state $i \in V_{\mathrm{Max}}$ and every action $b \in B^{(i)}$. If a game is bipartite, then we define its *bipartite Shapley operator* $F \colon \mathbb{T}^{V_{\mathrm{Min}}} \to \mathbb{T}^{V_{\mathrm{Min}}}$ as

$$\forall k \in V_{\mathrm{Min}}, (F(x))_k := \min_{a \in A^{(k)}} \Bigl(r_a + \sum_{i \in V_{\mathrm{Max}}} p^a_{ki} \max_{b \in B^{(i)}} \bigl(r^b + \sum_{l \in V_{\mathrm{Min}}} p^b_{il} x_l \bigr) \Bigr) . \tag{6.5}$$

Lemma 6.21. *A bipartite Shapley operator $F \colon \mathbb{T}^{V_{\mathrm{Min}}} \to \mathbb{T}^{V_{\mathrm{Min}}}$ is monotone, homogeneous, and preserves $\mathbb{R}^{V_{\mathrm{Min}}}$. Moreover, its restriction $F_{|\mathbb{R}^{V_{\mathrm{Min}}}}$ is piecewise affine.*

Sketch of the proof. The fact that F is monotone, homogeneous, and preserves $\mathbb{R}^{V_{\mathrm{Min}}}$ follows from its definition. Moreover, its restriction $F_{|\mathbb{R}^{V_{\mathrm{Min}}}}$ is piecewise affine because a coordinatewise minimum/maximum of piecewise affine functions is piecewise affine. □

Lemma 6.22. *Suppose that a stochastic mean payoff game is bipartite, let $T \colon \mathbb{T}^V \to \mathbb{T}^V$ be its Shapley operator and $F \colon \mathbb{T}^{V_{\mathrm{Min}}} \to \mathbb{T}^{V_{\mathrm{Min}}}$ be its bipartite Shapley operator. Then, for every $x \in \mathbb{T}^V$, every $k \in V_{\mathrm{Min}}$, and every $N \geqslant 1$ we have $(F^N(x_{V_{\mathrm{Min}}}))_k = (T^{2N}(x))_k$. Moreover, if (u, χ) is an invariant half-line of T, then $(u_{V_{\mathrm{Min}}}, 2\chi_{V_{\mathrm{Min}}})$ is an invariant half-line of F. In particular, for every $x \in \mathbb{R}^{V_{\mathrm{Min}}}$ and every $k \in V_{\mathrm{Min}}$ we have $\lim_{N \to \infty} \frac{1}{2N}(F^N(x))_k = \chi_k$.*

Proof. First, by the definition of Shapley operator and the fact that the game is bipartite, we have

$$\begin{aligned} \forall k \in V_{\mathrm{Min}}, (T^2(x))_k &= \min_{a \in A^{(k)}} \Bigl(r_a + \sum_{i \in V_{\mathrm{Max}}} p^a_{ki} (T(x))_i \Bigr) \\ &= \min_{a \in A^{(k)}} \Bigl(r_a + \sum_{i \in V_{\mathrm{Max}}} p^a_{ki} \max_{b \in B^{(i)}} \bigl(r^b + \sum_{l \in V_{\mathrm{Min}}} p^b_{il} x_l \bigr) \Bigr) = (F(x_{V_{\mathrm{Min}}}))_k . \end{aligned}$$

This proves the first claim for $N = 1$. The claim for other N follows by induction from the fact that

$$\begin{aligned} (T^{2N}(x))_k &= \Bigl(T^2(T^{2N-2}(x)) \Bigr)_k = \Bigl(F\Bigl((T^{2N-2}(x))_{V_{\mathrm{Min}}} \Bigr) \Bigr)_k \\ &= \Bigl(F(F^{N-1}(x_{V_{\mathrm{Min}}})) \Bigr)_k = (F^N(x_{V_{\mathrm{Min}}}))_k . \end{aligned}$$ □

Second, if (u, η) is an invariant half-line of T, then for $\gamma \geqslant 0$ large enough and every $k \in V_{\mathrm{Min}}$ we have

$$(F(u_{V_{\mathrm{Min}}} + 2\gamma\eta_{V_{\mathrm{Min}}}))_k = (T^2(u + 2\gamma\eta))_k = u_k + (\gamma + 1)(2\eta_k)$$

and $(u_{V_{\mathrm{Min}}}, 2\chi_{V_{\mathrm{Min}}})$ is an invariant half-line of F. The rest of the claim follows from Corollary 6.7 and Remark 6.11.

Definition 6.23. If the game is bipartite and $W \subset V_{\mathrm{Min}}$ is a subset of states controlled by Player Min, then we denote by $V(W) \subset V$ the set of vertices that are reachable from W in at most one step, i.e., $V(W) := W \cup \{ i \in V_{\mathrm{Max}} \colon \exists k \in W, \exists a \in A^{(k)}, p^a_{ki} > 0 \}$.

Proposition 6.24. *If $\lambda \in \mathbb{R}$ is a real parameter and $W \subset V_{\mathrm{Min}}$, then the set $\{ x \in \mathbb{T}^{V_{\mathrm{Min}}} \colon \lambda + x \leqslant F(x) \}$ has a nonempty stratum associated with W if and only if $V(W)$ is a dominion and every state of this dominion has an induced value that is not smaller than $\frac{1}{2}\lambda$.*

Proof. Suppose that $V(W)$ is a dominion such that its induced value is not smaller than $\frac{1}{2}\lambda$ for all states in $V(W)$. Then, by Theorem 6.16, there exists a point $x \in \mathbb{T}^V$ with support $V(W)$ such that $\frac{1}{2}\lambda + x \leqslant T(x)$. In particular, for every $k \in V_{\mathrm{Min}}$ we have

$$\begin{aligned}\lambda + x_k &\leqslant \frac{1}{2}\lambda + \min_{a \in A^{(k)}} (r_a + \sum_{i \in V_{\mathrm{Max}}} p_{ki}^a x_i) = \min_{a \in A^{(k)}} (r_a + \sum_{i \in V_{\mathrm{Max}}} p_{ki}^a (\frac{1}{2}\lambda + x_i)) \\ &\leqslant \min_{a \in A^{(k)}} \Big(r_a + \sum_{i \in V_{\mathrm{Max}}} p_{ki}^a \max_{b \in B^{(i)}} (r^b + \sum_{l \in V_{\mathrm{Min}}} p_{il}^b x_l)\Big) = (F(x_{V_{\mathrm{Min}}}))_k \,.\end{aligned}$$

Hence, $x_{V_{\mathrm{Min}}}$ satisfies $\lambda + x_{V_{\mathrm{Min}}} \leqslant F(x_{V_{\mathrm{Min}}})$ and the support of $x_{V_{\mathrm{Min}}}$ is equal to W. Conversely, suppose that $x \in \mathbb{T}^{V_{\mathrm{Min}}}$ is a point with support W and such that $\lambda + x \leqslant F(x)$. Define a point $\tilde{x} \in \mathbb{T}^V$ as

$$\tilde{x}_v := \begin{cases} \frac{1}{2}\lambda + x_v & \text{if } v \in W \\ \max_{b \in B^{(v)}} (r_b + \sum_{k \in V_{\mathrm{Min}}} p_{vk}^b x_k) & \text{if } v \in V(W) \cap V_{\mathrm{Max}} \\ -\infty & \text{otherwise}\,. \end{cases}$$

Observe that for every $k \in W$ we have

$$\begin{aligned}\frac{1}{2}\lambda + \tilde{x}_k = \lambda + x_k &\leqslant \min_{a \in A^{(k)}} \Big(r_a + \sum_{i \in V_{\mathrm{Max}}} p_{ki}^a \max_{b \in B^{(i)}} (r^b + \sum_{l \in V_{\mathrm{Min}}} p_{il}^b x_l)\Big) \\ &= \min_{a \in A^{(k)}} \Big(r_a + \sum_{i \in V(W) \cap V_{\mathrm{Max}}} p_{ki}^a \max_{b \in B^{(i)}} (r^b + \sum_{l \in V_{\mathrm{Min}}} p_{il}^b x_l)\Big) \\ &= \min_{a \in A^{(k)}} (r_a + \sum_{i \in V_{\mathrm{Max}}} p_{ki}^a \tilde{x}_i) = (T(\tilde{x}))_v \,.\end{aligned}$$

Furthermore, for every $i \in V(W) \cap V_{\mathrm{Max}}$ we have

$$\frac{1}{2}\lambda + \tilde{x}_i = \max_{b \in B^{(i)}} (r_b + \sum_{k \in V_{\mathrm{Min}}} p_{ik}^b (\frac{1}{2}\lambda + x_k)) = \max_{b \in B^{(i)}} (r_b + \sum_{k \in V_{\mathrm{Min}}} p_{ik}^b \tilde{x}_k) = (T(\tilde{x}))_i \,.$$

Hence $\frac{1}{2}\lambda + \tilde{x} \leqslant T(x)$. By Theorem 6.16, the support of $\tilde{x}$ is a dominion. Moreover, the support of $\tilde{x}$ contains W and is included in $V(W)$. Note that every dominion that contains W must also contain $V(W)$. Hence, the support of $\tilde{x}$ is equal to $V(W)$ and the claim follows from Theorem 6.16. ☐

Corollary 6.25. *We have the equalities*

$$\begin{aligned} 2\max_{v \in V_{\mathrm{Min}}} \chi_v = 2\max_{v \in V} \chi_v &= \max\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{T}^{V_{\mathrm{Min}}}, x \neq -\infty, \lambda + x \leqslant F(x)\}\,, \\ 2\min_{v \in V_{\mathrm{Min}}} \chi_v = 2\min_{v \in V} \chi_v &= \max\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{R}^{V_{\mathrm{Min}}}, \lambda + x \leqslant F(x)\}\,. \end{aligned}$$

The proof of Corollary 6.25 is similar to the proof of Corollary 6.18, but requires a simple lemma.

Lemma 6.26. *If a stochastic mean payoff game is bipartite, then $W := V(V_{\mathrm{Min}})$ is a dominion and all states in the game induced by W have the same value as in the original game.*

Proof. The fact that W is a dominion follows from the definition. To prove the rest of the claim, let $T\colon \mathbb{T}^V \to \mathbb{T}^V$ denote the Shapley operator of the original game and $\tilde{T}\colon \mathbb{T}^W \to \mathbb{T}^W$ denote the Shapley operator of the induced game. Observe that if a state $i \in V_{\mathrm{Max}}$ belongs to W, then all actions of Player Max are preserved in the induced game. In particular, if $(u,\eta) \in \mathbb{R}^V \times \mathbb{R}^V$ is an invariant half-line of T, then $(u_W, \eta_W) \in \mathbb{R}^W \times \mathbb{R}^W$ is an invariant half-line of $\tilde{T}$. This implies, by Remark 6.11, that all states in the game induced by W have the same value as in the original game. □

Sketch of the proof of Corollary 6.25. Let us start by showing the equalities $\max_{v \in V_{\mathrm{Min}}} \chi_v = \max_{v \in V} \chi_v$ and $\min_{v \in V_{\mathrm{Min}}} \chi_v = \min_{v \in V} \chi_v$. To do so, recall (from Theorem 6.1) that if $(\bar{\sigma}, \bar{\tau})$ is a couple of optimal policies, then we have the equality $g(\bar{\sigma}, \bar{\tau}) = \chi$. Hence, by Theorem 2.137, the maximal and minimal value of the game are achieved in some recurrent classes of the Markov chain induced by fixing $(\bar{\sigma}, \bar{\tau})$. Since the game is bipartite, this classes contain states controlled by both players. The rest of the proof is similar to the proof of Corollary 6.18. First, note that $W \coloneqq V(V_{\mathrm{Min}})$ is a dominion by Lemma 6.26. Hence, applying Proposition 6.24 to W gives the second equality. To prove the first equality, note that we have the inequality $2\max_{v \in V} \geqslant \sup\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{T}^{V_{\mathrm{Min}}}, x \neq -\infty, \lambda + x \leqslant F(x)\}$ by Lemma 6.13 and Proposition 6.24. Conversely, if $\tilde{W} \subset V$ denotes the set of all states with maximal value, then (by Lemma 6.14) $\tilde{W}$ is a dominion such that every state in the induced game has value $\max_{v \in V} \chi_v$. By applying Lemma 6.26 to the game induced by $\tilde{W}$ we see that $W \coloneqq V(V_{\mathrm{Min}} \cap \tilde{W})$ is a dominion and that every state in this dominion has induced value $\max_{v \in V} \chi_v$. The claim follows from Proposition 6.24. □

Let us now explain how to connect the bipartite games and their Shapley operators with the graphs and operators discussed in Section 5.3. Given a bipartite stochastic mean payoff game, we can represent it using a directed graph $\vec{\mathcal{G}}$ with Min, Max, and Random vertices. The sets of Min and Max vertices of $\vec{\mathcal{G}}$ are given by V_{Min} and V_{Max} respectively. The set of Random vertices of $\vec{\mathcal{G}}$ is created as follows. For every nondeterministic action $a \in A^{(k)}$ of player Min,[2] we create a vertex $v_a \in V_{\mathrm{Rand}}$. We add the edge (k, v_a) to $\vec{\mathcal{G}}$. Moreover, we add the edges (v_a, i) for every $i \in V_{\mathrm{Max}}$ such that $p_{ki}^a > 0$. Furthermore, we put $(p_{ki}^a)_i$ as the probability distribution associated with v_a. We do an analogous operation for every nondeterministic action of Player Max. Then, for every deterministic action $a \in A^{(k)}$ of Player Min we take $i \in V_{\mathrm{Max}}$ such that $p_{ki}^a = 1$ and add the edge (k,i) to $\vec{\mathcal{G}}$. This edge is equipped with the payoff

$$r_{ki} \coloneqq \min_{\{a \in A^{(k)},\, p_{ki}^a = 1\}} r_a \,.$$

This definition of the payoff comes from the fact that, a priori, there may be more than one deterministic action of Player Min at state k that leads to the state i. (However, these actions are redundant in the sense that it is always profitable for Player Min to choose such an action with the minimal payoff r_a.) We do an analogous operation for every deterministic action of Player Max. The following lemma connects the bipartite Shapley operators with the operators considered in Section 5.3.

Lemma 6.27. *Given a bipartite stochastic mean payoff game with rational probabilities, let $\vec{\mathcal{G}}$ be its graph as constructed above. Then, $\vec{\mathcal{G}}$ fulfills Assumption C. Furthermore, the operator encoded by $\vec{\mathcal{G}}$ (as given in Definition 5.36) is equal to the bipartite Shapley operator of the game (restricted to $\mathbb{R}^n$).*

[2]By a *nondeterministic* action, we mean an action whose outcome is randomized, i.e., such that the associated probability distribution has an entry in the open interval $]0,1[$. Otherwise, we say that an action is *deterministic*.

Proof. The claim follows immediately from the definitions. □

Conversely, given a graph $\vec{\mathcal{G}}$ as in (5.2), and its operator $F\colon \mathbb{R}^n \to \mathbb{R}^n$ of the form considered in (5.2), we can constructed a bipartite stochastic mean payoff game with F as its Shapley operator. This is done as follows (we use the same notation as in (5.2)). We put $V_{\mathrm{Min}} := [n]$ and $V_{\mathrm{Max}} := [m]$. Then, for every state $v \in [n]$ of Player Min and every outgoing edge $e \in \mathrm{Out}(v)$, we equip this state with an action $a := e$, whose associated probability distribution is equal to $(p^e_w)_{w\in[m]}$ and payoff is equal to r_e. Similarly, for every state $w \in [m]$ of Player Max and every outgoing edge $e' \in \mathrm{Out}(w)$ we equip this state with an action $b := e'$ whose associated probability distribution is equal to $(p^{e'}_v)_{v\in[n]}$ and payoff is equal to $r_{e'}$. Since we suppose that every vertex of $\vec{\mathcal{G}}$ has at least one outgoing edge, the resulting game is well defined (i.e., every state is equipped with at least one action). If $\tilde{F}\colon \mathbb{T}^n \to \mathbb{T}^n$ denotes the Shapley operator of this game, then by definition we have $\tilde{F}_{|\mathbb{R}^n} = F$. Therefore, the Shapley operator of the game is an extension of F from $\mathbb{R}^n$ to $\mathbb{T}^n$. Combining this observation with the results of previous chapter, we can characterize the tropicalizations of closed convex semialgebraic sets in terms of Shapley operators of stochastic mean payoff games. Before doing so, let us define the notion of a support of a tropically convex set.

Definition 6.28. If $\mathcal{S} \subset \mathbb{T}^n$ is a tropically convex set, then we define its *support* as the largest set $K \subset [n]$ such that the stratum $\mathcal{S}_K$ is nonempty (with the convention that if $\mathcal{S}$ is empty, then its support is equal to $\emptyset$).

Remark 6.29. We note that the support is well defined and unique. Indeed, if $x, y \in \mathcal{S}$ are two points, then $z := x \oplus y$ belongs to $\mathcal{S}$ by tropical convexity and the support of z is equal to the union of the supports of x and y. Hence, the support of $\mathcal{S}$ is the union of all supports of points in $\mathcal{S}$.

Theorem 6.30. *Fix a set $\mathcal{S} \subset \mathbb{T}^n$. Then, the following conditions are equivalent:*
(a) $\mathcal{S}$ is a tropicalization of a closed convex semialgebraic cone;
(b) $\mathcal{S}$ is a tropical cone that has semilinear strata and is closed in the topology of $\mathbb{T}^n$;
(c) there exists a bipartite stochastic mean payoff game such that its Shapley operator $F\colon \mathbb{T}^n \to \mathbb{T}^n$ satisfies $\mathcal{S} = \{x \in \mathbb{T}^n : x \leqslant F(x)\}$.

Remark 6.31. If we are given a game whose Shapley operator satisfies $\mathcal{S} = \{x \in \mathbb{T}^n : x \leqslant F(x)\}$, then (by Proposition 6.24) the support of $\mathcal{S}$ is equal to the largest set $K \subset [n]$ such that $V(K)$ is a winning dominion. Equivalently, this support is equal to $W \cap V_{\mathrm{Min}}$, where W is the largest winning dominion of the game. Moreover, by Corollary 6.25, $\mathcal{S}$ is nonempty if and only if this game has at least one winning state.

Proof of Theorem 6.30. We start by proving the implication from Theorem 6.30 (a) to Theorem 6.30 (b). If $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$ is a closed convex semialgebraic cone, then $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is a tropical cone that has semilinear strata (by the considerations of Remark 5.22). Furthermore, $\mathsf{val}(\boldsymbol{\mathcal{S}})$ is closed in the topology of $\mathbb{T}^n$ by Theorem 3.1. To prove the converse implication, let $\mathcal{S}$ be a tropical cone that has semilinear strata and is closed in the topology of $\mathbb{T}^n$. Remark 5.22 shows that $\mathcal{S}$ is a tropicalization of some convex semialgebraic cone $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n$. By the continuity of addition and multiplication in $\mathbb{K}$, the set $\mathrm{cl}_{\mathbb{K}}(\boldsymbol{\mathcal{S}})$ is also a convex cone. Since $\mathcal{S}$ is closed in the topology of $\mathbb{T}^n$, Theorem 3.1 shows that $\mathcal{S} = \mathsf{val}(\mathrm{cl}_{\mathbb{K}}(\boldsymbol{\mathcal{S}}))$. Thanks to the equivalence between Shapley operators and operators considered in Section 5.3 as discussed above, the equivalence between Theorem 6.30 (b) and Theorem 6.30 (c) is a variant of Proposition 5.34 (in which we replace $\mathbb{R}^n$

by $\mathbb{T}^n$). Hence, the proof is similar to the proof of Proposition 5.34. To prove the implication from Theorem 6.30 (c) to Theorem 6.30 (b), let $\mathcal{S} := \{x \in \mathbb{T}^n : x \leqslant F(x)\}$ for some Shapley operator F. Then, $\mathcal{S}$ is a tropical cone (this can be proven as in Proposition 5.34). Furthermore, $\mathcal{S}$ is definable in $\mathcal{L}_{\mathrm{ogb}}$ and hence it has semilinear strata (by Lemma 2.108). To see that it is closed in the topology of $\mathbb{T}^n$, note that the functions

$$\mathbb{T}^2 \ni (a,b) \to a+b \in \mathbb{T},\ \mathbb{T}^2 \ni (a,b) \to \max\{a,b\} \in \mathbb{T},$$
$$\mathbb{T}^2 \ni (a,b) \to \min\{a,b\} \in \mathbb{T},\ \mathbb{T} \ni a \to x \cdot a \in \mathbb{T} \text{ (for } x \geqslant 0)$$

are continuous. Hence, $F : \mathbb{T}^n \to \mathbb{T}^n$ is a continuous function and $\mathcal{S}$ is closed in the topology of $\mathbb{T}^n$. In remains to prove the implication from Theorem 6.30 (b) to Theorem 6.30 (c). If $\mathcal{S}$ is trivial, $\mathcal{S} = -\infty$, then $F(x) := (x_1-1, \dots, x_n-1)$ is an operator of a stochastic mean payoff game and satisfies the claim. Otherwise, let K denote the support of $\mathcal{S}$. Moreover, the set $\mathcal{S}_K \subset \mathbb{R}^K$ is a closed semilinear real tropical cone and hence, by Proposition 5.34, there exists a semilinear monotone homogeneous operator $\tilde{F} : \mathbb{R}^K \to \mathbb{R}^K$ such that $\mathcal{S}_K = \{x \in \mathbb{R}^K : x \leqslant \tilde{F}(x)\}$. By Lemma 5.38, $\tilde{F}$ is encoded by a graph and, by the remarks above, there is a bipartite stochastic mean payoff game whose Shapley operator is an extension of $\tilde{F}$ from $\mathbb{R}^K$ to $\mathbb{T}^K$. Moreover, the set of states of Player Min in this game is equal to K. We will also denote this extension by $\tilde{F} : \mathbb{T}^K \to \mathbb{T}^K$. We claim that we have the equality

$$\mathcal{S} = \{x \in \mathbb{T}^n : x_K \leqslant \tilde{F}(x_K) \wedge x_{[n]\setminus K} = -\infty\}\,. \tag{6.6}$$

Indeed, by the definition of K, there exists a point $y \in \mathcal{S}$ such that $y_k \neq 0$ for all $k \in K$ and $y_k = -\infty$ otherwise. This point also belongs to the right-hand side of (6.6). To prove the inclusion $\subset$, take any point $x \in \mathcal{S}$. Observe that if $\lambda \in \mathbb{R}$ is any real constant, then $x^{(\lambda)} := x \oplus (\lambda \odot y)$ belongs to $\mathcal{S}$, and its support is equal to K. Therefore, $x^{(\lambda)}$ belongs to the right-hand side of (6.6). Hence, by the continuity of $\tilde{F}$ (and taking $\lambda \to -\infty$), we get that x belongs to the right-hand side of (6.6). Conversely, if x belongs to the right-hand side of (6.6), then $x^{(\lambda)} := x \oplus (\lambda \odot y)$ satisfies $x_K^{(\lambda)} \leqslant \tilde{F}(x_K^{(\lambda)})$ (because the set $\{z \in \mathbb{T}^K : z \leqslant \tilde{F}(z)\}$ is a tropical cone) and its support is equal to K. Therefore, it belongs to $\mathcal{S}$. Since $\mathcal{S}$ is closed in the topology of $\mathbb{T}^n$, we get the inclusion $\supset$. Given the equality (6.6), we define $F : \mathbb{T}^n \to \mathbb{T}^n$ by

$$(F(x))_k := \begin{cases} (\tilde{F}(x_K))_k & \text{if } k \in K\,, \\ x_k - 1 & \text{otherwise} \end{cases}$$

and we have $\mathcal{S} = \{x \in \mathbb{T}^n : x \leqslant F(x)\}$. Moreover, it is easy to see that F is a Shapley operator of a stochastic mean payoff game. We take the game associated with $\tilde{F}$ and for every $k \in [n] \setminus K$ we add k to the set of states controlled by Player Min. Moreover, we add a state k' to the states of Player Max. The state k is equipped with one action a_k which is deterministic, $p_{kk'}^{a_k} := 1$, and has payoff $r_{a_k} := -1$. Similarly, the state k' is equipped with one action $b_{k'}$ which is deterministic, $p_{k'k}^{b_{k'}} := 1$, and has payoff $r_{b_{k'}} := 0$. ☐

Remark 6.32. The claim of Theorem 6.30 and Remark 6.31 states that, in theory, one could solve an arbitrary conic feasibility problem over $\mathbb{K}$ by reducing it to a stochastic mean payoff game. However, even though the proof of Theorem 6.30 can be made effective (by the Denef–Pas quantifier elimination, the construction from the proof of Lemma 5.52, and turning the proof of Theorem 5.27 into an algorithm), this construction is far from being polynomial-time. In the next chapter, we present a class of semidefinite feasibility problems for which we can construct the corresponding game in polynomial-time.

Remark 6.33. In the above, we have seen that a semilinear monotone homogeneous operator $F\colon \mathbb{R}^n \to \mathbb{R}^n$ can be extended to a continuous operator on $\mathbb{T}^n$. It can be shown that such extension is well defined and unique even if we omit the assumption that F is semilinear, see [BNS03].

CHAPTER 7

Equivalence between tropical semidefinite feasibility and stochastic mean payoff games

In this chapter we consider the algorithmic problems of solving stochastic mean payoff games and deciding the feasibility of a tropical Metzler spectrahedron and we want to show that these two problems are polynomial-time equivalent. Moreover, we give an application of this theorem to nonarchimedean semidefinite feasibility problem, showing that a large class of these problems can be solved by a reduction to stochastic mean payoff games. In the chapter, we will use some basic notions of complexity theory (such as polynomial-time reduction). We refer to the book [AB09] for the necessary background. Let us first define the algorithmic problems that we consider.

Definition 7.1. Suppose that we are given a stochastic mean payoff game in which all probabilities and payoffs are rational. Then, the Stochastic Mean Payoff Game Problem (Smpg) refers to the task of finding a pair of optimal strategies and the value of the game.

Remark 7.2. It is clear that we can equivalently suppose that all payoffs in the problem above are integer. Indeed, if we multiply all the payoffs by their common denominator, then the optimal policies do not change, and the value is multiplied by this denominator. Therefore, in later chapters we will often suppose that stochastic mean payoff games have integer payoffs.

Definition 7.3. Suppose that we are given symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in$

$(\mathbb{Q}\cup\{-\infty\})^{m\times m}$. Then, the Tropical Metzler Semidefinite Feasibility Problem (Tmsdfp) refers to the task of deciding if the tropical Metzler spectrahedral cone $\mathcal{S}(Q^{(1)},\dots,Q^{(n)})$ is nontrivial, i.e., contains a point different than $-\infty$.

The main result of this chapter consists of the following two theorems. The first one shows the equivalence between the problem of stochastic mean payoff games and deciding the feasibility of tropical Metzler spectrahedral cones.

Theorem 7.4. *The problems* Smpg *and* Tmsdfp *are polynomial-time equivalent. Furthermore, if either of these problems can be solved in pseudopolynomial time, then both of them can be solved in polynomial time.*

Then second theorem is an application of our results to nonarchimedean semidefinite programming and shows that solving generic nonarchimedean semidefinite feasibility problems for cones can be reduced to solving stochastic mean payoff games. Let us recall the following definition.

Definition 7.5. Let $\boldsymbol{Q}^{(1)},\dots,\boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ be a sequence of symmetric matrices and let $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)} \succcurlyeq 0\}$ be the associated spectrahedron. We say that $\boldsymbol{\mathcal{S}}$ is *nontrivial* if is contains a point different than 0. We say that $\boldsymbol{\mathcal{S}}$ is *strictly feasible* if there exists a point $\boldsymbol{x} \in \mathbb{K}^n_{>0}$ such that the matrix $\boldsymbol{Q}(\boldsymbol{x}) \coloneqq \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)}$ is positive definite.

Theorem 7.6. *Suppose that the matrices $\boldsymbol{Q}^{(k)}$ have rational valuations, i.e.,* $\mathsf{val}(\boldsymbol{Q}^{(k)}) \in (\mathbb{Q}\cup\{-\infty\})^{m\times m}$. *Then, given only the signed valuations* $\mathsf{sval}(\boldsymbol{Q}^{(k)})$ *of these matrices, we can construct (in polynomial-time) two stochastic mean payoff games, one called the "feasibility game" and the second called the "strict feasibility game." If $\overline{\chi}$ denotes the maximal value of the feasibility game and $\underline{\chi}$ denotes the minimal value of the strict feasibility game, then the following is true:*

- *if $\overline{\chi} < 0$, then $\boldsymbol{\mathcal{S}}$ is trivial;*
- *if $\overline{\chi} > 0$, then $\boldsymbol{\mathcal{S}}$ is nontrivial;*
- *if $\underline{\chi} < 0$, then $\boldsymbol{\mathcal{S}}$ has empty interior;*
- *if $\underline{\chi} > 0$, then $\boldsymbol{\mathcal{S}}$ has nonempty interior;*
- *if $\underline{\chi} > 0$ and the matrices $\boldsymbol{Q}^{(k)}$ satisfy Assumption B, then $\boldsymbol{\mathcal{S}}$ is strictly feasible.*

Furthermore, if the matrices $\boldsymbol{Q}^{(1)},\dots,\boldsymbol{Q}^{(n)}$ satisfy the conditions of Theorem 4.28, then $\boldsymbol{\mathcal{S}}$ is nontrivial if and only if $\overline{\chi} \geqslant 0$. If these matrices also satisfy Assumption B, then $\boldsymbol{\mathcal{S}}$ is strictly feasible if and only if $\underline{\chi} \geqslant 0$.

This chapter is organized as follows. In Section 7.1 we show how stochastic mean payoff games can be reduced to Tmsdfp. In Section 7.2 we show the converse reduction. This is the most direct for a special class of matrices that induce a well-formed linear matrix inequalities. This special case is discussed in Section 7.2.1, where we also give a simplified version of Theorem 7.6 that is adapted to this case. The remainder of the chapter is devoted to extending our results from this special case to general non-Metzler matrices and to proving Theorem 7.6. This extension is quite technical and requires preprocessing procedures explained in Section 7.2.2 and Section 7.3. The chapter is based on the paper [AGS18].

7.1 From stochastic mean payoff games to tropical spectrahedra

In this section, we reduce the problem of solving stochastic mean payoff games to the problem of deciding the feasibility of tropical Metzler spectrahedra. To do so, we need to define the class of stopping simple stochastic games.

Definition 7.7. We say that a stochastic mean payoff game is a *stopping simple stochastic game* if it satisfies the following properties:

- All probabilities of the game belong to the set $\{0, 1/2, 1\}$.
- There are two special states $k_0, k_1 \in V_{\mathrm{Min}}$ such that Player Min has only one action available at each of this states, $A^{(k_0)} = \{a_0\}$, $A^{(k_1)} = \{a_1\}$, this action is a loop, $p^{a_0}_{k_0,k_0} = p^{a_1}_{k_1 k_1} = 1$, and the respective payoffs are equal to 0 and 1, $r_{a_0} = 0$, $r_{a_1} = 1$.
- For every couple of policies (σ, τ) the Markov chain induced by (σ, τ) has only two recurrent classes: one equal to $\{k_0\}$ and the other equal to $\{k_1\}$.

We refer to the states $\{k_0, k_1\}$ as *sinks*.

Remark 7.8. If (σ, τ) is a pair of policies in a stopping simple stochastic game, then Theorem 2.137 shows that $(g(\sigma, \tau))_v$ is equal to the probability that the Markov chain induced by (σ, τ) and starting from v is reaches k_1. Therefore, we can suppose that all payoffs other that r_{a_1} are equal to zero (this does not change the expected average payoff).

The following reduction was proven by Andersson and Miltersen [AM09].

Theorem 7.9 ([AM09])**.** *The problem of solving stochastic mean payoff games is polynomial-time equivalent to the problem of finding values of stopping simple stochastic games.*

Remark 7.10. We point out that the Theorem 7.9 is stated in [AM09] without the word "stopping," but this is what the authors actually show.

Our reduction of stochastic mean payoff games to the feasibility of tropical spectrahedra builds upon Theorem 7.9. As a first step, we want to show that the problem of computing the value can be reduced to the problem of deciding if the value is greater than 1/2. This requires the following lemmas.

Lemma 7.11 ([Con92, Lemma 2])**.** *If $\chi \in [0,1]^V$ denotes the value of a stochastic mean payoff game, then every coordinate χ_v is a rational number with a denominator not greater than $4^{|V|-1}$.*

Lemma 7.12. *Fix a rational number $\alpha \in [0,1]$. Then, the problem of deciding if a given state of a stopping simple stochastic game has value at least α is polynomial-time reducible to the problem of deciding if a given state in a stopping simple stochastic game has a value at least* 1/2. *(Here, "polynomial-time" means polynomial in the size of the game and the number of bits needed to encode α.)*

Proof. Suppose that we are given a stopping stochastic game, a fixed state $v \in V$, and a fixed rational number $\alpha \in [0,1]$. We want to decide if $\chi_v \geqslant \alpha$. If $\alpha > \frac{1}{2}$, then we modify the game as follows: we change the action a_1 is such a way that when the sink with payoff 1 is reached, the game does not start to loop, but instead moves with probability $1 - \frac{1}{2\alpha}$ to the sink with payoff 0 and with probability $\frac{1}{2\alpha}$ to the (newly created) sink with payoff 1. Denote the value of the modified game by $\tilde{\chi}$. We have $\tilde{\chi}_v = \frac{1}{2\alpha}\chi_v$ and hence $\tilde{\chi}_v \geqslant 1/2 \iff \chi_v \geqslant \alpha$. Note that the modified game does not fulfill the conditions of Definition 7.7 because it has probability distributions different than $\{0, 1/2, 1\}$. Nevertheless, we may apply the construction of Zwick and Paterson (see Fig. 5.5 and the proof of Lemma 5.41) to the modified action a_1 and obtain (in polynomial-time) a stopping simple stochastic game that has $\tilde{\chi}_v$ as the value at state v. The case $0 \leqslant \alpha < 1/2$ is analogous—we modify the action a_0 in such a way that when the sink with payoff 0 is reached, the game does not start to loop, but instead moves with probability $\frac{1-2\alpha}{2-2\alpha}$ to the sink with payoff 1 and with probability $\frac{1}{2-2\alpha}$ to the (newly created) sink with payoff 0. We have $\tilde{\chi}_v = \chi_v + \frac{1-2\alpha}{2-2\alpha}(1 - \chi_v)$ and hence $\tilde{\chi}_v \geqslant 1/2 \iff \chi_v \geqslant \alpha$. To finish, we apply the construct of Zwick and Paterson as in the previous case. $\square$

Given the lemmas above, the reduction of computation problem to the decision problem follows from the rational search technique, see, e.g., [KM03, For07].

Lemma 7.13. *The problem of solving stochastic mean payoff games is polynomial-time equivalent to the problem of deciding if a given state in a stopping simple stochastic game has a value at least* $1/2$.

Proof. By Theorem 7.9, the problem of solving stochastic mean payoff games is reducible in polynomial time to the problem of finding value of a stopping simple stochastic game. This is trivially reducible to the problem of finding the value of one state. Given a state $v \in V$ of a stopping simple stochastic game, Lemma 7.11 shows that χ_v is a rational number in the interval $[0,1]$ with denominator not greater than $4^{|V|-1}$. Observe that if $\frac{a}{b}, \frac{c}{d} \in [0,1]$ are different rational numbers with denominators bounded by $4^{|V|-1}$, then they differ by at least

$$|\frac{a}{b} - \frac{c}{d}| = |\frac{ad-bc}{bd}| \geqslant \frac{1}{16^{|V|-1}} > \frac{1}{16^{|V|}} \, . \tag{7.1}$$

Let $\ell \in \{0,1,\dots,16^{|V|}-1\}$ be a number such that $\chi_v \in [\frac{\ell}{16^{|V|}}, \frac{\ell+1}{16^{|V|}}]$. Note that ℓ can be found by a binary search asking $O(|V|)$ queries of the form "is $\chi_v \geqslant \frac{m}{16^{|V|}}$?" for $m \in \{1,\dots,16^{|V|}-1\}$. Therefore, by Lemma 7.12, the task of finding ℓ is reducible in polynomial time to the task of deciding if a given state in a stopping simple stochastic game has a value at least $1/2$. Moreover, given ℓ, (7.1) shows that χ_v is the unique rational number in the interval $[\frac{\ell}{16^{|V|}}, \frac{\ell+1}{16^{|V|}}]$ with denominator not greater than $4^{|V|-1}$. Therefore, the exact value of χ_v can be found in $O(|V|)$ time using the algorithms presented in [KM03, Lemma 5] or [For07]. ☐

In the next lemma, we change the class of games, transforming stopping simple stochastic games into ergodic games. Let us give the necessary definition.

Definition 7.14. We say that a stochastic mean payoff game is a *constant value game* if the value of the game does not depend on the initial state, i.e., $\chi = \rho(1,1,\dots,1) \in \mathbb{R}^V$ for some $\rho \in \mathbb{R}$. We say that a game in *ergodic* if it is a constant value game and it remains a constant value game even if one changes the payoffs associated with actions in an arbitrary way.

Remark 7.15. Different authors use the term "ergodic game" in defferent senses. For instance [CIJ14] define the game to be ergodic if the Markov chain induced by any pair of policies (σ,τ) is irreducible (i.e., has only one recurrent class and every state belongs to this class). This corresponds to the ergodicity condition used in the policy iteration algorithm of Hoffman and Karp [HK66]. One could also define a game to be ergodic if the Markov chain induced by any pair of policies (σ,τ) has only one final class (skipping the requirement that every state is recurrent). In fact, this is the class of games that appears in Lemma 7.16 below. It is trivial to see that these two definitions are more restrictive than Definition 7.14. In the other direction, the authors of [BEGM15] use the term "ergodic game" for a constant value game and the term "ergodic graph" for what we call an ergodic game. Definition 7.14 appears in [AGH15] and is motivated by the fact that it has many equivalent characterizations that generalize the algebraic characterizations of ergodicity for Markov chains.

Lemma 7.16. *The problem of solving stochastic mean payoff games is polynomial-time reducible to the problem of deciding if the value of a bipartite ergodic stochastic mean payoff game is nonnegative.*

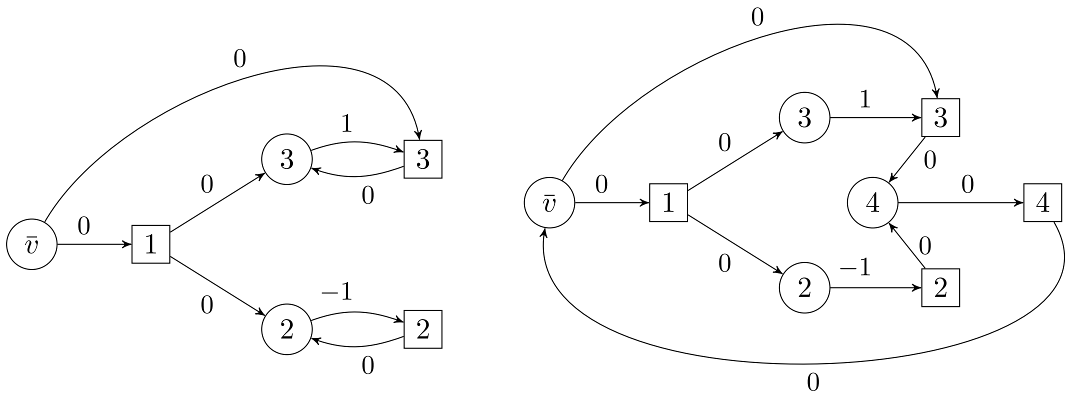

Figure 7.1: Transformation of stopping simple stochastic games, Lemma 7.16 Step IV.

Even more, the above is true if we further restrict the class of bipartite ergodic games by supposing that every probability in the game belongs to the set $\{0, 1/2, 1\}$*, that every payoff of the game belongs to the set* $\{-1, 0, 1\}$*, and that Player Max has only deterministic actions (i.e.,* $p^b_{ik} \in \{0, 1\}$ *for every action* b *of Player Max and every pair of states* $i \in V_{\mathrm{Max}}$*,* $k \in V_{\mathrm{Min}}$*).*

Proof. By Lemma 7.13, it is enough to provide the reduction for the problem of deciding if a value of a state in a stopping simple stochastic game is at least 1/2. To do so, take a stopping simple stochastic game, fix a state $\bar{v} \in V$ and let $\chi_{\bar{v}}$ denote its value. We modify the game using the following steps.

Step I: Let k_0 denote the sink with payoff 0. We change the payoff in this sink from 0 to -1. Note that, if (σ, τ) is any pair of policies, and $\tilde{g}(\sigma, \tau)$ denotes the expected average payoff of Player Max, in the modified game, then we have $\tilde{g}(\sigma, \tau) = g(\sigma, \tau) - (1 - g(\sigma, \tau)) = 2g(\sigma, \tau) - 1$. In particular, if $\tilde{\chi}$ denotes the value of the modified game, then we have $\tilde{\chi} = 2\chi - 1$ and $\chi_{\bar{v}} \geqslant 1/2 \iff \tilde{\chi}_{\bar{v}} \geqslant 0$.

Step II: We take the game obtained in the previous step and transform it in such a way that Player Max does not have any nondeterministic actions. More precisely, if $b \in B^{(i)}$ is an action of Player Max such that $p^b_{iv} = 1/2$ for some $v \in V$, then we add a new state k_b controlled by Player Min to the game and change the action b to b' by putting $p^{b'}_{ik_b} := 1$. Furthermore, we equip the state k_b with exactly one action $A^{(k_b)} = \{a\}$ and we put $p^a_{k_b v} := p^b_{iv}$ for all $v \in V$. We repeat this operation for every action of Player Max.

Step III: We take the game obtained in the previous step and transform it into a bipartite game. This is done in the following way. If $a \in A^{(k)}$ is an action of Player Min such that $p^a_{kl} > 0$ for some $l \in V_{\mathrm{Min}}$, then we add a new state i_a of Player Max to the game and modify the action a into $\hat{a}$ by putting $p^{\hat{a}}_{ki_a} := p^a_{kl}$, $p^{\hat{a}}_{kl} := 0$, and $p^{\hat{a}}_{kv} := p^a_{kv}$ for all other vertices. Moreover, we put $r_{\hat{a}} := r_a$. Furthermore, we equip the state i_a with exactly one action $B^{(i_a)} = \{b\}$ and this action is such that $p^b_{i_a l} := 1$ and $r_b := 0$. We repeat this operation for every action of Player Min and we do an analogous operation for every action of Player Max.

Step IV: We take the game obtained in the previous step and we perform the construction presented in Fig. 7.1.More precisely, if $(n-1, m-1) \in V_{\mathrm{Min}} \times V_{\mathrm{Max}}$ denotes the recurrent class with payoff $-1/2$ and (n, m) denotes the recurrent class with payoff 1/2, then we do the following operation. We add two states, $n+1$ (controlled by Player Min) and $m+1$ (controlled by Player Max). At $n+1$, Player Min has only one possible action: to go to $m+1$; after this action Player Min pays 0 to Player Max. Moreover, at $m+1$ Player Max also has only one action: to go to $\bar{v}$; after this action Player Max receives 0 from Player Min. Finally, at $m-1$ (resp. m) Player Max has only one possible action: to go to $n+1$; after this action he receives 0 from Player Min.

Take the game obtained by applying the steps above and let $\check{\chi}$ denote the value of the modified game. We want to show that this game is ergodic. To see that this is the case, let $(\check{\sigma}, \check{\tau})$ denote a pair of policies in the modified game. Observe that, since the original game is a stopping simple stochastic game, the Markov chain induced by $(\check{\sigma}, \check{\tau})$ reaches $\bar{v}$ independently of the initial state of the chain. Hence, $\bar{v}$ is recurrent in this chain and its recurrent class is the only recurrent class of the chain. In particular, by Theorem 2.137, the average payoff $\check{g}(\check{\sigma}, \check{\tau})$ does not depend on the initial state and this is true for every choice of the payoffs. Hence, the game is ergodic. Denote $\check{\chi} = \rho(1, 1, \dots, 1)$. Moreover, note that every state that we added to the game has only one action (and every state that was present in the original game has the same number of actions in the modified game). Therefore, there is a natural bijection between the policies of players in the original game and their policies in the modified game. If (σ, τ) is a pair of policies in the original game that corresponds to $(\check{\sigma}, \check{\tau})$, then Theorem 2.137 shows that $(\check{g}(\check{\sigma}, \check{\tau}))_{\bar{v}} = (\tilde{g}(\sigma, \tau))_{\bar{v}}/\theta_{\bar{v}}$, where $\theta_{\bar{v}}$ is the expected time of first return to $\bar{v}$. In particular, a policy $\check{\tau}$ guarantees a nonnegative payoff to Player Max for the initial state $\bar{v}$ if and only if τ does the same thing. Hence $\rho \geqslant 0 \iff \tilde{\chi}_{\bar{v}} \geqslant 0$. □

Proposition 7.17. *The problem of solving stochastic mean payoff games is polynomial-time reducible to* Tmsdfp.

Proof. By Lemma 7.16, it is enough to provide a reduction from the problem of deciding if a bipartite ergodic game has a nonnegative value. Fix such a game and suppose that it has all the properties mentioned in Lemma 7.16. Then, its graph $\vec{\mathcal{G}}$ (as defined in Definition 5.36) fulfills the conditions of Proposition 5.40. Therefore, by Proposition 5.40, the set $\mathcal{S} \coloneqq \{x \in \mathbb{T}^{V_{\mathrm{Min}}} \colon x \leqslant F(x)\}$ is a tropical Metzler spectrahedron, $\mathcal{S} = \mathcal{S}(Q^{(1)}, \dots, Q^{(|V_{\mathrm{Min}}|)}), Q^{(k)} \in \mathbb{T}_{\pm}^{|V_{\mathrm{Max}}| \times |V_{\mathrm{Max}}|}$ and the proof of Proposition 5.40 gives a polynomial-time construction of the matrices $Q^{(k)}$.

Furthermore, Corollary 6.25 gives the equalities

$$\begin{aligned} 2\rho &= \max\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{T}^{V_{\mathrm{Min}}}, x \neq -\infty, \lambda + x \leqslant F(x)\} \\ &= \max\{\lambda \in \mathbb{R} \colon \exists x \in \mathbb{R}^{V_{\mathrm{Min}}}, \lambda + x \leqslant F(x)\} \, , \end{aligned}$$

where $\rho(1, 1, \dots, 1)$ is the value of the game. Hence, we have the equivalence

$$\mathcal{S} \text{ is nontrivial} \iff \mathcal{S} \text{ contains a real point} \iff \rho \geqslant 0 \, .$$ □

7.2 From tropical spectrahedra to stochastic mean payoff games

In this section, we show that the feasibility of tropical spectrahedra can be reduced to the problem of solving stochastic mean payoff games. More precisely, we will show the following theorem whose formulation is very close to the formulation of Theorem 7.6. We define the sets $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ as in Definition 4.16.

Theorem 7.18. *Suppose that $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ is a sequence of symmetric tropical Metzler matrices such that $|Q^{(k)}| \in (\mathbb{Q} \cup \{-\infty\})^{m \times m}$ for all k. Denote*

$$\begin{aligned} \bar{\lambda} &\coloneqq \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \text{ is nontrivial}\}\,, \\ \underline{\lambda} &\coloneqq \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \text{ contains a real point}\}\,. \end{aligned}$$

If any of these supremums is finite, then it is attained. Moreover, we can construct, in polynomial-time, two stochastic mean payoff games: a "feasibility game" and a "strict feasibility game." If $\bar{\chi}$ denotes the maximal value of the feasibility game and $\underline{\chi}$ denotes the minimal value of the strict feasibility game, then we have

$$\bar{\chi} = \begin{cases} \bar{\lambda}/2 & \text{if } \bar{\lambda} \text{ is finite} \\ 1 & \text{if } \bar{\lambda} = +\infty \\ -1 & \text{if } \bar{\lambda} = -\infty \end{cases} \quad \text{and} \quad \underline{\chi} = \begin{cases} \underline{\lambda}/2 & \text{if } \underline{\lambda} > -\infty \\ 1 & \text{if } \bar{\lambda} = +\infty \\ -1 & \text{if } \bar{\lambda} = -\infty\,. \end{cases}$$

The proof is based on the following construction. Consider a tropical Metzler spectrahedron $\mathcal{S} \coloneqq \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ associated with the matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$. We construct a bipartite stochastic mean payoff game consisting of m states of Player Max and n states of Player Min, $V_{\text{Max}} \coloneqq [m], V_{\text{Min}} \coloneqq [n]$. For each state $k \in [n]$ of Player Min, the set $A^{(k)}$ of actions available to Player Min is created as follows:

- For every $i \in [m]$ such that $Q_{ii}^{(k)} \in \mathbb{T}_-$ we add an action $a \coloneqq (k, i)$ to $A^{(k)}$. That action is deterministic and goes to i, i.e., $p_{ki}^a \coloneqq 1$. Moreover, the associated payoff is given by $r_a \coloneqq -|Q_{ii}^{(k)}|$.
- For every $i, j \in [m]$ such that $i < j$ and $Q_{ij}^{(k)} \in \mathbb{T}_-$ we add an action $a \coloneqq (k, i, j)$ to $A^{(k)}$. This action is stochastic and we set $p_{ki}^a \coloneqq p_{kj}^a \coloneqq 1/2$. The associated payoff is given by $r_a \coloneqq -|Q_{ij}^{(k)}|$.

Similarly, for every state $i \in [m]$ of Player Max, and every $k \in [n]$ such that $Q_{ii}^{(k)} \in \mathbb{T}_+$, we add a deterministic action $b \coloneqq (i, k)$, $p_{ik}^b \coloneqq 1$ to $B^{(i)}$. This action has payoff $r_b \coloneqq Q_{ii}^{(k)}$.

Recall that in the games which we consider, every state has to be equipped with at least one action, i.e., the sets $A^{(k)}$ and $B^{(i)}$ must be nonempty. In consequence, our construction is valid provided that the following assumptions on the matrices $Q^{(k)}$ is satisfied:

Assumption D. For all $i \in [m]$, there exists $k \in [n]$ such that the diagonal coefficient $Q_{ii}^{(k)}$ belongs to $\mathbb{T}_+$.

Assumption E. For all $k \in [n]$, the matrix $Q^{(k)}$ has at least one coefficient in $\mathbb{T}_-$.

Definition 7.19. We say that a sequence $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ of symmetric tropical Metzler matrices defines a *well-fomed linear matrix inequality* if it satisfies Assumptions D and E.

In order to facilitate the reading of this section, we divide our discussion into two parts. First, we will assume that the matrices define a well-formed linear matrix inequality and analyze the games that are obtained by the construction above. In the second and more technical part, we show how these two assumption can be ensured by a suitable preprocessing.

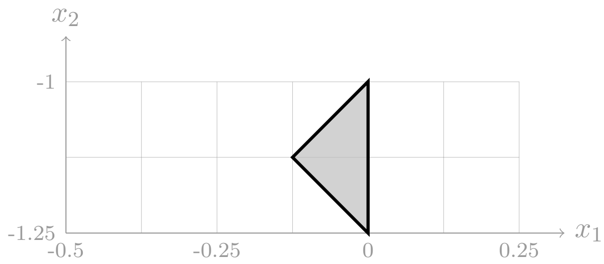

Figure 7.2: The tropical spectrahedron of Example 7.22.

7.2.1 Games obtained from well-formed linear matrix inequalities

In this subsection, we assume that the matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$ define a well-formed linear matrix inequality. Let $F\colon \mathbb{T}^n \to \mathbb{T}^n$ denote the bipartite Shapley operator of the game constructed from $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$. The next lemma gives the equivalence between sublevel sets of F and the sets $\mathcal{S}_\lambda \coloneqq \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ introduced in Definition 4.16.

Lemma 7.20. *For every $\lambda \in \mathbb{R}$ we have $\mathcal{S}_\lambda = \{x \in \mathbb{T}^n \colon \lambda + x \leqslant F(x)\}$.*

Proof. By definition, for all $k \in [n]$ we have

$$(F(x))_k = \min_{Q_{ij}^{(k)} \in \mathbb{T}_-} \Big(-|Q_{ij}^{(k)}| + \frac{1}{2}\big(\max_{Q_{ii}^{(l)} \in \mathbb{T}_+} (Q_{ii}^{(l)} + x_l) + \max_{Q_{jj}^{(l)} \in \mathbb{T}_+} (Q_{jj}^{(l)} + x_l)\big)\Big) . \tag{7.2}$$

Hence, a vector $x \in \mathbb{T}^n$ satisfies $\lambda + x \leqslant F(x)$ if and only if for all $i, j \in [m]$,

$$2\Big(\lambda + \max_{Q_{ij}^{(k)} \in \mathbb{T}_-} (|Q_{ij}^{(k)}| + x_k)\Big) \leqslant \max_{Q_{ii}^{(l)} \in \mathbb{T}_+} (Q_{ii}^{(l)} + x_l) + \max_{Q_{jj}^{(l)} \in \mathbb{T}_+} (Q_{jj}^{(l)} + x_l) .$$

By distinguishing whether i and j are equal in these inequalities, and recalling that $Q_{ij}^{(k)}$ is in $\mathbb{T}_- \cup \{-\infty\}$ for all $k \in [n]$ when $i \neq j$, we recover the constraints that describe $\mathcal{S}_\lambda$. ☐

The following theorem is an immediate corollary of Lemma 7.20 and Corollary 6.25.

Theorem 7.21. *The set $\mathcal{S}_\lambda$ is nontrivial if and only if $\lambda \geqslant 2\max_{k\in[n]} \chi_k$, where χ is the value of the game associated with $Q^{(1)}, \dots, Q^{(n)}$. Similarly, the set $\mathcal{S}_\lambda \cap \mathbb{R}^n$ is nonempty if and only if $\lambda \geqslant 2\min_{k\in[n]} \chi_k$. In particular, the tropical spectrahedron $\mathcal{S}$ is nontrivial if and only if the associated stochastic game has at least one state with nonnegative value and it contains a real point if and only if all states have nonnegative values.*

Example 7.22. Consider the matrices

$$\boldsymbol{Q}^{(1)} \coloneqq \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad \boldsymbol{Q}^{(2)} \coloneqq \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & t^{9/4} \end{bmatrix} , \quad \boldsymbol{Q}^{(3)} \coloneqq \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix} .$$

The matrices $(Q^{(1)}, Q^{(2)}, Q^{(3)}) \coloneqq (\mathsf{val}(\boldsymbol{Q}^{(1)}), \mathsf{val}(\boldsymbol{Q}^{(2)}), \mathsf{val}(\boldsymbol{Q}^{(3)}))$ satisfy Assumptions D and E. Furthermore, the game constructed out of these matrices is exactly the game from Example 6.2.

The associated tropical Metzler spectrahedron $\mathcal{S}$ is defined by the constraints

$$\begin{aligned} \max(-1+x_1, -5/4+x_3) &\geqslant x_2\,, \\ \max(x_1+x_3, -1/4+2x_3) &\geqslant 2x_1\,, \\ x_2 &\geqslant -7/4+x_3\,, \\ \max(5/4+x_1+x_2, 1+x_2+x_3) &\geqslant 2x_3\,. \end{aligned}$$

This tropical spectrahedron is depicted in Fig. 7.2.

As a corollary, we get a simplified version of Theorem 7.6.

Theorem 7.23. *Let $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ be as in Theorem 7.6. Furthermore, suppose that the matrices $Q^{(k)} := \mathsf{sval}(\boldsymbol{Q}^{(n)})$ are Metzler and define a well-form linear matrix inequality. Then, the game constructed from the matrices $Q^{(1)}, \dots, Q^{(n)}$ is, at the same time, both a feasibility game and a strict feasibility game as announced in Theorem 7.6.*

Proof. Let $\overline{\chi}$ be the maximal value of the game constructed from $Q^{(1)}, \dots, Q^{(n)}$, and $\underline{\chi}$ be its minimal value. If $\overline{\chi} > 0$, then by Theorem 7.21, there exists $\lambda > 0$ such that $\mathcal{S}_\lambda$ is nontrivial. By Lemma 4.17, if $x \in \mathcal{S}_\lambda$, then any lift $\boldsymbol{x} \in \mathsf{val}^{-1}(x) \cap \mathbb{K}^n_{\geqslant 0}$ belongs to $\boldsymbol{\mathcal{S}}$. Hence, $\boldsymbol{\mathcal{S}}$ is nontrivial. Similarly, if $\underline{\chi} > 0$, then there exists $\lambda > 0$ such that $\mathcal{S}_\lambda \cap \mathbb{R}^n$ is nonempty. Then, by Lemma 4.41 any lift of any point $x \in \mathcal{S}_\lambda \cap \mathbb{R}^n$ belongs to the interior of $\boldsymbol{\mathcal{S}}$. Moreover, if the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy Assumption B, then by Lemma 4.50, any point $\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0}$ such that $\mathsf{val}(\boldsymbol{x}) = x$ is a strictly feasible point of $\boldsymbol{\mathcal{S}}$. If the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy the conditions of Theorem 4.28 and $\overline{\chi} = 0$, then $\mathcal{S}_0$ is nontrivial by Theorem 7.21 and $\boldsymbol{\mathcal{S}}$ is nontrivial by Theorem 4.28. If these conditions are satisfied and $\underline{\chi} = 0$, then $\mathcal{S}_0 \cap \mathbb{R}^n$ is nonempty by Theorem 7.21 and $\boldsymbol{\mathcal{S}}$ has a nonempty interior by Corollary 4.42. If the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ additionally satisfy Assumption B, then $\boldsymbol{\mathcal{S}}$ has a strictly feasible point by Corollary 4.52. If $\overline{\chi} < 0$, then $\mathcal{S}_0$ is trivial by Theorem 7.21 and $\boldsymbol{\mathcal{S}}$ is trivial by Lemma 4.22. Finally, if $\underline{\chi} < 0$, then $\mathcal{S}_0 \cap \mathbb{R}^n$. By Lemma 4.22 we have $\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) \subset \mathcal{S}_0 \cap \mathbb{R}^n$ and hence $\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}$ is empty. In particular, $\boldsymbol{\mathcal{S}}$ has an empty interior. $\square$

Remark 7.24. We note that the proof above gives a slightly stronger property that the one announced in Theorem 7.6— it shows that if the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy the conditions of Theorem 4.28 and $\underline{\chi} = 0$, then $\boldsymbol{\mathcal{S}}$ has a nonempty interior. This does not carry over to non-Metzler case as discussed in Section 4.4.

Remark 7.25. Thanks to the analysis above, one could solve not only generic feasibility problems for cones, but also for non-conic spectrahedra. Indeed, if we suppose that the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy the conditions of Theorem 4.28, then this theorem combined with Lemma 7.20 and Proposition 6.24 shows that $\boldsymbol{\mathcal{S}}$ contains a point $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}$ such that $\boldsymbol{x}_1 \neq 0$ if and only if the state $1 \in V_{\mathrm{Min}}$ belongs to some winning dominion of the associated game. This can be checked by solving a series of stochastic mean payoff games. However, the generalization of this observation to matrices that do not fulfill Assumptions D and E is quite involved and requires to use some of the estimates that are obtained in Chapter 8. For the sake of simplicity, we do not develop all of the details of this extension in this dissertation, and we focus only on the conic case.

7.2.2 Preprocessing

We now turn to our discussion of Assumptions D and E. Even though these two assumptions have a similar interpretation in terms of games (the first one states that Player Max always has an action and the second states that Player Min always has an action), their impact on our results is quite different, and so we discuss them separately.Assumption D can be made without loss of generality, up to an easy preprocessing.

Lemma 7.26. *There is a polynomial-time algorithm that takes symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ as an input and outputs a set $K \subset [n]$ and a sequence of symmetric Metzler matrices $(R^{(k)})_{k \in K}$, $R^{(k)} \in \mathbb{T}_{\pm}^{p \times p}$ with $p \leqslant m$. The matrices $(R^{(k)})_{k \in K}$ satisfy Assumption D and are such that*

$$\mathcal{S}_{\lambda}(Q^{(1)}, \dots, Q^{(n)}) = \{x \in \mathbb{T}^n \colon \forall l \notin K, x_l = -\infty \wedge x_K \in \mathcal{S}_{\lambda}\big((R^{(k)})_{k \in K}\big)\}$$

for all $\lambda \in \mathbb{R}$. (With the convention that $\mathcal{S}_{\lambda}(Q^{(1)}, \dots, Q^{(n)})$ is trivial if $K = \emptyset$.)

Proof. Fix $\lambda \in \mathbb{R}$ and matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$. Suppose that Assumption D is not satisfied and fix $i \in [m]$ such that no diagonal coefficient $Q_{ii}^{(k)}$ is in $\mathbb{T}_{+}$. We distinguish three cases:

- If the set L of indices $l \in [n]$ such that $Q_{ii}^{(l)} \in \mathbb{T}_{-}$ is nonempty, the relation $Q_{ii}^{+}(x) \geqslant \lambda \odot Q_{ii}^{-}(x)$ enforces that $x_l = -\infty$ for all $x \in \mathcal{S}_{\lambda}$ and $l \in L$. Therefore, we have

$$\mathcal{S}_{\lambda}(Q^{(1)}, \dots, Q^{(n)}) = \{x \in \mathbb{T}^n \colon \forall l \in L, x_l = -\infty \wedge x_{[n] \setminus L} \in \mathcal{S}_{\lambda}\big((Q^{(k)})_{[n] \setminus L}\big)\}\,. \tag{7.3}$$

  (If $L = [n]$, then $\mathcal{S}_{\lambda}$ is trivial.)

- If the aforementioned set L is empty, then let $\tilde{L}$ be the set of indices $l \in [n]$ such that $Q^{(l)}$ contains an entry different than $-\infty$ on its ith row, namely $Q_{ij}^{(l)} \in \mathbb{T}_{-}$ with $i < j$. Then, the relation $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2}$ enforces $x_l = -\infty$ for any $x \in \mathcal{S}_{\lambda}$. Therefore, we get the equality as in (7.3), with L replaced by $\tilde{L}$. (If $\tilde{L} = [n]$, then $\mathcal{S}_{\lambda}$ is trivial).

- If both L and $\tilde{L}$ are empty, then the ith row and column of the matrices $Q^{(k)}$ are all identically equal to $-\infty$. Hence, we can remove all these rows and columns, and obtain matrices of order $m-1$ over the variables $x_1, \dots, x_n$ that describe the same set $\mathcal{S}_{\lambda}$. (If $m = 1$, then $\mathcal{S}_{\lambda} = \mathbb{T}^n$ and we can replace the matrices $Q^{(k)}$ by $R^{(k)} \coloneqq 0$.)

The observations above lead to the polynomial-time algorithm described in the claim. Indeed, given the matrices $Q^{(1)}, \dots, Q^{(n)}$ that do not satisfy Assumption D, we can repeatedly apply the reductions above till we find the claimed matrices $(R^{(k)})_{k \in K}$ or decide that $\mathcal{S}_{\lambda}$ is trivial. Furthermore, we point out that these reductions do not depend on $\lambda \in \mathbb{R}$. □

The importance of Assumption E depends on the question that one wants to answer. If we are only interested in knowing whether $\mathcal{S}_{\lambda}$ is trivial, then this assumption is insignificant, as observed in the next lemma.

Lemma 7.27. *Suppose that the matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ do not satisfy Assumption E. Then, the set $\mathcal{S}_{\lambda}(Q^{(1)}, \dots, Q^{(n)})$ is nontrivial for all $\lambda \in \mathbb{R}$.*

Proof. Suppose that $Q^{(l)}$ has no coefficient in $\mathbb{T}_{-}$. Then, this matrix is diagonal (because it is a Metzler matrix). Therefore, the point $x \in \mathbb{T}^n$ defined as $x_l \coloneqq 0$ and $x_k \coloneqq -\infty$ for $k \neq l$ belongs to $\mathcal{S}_{\lambda}$ for all $\lambda \in \mathbb{R}$. □

On the other hand, it will be also interesting to decide if $\mathcal{S}_\lambda$ contains a real point, i.e., if $\mathcal{S}_\lambda \cap \mathbb{R}^n$ is nonempty. In this case, Assumption E is no longer trivial.

Lemma 7.28. *There is a polynomial-time algorithm that takes symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_\pm^{m \times m}$ as an input and either decides that $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is nonempty for all $\lambda \in \mathbb{R}$ or outputs a set $K \subset [n]$ and a sequence of symmetric Metzler matrices $(R^{(k)})_{k \in K}$, $R^{(k)} \in \mathbb{T}_\pm^{p \times p}$ with $p \leqslant m$ satisfying Assumptions D and E. These set and matrices are such that $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is nonempty if and only if $\mathcal{S}_\lambda\big((R^{(k)})_{k \in K}\big) \cap \mathbb{R}^K$ is nonempty. (With the convention that if $K = \emptyset$, then $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is empty.)*

Proof. First, if the matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_\pm^{m \times m}$ do not satisfy Assumption D, then we can apply the algorithm of Lemma 7.26. This algorithm either decides that $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is empty for all $\lambda \in \mathbb{R}$ or outputs a sequence of matrices that satisfy Assumption D and describes the same family $(\mathcal{S}_\lambda)_\lambda$. From now on we assume that $Q^{(1)}, \dots, Q^{(n)}$ satisfy Assumption D. As previously, suppose $Q^{(l)}$ has no coefficient in $\mathbb{T}_-$ and observe that this matrix is diagonal (because it is a Metzler matrix). Moreover, note that we have $n \geqslant 2$ because Assumption D is satisfied. Let $I \subset [m]$ denote the set of indices such that $i \in I$ if $Q_{ii}^{(l)} \in \mathbb{T}_+$ (i.e., $i \in I$ if $Q_{ii}^{(l)}$ is finite). If $I = [m]$, then the point defined as $x_k \coloneqq 0$ for $k \neq l$ and $x_l \coloneqq M \in \mathbb{R}$ belongs to $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ for M large enough. If $I \subsetneq [m]$, then we remove all the rows and columns indexed by I from from all the matrices $Q^{(k)}$. Denote the new sequence of matrices by $\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n)}$. We claim that $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ contains a real point if and only if $\mathcal{S}_\lambda\big((\tilde{Q}^{(k)})_{k \neq l}\big)$ does. Let $\tilde{x} \in \mathbb{R}^{n-1}$ be any point and let $x \in \mathbb{R}^n$ be a point defined as $x_k \coloneqq \tilde{x}_k$ for $k \neq l$ and $x_l \coloneqq M$ for some $M \in \mathbb{R}$. Observe that for every $i \in [m] \setminus I$ we have

$$\max_{\tilde{Q}_{ii}^{(k)} \in \mathbb{T}_+} (\tilde{Q}_{ii}^{(k)} + \tilde{x}_k) = \max_{Q_{ii}^{(k)} \in \mathbb{T}_+} (Q_{ii}^{(k)} + x_k) \quad \text{and} \quad \max_{Q_{ii}^{(k')} \in \mathbb{T}_-} (Q_{ii}^{(k')} + x_k) = \max_{\tilde{Q}_{ii}^{(k')} \in \mathbb{T}_-} (\tilde{Q}_{ii}^{(k')} + \tilde{x}_{k'}) \, .$$

Similarly, for every $\{i, j\} \subset [m] \setminus I$, $i \neq j$ we have

$$\max_{\tilde{Q}_{ij}^{(k)} \in \mathbb{T}_-} (|\tilde{Q}_{ij}^{(k)}| + \tilde{x}_k) = \max_{\tilde{Q}_{ij}^{(k)} \in \mathbb{T}_-} (|Q_{ij}^{(k)}| + x_k) \, .$$

Thus, if x belongs to $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$, then $\tilde{x}$ belongs to $\mathcal{S}_\lambda\big((\tilde{Q}^{(k)})_{k \neq l}\big)$. Conversely, suppose that $\tilde{x}$ belongs to $\mathcal{S}_\lambda\big((\tilde{Q}^{(k)})_{k \neq l}\big)$. If $i \in I$, then for sufficiently large $M \in \mathbb{R}$ we have

$$\begin{aligned}
\max_{Q_{ii}^{(k)} \in \mathbb{T}_+} (Q_{ii}^{(k)} + x_k) &= (Q_{ii}^{(l)} + M) \oplus \max_{Q_{ii}^{(k)} \in \mathbb{T}_+, k \neq l} (Q_{ii}^{(k)} + \tilde{x}_k) \\
&\geqslant \lambda + \max_{Q_{ii}^{(k')} \in \mathbb{T}_-, k \neq l} (Q_{ii}^{(k')} + \tilde{x}_k) = \lambda + \max_{Q_{ii}^{(k')} \in \mathbb{T}_-} (Q_{ii}^{(k')} + x_k) \, .
\end{aligned}$$

Similarly, if $i \in I$, $j \neq i$, then $\max_{Q_{jj}^{(k)} \in \mathbb{T}_+} (Q_{jj}^{(k)} + x_k)$ is a real number by Assumption D and we can assume that $\max_{Q_{jj}^{(k)} \in \mathbb{T}_+} (Q_{jj}^{(k)} + x_k) > c$ for some $c \in \mathbb{R}$ and all $M \geqslant 0$. Therefore, for sufficiently large $M \geqslant 0$ we have

$$\begin{aligned}
\max_{Q_{ii}^{(k)} \in \mathbb{T}_+} (Q_{ii}^{(k)} + x_k) + \max_{Q_{jj}^{(k)} \in \mathbb{T}_+} (Q_{jj}^{(k)} + x_k) &\geqslant (Q_{ii}^{(l)} + M) \oplus \max_{Q_{ii}^{(k)} \in \mathbb{T}_+, k \neq l} (Q_{ii}^{(k)} + \tilde{x}_k) + c \\
&\geqslant 2\lambda + \max_{\tilde{Q}_{ij}^{(k)} \in \mathbb{T}_-, k \neq l} (|Q_{ij}^{(k)}| + \tilde{x}_k) = 2\lambda + \max_{\tilde{Q}_{ij}^{(k)} \in \mathbb{T}_-} (|Q_{ij}^{(k)}| + x_k) \, .
\end{aligned}$$

Hence $x \in \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$. As previously, the observations above lead to a polynomial-time algorithm. Indeed, we can repeatedly apply the construction above until we decide that $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \cap \mathbb{R}^n$ is (non)empty for all λ, or we construct the matrices $(R^{(k)})_{k \in K}$ that satisfy the claim. Moreover, these reductions do not depend on the value of $\lambda \in \mathbb{R}$. ☐

The considerations of this section allow us to prove Theorems 7.4 and 7.18.

Proof of Theorem 7.4. Proposition 7.17 gives a reduction from Smpg to Tmsdfp. Conversely, Lemmas 7.26 and 7.27 and Theorem 7.21 give a reduction from Tmsdfp to Smpg. Moreover, Theorem 7.9 shows that a pseudopolynomial-time algorithm for Smpg would give a polynomial-time algorithm for Smpg. This implies that it would also give a polynomial time algorithm for Tmsdfp. Conversely, the proof of Proposition 7.17 reduces Smpg to Tmsdfp restricted to matrices that have entries in $\{0, \ominus 0, \pm 1, \ominus(\pm 1), -\infty\}$. Hence, a pseudopolynomial-time algorithm for Tmsdfp would give a polynomial-time algorithm that solves Smpg. ☐

Proof of Theorem 7.18. Let us first prove the claim for $\bar{\lambda}$. We take the matrices $Q^{(1)}, \dots, Q^{(n)}$ and apply the preprocessing of Lemma 7.26 to them. If this preprocessing detects that the set $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ is trivial for all λ (i.e., that $\bar{\lambda} = -\infty$), then we can take the feasibility game to be any game having value equal to -1 (e.g., a game on two states in which every player has only one action and all payoffs are equal to -1). Otherwise, we can replace the matrices $Q^{(1)}, \dots, Q^{(n)}$ by the matrices $(R^{(k)})_k$ given by Lemma 7.26 and this does not change the value of $\bar{\lambda}$. Hence, from now on we can assume that $Q^{(1)}, \dots, Q^{(n)}$ satisfy Assumption D. If $Q^{(1)}, \dots, Q^{(n)}$ do not satisfy Assumption E, then Lemma 7.27 shows that $\bar{\lambda} = +\infty$. As previously, we can take the feasibility game to be any game having all payoffs equal to 1. Otherwise, the matrices $Q^{(1)}, \dots, Q^{(n)}$ satisfy both assumptions. Then, the construction presented at the beginning of Section 7.2 gives the demanded feasibility game. Indeed, if $\bar{\chi}$ denotes the maximal value of this game, then Lemma 7.20 and Corollary 6.25 shows that $\bar{\lambda} = 2\bar{\chi}$. The proof for $\underline{\lambda}$ is analogous, using the preprocessing of Lemma 7.28 instead of Lemmas 7.26 and 7.27. ☐

7.3 Extension to non-Metzler matrices

In the previous sections, we dealt only with a particular variant of tropical semidefinite feasibility problem—we assumed that the matrices are Metzler and that we are interested in solving a conic problem. In this section, we abandon these assumption. The following proposition shows that the problems for non-Metzler matrices can be reduced to the corresponding problems with Metzler matrices. Since we want to use this proposition to show Theorem 7.6, we need the following notation. We let $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_\pm^{m \times m}$ be symmetric tropical matrices and we denote by $P^{(1)}, \dots, P^{(n)} \in \mathbb{T}_\pm^{m \times m}$ the matrices defined as

$$P_{ij}^{(k)} \coloneqq \begin{cases} Q_{ij}^{(k)} & \text{if } i = j \\ \ominus |Q_{ij}^{(k)}| & \text{if } i \neq j \, . \end{cases}$$

(We note that $P^{(k)} = Q^{(k)}_{\Sigma, \Diamond}$, where $\Sigma \coloneqq \{(i,j) \in [m]^2 \colon i < j\}$ and $\Diamond \coloneqq \emptyset$ if we use the notation introduced in Definition 4.23 and (4.6).) Furthermore, let $\hat{\mathcal{S}}_\lambda \coloneqq \mathcal{S}_\lambda(P^{(1)}, \dots, P^{(n)})$ for all $\lambda \in \mathbb{R}$.

Proposition 7.29. *We can construct (in polynomial-time) symmetric tropical Metzler matrices $\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')}, \tilde{Q}^{(k)} \in \mathbb{T}_{\pm}^{m' \times m'}$ such that the set $\mathcal{S}$ is nontrivial if and only if $\mathcal{S}(\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')})$ is nontrivial. Similarly, the set $\mathcal{S}$ contains a real point if and only if $\mathcal{S}(\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')})$ contains a real point. Even more, if $\lambda > 0$ and the set $\mathcal{S}_\lambda(\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')})$ is nontrivial, then $\hat{\mathcal{S}}_\lambda$ is nontrivial, and if $\mathcal{S}_\lambda(\tilde{Q}^{(1)}, \dots, \tilde{Q}^{(n')})$ contains a real point, then $\hat{\mathcal{S}}_\lambda$ contains a real point.*

Proof. Let $I \subset [m]^2$, denote the set of pairs $(i,j) \in [m] \times [m]$, $i < j$ such that the tropical polynomial $Q_{ij}(x)$ is nonzero (i.e., the set of pairs (i,j) such that at least one $k \in [n]$ satisfies $Q_{ij}^{(k)} \neq -\infty$). Denote $n' \coloneqq |I|$ and $I = \{(i_1, j_1), \dots, (i_{n'}, j_{n'})\}$. For every such pair we introduce a variable y_{ij}, and we consider the set $\tilde{\mathcal{S}}_\lambda$ of points $(x,y) \in \mathbb{T}^{n+n'}$ that fulfill the following conditions:

- for all $i \in [m]$, $Q_{ii}^{+}(x) \geqslant \lambda \odot Q_{ii}^{-}(x)$;
- for all $(i,j) \in I$, $i < j$, $y_{ij} \oplus Q_{ij}^{+}(x) \geqslant \lambda \odot Q_{ij}^{-}(x)$;
- for all $(i,j) \in I$, $i < j$, $y_{ij} \oplus Q_{ij}^{-}(x) \geqslant \lambda \odot Q_{ij}^{+}(x)$;
- for all $(i,j) \in I$, $i < j$, $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (\lambda \odot y_{ij})^{\odot 2}$.

Observe that the set $\tilde{\mathcal{S}}_\lambda$ can be described as $\tilde{\mathcal{S}}_\lambda = \mathcal{S}_\lambda\big((R^{(k)})_{k\in[n]}, (T^{(k)})_{k \in [n']}\big)$, where the matrices $(R^{(k)})_{k\in[n]}, (T^{(k)})_{k \in [n']} \in \mathbb{T}_{\pm}^{m' \times m'}$, $m' \coloneqq m + 2n'$ are defined as follows. For $k \in [n]$ we set $R^{(k)}$ to be the diagonal matrix

$$R^{(k)} \coloneqq \operatorname{tdiag}\big((Q_{ii}^{(k)})_{i \in [m]}, (Q_{i_l j_l}^{(k)})_{l \in [n']}, (\ominus Q_{i_l j_l}^{(k)})_{l \in [n']}\big) \, .$$

Then, for every $k \in [n']$ we define the matrix $T^{(k)}$ as

$$T_{ij}^{(k)} \coloneqq \begin{cases} \ominus 0 & \text{if } \{i,j\} = \{i_k, j_k\} \, , \\ 0 & \text{if } i = j = m + k \text{ or } i = j = m + n' + k \\ -\infty & \text{otherwise} \, . \end{cases}$$

To prove the first part of the claim, suppose that $\mathcal{S}$ is nontrivial and take a point $x \in \mathcal{S} \setminus \{-\infty\}$. For every $(i,j) \in I$ such that $Q_{ij}^{-}(x) = Q_{ij}^{+}(x) \neq -\infty$ we put $y_{ij} \coloneqq Q_{ij}^{-}(x)$. For every remaining $(i,j) \in I$ we have $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (Q_{ij}^{+}(x) \oplus Q_{ij}^{-}(x))^{\odot 2}$ and we put $y_{ij} \coloneqq Q_{ij}^{+}(x) \oplus Q_{ij}^{-}(x)$. It is clear that we have $(x,y) \in \tilde{\mathcal{S}}_0$. Even more, if $x \in \mathbb{R}^n$, then $Q_{ij}^{+}(x) \oplus Q_{ij}^{-}(x) \neq -\infty$ by the definition of I and our construction gives a point $(x,y) \in \tilde{\mathcal{S}}_0 \cap \mathbb{R}^{n+n'}$. Conversely, let $(x,y) \in \tilde{\mathcal{S}}_0 \setminus \{-\infty\}$. Note that in this case we have $x \neq -\infty$ by the definition of $\tilde{\mathcal{S}}_0$. For every $(i,j) \in I$ we consider two cases. If $y_{ij} \geqslant Q_{ij}^{-}(x)$, then we have $y_{ij} \geqslant Q_{ij}^{+}(x)$ and hence $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant y_{ij}^{\odot 2} \geqslant (Q_{ij}^{+}(x) \oplus Q_{ij}^{-}(x))^{\odot 2}$. If $y_{ij} < Q_{ij}^{-}(x)$, then we have $Q_{ij}^{+}(x) \geqslant Q_{ij}^{-}(x)$ and $Q_{ij}^{-}(x) \geqslant Q_{ij}^{+}(x)$. Hence $Q_{ij}^{+}(x) = Q_{ij}^{-}(x)$. Therefore $x \in \mathcal{S} \setminus \{-\infty\}$.

To prove the second part of the claim, let $(x,y) \in \tilde{\mathcal{S}}_\lambda \setminus \{-\infty\}$ for some $\lambda > 0$. As previously, this implies that $x \neq -\infty$. Furthermore, we have two cases. If $y_{ij} < Q_{ij}^{-}(x)$, then $Q_{ij}^{+}(x) \geqslant \lambda \odot Q_{ij}^{-}(x) > Q_{ij}^{-}(x)$ and $Q_{ij}^{-}(x) \geqslant \lambda \odot Q_{ij}^{+}(x) \geqslant Q_{ij}^{+}(x)$, which is impossible. Hence $y_{ij} \geqslant Q_{ij}^{-}(x)$. This implies that $y_{ij} \geqslant \lambda \odot Q_{ij}^{+}(x) \geqslant Q_{ij}^{+}(x)$ and thus $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (\lambda \odot y_{ij})^{\odot 2} \geqslant \big(\lambda \odot (Q_{ij}^{+}(x) \oplus Q_{ij}^{-}(x))\big)^{\odot 2}$. In particular, we have $x \in \hat{\mathcal{S}}_\lambda$. □

7.4 Nonarchimedean semidefinite feasibility problems

In this section we prove Theorem 7.6. To do so, let $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ be symmetric matrices that satisfy the conditions of Theorem 7.6. Let $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}_{\geqslant 0}^n \colon \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} \succcurlyeq$

$0\}$ be the associated spectrahedron, and denote $Q^{(k)} \coloneqq \mathsf{sval}(\boldsymbol{Q}^{(k)})$. We let $\mathcal{S} \coloneqq \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ and, for all $\lambda \in \mathbb{R}$, define $\hat{\mathcal{S}}_\lambda$ as in Section 7.3. The proof of Theorem 7.6 is based on the results presented in Section 4.2.

Proof of Theorem 7.6. In the proof we use the notation introduced above. First, we construct the Metzler matrices $(\tilde{Q}^{(k)})_k$ as given by Proposition 7.29. Then, Theorem 7.18 gives a feasibility game and a strict feasibility game associated with $\tilde{Q}^{(k)}$. We will show that these games satisfy the claim. To see this, let

$$\bar{\lambda} \coloneqq \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_\lambda\big((\tilde{Q}^{(k)})_k\big) \text{ is nontrivial}\}\,,$$
$$\underline{\lambda} \coloneqq \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_\lambda\big((\tilde{Q}^{(k)})_k\big) \text{ contains a real point}\}\,.$$

If $\bar{\lambda} < 0$, then $\mathcal{S}\big((\tilde{Q}^{(k)})_k\big)$ is trivial. Hence $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ is trivial by Proposition 7.29 and $\boldsymbol{\mathcal{S}}$ is trivial because $\mathsf{val}(\boldsymbol{\mathcal{S}}) \subset \mathcal{S}$ by Lemma 4.22. Similarly, if $\underline{\lambda} < 0$, then $\mathcal{S}\big((\tilde{Q}^{(k)})_k\big)$ does not contain a real point. Therefore, by Proposition 7.29, the set $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ does not contain a real point, and $\boldsymbol{\mathcal{S}}$ has empty interior because $\mathsf{val}(\boldsymbol{\mathcal{S}} \cap \mathbb{K}^n_{>0}) \subset \mathcal{S} \cap \mathbb{R}^n$ by Lemma 4.22. To prove the claims for the remaining inequalities, let $\Sigma \coloneqq \{(i,j) \in [m]^2 \colon i < j\}$ and $\lozenge \coloneqq \emptyset$. If $\bar{\lambda} > 0$, then Proposition 7.29 shows that $\hat{\mathcal{S}}_\lambda$ is nontrivial for some $\lambda > 0$. Hence, the set $\mathcal{T}_{\Sigma,\lozenge} \subset \mathbb{T}^n$ (as defined in Lemma 4.26) is nontrivial. Moreover, by Lemma 4.26, we have $\mathcal{T}_{\Sigma,\lozenge} \subset \mathsf{val}(\boldsymbol{\mathcal{S}})$ and hence $\boldsymbol{\mathcal{S}}$ is nontrivial. Furthermore, if $\underline{\lambda} > 0$, then the same reasoning shows that $\hat{\mathcal{S}}_\lambda$ contains a real point and that $\mathcal{T}_{\Sigma,\lozenge}$ contains a real point. Hence, $\boldsymbol{\mathcal{S}}$ has a nonempty interior by Lemma 4.44. Furthermore, if the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy Assumption B, then $\boldsymbol{\mathcal{S}}$ is strictly feasible by Lemma 4.51.

If the matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$ satisfy the conditions of Theorem 4.28, then this theorem gives the equality $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$. Therefore, $\boldsymbol{\mathcal{S}}$ is nontrivial if and only if $\mathcal{S}$ is nontrivial. By Proposition 7.29, this is equivalent to $\bar{\lambda} \geqslant 0$. Furthermore, if these matrices satisfy Assumption B, then Corollary 4.52 shows that $\boldsymbol{\mathcal{S}}$ is strictly feasible if and only if $\mathcal{S}$ contains a real point. By Proposition 7.29, this is equivalent to $\underline{\lambda} \geqslant 0$. ☐

CHAPTER 8

Condition number of stochastic mean payoff games

In the final chapter of this work, we study the notion of a condition number of stochastic mean payoff games and nonarchimedean semidefinite programming. We do that from different perspectives. First, in Section 8.1, we study the relation between nonarchimedean and archimedean problems. More precisely, we replace the formal parameter t involved in the definition of the Puiseux series by a large real number and investigate how large this number should be in order to obtain an archimedean problem that behaves in the same way as the nonarchimedean one. In turns out that this parameter decreases as the value of the associated stochastic mean payoff game increases. Then, in Section 8.1.1, we give a geometric description of this behavior, interpreting the maximal value of the game as a radius of a Hilbert ball included in a tropical cone. Subsequently, in Section 8.2, we move to study the value iteration and define the condition number of this algorithm. The analysis of Section 8.2 is valid for abstract monotone and homogeneous operators. The main technical result of this chapter is given in Section 8.3, where we estimate the condition number of stochastic mean payoff games. In Section 8.4 we use these estimates to show that value iteration can solve constant value stochastic mean payoff games in pseudopolynomial time when the number of randomized actions is fixed. In Section 8.5 we specify these complexity bounds to the task of checking the feasibility of tropical Metzler spectrahedral cones and we present numerical results.

8.1 Archimedean feasibility problems

In this section, we relate the tropical feasibility problem with the archimedean feasibility problem. For simplicity of exposition, we consider the case of conic spectrahedra. As in the previous sections, we suppose that $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ are symmetric Metzler matrices, set $\boldsymbol{Q}(\boldsymbol{x}) \coloneqq \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)}$, and consider the spectrahedron $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$. For any fixed value of the parameter $t \in \mathbb{R}$, we also consider the real spectrahedron $\boldsymbol{\mathcal{S}}(t) \subset \mathbb{R}^n_{\geqslant 0}$ described by $\boldsymbol{Q}^{(1)}(t), \dots, \boldsymbol{Q}^{(n)}(t)$ in the same way. We want to study the feasibility problem of $\boldsymbol{\mathcal{S}}(t)$ as t goes to infinity.

First, we point out that for sufficiently large t, $\boldsymbol{\mathcal{S}}(t) \subset \mathbb{R}_{\geqslant 0}$ is feasible if and only if $\boldsymbol{\mathcal{S}} \subset \mathbb{K}_{\geqslant 0}$ is feasible. This is an immediate corollary of the following general statement.

Proposition 8.1. *Let $\phi(x_1, \dots, x_n)$ be any $\mathcal{L}_{\mathrm{or}}$-formula. Then, for every $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{K}$ there exists $T > 0$ such that for all $t \geqslant T$ the formula $\phi(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ is true in $\mathbb{K}$ if and only if $\phi(\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t))$ is true in $\mathbb{R}$. In particular, the formula $\phi(\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t))$ is either true (in $\mathbb{R}$) for all sufficiently large t or false for all sufficiently large t.*

Sketch of the proof. First, suppose that ϕ is atomic. As observed in the proof of Lemma 2.111, $\mathcal{L}_{\mathrm{or}}$-terms are interpreted as polynomials with natural coefficients. In other words, after fixing the variables $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$, the expression $\phi(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ is of the form $\boldsymbol{c} = 0$ or $\boldsymbol{c} \geqslant 0$ for some $\boldsymbol{c} \in \mathbb{K}$. Furthermore, for all $t > 0$ such that $\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t)$ are absolutely convergent, the expression $\phi(\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t))$ is, respectively, of the form $\boldsymbol{c}(t) = 0$ or $\boldsymbol{c}(t) \geqslant 0$ with the same $\boldsymbol{c} \in \mathbb{K}$. Hence, the claim follows from the definition of order in $\mathbb{K}$. Since the claim is true for atomic formulas, it follows that it is also true for negated atomic formulas, their conjunctions, and disjunctions. Therefore, the claim is true for all quantifier-free formulas. To finish, if ϕ is an arbitrary formula, then quantifier elimination in real closed fields (Theorem 2.110) shows that there exists a quantifier-free $\mathcal{L}_{\mathrm{or}}$-formula ψ such that $\phi(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ is true in $\mathbb{K}$ if and only if $\psi(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ is true in $\mathbb{K}$ and $\phi(\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t))$ is true in $\mathbb{R}$ if and only if $\psi(\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t))$ is true in $\mathbb{R}$ (for all $t > 0$ such that $\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t)$ are absolutely convergent). □

Corollary 8.2. *The set $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ is nontrivial (res. trivial) if and only if $\boldsymbol{\mathcal{S}}(t) \subset \mathbb{R}_{\geqslant 0}$ is nontrivial (resp. trivial) for all sufficiently large $t > 0$.*

Sketch of the proof. Let $y^{(k)}_{ij}$ be a variable symbol for all $k \in [n]$ and $i, j \in [m]$, $i \leqslant j$. Then, the statement "there exist $x_1 \geqslant 0, \dots, x_n \geqslant 0$, $(x_1, \dots, x_n) \neq 0$ such that the matrix

$$\begin{bmatrix} x_1 y^{(1)}_{11} + \dots + x_n y^{(n)}_{11} & \dots & x_1 y^{(1)}_{1m} + \dots + x_n y^{(n)}_{1m} \\ \vdots & \ddots & \vdots \\ x_1 y^{(1)}_{1m} + \dots + x_n y^{(n)}_{1m} & \dots & x_1 y^{(1)}_{mm} + \dots + x_n y^{(n)}_{mm} \end{bmatrix}$$

is positive semidefinite" is an $\mathcal{L}_{\mathrm{or}}$-formula and the claim follows from Proposition 8.1 by putting $y^{(k)}_{ij} \coloneqq \boldsymbol{Q}^{(k)}_{ij}$. □

The aim of this section is to give a bound on the size of the parameter t that satisfies the conclusions of Corollary 8.2. We denote $\mathsf{sval}(\boldsymbol{Q}^{(k)}) = Q^{(k)}$ for all k. We also introduce a threshold $T > 1$ such that for all $t \geqslant T$, every series $\boldsymbol{Q}^{(k)}_{ij}(t)$ is absolutely convergent, and the signs of $\boldsymbol{Q}^{(k)}_{ij}(t)$ and $Q^{(k)}_{ij}$ are the same. For any $t \geqslant T$, we define

$$\delta(t) \coloneqq \max_{Q^{(k)}_{ij} \neq -\infty} \left| |Q^{(k)}_{ij}| - \log_t |\boldsymbol{Q}^{(k)}_{ij}(t)| \right| .$$

We have the following observation.

Lemma 8.3. *For all $k \in [n]$ and $i, j \in [m]$ we have $t^{|Q_{ij}^{(k)}|-\delta(t)} \leqslant |\boldsymbol{Q}_{ij}^{(k)}(t)| \leqslant t^{|Q_{ij}^{(k)}|+\delta(t)}$ (with the convention that $t^{-\infty} = 0$). Furthermore, $\lim_{t\to\infty} \delta(t) = 0$.*

Proof. The second claim follows from the definition of valuation. To prove the first one, note that $\delta(t) \geqslant |Q_{ij}^{(k)}| - \log_t |\boldsymbol{Q}_{ij}^{(k)}(t)|$ and $\delta(t) \geqslant -|Q_{ij}^{(k)}| + \log_t |\boldsymbol{Q}_{ij}^{(k)}(t)|$, which implies that $|Q_{ij}^{(k)}| - \delta(t) \leqslant \log_t |\boldsymbol{Q}_{ij}^{(k)}(t)| \leqslant |Q_{ij}^{(k)}| + \delta(t)$. Since $t > 1$, we obtain the claim. $\square$

We are now ready to state the main result of this section. As in the previous chapters, we state the result in terms of the sets $\mathcal{S}_\lambda$ introduced in Definition 4.16.

Theorem 8.4. *Let $m \geqslant 2$, and let*

$$\bar{\lambda} \coloneqq \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \text{ is nontrivial}\} .$$

Furthermore, suppose that $\bar{\lambda}$ is finite and nonzero. Take any $t \geqslant T > 1$ such that $2\delta(t) < |\bar{\lambda}|$ and

$$t > (2(m-1)n)^{1/(|\bar{\lambda}|-2\delta(t))} .$$

Then, the spectrahedron $\boldsymbol{\mathcal{S}}(t)$ is nontrivial if and only if $\bar{\lambda}$ is positive. Even more, if $\bar{\lambda} = +\infty$, then $\boldsymbol{\mathcal{S}}(t)$ is nontrivial for all $t \geqslant T$ and if $\bar{\lambda} = -\infty$, then $\boldsymbol{\mathcal{S}}(t)$ is trivial for all $t \geqslant T$.

Proof. Suppose that $\bar{\lambda} > 0$. Hence, for every $0 < \lambda < \bar{\lambda}$ there is a point $x \in \mathbb{T}^n$, $x \neq -\infty$ such that

$$\forall (i,j) \in [m]^2, \ \lambda + Q_{ij}^{-}(x) \leqslant \frac{1}{2}Q_{ii}^{+}(x) + \frac{1}{2}Q_{jj}^{+}(x) .$$

Thus, we have $t^{2\lambda}t^{2Q_{ij}^{-}(x)} \leqslant t^{Q_{ii}^{+}(x)}t^{Q_{jj}^{+}(x)}$. Take the point $\boldsymbol{x} = (t^{x_1}, \dots, t^{x_n})$, where $t^{-\infty} = 0$. By Lemma 8.3 we have

$$\forall (i,j) \in [m]^2, \ \boldsymbol{Q}_{ij}^{-}(t)(\boldsymbol{x}) \leqslant \sum_{Q_{ij}^{(k)} \in \mathbb{T}_{-}} t^{|Q_{ij}^{(k)}|+\delta(t)+x_k} \leqslant nt^{Q_{ij}^{-}(x)+\delta(t)}$$

and

$$\forall i \in [m], \ \boldsymbol{Q}_{ii}^{+}(t)(\boldsymbol{x}) \geqslant \sum_{Q_{ii}^{(k)} \in \mathbb{T}_{+}} t^{Q_{ii}^{(k)}-\delta(t)+x_k} \geqslant t^{Q_{ii}^{+}(x)-\delta(t)}.$$

Therefore, for all $i \in [m]$ we have

$$\boldsymbol{Q}_{ii}^{+}(t)(\boldsymbol{x}) \geqslant t^{Q_{ii}^{+}(x)-\delta(t)} \geqslant t^{\lambda+Q_{ii}^{-}(x)-\delta(t)} \geqslant \frac{t^{\lambda-2\delta(t)}}{n}\boldsymbol{Q}_{ii}^{-}(t) .$$

By our definition of t, we can choose $\lambda < \bar{\lambda}$ that satisfies $t \geqslant (2n)^{1/(\lambda-2\delta(t))}$. Thus, we have $\boldsymbol{Q}_{ii}^{+}(t)(\boldsymbol{x}) \geqslant 2\boldsymbol{Q}_{ii}^{-}(t)$ and hence $\boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant \frac{1}{2}\boldsymbol{Q}_{ii}^{+}(\boldsymbol{x})$. Therefore, for any $i < j$ we have

$$\begin{aligned}
\boldsymbol{Q}_{ii}(\boldsymbol{x})\boldsymbol{Q}_{jj}(\boldsymbol{x}) &\geqslant \frac{1}{4}\boldsymbol{Q}_{ii}^{+}(\boldsymbol{x})\boldsymbol{Q}_{jj}^{+}(\boldsymbol{x}) \geqslant \frac{1}{4}t^{Q_{ii}^{+}(x)+Q_{jj}^{+}(x)-2\delta(t)} \\
&\geqslant \frac{1}{4}t^{2|Q_{ij}(x)|+2\lambda-2\delta(t)} \geqslant \frac{t^{2\lambda-4\delta(t)}}{4n^2}(\boldsymbol{Q}_{ij}(\boldsymbol{x}))^2 .
\end{aligned}$$

Moreover, by augmenting $\lambda < \bar{\lambda}$ if necessary, we can suppose that $t \geqslant (2(m-1)n)^{1/(\lambda-2\delta(t))}$. Hence, we have $(\boldsymbol{Q}_{ii}(t)(\boldsymbol{x}))(\boldsymbol{Q}_{jj}(t)(\boldsymbol{x})) \geqslant (m-1)^2(\boldsymbol{Q}_{ij}(t)(\boldsymbol{x}))^2$ for all $i < j$ and the point $\boldsymbol{x}$

belongs to $\boldsymbol{\mathcal{S}}(t)$ by Lemma 4.8 (as noted in the proof of this lemma, it holds over any real closed field).

Conversely, suppose that $\bar{\lambda}$ is negative, but $\boldsymbol{\mathcal{S}}(t)$ is nontrivial. Take any nonzero point $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}(t)$ and a point $x \in \mathbb{T}^n$ defined as $x_k = \log_t(\boldsymbol{x}_k)$ for all $k \in [n]$ (where $\log_t(0) = -\infty$). Since $\bar{\lambda}$ is negative, for every $\underline{\lambda} < \lambda < 0$ there is a pair $(i,j) \in [m]^2$ such that

$$\lambda + Q_{ij}^-(x) > \frac{1}{2}Q_{ii}^+(x) + \frac{1}{2}Q_{jj}^+(x)\,.$$

Hence $t^{2\lambda}t^{2Q_{ij}^-(x)} > t^{Q_{ii}^+(x)}t^{Q_{jj}^+(x)}$. Similarly to the previous case, observe that we have

$$\boldsymbol{Q}_{ij}^-(t)(\boldsymbol{x}) \geqslant \sum_{Q_{ij}^{(k)} \in \mathbb{T}_-} t^{Q_{ij}^{(k)} - \delta(t) + x_k} \geqslant t^{Q_{ij}^-(x) - \delta(t)}$$

and

$$\boldsymbol{Q}_{ii}^+(t)(\boldsymbol{x}) \leqslant \sum_{Q_{ii}^{(k)} \in \mathbb{T}_+} t^{Q_{ii}^{(k)} + \delta(t) + x_k} \leqslant n t^{Q_{ii}^+(x) + \delta(t)}\,.$$

Therefore, we have

$$t^{2\lambda + 2\delta(t)}(\boldsymbol{Q}_{ij}^-(t)(\boldsymbol{x}))^2 > t^{Q_{ii}^+(x)}t^{Q_{jj}^+(x)} \geqslant \frac{t^{-2\delta(t)}}{n^2}(\boldsymbol{Q}_{ii}^+(t)(\boldsymbol{x}))(\boldsymbol{Q}_{jj}^+(t)(\boldsymbol{x}))\,.$$

Thus, if we take λ such that $\lambda + 2\delta(t) < 0$ and $t \geqslant n^{1/|\lambda + 2\delta(t)|}$, then we have $(\boldsymbol{Q}_{ij}^-(t)(\boldsymbol{x}))^2 > (\boldsymbol{Q}_{ii}^+(t)(\boldsymbol{x}))(\boldsymbol{Q}_{jj}^+(t)(\boldsymbol{x}))$, which gives a contradiction. □

Remark 8.5. It is easy to see from the proof that if the matrices $\boldsymbol{Q}^{(k)}$ are diagonal (which holds, in particular, if $m = 1$), then the term $2(m-1)$ is not needed, and the bound for t takes the form $t > n^{1/(|\bar{\lambda}| - 2\delta(t))}$.

Remark 8.6. We recall that if the matrices $Q^{(1)}, \dots, Q^{(n)}$ define a well-formed linear matrix inequality, then $\bar{\lambda}$ is equal to $2\bar{\chi}$, where $\bar{\chi}$ is the value of the associated stochastic mean payoff game (see Section 7.2.1).

In the light of Theorem 8.4, the number $1/\bar{\lambda}$ can be thought of as a condition number of the nonarchimedean problem—the smaller it is, the smaller is the value of the parameter t that turns the problem into a nonarchimedean one. In the next section, we give a geometric interpretation of this value.

8.1.1 Geometric interpretation of the condition number

In this section, we give a geometric interpretation of the number $\bar{\lambda}$. More precisely, we relate this number to with the radius of the largest ball (in the Hilbert seminorm) contained in a certain tropical Metzler spectrahedral cone in dimension $2n$. We start with the definition of the Hilbert seminorm and the balls in this seminorm.

Definition 8.7. We define the *Hilbert seminorm* of a vector $x \in \mathbb{R}^n$ as $\|x\|_{\mathrm{H}} := \mathbf{t}(x) - \mathbf{b}(x)$, where $\mathbf{t}(x) := \max_{k \in [n]} x_k$ and $\mathbf{b}(x) := \min_{k \in [n]} x_k$.

Lemma 8.8. *Hilbert seminorm is, indeed, a seminorm. More precisely, it satisfies the conditions* $\|x\|_{\mathrm{H}} \geqslant 0$, $\|\lambda x\|_{\mathrm{H}} = |\lambda| x$, *and* $\|x + y\|_{\mathrm{H}} \leqslant \|x\|_{\mathrm{H}} + \|y\|_{\mathrm{H}}$ *for all* $x, y \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$.

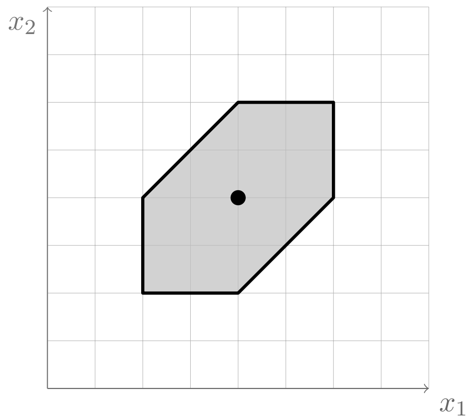

Figure 8.1: Hilbert ball (for $x_0 = 0$).

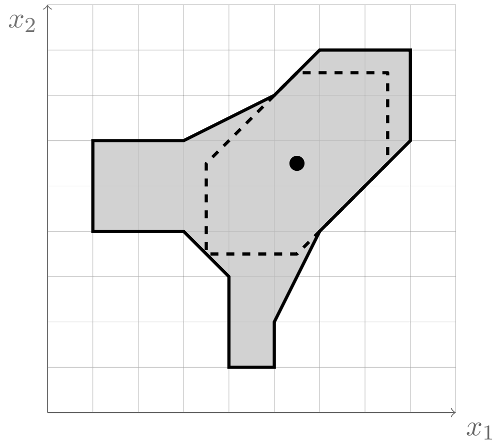

Figure 8.2: Tropical spectrahedron and its maximal inner Hilbert ball (for $x_0 = 0$).

Proof. The first two conditions are trivially satisfied. The prove the third one, note that for all $k, l \in [n]$ we have

$$|x_k + y_k - x_l - y_l| \leqslant |x_k - x_l| + |y_k - y_l| \, .$$

Hence,

$$\|x + y\|_{\mathrm{H}} = \max_{k,l \in [n]} |x_k + y_k - x_l - y_l| \leqslant \max_{k,l \in [n]} |x_k - x_l| + \max_{k,l \in [n]} |y_k - y_l| = \|x\|_{\mathrm{H}} + \|y\|_{\mathrm{H}} \, . \quad \square$$

Remark 8.9. We point out that $\| \cdot \|_{\mathrm{H}}$ is not a norm, because it does not satisfy the condition $\|x\|_{\mathrm{H}} = 0 \iff x = 0$. Indeed, we have $\|x\|_{\mathrm{H}} = 0$ if and only if x has equal coordinates, $x = \lambda(1, 1, \dots, 1)$.

Definition 8.10. If $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_{\geqslant 0}$, then we define the *Hilbert ball* centered at x with radius r, denoted $B_{\mathrm{H}}(x, r)$, as

$$B_{\mathrm{H}}(x, r) \coloneqq \{ y \in \mathbb{R}^n \colon \|x - y\|_{\mathrm{H}} \leqslant r \} \, .$$

If $x \in \mathbb{T}^n$ is a point with nonempty support $K \subset [n]$, then we extend this definition by putting

$$B_{\mathrm{H}}(x, r) \coloneqq \{ y \in \mathbb{T}^n \colon \text{the support of } y \text{ is equal to } K \text{ and } \|x_K - y_K\|_{\mathrm{H}} \leqslant r \} \, .$$

Since $\| \cdot \|_{\mathrm{H}}$ is invariant by adding a the same constant to all coordinates, Hilbert balls are unbounded.

Example 8.11. Figure 8.1 depicts a Hilbert ball (its center is marked by a dot). Figure 8.2 depicts the tropical Metzler spectrahedral cone from Example 4.15 and the largest Hilbert ball that is included inside this cone. This example also shows that such ball is not unique (because one can slide the center of the ball in the direction of the top right corner).

In the next proposition we show that Hilbert balls are tropical cones. This proposition is well known, as it is a particular application of the tropical spectral theory, see e.g. [BCOQ92, Chapter 3.7], [Ser07]. We present a self-contained proof for the sake of completeness.

Proposition 8.12. *The set $B_{\mathrm{H}}(x,r)\cup\{-\infty\}$ is a tropical cone. Moreover, we have the equality*

$$B_{\mathrm{H}}(x,r)\cup\{-\infty\} = \{\bigoplus_{k\in[n]} \lambda_k \odot (r\epsilon_k + x) \colon \lambda_1,\dots,\lambda_n \in \mathbb{T}\}\,, \tag{8.1}$$

where ϵ_k denotes the kth vector of the standard basis in $\mathbb{R}^n$.

Proof. Let $K\subset[n]$ denote the support of x. To prove that $B_{\mathrm{H}}(x,r)\cup\{-\infty\}$ is a tropical cone, we first show that $B_{\mathrm{H}}(x,r)\cup\{-\infty\}$ is equal to the set of all $y\in\mathbb{T}^n$ that satisfy the following system of inequalities:

$$\begin{aligned} \forall k,l\in K, k\neq l,\ r+y_k &\geqslant (x_k-x_l)+y_l \\ \forall k\notin K,\ y_k &\geqslant -\infty \\ \forall k\notin K,\ -\infty &\geqslant y_k\,. \end{aligned} \tag{8.2}$$

Indeed, $y=-\infty$ satisfies this system. Moreover, if $y\in B_{\mathrm{H}}(x,r)$, then $y_k=-\infty$ for all $k\notin K$. Furthermore, for all $k,l\in K$ we have $|x_k-y_k-x_l+y_l|\leqslant r$, which implies that $(x_k-x_l)+y_l\leqslant r+y_k$. Conversely, if $y\in\mathbb{T}^n$ satisfies the system above, then it is either equal to $-\infty$, or its support is equal to K. Indeed, its support cannot be greater than K by the last two sets of inequalities. Moreover, if the support of y is nonempty but smaller than K, then there are two coordinates $k,l\in K$ such that $y_k=-\infty$ and $y_l\neq-\infty$, which gives a contradiction with the first inequality in (8.2). If support of y is equal to K, then the inequality $r+y_k\geqslant(x_k-x_l)+y_l$ implies $x_k-y_k-x_l+y_l\leqslant r$ and, by exchanging k and l, $|x_k-y_k-x_l+y_l|\leqslant r$. Thus

$$\|x_K-y_K\|_{\mathrm{H}} = \max_{k,l\in K}|x_k-y_k-x_l+y_l|\leqslant r$$

and $y\in B_{\mathrm{H}}(x,r)$. Hence, the system of inequalities (8.2) describes $B_{\mathrm{H}}(x,r)\cup\{-\infty\}$. Moreover, it is immediate to check that each of these inequalities defines a tropical cone. Hence, $B_{\mathrm{H}}(x,r)\cup\{-\infty\}$ is also a tropical cone.

To prove the second part of the claim, fix $l\in[n]$ and consider the point $y^{(l)}:=r\epsilon_l+x$. Note that the support of y coincides with K. Furthermore, for all $k',l'\in K$ we have

$$|x_{k'}-y^{(l)}_{k'}-x_{l'}+y^{(l)}_{l'}| = \begin{cases} r & \text{if } k'\neq l' \text{ and } l\in\{k',l'\}\\ 0 & \text{otherwise}\,. \end{cases}$$

Hence $\|x_K-y^{(l)}_K\|_{\mathrm{H}} = \max_{k',l'\in K}|x_{k'}-y^{(l)}_{k'}-x_{l'}+y^{(l)}_{l'}|\leqslant r$ and $y^{(l)}\in B_{\mathrm{H}}(x,r)$. Therefore, using the fact that $B_{\mathrm{H}}(x,r)\cup\{-\infty\}$ is a tropical cone, we have

$$\{\bigoplus_{k\in[n]}\lambda_k\odot(r\epsilon_k+x)\colon\lambda_1,\dots,\lambda_n\in\mathbb{T}\}\subset B_{\mathrm{H}}(x,r)\cup\{-\infty\}\,.$$

To prove the opposite inclusion, take $y\in B_{\mathrm{H}}(x,r)$ and let $\lambda_l:=y_l-x_l-r$ for all $l\in K$ and $\lambda_l:=-\infty$ otherwise. Then, for any $l\in K$ the vector $\lambda_l\odot(r\epsilon_l+x)$ is given by

$$\big(\lambda_l\odot(r\epsilon_l+x)\big)_k = \begin{cases} y_k & \text{if } k=l\\ y_l-x_l+x_k-r & \text{if } k\neq l, k\in K\\ -\infty & \text{if } k\notin K\,. \end{cases}$$

By (8.2) we have $y_k\geqslant\max_{l\in K}(y_l-x_l+x_k-r)$ for all $k\in K$. Hence $y=\bigoplus_{k\in[n]}\lambda_k\odot(r\epsilon_k+x)$. $\square$

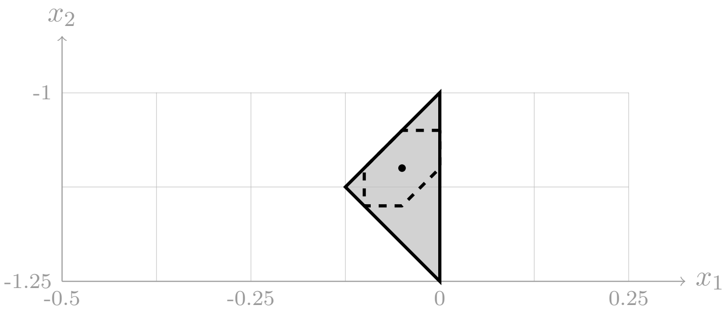

Figure 8.3: The largest inner Hilbert ball of the tropical spectrahedron from Example 7.22.

We can now interpret the quantity $\bar{\lambda}$ is terms of the Hilbert balls included inside a spectrahedron. As before, let $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}^{m \times m}$ be a sequence of symmetric Metzler matrices and let

$$\bar{\lambda} \coloneqq \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)}) \text{ is nontrivial}\} .$$

Lemma 8.13. *Suppose that $\bar{\lambda} \geqslant 0$. Then, for every $\lambda \in \mathbb{R}$ such that $0 \leqslant \lambda \leqslant \bar{\lambda}$, the set $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ contains a Hilbert ball of radius λ.*

Proof. Let $\mathcal{S} \coloneqq \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ and let $x \in \mathbb{T}^n \setminus \{-\infty\}$ be a point that belongs to the set $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ (we recall that if $\bar{\lambda}$ is finite, then such a point exists even for $\lambda = \bar{\lambda}$, see Theorem 7.18). Therefore, for every $i \in [m]$ we have $Q_{ii}^{+}(x) \geqslant \lambda \odot Q_{ii}^{-}(x)$ and for every pair $i, j \in [m]$, $i < j$ we have $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2}$. Fix $k \in [n]$ and consider the point $y \coloneqq \lambda \epsilon_k + x$. By the definition of y, for every $i, j \in [m]$, $i \leqslant j$ we have $Q_{ij}^{-}(y) \leqslant \lambda \odot Q_{ij}^{-}(x)$ and $Q_{ii}^{+}(y) \geqslant Q_{ii}^{+}(x)$. Hence $y \in \mathcal{S}$. Since the choice of k was arbitrary and $\mathcal{S}$ is a tropical cone (as proved in Example 5.12), Proposition 8.12 shows that $B_{\mathrm{H}}(x, \lambda) \subset \mathcal{S}$. □

The next example shows that the converse of Lemma 8.13 is, in general, false.

Example 8.14. Consider the tropical Metzler spectrahedral cone $\mathcal{S}$ from Example 7.22. As noted in Example 6.2, the associated game has constant value and this value is given by $\chi = 1/56$. Hence, by Theorem 7.21 we have $\bar{\lambda} = 1/28$. However, the largest Hilbert ball contained in $\mathcal{S}$ has radius $1/20$, as depicted in Fig. 8.3. The ball is centered at $(-1/20, -11/10, 0)$.

Even though the converse of Lemma 8.13 does not hold in general, it it true if the entries of the matrices $Q^{(1)}, \dots, Q^{(n)}$ have a special sign pattern.

Assumption F. For every $k \in [n]$ and every pair $i, j \in [m]$ such that $i < j$ and $|Q_{ij}^{(k)}|$ is finite, we have $Q_{ii}^{(k)} = Q_{jj}^{(k)} = -\infty$.

Lemma 8.15. *Suppose that the matrices $Q^{(1)}, \dots, Q^{(n)}$ satisfy Assumption F. Then, the set $\mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ contains a Hilbert ball of radius $\lambda \in \mathbb{R}_{\geqslant 0}$ if and only if $\mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ is nontrivial.*

Proof. The "if" part of the claim follows from Lemma 8.13. To prove the opposite inclusion, denote $\mathcal{S} \coloneqq \mathcal{S}(Q^{(1)}, \dots, Q^{(n)})$ and suppose that $B_{\mathrm{H}}(x, \lambda) \subset \mathcal{S}$. We will show that $x \in \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$. To do so, first take any $i \in [m]$ and consider the inequality $Q_{ii}^{+}(x) \geqslant Q_{ii}^{-}(x)$. Take $k \in [n]$ such that $Q_{ii}^{-}(x) = Q_{ii}^{(k)} + x_k$ and consider the point $y \coloneqq \lambda \epsilon_k + x$. By our choice of k we have $Q^+_{ii}(y) = Q^+_{ii}(x)$ and $Q^-_{ii}(y) = \lambda \odot Q^-_{ii}(x)$. Since $y \in B_{\mathrm{H}}(x,\lambda)$, we have $y \in \mathcal{S}$ and hence $Q^+_{ii}(x) \geqslant \lambda \odot Q^-_{ii}(x)$. Similarly, for all $i,j \in [m]$, $i<j$ we consider the inequality $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant (Q_{ij}(x))^{\odot 2}$. If $Q_{ij}(x) = -\infty$, then we trivially have $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2}$. Otherwise, we take $k \in [n]$ such that $Q_{ij}(x) = Q^{(k)}_{ij} + x_k \neq -\infty$ and set $y \coloneqq \lambda\epsilon_k + x$. By Assumption F, we have $Q^+_{ii}(y) = Q^+_{ii}(x)$, $Q^+_{jj}(y) = Q^+_{jj}(x)$, and $Q_{ij}(y) = \lambda \odot Q_{ij}(x)$. As previously, this gives $Q^+_{ii}(x) \odot Q^+_{jj}(x) \geqslant (\lambda \odot Q_{ij}(x))^{\odot 2}$. Hence $x \in \mathcal{S}_\lambda(Q^{(1)},\dots,Q^{(n)})$. □

Given Lemma 8.15, we can now take arbitrary symmetric Metzler matrices $Q^{(1)},\dots,Q^{(n)}$ and construct another tropical Metzler spectrahedral cone such that $\bar{\lambda}$ corresponds to the radius of its largest Hilbert ball. To do so, we define the matrices $\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)} \in \mathbb{T}_\pm^{(m+n)\times(m+n)}$ as

$$\tilde{Q}^{(k)}_{ij} \coloneqq \begin{cases} Q^{(k)}_{ij} & \text{if } i,j \leqslant m,\ k \leqslant n, \text{ and } Q^{(k)}_{ij} \in \mathbb{T}_-\,,\\ Q^{(k-n)}_{ij} & \text{if } i = j \leqslant m,\ k > n, \text{ and } Q^{(k-n)}_{ij} \in \mathbb{T}_+\,,\\ 0 & \text{if } i = j = m+k \text{ and } k \leqslant n\,,\\ \ominus 0 & \text{if } i = j = m+k-n \text{ and } k > n\,,\\ -\infty & \text{otherwise}\,. \end{cases}$$

In this way, the set $\mathcal{S}_\lambda(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$ is described by the constraints

$$\begin{aligned} \forall i \leqslant m,\ Q^+_{ii}(x_{n+1},\dots,x_{2n}) \geqslant \lambda \odot Q^-_{ii}(x_1,\dots,x_n)\,,\\ \forall k \leqslant n,\ x_k \geqslant \lambda \odot x_{n+k}\\ \forall i,j \in [m], i<j,\ Q^+_{ii}(x_{n+1},\dots,x_{2n}) \odot Q^+_{jj}(x_{n+1},\dots,x_{2n}) \geqslant (\lambda \odot Q^-_{ij}(x_1,\dots,x_n))^{\odot 2}\,. \end{aligned} \tag{8.3}$$

Furthermore, the matrices $\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)}$ satisfy Assumption F.

Proposition 8.16. *The set $\mathcal{S}(Q^{(1)},\dots,Q^{(n)})$ is a projection of $\mathcal{S}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$ on the first n coordinates. Furthermore, for any $\lambda > 0$, the set $\mathcal{S}_\lambda(Q^{(1)},\dots,Q^{(n)})$ is nontrivial if and only if $\mathcal{S}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$ contains a Hilbert ball of radius $\lambda/2$.*

Proof. Denote $\mathcal{S} \coloneqq \mathcal{S}(Q^{(1)},\dots,Q^{(n)})$ and $\tilde{\mathcal{S}} \coloneqq \mathcal{S}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$. To prove that $\mathcal{S}$ is the projection of $\tilde{\mathcal{S}}$, note that for every $i \in [m]$ we have $Q^+_{ii}(x_1,\dots,x_n) \geqslant Q^+_{ii}(x_{n+1},\dots,x_{2n})$. Hence, $\mathcal{S}$ is included in the projection of $\tilde{\mathcal{S}}$. Moreover, for any $(x_1,\dots,x_n) \in \mathcal{S}$ we can define $x_{n+k} \coloneqq x_k$ for all $k \in [n]$ and the point $(x_1,\dots,x_{2n})$ belongs to $\tilde{\mathcal{S}}$. This proves the first claim. To prove the second one, suppose that $\mathcal{S}_\lambda \coloneqq \mathcal{S}_\lambda(Q^{(1)},\dots,Q^{(n)})$ is nontrivial, take a point $x \in \mathcal{S}_\lambda \setminus \{-\infty\}$ and define $x_{n+k} \coloneqq -\frac{\lambda}{2} + x_k$ for all $k \in [n]$. Then $Q^+_{ii}(x_{n+1},\dots,x_{2n}) = -\frac{\lambda}{2} + Q^+_{ii}(x_1,\dots,x_n)$ for all $i \in [m]$. Hence, $(x_1,\dots,x_{2n})$ belongs to $\mathcal{S}_{\lambda/2}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$ and $\tilde{\mathcal{S}}$ contains a Hilbert ball of radius $\lambda/2$ by Lemma 8.15. Conversely, if $\tilde{\mathcal{S}}$ contains a Hilbert ball of radius $\lambda/2$, then $\mathcal{S}_{\lambda/2}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$ is nontrivial by Lemma 8.15 and the fact that the matrices $\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)}$ satisfy Assumption F. Take a point $x \in \mathbb{T}^{2n}\setminus\{-\infty\}$ that belongs to $\mathcal{S}_{\lambda/2}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$. Then, we have $Q^+_{ii}(x_1,\dots,x_n) \geqslant \frac{\lambda}{2} + Q^+_{ii}(x_{n+1},\dots,x_{2n})$ and $(x_1,\dots,x_n)$ belongs to $\mathcal{S}_\lambda$. □

Remark 8.17. The proof shows that every point in $\mathcal{S}_\lambda(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)}) \setminus \{-\infty\}$ is the projection of the center of some Hilbert ball of radius $\lambda/2$ contained in $\mathcal{S}(\tilde{Q}^{(1)},\dots,\tilde{Q}^{(2n)})$.

8.2 Value iteration for constant value games

In the previous sections we showed a link between nonarchimedean semidefinite feasibility problems and stochastic mean payoff games. We now want to propose an algorithm that solves a class of these problems and analyze its complexity. Suppose that we are given a stochastic mean payoff game. Corollary 6.7 shows that the value of this game is equal to the limit $\lim_{n\to\infty} T^N(0)/N$, where T is the Shapley operator of the game. We want to turn this observation into a simple iterative algorithm that computes the value and propose a condition number for this algorithm.

Throughout this section, we will further suppose that the game has a constant value (i.e., such that its value does not depend on the initial state). In this case, we may consider this value to be a scalar (instead of a vector as discussed in Section 6.1) and denote it by $\eta \in \mathbb{R}$. Furthermore, the invariant half-line of a Shapley operator of such game has a simple form. More precisely, Corollary 6.7 shows that there exists $u \in \mathbb{R}^n$ such that $T(u) = \eta + u$. In other words, T has an additive eigenvalue.

Let us motivate the fact that we restrict ourselves to constant value games. First, by Lemma 7.16, this smaller class already captures the of complexity of general stochastic mean payoff games. However, the reduction from general games to ergodic games destroys a lot of the properties of these games, and it often a nontrivial task to extend an algorithm that works in the ergodic case to the general games (see [ACTDG12, BEGM15] for two such examples). In this way, supposing that the game is ergodic can significantly simplify the analysis. Furthermore, the remarks in the previous paragraph show that Shapley operators of constant value games belong to a wider class of functions—monotone and homogenous operators that have an additive eigenvalue. Operators of this type arise also in other classes of games, such as the entropy games of [ACD$^+$16, AGGCG17] (and the operators arising in these games are not piecewise-affine). Therefore, to cover these classes of games, in this section we consider a general operator $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that fulfills the following assumption.

Assumption G. We suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is monotone and (additively) homogeneous. Furthermore, we suppose that f has an additive eigenvalue, i.e., that there exists a couple $(\eta, u) \in \mathbb{R} \times \mathbb{R}^n$ such that $f(u) = \eta + u$.

As previously, we refer to the vector u as a *bias* vector of f. The following lemma is an improved version of Corollary 6.7.

Lemma 8.18. *For every $N \geqslant 1$ we have*

$$\| \frac{f^N(0)}{N} - \eta \|_\infty \leqslant \frac{\|u\|_{\mathrm{H}}}{N} \, .$$

Proof. We have $u - \mathbf{b}(u) \geqslant 0 \geqslant u - \mathbf{t}(u)$. Hence, for every $N \geqslant 1$ we have

$$-\mathbf{b}(u) + f^N(u) \geqslant f^N(0) \geqslant -\mathbf{t}(u) + f^N(u) \, .$$

Since $f^N(u) = N\eta + u$, we get $u - \mathbf{b}(u) \geqslant f^N(0) - N\eta \geqslant u - \mathbf{t}(u)$. Thus $\|f^N(0) - N\eta\|_\infty \leqslant \|u\|_{\mathrm{H}}$. □

Remark 8.19. Since f is nonexpansive in the supremum norm (Lemma 5.31), we have $\|\frac{f^N(x)}{N} - \frac{f^N(0)}{N}\|_\infty \leqslant \frac{\|x\|_\infty}{N}$ for all $x \in \mathbb{R}^n$ and $N \geqslant 1$ and the result of Lemma 8.18 implies that $\lim_{N\to\infty} f^N(x)/N = \eta$.

In order to create algorithms, it is useful to consider the sequences $\mathbf{t}(f^N(0))$ and $\mathbf{b}(f^N(0))$. This requires the following definition.

Definition 8.20. We say that a sequence $(a_n)_{n\geqslant 1}$ of real numbers is *subadditive* if $a_{n+m} \leqslant a_n + a_m$ for all $n, m \geqslant 1$. Similarly, we say that $(a_n)_{n\geqslant 1}$ is *superadditive* if $a_{n+m} \geqslant a_n + a_m$ for all $n, m \geqslant 1$.

Lemma 8.21. *The sequence* $\Big(\mathbf{t}(f^N(0))\Big)_N$ *is subadditive and the sequence* $\Big(\mathbf{b}(f^N(0))\Big)_N$ *is superadditive.*

Proof. We prove the claim for $\mathbf{t}(f^N(0))$ (the other part is analogous). Since f is monotone and homogeneous, we have

$$f^{N+M}(0) = f^N(f^M(0)) \leqslant f^N\Big(\mathbf{t}(f^M(0))\Big) = f^N(0) + \mathbf{t}(f^M(0)) \leqslant \mathbf{t}(f^N(0)) + \mathbf{t}(f^M(0)) . \quad \square$$

The following lemma is a basic property of subadditive sequences.

Lemma 8.22 (Fekete's lemma, [Sch03, Theorem 2.2])**.** *If the sequence* $(a_n)_{n\geqslant 1}$ *is subadditive, then*

$$\inf_{n\geqslant 1} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n}{n} .$$

In particular, the limit on the right-hand side exists (it may be equal to $-\infty$*). Similarly, if* $(a_n)_{n\geqslant 1}$ *is superadditive, then* $\sup_{n\geqslant 1} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n}{n}$.

As a corollary, we get the following result.

Corollary 8.23. *For every* $N \geqslant 1$ *we have the inequality*

$$\eta - \frac{\|u\|_{\mathrm{H}}}{N} \leqslant \frac{\mathbf{b}(f^N(0))}{N} \leqslant \eta \leqslant \frac{\mathbf{t}(f^N(0))}{N} \leqslant \eta + \frac{\|u\|_{\mathrm{H}}}{N} .$$

Proof. The inequality

$$\frac{\mathbf{t}(f^N(0))}{N} \leqslant \eta + \frac{\|u\|_{\mathrm{H}}}{N}$$

follows from Lemma 8.18. In particular, we have

$$\eta \geqslant \limsup_{N\to\infty} \frac{\mathbf{t}(f^N(0))}{N} \geqslant \liminf_{N\to\infty} \frac{\mathbf{t}(f^N(0))}{N} \geqslant \liminf_{N\to\infty} \frac{(f^N(0))_1}{N} = \eta .$$

Hence $\lim_{N\to\infty} \mathbf{t}(f^N(0))/N = \eta$. Therefore, Lemmas 8.21 and 8.22 show that $\mathbf{t}(f^N(0))/N \geqslant \eta$ for all N. The proof of the remaining inequalities is analogous. $\square$

Remark 8.24. It is shown in [GG04, Theorem 8] that a stronger statement holds—namely, there exist fixed coordinates $k, l \in [n]$ such that $(f^N(0))_k/N \geqslant \eta \geqslant (f^N(0))_l/N$ for all $N \geqslant 1$. However, we do not need this property for our purposes.

In the next corollary we denote by R the infimum of the Hilbert seminorm of bias vectors of f, i.e.,

$$R := \inf\{\|u\|_{\mathrm{H}} \colon f(u) = \eta + u\} .$$

The following theorem estimates the complexity of a value iteration algorithm.

Theorem 8.25. *For every $N \geqslant 1$, if $\mathbf{b}(f^N(0)) > 0$, then $\eta > 0$. Similarly, if $\mathbf{t}(f^N(0)) < 0$, then $\eta < 0$. Even more, if $\eta \neq 0$, then one of these two possibilities arises for some $N \leqslant \lceil 1 + R/|\eta| \rceil$. In particular, if η is a rational number with denominator D and we have $\mathbf{b}(f^N(0)) \leqslant 0 \leqslant \mathbf{t}(f^N(0))$ for all $N \leqslant \lceil 1 + DR \rceil$, then $\eta = 0$.*

Proof. The first part is obvious from Corollary 8.23. To prove the second one, suppose that $\eta \neq 0$ and let $u \in \mathbb{R}^n$ be a bias vector such that $\|u\|_{\mathrm{H}} \leqslant R + \frac{1}{2}|\eta|$. Furthermore, let $N = \lceil 1 + R/|\eta| \rceil$. Then $N > \|u\|_{\mathrm{H}}/|\eta|$. Therefore, for $\eta > 0$ Corollary 8.23 shows that

$$\frac{\mathbf{b}(f^N(0))}{N} \geqslant \eta - \frac{\|u\|_{\mathrm{H}}}{N} > 0 \, .$$

Similarly, if $\eta < 0$, then $\mathbf{t}(f^N(0)) < 0$. If η is rational and has denominator D, then we have either $\eta = 0$ or $|\eta| \geqslant 1/D$ and the last claim follows. $\square$

In the light of Theorem 8.25, the quantities $R/|\eta|$ and DR may be thought of as natural condition numbers of the problem of finding the sign of a value. The first bound is more precise whenever $\eta \neq 0$. However, it is infinite when $\eta = 0$. On the other hand, we are able to give bounds on the number DR for the class of stochastic mean payoff games. As a result, this condition number can be used to develop an algorithm that solves stochastic mean payoff games even if their value is equal to 0. These estimates are given in Section 8.3. Before that, let us point out that a simple extension of the algorithm above can not only compute the sign of the value, but also produce a certificate that justifies its correctness.

Lemma 8.26 ([GG04, Lemma 2])**.** *Suppose that we are given $N \geqslant 1$ such that $\mathbf{b}(f^N(0))/N \geqslant \lambda$ and define*

$$\hat{u} \coloneqq 0 \oplus \Big(-\lambda + f(0)\Big) \oplus \cdots \oplus \Big(-(N-1)\lambda + f^{N-1}(0)\Big) \, .$$

Then $\lambda + \hat{u} \leqslant f(\hat{u})$.

Proof. Since f is order preserving, we have $f(x \oplus y) \geqslant f(x) \oplus f(y)$ and thus

$$f(\hat{u}) \geqslant f(0) \oplus \Big(-\lambda + f^2(0)\Big) \oplus \cdots \oplus \Big(-(N-1)\lambda + f^N(0)\Big) \, .$$

Moreover, we have $\mathbf{b}(f^N(0))/N \geqslant \lambda$ and hence

$$f(\hat{u}) \geqslant \lambda + \bigg(\Big(-\lambda + f(0)\Big) \oplus \Big(-2\lambda + f^2(0)\Big) \oplus \cdots \oplus \Big(-(N-1)\lambda + f^{N-1}(0)\Big) \oplus 0\bigg) \, .$$

In other words, $f(\hat{u}) \geqslant \lambda + \hat{u}$. $\square$

Remark 8.27. We point out that, as in Lemma 6.17, if $\lambda + \hat{u} \leqslant f(\hat{u})$, then $N\lambda + \hat{u} \leqslant f^N(\hat{u})$, and $\eta \geqslant \lambda$ by Remark 8.19. In particular, $\hat{u}$ certifies that the value η is not smaller than λ.

Remark 8.28. The result of Lemma 8.26 can be easily dualized. If $\mathbf{t}(f^N(0))/N \leqslant \lambda$, then we define

$$\check{u} \coloneqq \min\Big\{0, \Big(-\lambda + f(0)\Big), \dots, \Big(-(N-1)\lambda + f^{N-1}(0)\Big)\Big\} \, ,$$

and we get $\lambda + \check{u} \geqslant f(\check{u})$. This point certifies that the value of η is not greater than λ.

procedure ValueIteration(f)
  ▷ *we suppose that the value is nonzero,* $\eta \neq 0$
  $\nu := 0 \in \mathbb{R}^n$, $x := 0 \in \mathbb{R}^n$, $y := 0 \in \mathbb{R}^n$
  while `True` **do**
    $\nu := f(\nu)$
    if $\mathbf{b}(\nu) > 0$ **then**
      return $\eta > 0$ and $x \leqslant f(x)$
    end
    if $\mathbf{t}(\nu) < 0$ **then**
      return $\eta < 0$ and $y \geqslant f(y)$
    end
    $x := \max\{x, \nu\}$
    $y := \min\{y, \nu\}$
  ▷ *operations* max *and* min *are taken entrywise*
  done
end

Figure 8.4: Procedure that computes the sign of the value in exact arithmetic.

The results of this section can be summarized by the algorithm ValueIteration given in Fig. 8.4. In the model of computation in which the function f can be evaluated exactly, Theorem 8.25 and Lemma 8.26 show that this algorithm is correct and stops after at most $\lceil 1 + R/|\eta| \rceil$ iterations of the loop. In the next section we study a more realistic situation where one cannot (or does not want to) evaluate f exactly. Furthermore, we give a variant of the algorithm in which one computes a more precise approximation of η.

8.2.1 Oracle-based approximation algorithms

In this section we study the complexity of approximating the value η using an oracle that approximates f. There are two motivations to use oracles in order to approximate f. Firstly, one may be interested in studying operators f that cannot be evaluated exactly on a Turing machine (such as the operators associated with the entropy games). Secondly, even if the operator f can be evaluated exactly (as is the case of the Shapley operators of stochastic mean payoff games), using an approximation oracle may significantly reduce the amount of memory needed in the execution of the algorithm. Indeed, while a naive implementation of value iteration for the Shapley operators may require exponential memory, we will show in Section 8.4 that the use of a simple approximation oracle results in an algorithm that requires only a polynomial amount of memory. Throughout this section, we will suppose that an upper bound on R is known beforehand. In other words, we will denote by $R_{\mathrm{ub}} \in \mathbb{N}^*$ any positive natural number that satisfies $R_{\mathrm{ub}} \geqslant R$. Our results are based on the following lemma.

Lemma 8.29. *Let $\delta \geqslant 0$ and suppose that we are given a function $\tilde{f} \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\|\tilde{f}(x) - f(x)\|_\infty \leqslant \delta$ for all $x \in \mathbb{R}^n$. Then, for every $N \geqslant 1$ we have the inequality*

$$\|\frac{\tilde{f}^N(0)}{N} - \frac{f^N(0)}{N}\|_\infty \leqslant \delta \,.$$

Moreover, we also have

$$|\frac{\mathbf{t}(\tilde{f}^N(0))}{N} - \frac{\mathbf{t}(f^N(0))}{N}| \leqslant \delta \quad \textit{and} \quad |\frac{\mathbf{b}(\tilde{f}^N(0))}{N} - \frac{\mathbf{b}(f^N(0))}{N}| \leqslant \delta\,.$$

Proof. By induction and the fact that f is nonexpansive in the supremum norm (Lemma 5.31), we have

$$\begin{aligned}&\|\tilde{f}(\tilde{f}^{N-1}(0)) - f(f^{N-1}(0))\|_\infty \\ &\qquad \leqslant \|\tilde{f}(\tilde{f}^{N-1}(0)) - f(\tilde{f}^{N-1}(0))\|_\infty + \|f(\tilde{f}^{N-1}(0)) - f(f^{N-1}(0))\|_\infty \\ &\qquad \leqslant \delta + \|\tilde{f}^{N-1}(0) - f^{N-1}(0)\|_\infty \leqslant N\delta\,.\end{aligned}$$

To prove the second part of the claim, let $k,l \in [n]$ be such that $\mathbf{t}(\tilde{f}^N(0)) = \big(\tilde{f}^N(0)\big)_k$, $\mathbf{t}(f^N(0)) = \big(f^N(0)\big)_l$. Then

$$\mathbf{t}(\tilde{f}^N(0)) \leqslant N\delta + \big(f^N(0)\big)_k \leqslant N\delta + \mathbf{t}(f^N(0))$$

and

$$\mathbf{t}(f^N(0)) \leqslant N\delta + \big(\tilde{f}^N(0)\big)_l \leqslant N\delta + \mathbf{t}(\tilde{f}^N(0))\,.$$

The proof of the last inequality is analogous. ☐

Definition 8.30. We denote by $\textsc{Oracle}(f,x,K)$ an algorithm that, given $K \in \mathbb{N}^*$ and $x \in \mathbb{Q}^n$, outputs a rational vector $y \in \mathbb{Q}^n$ such that the denominator of every coordinate y_k is equal to K and $\|f(x) - y\|_\infty \leqslant 1/K$.

Remark 8.31. We point out that the oracle given above gives rise to a function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\tilde{f}(x) \coloneqq \begin{cases}\textsc{Oracle}(f,x,K) & \text{if } x \in \mathbb{Q}^n \\ f(x) & \text{otherwise}\,.\end{cases}$$

The function $\tilde{f}$ satisfies $\|f(x) - \tilde{f}(x)\|_\infty \leqslant 1/K$ for all $x \in \mathbb{R}^n$ and we use this function in the proofs below.

We now present value iteration algorithms that can decide the sign of the value η up to a given precision. Our first algorithm is ValueSign presented in Fig. 8.5. It decides the sign but does not produce a certificate that justifies its claim. A modification, ValueSignCertified that produces such a certificate is given in Fig. 8.6. However, in order to produce the certificate the algorithm ValueSignCertified requires a much more accurate oracle that ValueSign.

Lemma 8.32. *The procedures* ValueSign *and* ValueSignCertified *are correct. Moreover, if* $|\eta| \geqslant 1/K$*, then they finish in at most* $\lceil 1 + 2R/|\eta| \rceil$ *iterations of the loop.*

Proof. First, let us prove that both procedures correctly recognize the sign of η. Let $\varepsilon > 0$ and let $u \in \mathbb{R}^n$ be a bias of f such that $\|u\|_{\mathrm{H}} \leqslant R + \varepsilon$. For both procedures, by Corollary 8.23 and Lemma 8.29, for all $N \geqslant 1$ we have

$$-\frac{1}{4K} + \frac{\mathbf{b}(\tilde{f}^N(0))}{N} \leqslant \eta \leqslant \frac{\mathbf{t}(\tilde{f}^N(0))}{N} + \frac{1}{4K}\,.$$

procedure ValueSign(f, K)
  $\nu \coloneqq 0 \in \mathbb{R}^n$
  for $i = 1, 2, \dots, 2KR_{\text{ub}}$ **do**
    $\nu \coloneqq$ Oracle$(f, \nu, 4K)$
    if $\mathbf{b}(\nu) \geqslant \frac{1}{2K}$ **then**
      the value is positive, $\eta > 0$
    end
    if $\mathbf{t}(\nu) \leqslant -\frac{1}{2K}$ **then**
      the value is negative, $\eta < 0$
    end
  done
  the value η belongs to the interval $[-\frac{1}{K}, \frac{1}{K}]$
end

Figure 8.5: Procedure that computes the sign of the value.

procedure ValueSignCertified(f, K)
  $\nu \coloneqq 0 \in \mathbb{R}^n$, $x \coloneqq 0 \in \mathbb{R}^n$, $y \coloneqq 0 \in \mathbb{R}^n$
  $\lambda \coloneqq \frac{1}{3K}$, $\mu \coloneqq -\frac{1}{3K}$
  for $i = 1, 2, \dots, 2KR_{\text{ub}}$ **do**
    $\nu \coloneqq$ Oracle$(f, \nu, 12K^2 R_{\text{ub}})$
    if $\mathbf{b}(\nu) \geqslant \frac{1}{2K}$ **then**
      the value η is positive and we have $\frac{1}{6K} + x \leqslant f(x)$
    end
    if $\mathbf{t}(\nu) \leqslant -\frac{1}{2K}$ **then**
      the value η is negative and we have $-\frac{1}{6K} + y \geqslant f(y)$
    end
    $x \coloneqq \max\{x, -(i-1)\lambda + \nu\}$
    $y \coloneqq \min\{y, -(i-1)\mu + \nu\}$
  ▷ *operations* max *and* min *are taken entrywise*
  done
  the value η belongs to the interval $[-\frac{1}{K}, \frac{1}{K}]$
end

Figure 8.6: Procedure that computes the sign of the value and certifies its claim.

This proves that the stopping conditions inside the loop provide correct information about the sign of η. Moreover, if the algorithm terminates after $N = 2KR_{\mathrm{ub}}$ iterations of the loop, then we have

$$\eta \leqslant \frac{\mathbf{b}(\tilde{f}^N(0))}{N} + \frac{\|u\|_{\mathrm{H}}}{N} + \frac{1}{4K} \leqslant \frac{\|u\|_{\mathrm{H}}}{N} + \frac{1}{2K} \leqslant \frac{R+\varepsilon}{2KR} + \frac{1}{2K} = \frac{1}{K} + \frac{\varepsilon}{2KR}\,.$$

Since the choice of $\varepsilon > 0$ is arbitrary, we have $\eta \leqslant 1/K$. Similarly, $\eta \geqslant -1/K$.

Second, suppose that $|\eta| \geqslant 1/K$. If $\eta > 0$, then for $N = \lceil 1 + 2R/|\eta| \rceil$ we have

$$\frac{\mathbf{b}(\tilde{f}^N(0))}{N} \geqslant \eta - \frac{\|u\|_{\mathrm{H}}}{N} - \frac{1}{4K} \geqslant \frac{1}{2}\eta - \frac{1}{4K} - \frac{\varepsilon\eta}{2R} \geqslant \frac{1}{4K} - \frac{\varepsilon\eta}{2R}\,.$$

Since ε is arbitrary, we obtain the claim. The case $\eta < 0$ is analogous.

It remains to prove that the vectors x, y constructed by ValueSignCertified have the desired properties. To do so, let $\delta \coloneqq (12K^2R_{\mathrm{ub}})^{-1}$ and suppose that $\mathbf{b}(\tilde{f}^N(0))/N \geqslant \frac{1}{2K}$ for some $N \leqslant 2KR_{\mathrm{ub}}$. Then, we have

$$\frac{\mathbf{b}(f^N(0))}{N} \geqslant \frac{1}{2K} - \delta \geqslant \lambda + \delta$$

because $\frac{1}{2K} - \lambda = \frac{1}{6K} \geqslant 2\delta = \frac{1}{6K^2R_{\mathrm{ub}}}$. Moreover, by Lemma 8.29 we have

$$\begin{aligned} x &= 0 \oplus \Big(-\lambda + \tilde{f}(0)\Big) \oplus \cdots \oplus \Big(-(N-1)\lambda + \tilde{f}^{N-1}(0)\Big) \\ &\geqslant 0 \oplus \Big(-(\lambda+\delta) + f(0)\Big) \oplus \cdots \oplus \Big(-(N-1)(\lambda+\delta) + f^{N-1}(0)\Big)\,. \end{aligned}$$

Let $\hat{x}$ denote the vector given by the last expression. Then, Lemma 8.26 shows that $\lambda + \delta + \hat{x} \leqslant f(\hat{x})$. Since f is monotone, we also have $f(\hat{x}) \leqslant f(x)$. Moreover, $\hat{x} \geqslant -(N-1)\delta + x$ and therefore $\lambda - (N-2)\delta + x \leqslant f(x)$. To finish, note that

$$\lambda - N\delta \geqslant \frac{1}{3K} - \frac{1}{6K} = \frac{1}{6K}\,.$$

The proof of the case $\mathbf{t}(\tilde{f}^N(0))/N \leqslant -\frac{1}{2K}$ is analogous. ☐

Our next algorithm, ApproximateValue, computes the approximation of η up to a given precision. Then, once an approximation of η is known, it certifies that the approximation is correct. Contrary to the previous algorithms, ApproximateValue has to perform a fixed number of iterations, independent of the value of η.

Lemma 8.33. *The procedure* ApproximateValue *is correct.*

Proof. The proof is similar to the proof of Lemma 8.32. Let $\delta \coloneqq (8K^2R_{\mathrm{ub}})^{-1}$ and $N \coloneqq 2KR_{\mathrm{ub}}$. By Lemma 8.29 we have

$$\eta \geqslant \frac{\mathbf{b}(f^N(0))}{N} \geqslant \frac{\mathbf{b}(\tilde{f}^N(0))}{N} - \delta = \frac{\mathbf{b}(\nu)}{2KR_{\mathrm{ub}}} - \frac{1}{8K^2R_{\mathrm{ub}}} = \frac{a - 2KR_{\mathrm{ub}}}{16K^3R_{\mathrm{ub}}^2} = \lambda + \delta\,.$$

Hence, as in the proof of Lemma 8.32, we have $\lambda - (N-2)\delta + x \leqslant f(x)$. Furthermore, if $u \in \mathbb{R}^n$ is a bias vector such that $\|u\|_{\mathrm{H}} \leqslant R + \varepsilon$, then Corollary 8.23 and Lemma 8.29 imply that

$$\eta \leqslant \frac{\mathbf{b}(\tilde{f}^N(0))}{N} + \frac{\|u\|_{\mathrm{H}}}{N} + \delta \leqslant \lambda + 3\delta + \frac{R}{N} + \frac{\varepsilon}{N} \leqslant \lambda + 3\delta + \frac{1}{2K} + \frac{\varepsilon}{2KR}\,.$$

procedure ApproximateValue(f, K)
  $\nu := 0 \in \mathbb{R}^n$
  for $i = 1, 2, \dots, 2KR_{\mathrm{ub}}$ **do**
    $\nu := \textsc{Oracle}(f, \nu, 8K^2R_{\mathrm{ub}})$
  done
  ▷ *we denote* $\mathbf{b}(\nu) = \frac{a}{8K^2R_{\mathrm{ub}}}$ *and* $\mathbf{t}(\nu) = \frac{b}{8K^2R_{\mathrm{ub}}}$, $a, b \in \mathbb{Z}$
  the value η belongs to the interval $[\frac{a-2KR_{\mathrm{ub}}}{16K^3R_{\mathrm{ub}}^2}, \frac{a+2KR_{\mathrm{ub}}+8K^2R_{\mathrm{ub}}^2}{16K^3R_{\mathrm{ub}}^2}]$
  $\lambda := \frac{a-4KR_{\mathrm{ub}}}{16K^3R_{\mathrm{ub}}^2}$, $\mu := \frac{b+4KR_{\mathrm{ub}}}{16K^3R_{\mathrm{ub}}^2}$
  $\nu := 0 \in \mathbb{R}^n$
  $x := 0 \in \mathbb{R}^n$, $y := 0 \in \mathbb{R}^n$
  for $i = 1, 2, \dots, 2KR_{\mathrm{ub}}$ **do**
    $x := \max\{x, -(i-1)\lambda + \nu\}$
    $y := \min\{y, -(i-1)\mu + \nu\}$
  ▷ *operations* max *and* min *are taken entrywise*
    $\nu := \textsc{Oracle}(f, \nu, 8K^2R_{\mathrm{ub}})$
  done
  we have $\eta - \frac{1}{K} + x \leqslant f(x)$ and $\eta + \frac{1}{K} + y \geqslant f(y)$
end

Figure 8.7: Approximating the value.

Since $\varepsilon > 0$ is arbitrary, we have $\eta \leqslant \lambda + 3\delta + \frac{1}{2K} = \frac{a+2KR_{\mathrm{ub}}+8K^2R_{\mathrm{ub}}^2}{16K^3R_{\mathrm{ub}}^2}$. Moreover, note that

$$\eta - \frac{1}{K} \leqslant \lambda + 3\delta - \frac{1}{2K} = \lambda + 3\delta - 2N\delta \leqslant \lambda - (N-2)\delta \, .$$

Therefore $\eta - \frac{1}{K} + x \leqslant f(x)$. The proof of the fact that $\eta + \frac{1}{K} + y \geqslant f(y)$ is analogous. □

Remark 8.34. We point out that the length of the interval returned by ApproximateValue is not greater than $\frac{3}{4K}$. Furthermore, if one only wants to compute the approximation of η, then the second loop of the algorithm can be omitted.

Remark 8.35. The same reasoning shows that η belongs to the interval

$$[\frac{b - 2KR_{\mathrm{ub}} - 8K^2R_{\mathrm{ub}}^2}{16K^3R_{\mathrm{ub}}^2}, \frac{b + 2KR_{\mathrm{ub}}}{16K^3R_{\mathrm{ub}}^2}] \, .$$

In particular, we have $b - 2KR_{\mathrm{ub}} - 8K^2R_{\mathrm{ub}}^2 \leqslant a + 2KR_{\mathrm{ub}} + 8K^2R_{\mathrm{ub}}^2$, which is equivalent to

$$\mathbf{t}(\nu) - \mathbf{b}(\nu) \leqslant 2R_{\mathrm{ub}} + \frac{1}{2K} \, .$$

Hence $\mathbf{t}(\nu) - \mathbf{b}(\nu) < 2R_{\mathrm{ub}} + 1$. We will use this property later to check if a given stochastic mean payoff game has constant value.

8.3 Estimates of condition number

In this section we give an estimate for the condition number DR for the class of operators that arise in the study of stochastic mean payoff games. More precisely, we will suppose that the function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is semilinear on top of being monotone and homogeneous. Let us recall that if f is semilinear, monotone, homogeneous, then its affine pieces are of the form $Ax+b$, where A is a stochastic matrix (Lemma 5.32). We use the following definition.

Definition 8.36. If $A \in \mathbb{Q}^{n\times n}$ is a stochastic matrix, then we say that the kth row $(A_{kl})_{l\in[n]}$ of A is *nondeterministic* if it contains an entry different than 0 or 1, $\exists l^*, A_{kl^*} \in]0,1[$. We say that two rows $(A_{kl})_{l\in[n]}$, $(A_{k'l})_{l\in[n]}$ of A are *nonidentical* if they differ by at least one entry, $\exists l^*, A_{kl^*} \neq A_{k'l^*}$.

The following theorem estimates the condition number DR.

Theorem 8.37. *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise affine, monotone, and homogeneous operator. Let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ be a piecewise description of f. Furthermore, suppose that f satisfies Assumption G and that the matrices $(A^{(s)})_{s\in[p]}$ and vectors $(b^{(s)})_{s\in[p]}$ are rational, $A^{(s)} \in \mathbb{Q}^{n\times n}$, $b^{(s)} \in \mathbb{Q}^n$. Let M denote a common denominator of all the numbers $(A^{(s)}_{kl})_{s,k,l}$. Similarly, let L denote a common denominator of all the numbers $(b^{(s)}_k)_{s,k}$ and let $W := \max_{s,k} |b^{(s)}_k|$. Furthermore, let $n_r \in \mathbb{N}$ be such that every matrix $A^{(s)}$ has at most n_r nondeterministic rows. Then, the value η is rational and its denominator D satisfies $D \leqslant nLM^{\min\{n_r,n-1\}}$ and we have $R \leqslant 10n^2WM^{\min\{n_r,n-1\}}$. Even more, if $\tilde{n}_r \in \mathbb{N}$ is such that every matrix $A^{(s)}$ has at most $\tilde{n}_r$ pairwise nonidentical nondeterministic rows, then we also have $D \leqslant 2nLM^{\tilde{n}_r}$ and $R \leqslant 40n^2WM^{\tilde{n}_r}$.*

Before giving the proof of Theorem 8.37, let us discuss its implications for the stochastic mean payoff games.

Example 8.38. Suppose that we are given a stochastic mean payoff game that with n states controlled by Player Min and m states controlled by Player Max. Moreover, suppose that the game has rational probabilities $p^a_{kv}, p^b_{iv} \in \mathbb{Q}$ and integer payoffs $r_a, r_b \in \mathbb{Z}$. Finally, suppose that the game is a constant value game. Let M denote the least common denominator of all the numbers (p^a_{kv}, p^b_{iv}), and let W denote the maximal absolute value of payoffs $(|r_a|, |r_b|)$. Let us say that a state of the game is *nondeterministic* if there is an action associated with this state is nondeterministic (i.e., is such that the associated probability distribution has an entry in $]0,1[$). Let us consider a few possible situations.

1. Take f to be the Shapley operator $T_{|\mathbb{R}^{n+m}}\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ (as defined in Definition 6.3). Note that a piecewise description $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ of T can be obtained by fixing the terms that achieve min and max in (6.2). Moreover, in this description the vectors $b^{(s)}$ are integer, and M is a common denominator of the entries of matrices $A^{(s)}$. Since η is the value of the game (by Remark 6.11), we get from Theorem 8.37 that this value is rational and that its denominator is bounded by $D_{\mathrm{ub}} := (n+m)M^{\min\{s,n+m-1\}}$, where s is the number of nondeterministic states of the game. Furthermore, we have $R \leqslant R_{\mathrm{ub}}$, where $R_{\mathrm{ub}} := 10(n+m)^2WM^{\min\{s,n+m-1\}}$.

2. Suppose that the game is bipartite and take f to be the bipartite Shapley operator $F_{|\mathbb{R}^n}\colon \mathbb{R}^n \to \mathbb{R}^n$ as defined in Definition 6.20. Similarly to the previous situation, we have that η is twice as big as the value of the game (by Lemma 6.22) and that a piecewise description $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ of F can be obtained by fixing the terms that achieve min and max in (6.5). However, in this description we multiply the probabilities by the probabilities, and probabilities by the payoffs. Hence, M^2 is a common denominator of the entries of matrices $A^{(s)}$ and M is a common denominator of the entries of vectors $b^{(s)}$. Moreover, the absolute value of numbers $(b^{(s)}_k)$ is bounded by $2W$. In particular, we get from Theorem 8.37 that the denominator of η is not greater than $D_{\mathrm{ub}} := nM^{2n-1}$ and that R is bounded by $R_{\mathrm{ub}} := 20n^2WM^{2n-2}$.

3. Suppose, as above, that the game is bipartite, but Player Max has only deterministic actions (as in the case of games obtained from tropical Metzler matrices discussed in Section 7.2). In this case M is a common denominator of the entries of matrices $A^{(s)}$ and the entries of $b^{(s)}$. The bounds become $D \leqslant D_{\mathrm{ub}} := nM^{\min\{s+1,n\}}$ and $R \leqslant R_{\mathrm{ub}} := 20n^2WM^{\min\{s,n-1\}}$, where s is the number of nondeterministic states of Player Min.

4. The situation is slightly different if we suppose that the game is bipartite, but Player Min has only deterministic actions (because of the asymmetry between Players Min and Max in Definition 6.20). In this case, the vectors $b^{(s)}$ are integer. Moreover, we can measure the degree of randomness of the game by letting s to be the number of nonidentical distributions $(p^b_{i\cdot})_{b,i}$ associated with the actions of Player Max. Then, we use the last part of Theorem 8.37 to obtain the bounds $D \leqslant D_{\mathrm{ub}} := 2nM^{\min\{s,n-1\}}$ and $R \leqslant R_{\mathrm{ub}} := 80n^2WM^{\min\{s,n-1\}}$.

We now want to present the proof of Theorem 8.37. The following proposition allows us to reduce the case of general semilinear operators to the case of affine operators.

Proposition 8.39. *Let f be as in Theorem 8.37. Then, there exist $s \in [p]$, $\tilde{u} \in \mathbb{R}^n$, and $\gamma \geqslant 0$ such that $b^{(s)} + A^{(s)}\tilde{u} = \eta + \tilde{u}$ and such that $u := \gamma\eta + \tilde{u}$ is a bias of f. Furthermore, $\tilde{u}$ has the property that $\pi^{\mathsf{T}}\tilde{u} = 0$ for every stationary distribution $\pi \in [0,1]^n$ of $A^{(s)}$.*

The proof of Proposition 8.39 follows from the proof of Kohlberg's theorem. Therefore, we present in Appendix B.3 where we discuss Kohlberg's theorem in more detail. From now on, we focus on the case of affine operators f. We let $f(x) := r + Px$, where $P \in [0,1]^{n\times n}$ is a stochastic matrix, we suppose that f satisfies the conditions of Theorem 8.37. As in this theorem, we denote by M a common denominator of all the numbers (P_{kl}), we use L to denote a common denominator of the numbers (r_k), and we put $W := \max_k |r_k|$.

The following lemma shows that η is rational and relates its denominator D with a stationary distribution of the Markov chain associated with P.

Lemma 8.40. *We have the equality*

$$\eta = \sum_{k\in C} r_k\pi_k \,,$$

where C is any recurrent class of a Markov chain associated with P, and $(\pi_k)_{k\in C}$ is the stationary distribution of this class. In particular, η is rational and its denominator D is at most L times greater than the least common denominator of the numbers $(\pi_k)_{k\in C}$.

Proof. Define $\tilde{\pi} \in \mathbb{R}^n$ as $\tilde{\pi}_k = \pi_k$ for $k \in C$ and $\tilde{\pi} = 0$ otherwise. Then $\tilde{\pi}^{\mathsf{T}}P = \tilde{\pi}$ by the definition of a recurrent class. Moreover, we have $r + Pu = \eta + u$. Hence $\tilde{\pi}^{\mathsf{T}}(r + Pu) = \eta + \tilde{\pi}^{\mathsf{T}}u$ and $\sum_{k\in C} r_k\pi_k = \tilde{\pi}^{\mathsf{T}}r = \eta$. □

Lemma 8.40 reduces the problem of estimating D into the problem of estimating the least common denominator of a stationary distribution of a Markov chain with rational transition matrix. Since stationary distributions can be computing using linear algebra (Remark 2.135), one may bound this common denominator using Hadamard's inequality. Such estimations were done, e.g., in [BEGM15, AGH18]. However, these estimates are not optimal. An improved bound can be obtained using the combinatorial formula of Freidlin and Wentzell, see [FW12, Chapter 6, Lemma 3.1] or [Cat99, Lemma 3.2]. This formula requires the following definition.

Definition 8.41. Suppose that $P \in [0,1]^{n\times n}$ is a stochastic matrix and let $\vec{\mathcal{G}} := ([n], E)$ be the associated graph (given by Definition 2.128). We say that a subgraph $\vec{\mathcal{H}} := ([n], E')$, $E' \subset E$ is an *arborescence rooted at* $k \in [n]$ if is fulfills the following conditions:

- for every $l \in [n]$, $l \neq k$, there is a directed path from l to k in $\vec{\mathcal{H}}$;
- for every $l \in [n]$, $l \neq k$, there is exactly one edge of the form $(l, l') \in E$ that belongs to E';
- the set E' does not contain any edge of the form (k, l).

In other words, an arborescence is a tree in which every edge is directed towards k. We point out that an arborescence has exactly $n-1$ edges.

Definition 8.42. If $\vec{\mathcal{H}} := ([n], E')$, $E' \subset E$, $E' \neq \emptyset$ is a subgraph of $\vec{\mathcal{G}}$, then we define its *weight* as

$$p(\vec{\mathcal{H}}) := \prod_{(k,l) \in E'} P_{kl} \, .$$

We can now present the formula of Freidlin and Wentzell.

Lemma 8.43 ([FW12, Chapter 6, Lemma 3.1])**.** *Suppose that P is irreducible and let $\vec{\mathcal{A}}_k$ denote the set of all arborescences rooted at k. Then, the unique stationary distribution of this matrix is given by*

$$\forall k \in [n], \ \pi_k = \frac{\sum_{\vec{\mathcal{H}} \in \vec{\mathcal{A}}_k} p(\vec{\mathcal{H}})}{\sum_{l \in [n]} \sum_{\vec{\mathcal{H}} \in \vec{\mathcal{A}}_l} p(\vec{\mathcal{H}})} \, .$$

This formula allows us to give an estimate on the denominator D.

Theorem 8.44. *If $(\pi_k)_{k \in C}$ is the stationary distribution of the Markov chain associated with P, then the numbers $(\pi_k)_{k \in C}$ are rational and their common denominator is not greater than $nM^{\min\{n_r, n-1\}}$, where n_r is the number of nondeterministic rows of P. In particular, we have $D \leqslant nLM^{\min\{n_r, n-1\}}$.*

Proof. The second part follows from the first by Lemma 8.40. To prove the first part, note that, up to passing to a submatrix induced by the recurrent class, it is enough to prove the claim in the case where the matrix P is irreducible. In this case, denote $P_{kl} = \frac{a_{kl}}{b_{kl}}$, where $a_{kl}, b_{kl} \in \mathbb{N}$ and

$$b_{kl} := \begin{cases} 1 & \text{if the } k\text{th row of } P \text{ is deterministic} \, , \\ M & \text{otherwise} \, . \end{cases}$$

Furthermore, let $\vec{\mathcal{G}} := ([n], E)$ be the associated graph. For any arborescence $\vec{\mathcal{H}} = ([n], E')$, $E' \subset E$ we define

$$\tilde{p}(\vec{\mathcal{H}}) := \prod_{(k,l) \in E'} b_{kl}$$

and we put $Z := \operatorname{lcm}\{\tilde{p}(\vec{\mathcal{H}}) \colon \vec{\mathcal{H}} \text{ is an arborescence}\}$ (where lcm denotes the least common multiple). By Lemma 8.43, the stationary distribution is given by

$$\forall k \in [n], \ \pi_k = \frac{Z \sum_{\vec{\mathcal{H}} \in \vec{\mathcal{A}}_k} p(\vec{\mathcal{H}})}{Z \sum_{l \in [n]} \sum_{\vec{\mathcal{H}} \in \vec{\mathcal{A}}_l} p(\vec{\mathcal{H}})} \, . \tag{8.4}$$

Note that, by the definition of Z, the expression in (8.4) is a quotient of two natural numbers. In other words, the least common denominator of $(\pi_k)_{k \in [n]}$ is not greater than

$$Z \sum_{l \in [n]} \sum_{\vec{\mathcal{H}} \in \vec{\mathcal{A}}_l} p(\vec{\mathcal{H}}) \, .$$

Furthermore, observe that for any $l \in [n]$ we have

$$\sum_{\vec{\mathcal{H}} \in \vec{\mathcal{A}}_l} p(\vec{\mathcal{H}}) \leqslant \prod_{k' \neq l} \Big(\sum_{l' \in [n]} P_{k'l'} \Big) = 1 \, .$$

Hence, the least common denominator of $(\pi_k)_{k \in [n]}$ is not greater than nZ. To finish the proof, note that Z divides both M^{n-1} and M^{n_r}. ☐

Remark 8.45. As noted above, it is useful to compare our bound with the bounds obtained by Hadamard's inequality. Using this technique, [BEGM15, Lemma 6] gives a bound $(n_r + 1)n(2M)^{n_r+3}$ for the least common denominator of $(\pi_k)_{k\in[n]}$, which is worse than the bound presented above. Moreover, it can be shown that our bound is optimal in the sense that for every $n \geqslant 2$ and every $1 > \varepsilon > 0$ we can find M and a Markov chain such that the least common denominator of $(\pi_k)_{k \in [n]}$ is greater than $(1-\varepsilon)nM^{\min\{n_r, n-1\}}$.

Remark 8.46. Bounds of similar nature, but concerning the probabilities of absorption instead of stationary distribution, were studied in the context of simple stochastic games [Con92, ACS14] (we have used these bounds in Lemma 7.11). As in case considered above, these bounds are obtained by Hadamard's inequality and are suboptimal. We note that it is possible to extend the formula of Freidlin and Wentzell in order to obtain a similar formula for absorption probabilities, and this gives the optimal bounds. However, this result is not needed in this dissertation.

We now want to estimate the second part of our condition number, namely the estimate R of Hilbert seminorm of bias vectors. This requires the following definition and a lemma from the theory of Markov chains. Hereafter, I denotes the identity matrix (the dimension of this matrix is always clear from the context).

Definition 8.47. Given a Markov chain on a space V, we say that a nonempty subset of sets $U \subset V$ is *open* if the chain starting at any state $v \in U$ can go to $V \setminus U$ with nonzero probability, i.e.,

$$\forall v \in U, \; \mathbb{P}(\exists s, X_s \notin U \mid X_0 = v) > 0 \, .$$

Lemma 8.48. *Let $U \subset [n]$ denote any open set of a Markov chain associated with the stochastic matrix P. Let $\tilde{P} \in [0,1]^{U \times U}$ be the submatrix of P formed by the rows and columns of P indexed by U. Then, the matrix $(I - \tilde{P})$ is invertible and $(I-\tilde{P})^{-1}$ has nonnegative entries. Furthermore, every entry of the matrix $(I-\tilde{P})^{-1}$ is not greater than $M^{n_r(U)}$, where $n_r(U)$ denotes the number of nondeterministic rows of P with indexes in U.*

Remark 8.49. Since the estimate above is similar to the one of Theorem 8.44, one may wonder if it is possible to obtain it using arborescences. This is indeed possible, by replacing the formula of Lemma 8.43 by a more general statement of [Cat99, Lemma 3.1]. However, the probabilistic proof given below is slightly shorter.

Proof of Lemma 8.48. The fact that $(I - \tilde{P})$ is invertible follows from [KS76, Theorem 3.5.4]. Even more, the entries of the matrix $(I - \tilde{P})^{-1}$ have the following probabilistic interpretation. Denote $C \coloneqq [n] \setminus U$ and let $(X_0, X_1, \dots)$ be a Markov chain with transition matrix P. We denote by $T \coloneqq \inf\{m \geqslant 1 \colon X_m \in C\}$ the first (nontrivial) time the chain visits C. Furthermore, for every pair $v, w \in U$ we denote by

$$\zeta_{vw} \coloneqq \mathbb{E}\Big(\sum_{m=0}^{T-1} \mathbf{1}_{\{X_m = w\}} \Big| X_0 = v \Big)$$

the expected number of visits in w before reaching C provided that the Markov chain starts at v. Then, it is shown in [KS76, Theorem 3.5.4] that $(I - \tilde{P})^{-1}_{vw} = \zeta_{vw}$ for every pair $v, w \in U$. In particular, the entries of $(I - \tilde{P})^{-1}$ are nonnegative. Furthermore, let q_{vw} denote the probability that a chain starting at v reaches w before going back to v and before reaching C,

$$q_{vw} := \mathbb{P}(\exists s < T,\ X_1 \neq v, X_2 \neq v, \dots, X_{s-1} \neq v, X_s = w \mid X_0 = v) .$$

Note that under this notation q_{vv} is the probability that Markov chain starting at v comes back to v before reaching C. Then, we have

$$\zeta_{vw} = \sum_{s=1}^{\infty} \mathbb{P}(\sum_{m=0}^{T-1} \mathbf{1}_{\{X_m = w\}} \geqslant s \mid X_0 = v) = \sum_{s=1}^{\infty} q_{vw} q_{ww}^{s-1} \leqslant \sum_{s=1}^{\infty} q_{ww}^{s-1} = \frac{1}{1 - q_{ww}} .$$

Furthermore, since $1 - q_{ww}$ is the probability that Markov chain starting at w reaches C before it comes back to w, we can bound it from below as follows. Let $\vec{\mathcal{G}}$ be the graph associated with P and let $(k_0, k_1, \dots, k_s)$ be the shortest directed path in this graph that starts from $k_0 = w$ and ends in C, $k_s \in C$. (Such a path exists because U is open.) Since this is the shortest path, every state on this path is different. Moreover, note that we have $P_{k_i k_{i+1}} \geqslant 1/M$ for every $i \in \{0, \dots, s-1\}$. Even more, if the state k_i is deterministic, then $P_{k_i k_{i+1}} = 1$. In particular, we have

$$1 - q_{ww} \geqslant \mathbb{P}(X_1 = k_1, \dots, X_s = k_s \mid X_0 = w) = \prod_{i=0}^{s-1} P_{k_i k_{i+1}} \geqslant \frac{1}{M^{n_r(U)}} .$$

Hence $\zeta_{vw} \leqslant M^{n_r(U)}$. □

We can now use Lemma 8.48 to bound R. We start with the case where the matrix P is irreducible.

Lemma 8.50. *Suppose that P is irreducible and let $u \in \mathbb{R}^n$ be any bias vector of f. Then $\|u\|_{\mathrm{H}} \leqslant 4nWM^{\min\{n_r, n-1\}}$.*

Proof. If $n = 1$, then any scalar is a bias vector and its Hilbert seminorm is equal to 0. Let $n \geqslant 2$ and note that a vector $u \in \mathbb{R}^n$ is a bias of f if and only if $r + Pu = \eta + u$, which is equivalent to $(I - P)u = \eta - r$, where I is the identity matrix. Moreover, since P is irreducible, P has a unique stationary distribution by Proposition 2.134. Hence, by Remark 2.135, the kernel of $(I - P)^{\intercal}$ has dimension 1. Since this matrix is square, the kernel of $(I - P)$ also has dimension 1. In particular, the set of bias vectors of f forms an affine space of dimension one. Let u be a fixed bias vector. Note that $(1, 1, \dots, 1)$ is an eigenvector of P and hence the set of all bias vectors is a line of the form $\{\lambda + u \colon \lambda \in \mathbb{R}\}$. In particular, we have $\mathbf{t}(\lambda + u) = \lambda + \mathbf{t}(u)$ and $\mathbf{b}(\lambda + u) = \lambda + \mathbf{b}(u)$, which implies that the Hilbert seminorm does not depend on the choice of bias vector. Therefore, we can take a bias u such that $u_n = 0$. Let $\tilde{u}, \tilde{r}$ denote the vectors u, r with the nth coordinate removed and let $\tilde{P}$ denote the matrix P with the nth row and column removed. We have $(I - \tilde{P})\tilde{u} = \eta - \tilde{r}$. Moreover, by Lemma 8.40, we have the inequality $|\eta| \leqslant W$. Furthermore, note that $[n-1] \subset [n]$ is open because P is irreducible. Hence, by Lemma 8.48,

$$\begin{aligned} \|\tilde{u}\|_\infty = \|(I - \tilde{P})^{-1}(\eta - \tilde{r})\|_\infty &\leqslant 2W \|(I - \tilde{P})^{-1} e\|_\infty \\ &\leqslant 2(n-1)WM^{n_r([n-1])} \leqslant 2nWM^{\min\{n_r, n-1\}} . \end{aligned}$$

Moreover, for any $k, l \in [n]$ we have $|u_k - u_l| \leqslant |u_k - u_n| + |u_n - u_l| = |u_k| + |u_l| \leqslant 2\|\tilde{u}\|_\infty$ and the claim follows. □

In order to extend the result of Lemma 8.50 to matrices that are not irreducible, we use the fact that f admits a special bias vector that is orthogonal to every stationary distribution of P (Proposition 8.39).

Lemma 8.51. *If $P \in \mathbb{Q}^{n \times n}$ is any stochastic matrix (not necessarily irreducible) and let u be a bias vector of f that is orthogonal to every stationary distribution of P. Then $\|u\|_{\mathrm{H}} \leqslant 10n^2 W M^{\min\{n_r, n-1\}}$.*

Proof. If $n = 1$, then $\|u\|_{\mathrm{H}} = 0$ and the claim is trivial. Suppose that $n \geqslant 2$, let $C \subset [n]$ be any recurrent class of a Markov chain associated with P and let $\tilde{\pi} \in \left]0,1\right]^C$ be its stationary distribution. Moreover, let $\tilde{r}, \tilde{u} \in \mathbb{R}^C$ denote the vectors r, u restricted to coordinates indexed by C and let $\tilde{P} \in [0,1]^{C \times C}$ be the submatrix of P formed by the rows and columns indexed by C. By the definition of recurrent class we have $\tilde{r} + \tilde{P}\tilde{u} = \eta + \tilde{u}$. Hence, by Lemma 8.50, $\|\tilde{u}\|_{\mathrm{H}} \leqslant 4|C| W M^{\min\{n_r(C), |C|-1\}}$, where $n_r(C)$ denotes the number of nondeterministic rows of P with indexes in C. Furthermore, as already observed in the proof of Lemma 8.40, if we define $\pi \in [0,1]^n$ as $\pi_k := \tilde{\pi}_k$ for $k \in C$ and $\pi_k := 0$ otherwise, then π is a stationary distribution of P. In particular, we have $\pi^{\intercal} u = 0$ and hence $\tilde{\pi}^{\intercal}\tilde{u} = 0$. Hence, for every $k \in C$ we have

$$|u_k| = |\sum_{l \in C} \pi_l u_k| = |\sum_{l \in C} \pi_l (u_k - u_l)| \leqslant \|\tilde{u}\|_{\mathrm{H}} \leqslant 4|C| W M^{\min\{n_r(C), |C|-1\}} . \tag{8.5}$$

We now want to bound $|u_k|$ for a transient state $k \in [n]$. To do this, let $U \subset [n]$ be the set of all transient states, and suppose that this set is nonempty. Let $\tilde{U} := [n] \setminus U$ denote the set of all recurrent states. Let $\hat{r}, \hat{u} \in \mathbb{R}^U$ denote the vectors r, u restricted to the coordinates indexed by U and let $\hat{P} \in [0,1]^{U \times U}$ denote the submatrix of P formed by the rows and columns indexed by U. Furthermore, we denote by $\bar{u} \in \mathbb{R}^{\tilde{U}}$ the vector u restricted to coordinates indexed by $\tilde{U}$ and by $\overline{P} \in [0,1]^{U \times \tilde{U}}$ the (rectangular) submatrix of P formed by the rows indexed by U and columns indexed by $\tilde{U}$. Moreover, note that U is open. We have $\hat{r} + \hat{P}\hat{u} + \overline{P}\bar{u} = \eta + \hat{u}$. Hence, by Lemma 8.48 and (8.5), and the fact that $|\eta| \leqslant W$ (Lemma 8.40) we have

$$\begin{aligned}
\|\hat{u}\|_\infty = \|(I - \hat{P})^{-1}(\eta - \hat{r} - \overline{P}\bar{u})\|_\infty &\leqslant (2W + \|\bar{u}\|_\infty)\|(I - \hat{P})^{-1} e\|_\infty \\
&\leqslant (2W + 4|\tilde{U}| W M^{\min\{n_r(\tilde{U}), |\tilde{U}|-1\}})|U| M^{n_r(U)} \\
&\leqslant (2n + 4n^2) W M^{\min\{n_r, n-1\}} \leqslant 5n^2 W M^{\min\{n_r, n-1\}} .
\end{aligned}$$

The claim follows from the inequality $\|u\|_{\mathrm{H}} \leqslant 2\|u\|_\infty$. ☐

The next lemma extends these estimates to the case where we fix the number of nonidentical nondeterministic rows.

Lemma 8.52. *Let $\tilde{n}_r$ denote the number of pairwise nonidentical nondeterministic rows of P. Then, we have $D \leqslant 2nLM^{\tilde{n}_r}$. Moreover, if u is a bias of f that is orthogonal to every stationary distribution of P, then $\|u\|_{\mathrm{H}} \leqslant 40n^2 W M^{\tilde{n}_r}$.*

Proof. Let $\tilde{n}$ denote the number of pairwise nonidentical rows of P. Up to permuting the rows of P and the coordinates of r, we can assume that the first $\tilde{n}$ rows of P consist of all the pairwise nonidentical rows. Moreover, for every row $k \in [n]$, let $\sigma(k) \in [\tilde{n}]$ denote the index such that $P_{kl} = P_{\sigma(k)l}$ for all $l \in [n]$. Observe that for all $k \in [n]$ we have the equalities $\eta + u_k = (r + Pu)_k = r_k + (Pu)_k = r_k + (Pu)_{\sigma(k)}$ and $\eta + u_{\sigma(k)} = (r + Pu)_{\sigma(k)} = r_{\sigma(k)} + (Pu)_{\sigma(k)}$ Hence

$$u_k - r_k = u_{\sigma(k)} - r_{\sigma(k)} \tag{8.6}$$

for all $k \in [n]$. We now define a matrix $\tilde{P} \in [0,1]^{(n+\tilde{n})\times(n+\tilde{n})}$ and a vector $\tilde{r} \in \mathbb{R}^{n+\tilde{n}}$ as

$$\tilde{P} := \left[\begin{array}{c|c} 0 & \epsilon_{\sigma(1)} \\ \vdots & \vdots \\ 0 & \epsilon_{\sigma(n)} \\ \hline P_{1,1} \dots P_{1,n} & 0 \\ \vdots & \vdots \\ P_{\tilde{n},1} \dots P_{\tilde{n},n} & 0 \end{array}\right] \quad \text{and} \quad \tilde{r} := \left[\begin{array}{c} r_1 \\ \vdots \\ r_n \\ \hline 0 \\ \vdots \\ 0 \end{array}\right].$$

where $\epsilon_l \in \mathbb{R}^{\tilde{n}}$ denotes the lth vector of the standard basis in $\mathbb{R}^{\tilde{n}}$. Furthermore, let $\tilde{f}(x) := \tilde{r} + \tilde{P}x$ and define a vector $\tilde{u} \in \mathbb{R}^{n+\tilde{n}}$ as $\tilde{u}_k := u_k$ for all $k \leqslant n$ and $\tilde{u}_{n+k} := \frac{1}{2}\eta + u_k - r_k$ for all $k \leqslant \tilde{n}$. By (8.6) we have

$$\tilde{r} + \tilde{P}\tilde{u} = \left[\begin{array}{c} r_1 + \tilde{u}_{n+\sigma(1)} \\ \vdots \\ r_n + \tilde{u}_{n+\sigma(n)} \\ \hline (Pu)_1 \\ \vdots \\ (Pu)_{\tilde{n}} \end{array}\right] = \left[\begin{array}{c} \frac{1}{2}\eta + u_1 \\ \vdots \\ \frac{1}{2}\eta + u_n \\ \hline \eta + u_1 - r_1 \\ \vdots \\ \eta + u_n - r_n \end{array}\right] = \frac{1}{2}\eta + \tilde{u}\,.$$

In particular, $\tilde{f}$ satisfies Assumption G. Hence, by Theorem 8.44, the denominator D of η can be estimated by

$$D \leqslant (n+\tilde{n})LM^{\tilde{n}_r} \leqslant 2nLM^{\min\{\tilde{n}_r, n-1\}}\,.$$

To prove the second estimate, let $\tilde{\pi} \in [0,1]^{n+\tilde{n}}$ be a stationary distribution of $\tilde{P}$. Therefore, for all $l \leqslant n$ we have

$$\tilde{\pi}_l = \sum_{k=1}^{\tilde{n}} P_{kl}\tilde{\pi}_{n+k}$$

and for all $l \leqslant \tilde{n}$ we have

$$\tilde{\pi}_{n+l} = \sum_{\{k \in [n] \colon \sigma(k) = l\}} \tilde{\pi}_k\,.$$

In particular, $\sum_{l=1}^{\tilde{n}} \tilde{\pi}_{n+l} = \sum_{k=1}^{n} \tilde{\pi}_k$ and thus $\sum_{l=1}^{\tilde{n}} \tilde{\pi}_{n+l} = \sum_{k=1}^{n} \tilde{\pi}_k = 1/2$. Define $\pi \in [0,1]^n$ as $\pi_k := 2\tilde{\pi}_k$ for all $k \in [n]$. Then, the coordinates of the vector π sum to one and we have

$$\pi_l = \sum_{k=1}^{\tilde{n}} \sum_{\{k' \in [n] \colon \sigma(k') = k\}} P_{kl}\pi_{k'} = \sum_{k=1}^{n} P_{kl}\pi_k\,.$$

Hence, π is a stationary distribution of P. As a result, using (8.6) and the fact that $\pi^\intercal r = \eta$ (as noted in the proof of Lemma 8.40), we have

$$\begin{aligned} \tilde{\pi}^\intercal \tilde{u} &= \frac{1}{2}\pi^\intercal u + \sum_{l=1}^{\tilde{n}} \tilde{\pi}_{n+l}\tilde{u}_{n+l} = \frac{1}{2}\sum_{l=1}^{\tilde{n}} \sum_{\{k \in [n] \colon \sigma(k) = l\}} \pi_k(\frac{1}{2}\eta + u_l - r_l) \\ &= \frac{1}{4}\eta + \frac{1}{2}\sum_{l=1}^{\tilde{n}} \sum_{\{k \in [n] \colon \sigma(k) = l\}} \pi_k(u_k - r_k) = \frac{1}{4}\eta + \frac{1}{2}\sum_{k=1}^{n} \pi_k(u_k - r_k) = -\frac{1}{4}\eta\,. \end{aligned}$$

Therefore, the vector $\hat{u} \in \mathbb{R}^{n+\tilde{n}}$ defined as $\hat{u} := \frac{1}{4}\eta + \tilde{u}$ is a bias of f that is orthogonal to every stationary distribution of $\tilde{P}$. Moreover, we have $\|u\|_{\mathrm{H}} \leqslant \|\tilde{u}\|_{\mathrm{H}} = \|\hat{u}\|_{\mathrm{H}}$. Hence, by Lemma 8.51 we have the inequality

$$\|u\|_{\mathrm{H}} \leqslant 10(n+\tilde{n})^2 W M^{\tilde{n}_r} \leqslant 40 n^2 W M^{\tilde{n}_r} . \qquad \square$$

We are now ready to present the proof of Theorem 8.37.

Proof of Theorem 8.37. Let $s \in [p], \tilde{u} \in \mathbb{R}^n, \gamma \geqslant 0$ be the triple given by Proposition 8.39 and let $\tilde{f}(x) := b^{(s)} + A^{(s)}x$. Then, by Proposition 8.39 we have $\tilde{f}(\tilde{u}) = \eta + \tilde{u}$. Hence, by Lemma 8.40 and Theorem 8.44 the value of η is rational and its denominator is bounded by $D \leqslant nLM^{\min\{n_r, n-1\}}$. Furthermore, since $\tilde{u}$ is orthogonal to every stationary distribution of $A^{(s)}$, Lemma 8.51 shows that $\|\tilde{u}\|_{\mathrm{H}} \leqslant 10n^2 W M^{\min\{n_r, n-1\}}$. Moreover, if $u := \gamma\eta + \tilde{u}$, then $\mathbf{t}(u) = \gamma\eta + \mathbf{t}(\tilde{u})$ and $\mathbf{b}(u) = \gamma\eta + \mathbf{b}(\tilde{u})$. Hence $\|u\|_{\mathrm{H}} = \|\tilde{u}\|_{\mathrm{H}}$. Furthermore, u is a bias of f by Proposition 8.39 and the first claim follows. The second claim is proven in the same way, replacing Theorem 8.44 and Lemma 8.51 by Lemma 8.52. $\square$

8.4 Parametrized complexity bounds for stochastic mean payoff games

In this section, we combine the analysis of value iteration done in Section 8.2 with the estimates obtained in Section 8.3. As a result, we show that value iteration solves the constant value games in pseudopolynomial time when the number of nondeterministic actions is fixed. We also extend this result to the task of finding the set of states with maximal (or minimal) value in a general stochastic mean payoff game.

We use the notation introduced in Example 8.38. More precisely, we suppose that we are given a stochastic mean payoff game that with n states controlled by Player Min and m states controlled by Player Max. Moreover, we suppose that the game has rational probabilities $p^a_{kv}, p^b_{iv} \in \mathbb{Q}$ and integer payoffs $r_a, r_b \in \mathbb{Z}$. Let M denote the least common denominator of all the numbers (p^a_{kv}, p^b_{iv}), and let W denote the maximal absolute value of payoffs $(|r_a|, |r_b|)$. Furthermore, let s denote the number of nondeterministic states of the game, $\chi \in \mathbb{R}^{n+m}$ denote its value, and $a_{\mathrm{ct}} \in \mathbb{N}^*$ denote the total number of actions. We put

$$\begin{aligned} D_{\mathrm{ub}} &:= (n+m)M^{\min\{s, n+m-1\}} , \\ R_{\mathrm{ub}} &:= 10(n+m)^2 W M^{\min\{s, n+m-1\}} . \end{aligned}$$

Finally, let $T \colon \mathbb{T}^n \to \mathbb{T}^n$ be the Shapley operator of this game as defined in Definition 6.3. We recall that the restriction $f(x) := T_{|\mathbb{R}^{n+m}}(x) \colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is a semilinear, monotone, homogeneous operator. Before proceeding, let us observe that f preserves $\mathbb{Q}^n$ and hence the values $f^N(0)$ can be computed explicitly in the Turing machine model of computation. However, as noted in the previous sections, it is still desirable to use approximations of f, as they greatly reduce the amount of memory needed by the algorithms presented in Section 8.2. The algorithm that approximates f is very simple—we first compute the value of $f(x)$ exactly and then approximate the result using integer division. This is presented in Fig. 8.8.

Lemma 8.53. *The procedure* Oracle *presented in Fig. 8.8 is correct (i.e., satisfies the properties of Definition 8.30).*

procedure ORACLE(f, x, K)
  $\nu := f(x) \in \mathbb{Q}^n$
  $\triangleright$ *for all* $k \in [n]$ *we denote* $\nu_k = a_k/b_k$, *where* $a_k, b_k \in \mathbb{Z}$ *and* $b_k > 0$
  for $k = 1, 2, \ldots, n$ **do**
    find $p_k, r_k \in \mathbb{N}, r_k < b_k$ such that $K|a_k| = p_k b_k + r_k$
  done
  return the vector $(\frac{\mathsf{sign}(a_1)p_1}{K}, \ldots, \frac{\mathsf{sign}(a_n)p_n}{K})$
end

Figure 8.8: Approximation oracle for operators that preserve $\mathbb{Q}^n$ and can be computed explicitly.

Proof. To prove the claim, observe that for every $k \in [n]$ we have

$$|\frac{a_k}{b_k} - \frac{\mathsf{sign}(a_k)p_k}{K}| = |\frac{|a_k|}{b_k} - \frac{p_k}{K}| = |\frac{|a_k|K - p_k b_k}{b_k K}| = |\frac{r_k}{b_k K}| < \frac{1}{K}\,. \qquad \square$$

We will also need the following lemma that gives a separation bound between maximal and minimal value of a stochastic mean payoff game.

Lemma 8.54. *Let* $\bar{\chi} := \max_k \chi_k$ *and* $\underline{\chi} := \min_k \chi_k$ *denote the maximal and minimal value of the game. Then, these numbers are rational and have denominators bounded by* D_{ub}.

Proof. Let (σ, τ) denote a couple of optimal policies and consider the Markov chain induced by fixing these policies. Then, by Theorem 2.137, this chain contains two recurrent classes C_1, C_2 such that $\underline{\chi} = \sum_{w \in C_1} r_{\xi(w)} \pi_w^{(1)}$ and $\bar{\chi} = \sum_{w \in C_2} r_{\xi(w)} \pi_w^{(2)}$, where the function ξ is defined as $\xi(k) := \sigma(k)$ for $k \in [n]$ and $\xi(i) := \tau(i)$ for $i \in [m]$. Moreover, Theorem 8.44 shows that the common denominator of $(\pi_w^{(1)})_{w \in C_1}$ is not greater than D_{ub}. The same is true for $(\pi_w^{(2)})_{w \in C_2}$. $\square$

8.4.1 Solving constant value games

Our first results concern the complexity of solving constant value games. Suppose that the game introduced above has constant value and denote $\chi = \eta(1, 1, \ldots, 1)$. Then, the following lemma summarizes the conclusions of the previous sections.

Lemma 8.55. *There exists a bias vector* $u \in \mathbb{R}^{n+m}$ *such that* $T(u) = \eta + u$. *Furthermore,* η *is a rational number with denominator not greater than* D_{ub}. *Moreover, we have* $R_{\mathrm{ub}} \geqslant R$, *where* $R := \inf\{\|u\|_{\mathrm{H}} \colon T(u) = \eta + u\}$.

Proof. The existence of bias follows from Corollary 6.7 and the fact that $\chi = \eta(1, 1, \ldots, 1)$. Indeed, Corollary 6.7 shows that there exist $\tilde{u} \in \mathbb{R}^{n+m}$ and $\gamma \geqslant 0$ such that $T(\gamma\eta + \tilde{u}) = (\gamma + 1)\eta + \tilde{u}$. Hence, by putting $u := \gamma\eta + \tilde{u}$ we get $T(u) = \eta + u$. The remaining claims follow from Theorem 8.37, as explained in Example 8.38. $\square$

As a corollary, we get the following theorem about the complexity of determining the sign of η. In the following, we refer to the addition, multiplication, comparison (of rational numbers), and integer division as *arithmetic operations.*

Theorem 8.56. *Suppose that* $\eta \neq 0$. *Then, the sign of* η *can be found in* $O((1 + \frac{R}{|\eta|})a_{\mathrm{ct}}(n+m))$ *arithmetic operations. Moreover, every number that appears during these operations can be represented using* $O\Big((n+m)\log((n+m)MW)\Big)$ *bits of memory.*

Proof. Since $\eta \neq 0$, we have $W \neq 0$. By Lemma 8.55, if $\eta \neq 0$, then $\eta \geqslant 1/D_{\mathrm{ub}}$. Hence, we can put $K := D_{\mathrm{ub}} + 1$ and use the procedure $\textsc{ValueSignCertified}(T, K)$ with the oracle $\textsc{Oracle}(T, x, 12K^2 R_{\mathrm{ub}})$ given in Fig. 8.8. By Lemmas 8.32 and 8.55, the procedure finishes after at most $\lceil 1+2R/|\eta| \rceil \leqslant 2(1+R/|\eta|)$ iterations of the loop and correctly decides the sign of η. Each iteration of the loop consists of one call to the oracle and $O(n+m)$ arithmetic operations. Each call to the oracle consists of an evaluation of T and $O(n+m)$ arithmetic operations. Finally, each evaluation of T consists of $O(a_{\mathrm{ct}}(n+m))$ arithmetic operations (for every action we do $O(n+m)$ multiplications of the form $p^a_{kw} x_w$ or $p^b_{iw} x_w$). This proves the first claim. To prove the second claim, note that every rational number that appears during the execution of the algorithm, outside the calls for oracle, may be represented as fraction with a denominator equal to $12K^2 R_{\mathrm{ub}}$. Furthermore, note that $-W(1,1,\dots,1) \leqslant T(0) \leqslant W(1,1,\dots,1)$. Since T is monotone and homogeneous, this implies that $-NW(1,1,\dots,1) \leqslant T^N(0) \leqslant NW(1,1,\dots,1)$ for every $N \geqslant 1$. Since $N \leqslant 2KR_{\mathrm{ub}}$, by Lemma 8.29 we have $\|\nu\|_\infty \leqslant 2KR_{\mathrm{ub}}W + 2KR_{\mathrm{ub}}(12K^2R_{\mathrm{ub}})^{-1}$ for every vector ν that appears during the execution of the algorithm. Moreover, the coordinates of ν have denominator $12K^2R_{\mathrm{ub}}$, and hence every numerator of every coordinate of ν has an absolute value not greater than $24K^3R_{\mathrm{ub}}^2 W + 2KR_{\mathrm{ub}} = O(K^3 R_{\mathrm{ub}}^2 W)$. This also implies that the vectors $-(i-1)\lambda + \nu$, $-(i-1)\mu + \nu$ that appear in the definitions of x and y have numerators with absolute values bounded by $O(K^3R_{\mathrm{ub}}^2W)$ and the same is true for coordinates of vectors x, y. To bound the that appear during the call to the oracle, note that during the evaluation of $T(\nu)$, every rational number may be represented as a fraction with denominator $12K^2MR_{\mathrm{ub}}$. In particular, every numerator of every number that appears during this evaluation is bounded by $O(MK^3R_{\mathrm{ub}}^2W)$. This also implies that the numbers appearing in the integer division used in the oracle are bounded by $O(MK^5R_{\mathrm{ub}}^3W)$. Hence, all these numbers can be represented using $O(\log(MKR_{\mathrm{ub}}W)) = O(\log(MD_{\mathrm{ub}}R_{\mathrm{ub}}W)) = O\Big((n+m)\log((n+m)MW)\Big)$ bits of memory. □

Remark 8.57. We point out that in order to execute the algorithm we need to store $O(a_{\mathrm{ct}}(n+m))$ numbers. Therefore, the algorithm can be implemented to run in polynomial memory. Moreover, the algorithm given above produces a certificate that justifies its claim.

The next theorem shows that stochastic mean payoff games with constant value can be solved in pseudopolynomial time for every fixed number of nondeterministic states.

Theorem 8.58. *The value and a couple of optimal policies of a stochastic mean payoff game with constant value can be found in $O(a_{\mathrm{ct}}(n+m)^5 W M^{3\min\{s,n+m-1\}})$ arithmetic operations. Moreover, every number that appears during these operations can be represented using $O\Big((n+m)\log((n+m)MW)\Big)$ bits of memory.*

Proof. We let $K := 2D_{\mathrm{ub}}^2$ and we use the procedure $\textsc{ApproximateValue}(T, K)$ with the oracle $\textsc{Oracle}(T, x, 8K^2R_{\mathrm{ub}})$ given in Fig. 8.8. By repeating the proof of Theorem 8.56, we see that this procedure requires $O(KR_{\mathrm{ub}}a_{\mathrm{ct}}(n+m)) = O(a_{\mathrm{ct}}(n+m)^5WM^{3\min\{s,n+m-1\}})$ arithmetic operations and that every number used in the computations can be represented using the claimed amount of memory. Moreover, this procedure outputs an interval of the form

$$[\frac{a - 2KR_{\mathrm{ub}}}{16K^3R_{\mathrm{ub}}^2}, \frac{a + 2KR_{\mathrm{ub}} + 8K^2R_{\mathrm{ub}}^2}{16K^3R_{\mathrm{ub}}^2}]$$

that contains η. Furthermore, as in the proof of Theorem 8.56, $|a|$ is bounded by $O(K^3R_{\mathrm{ub}}^2W)$. Since the interval has length smaller than $1/K < 1/D_{\mathrm{ub}}^2$ (as noted in Remark 8.34) and the denominator of η is not greater than D_{ub} (by Lemma 8.55), we conclude that η is the rational number with the smallest denominator that belongs to this interval (as discussed in the proof of Lemma 7.13). Therefore, the exact value of η can be found in $O\big(\log(KR_{\mathrm{ub}}W)\big) = O\Big((n+m)\log\big((n+m)WM\big)\Big)$ complexity using the rational search technique, see [KM03, Lemma 5] or [For07]. To find an optimal policy for Player Max, we take the vector $x \in \mathbb{Q}^{n+m}$ returned by ApproximateValue(T, K) and such that $\eta - \frac{1}{K} + x \leqslant T(x)$. For every state $w \in [m]$ controlled by Player Max, we find an action $\tau(w)$ that realizes the maximum in $(T(x))_w = \max_{b \in B^{(w)}}(r_b + \sum_{v \in [n] \uplus [m]} p^b_{wv} x_v)$. This defines a policy τ. Moreover, by definition, if T^τ denote the Shapley operator of the game obtained by fixing τ, then we have $\eta - \frac{1}{K} + x \leqslant T(x) = T^\tau(x)$. Let $\tilde{\chi} \in \mathbb{R}^{n+m}$ denote the value of the game obtained by fixing τ. Then, Corollary 6.18 shows that $\min_v \tilde{\chi}_v \geqslant \eta - \frac{1}{K}$. Moreover, we have $\tilde{\chi}_v \leqslant \eta$ for all states $v \in [n] \uplus [m]$. Hence $\eta - \frac{1}{K} \leqslant \min_v \tilde{\chi}_v \leqslant \eta$. By Lemma 8.54, $\min_v \tilde{\chi}_v$ is a rational number with denominator not greater than D_{ub}. Moreover, the interval $[\eta - \frac{1}{K}, \eta]$ has length smaller than $1/D_{\mathrm{ub}}^2$. Hence, it contains at most one rational number with denominator bounded by D_{ub} and $\min_v \tilde{\chi}_v = \eta$. Since $\tilde{\chi}_v \leqslant \eta$ for all $v \in [n] \uplus [m]$, we have $\tilde{\chi}_v = \eta$ for all states and τ is an optimal policy for Player Max. Moreover, we note that the construction of τ requires one evaluation of T (i.e., it is cheaper that one call to the oracle). The construction of an optimal policy for Player Min is analogous. □

Remark 8.59. As previously, in order to execute the algorithm we need to store $O\big(a_{\mathrm{ct}}(n+m)\big)$ numbers. Therefore, the algorithm can be implemented to run in polynomial memory.

Remark 8.60. If the game is bipartite and one uses a bipartite Shapley operator $F \colon \mathbb{T}^{V_{\mathrm{Min}}} \to \mathbb{T}^{V_{\mathrm{Min}}}$ to describe it, then the analysis of Theorems 8.56 and 8.58 extends to this case. Indeed, if the value of the game is constant and given by $\chi = \eta(1, 1, \dots, 1)$, then Lemmas 6.22 and 8.55 show that there exists a vector $u \in \mathbb{R}^{V_{\mathrm{Min}}}$ such that $F(u) = 2\eta + u$. This enables us to repeat the analysis above, taking $D_{\mathrm{ub}}, R_{\mathrm{ub}}$ as in Example 8.38. Moreover, slightly better complexity bounds can be achieved when the cost of evaluating the Shapley operator is smaller than $O\big(a_{\mathrm{ct}}(n+m)\big)$. For instance, a general bipartite Shapley operator can be evaluated in $O(a_{\mathrm{ct}}^{\mathrm{Max}} n + a_{\mathrm{ct}}^{\mathrm{Min}} m)$ arithmetic operations, where $a_{\mathrm{ct}}^{\mathrm{Max}}, a_{\mathrm{ct}}^{\mathrm{Max}}$ are the numbers of actions of Player Max and Min respectively. If Player Max has only deterministic actions, then this cost drops to $O\big(a_{\mathrm{ct}}^{\mathrm{Max}} + a_{\mathrm{ct}}^{\mathrm{Min}} m\big)$. Additionally, if the probabilities associated with actions belong to $\{0, 1/2, 1\}$ (as in the case of games that arise from tropical Metzler spectrahedra), then this cost drops to $O(a_{\mathrm{ct}})$. It is therefore convenient to denote by c_{eval} the cost of evaluating the operator.[1] Then, for $\eta \neq 0$, finding the sign of the value of the game can be done in $O\big((1 + \frac{R}{|\eta|})c_{\mathrm{eval}}\big)$ arithmetic operations, while solving a constant value game can be done in $O\big(D_{\mathrm{ub}}^2 R_{\mathrm{ub}} c_{\mathrm{eval}}\big)$ arithmetic operations. Furthermore, the numbers that arise in the algorithm can be stored using $O(\log(MD_{\mathrm{ub}}R_{\mathrm{ub}}W))$ bits of memory.

Remark 8.61. It is interesting to compare the bounds obtained above with the bounds given by [BEGM15], where the same problem is solved using the pumping algorithm. The model of the game used in [BEGM15] is slightly different than ours—in this model, a nondeterministic action is represented by a state with only one action. This allows to represent the game as a graph, whose set of edges E correspond to the different possible transitions occurring in the game. In this model, the complexity of solving games with constant value is given by $O(|V|^4|E|WM^{3\min\{s,|V|-1\}})$, where V denotes the set of all states of the game, and $s \leqslant |V|$

[1] We suppose that evaluating the operator is more expensive than the rounding procedures given in Fig. 8.8.

denotes the number of states where a nondeterministic action takes place. This is better than the bounds obtained in [BEGM15]. Indeed, even if we take into account that the accuracy parameter used in [BEGM15] can be improved by the estimates presented in Section 8.3, the complexity bound presented in [BEGM15] is worse, as it is given by $O\big(|V|^2|E|W(s2^s|V|^3M^{4s} + \log W)\big)$. We also note that in [BEGM17] the same authors gave another algorithm to solve this problem. The algorithm presented in [BEGM17] is based on a variant of the ellipsoid method an its complexity bounds are not given explicitly.

8.4.2 Finding the states with maximal value

In this section, we consider games that may not have a constant value. We show that a modification of the algorithms presented before can decide if a game has constant value or not and, in the latter case, find the states that have the maximal (or minimal) value. In order to do so, we need to prove an analogue of Corollary 8.23 for games that do not have a constant value. This requires to use the specific properties of Shapley operators given in Section 6.2. We let $\bar{\chi} \coloneqq \max_w \chi_w$ and $\underline{\chi} \coloneqq \min_w \chi_w$. Moreover, even though we do not suppose that the value is constant, we still use the quantities $D_{\mathrm{ub}} \coloneqq (n+m)M^{\min\{s,n+m-1\}}$, $R_{\mathrm{ub}} \coloneqq 10(n+m)^2WM^{\min\{s,n+m-1\}}$ in our complexity bounds. Our method is based on the next lemma.

Lemma 8.62. *Let $W \subset [n] \uplus [m]$ denote the set of all states of the game with maximal value. Then, for every $N \geqslant 1$ and $w \in W$ we have*

$$\frac{(T^N(0))_w}{N} \geqslant \bar{\chi} - \frac{R_{\mathrm{ub}}}{N} \, .$$

Similarly, if $W' \subset [n] \uplus [m]$ denotes the set of all states of the game with minimal value, then for all $N \geqslant 1$ and $w' \in W'$ we have

$$\frac{(T^N(0))_{w'}}{N} \leqslant \underline{\chi} + \frac{R_{\mathrm{ub}}}{N} \, .$$

Proof. Let $W \subset [n] \uplus [m]$ denote the set of all states of the game with maximal value. As shown in Lemma 6.14, W is a dominion (for Player Max), and the game induced by W is a constant value game with value equal to $\bar{\chi}$. Let $\tilde{T} \colon \mathbb{T}^W \to \mathbb{T}^W$ denote the Shapley operator of the induced game. Then, Lemma 8.55 shows that for every $\varepsilon > 0$ there exists a bias vector $\tilde{u} \in \mathbb{R}^W$ of $\tilde{T}$, $\tilde{T}(\tilde{u}) = \bar{\chi} + \tilde{u}$, such that $\|\tilde{u}\|_{\mathrm{H}} \leqslant R_{\mathrm{ub}} + \varepsilon$. Let $u \in \mathbb{R}^{n+m}$ be a vector defined as $u_w \coloneqq \tilde{u}_w$ for every $w \in W$ and $u_w \coloneqq -\infty$ otherwise. As in the proof of Theorem 6.16, observe that for all $w \in W$ we have $\bar{\chi} + u_w = (\tilde{T}(\tilde{u}))_w = (T(u))_w$ (because W is a dominion for Player Max) and $\bar{\chi} + u_w = -\infty \leqslant (T(u))_w$ for $w \notin W$. Hence $\bar{\chi} + u \leqslant T(u)$. Furthermore, we have $0 \geqslant -\mathbf{t}(u) + u$. Hence

$$T^N(0) \geqslant -\mathbf{t}(u) + T^N(u) \geqslant N\bar{\chi} - \mathbf{t}(u) + u$$

for all $N \geqslant 1$. Hence, for every state $w \in W$ we have

$$\frac{(T^N(0))_w}{N} \geqslant \bar{\chi} + \frac{-\mathbf{t}(u) + u_w}{N} \geqslant \bar{\chi} - \frac{\|\tilde{u}\|_{\mathrm{H}}}{N} \geqslant \bar{\chi} - \frac{R_{\mathrm{ub}}}{N} - \frac{\varepsilon}{N} \, .$$

Since $\varepsilon > 0$ was arbitrary, we obtain the claim. The proof of the other inequality is analogous.[2] $\square$

procedure CONSTANTVALUE(f, K)
  $\nu \coloneqq 0 \in \mathbb{R}^n$
  for $i = 1, 2, \dots, 2KR_{\mathrm{ub}}$ **do**
    $\nu \coloneqq$ ORACLE$(f, \nu, 8K^2 R_{\mathrm{ub}})$
  done
  if $\mathbf{t}(\nu) - \mathbf{b}(\nu) < 2R_{\mathrm{ub}} + 1$ **then**
    $\mathrm{ConstVal} \coloneqq$ `True`
  ▷ *the game has constant value*
  else
    $\mathrm{ConstVal} \coloneqq$ `False`
  ▷ *any state w such that $\nu_w = \mathbf{b}(\nu)$ satisfies $\chi_w < \overline{\chi}$*
  ▷ *any state w' such that $\nu_{w'} = \mathbf{t}(\nu)$ satisfies $\chi_w > \underline{\chi}$*
  end
  return $(\mathrm{ConstVal}, w, w')$, where $\nu_w = \mathbf{b}(\nu), \nu_{w'} = \mathbf{t}(\nu)$
end

Figure 8.9: Procedure that checks if a stochastic mean payoff game has constant value.

Thanks to Lemma 8.62, we can now give a procedure that checks if a game has constant value. This is given in the procedure CONSTANTVALUE presented in Fig. 8.9.

Lemma 8.63. *If we put $f \coloneqq T$ and $K \coloneqq 3D_{\mathrm{ub}}^2$, then the procedure* CONSTANTVALUE(f, K) *is correct.*

Proof. Let $N \coloneqq 2KR_{\mathrm{ub}}$, $\delta \coloneqq (8K^2 R_{\mathrm{ub}})^{-1}$, and take the vector ν obtained at the end of the loop. If the game has constant value, then Remark 8.35 shows that $\mathbf{t}(\nu) - \mathbf{b}(\nu) < 2R_{\mathrm{ub}} + 1$. Conversely, if $\overline{\chi} \neq \underline{\chi}$, then by Lemma 8.54 we have $\overline{\chi} - \underline{\chi} \geqslant 1/D_{\mathrm{ub}}^2$. Furthermore, by Lemmas 8.29 and 8.62, we have

$$\frac{\mathbf{t}(\nu)}{N} \geqslant \frac{\mathbf{t}(f^N(0))}{N} - \delta \geqslant \overline{\chi} - \frac{R_{\mathrm{ub}}}{N} - \delta\,.$$

Similarly, $\mathbf{b}(\nu) \leqslant N\underline{\chi} + R_{\mathrm{ub}} + N\delta$. Therefore

$$\begin{aligned}\mathbf{t}(\nu) - \mathbf{b}(\nu) &\geqslant N(\overline{\chi} - \underline{\chi}) - 2R_{\mathrm{ub}} - 2N\delta \\ &\geqslant \frac{2KR_{\mathrm{ub}}}{D_{\mathrm{ub}}^2} - 2R_{\mathrm{ub}} - \frac{1}{4K} > 4R_{\mathrm{ub}} - 1 \geqslant 2R_{\mathrm{ub}} + 1\,.\end{aligned}$$

Hence, the procedure correctly decides if the game has constant value. Moreover, if w is such that $\nu_w = \mathbf{t}(\nu)$, then by Lemma 8.29 we have

$$\begin{aligned}\frac{(f^N(0))_w}{N} &\geqslant \frac{\nu_w}{N} - \delta = \frac{\mathbf{t}(\nu)}{N} - \delta \geqslant \overline{\chi} - \frac{R_{\mathrm{ub}}}{N} - 2\delta \\ &\geqslant \underline{\chi} + \frac{1}{D_{\mathrm{ub}}^2} - \frac{R_{\mathrm{ub}}}{N} - 2\delta = \underline{\chi} + \frac{1}{D_{\mathrm{ub}}^2} - \frac{1}{6D_{\mathrm{ub}}^2} - \frac{1}{36D_{\mathrm{ub}}^2 R_{\mathrm{ub}}} \\ &> \underline{\chi} + \frac{1}{6D_{\mathrm{ub}}^2} = \underline{\chi} + \frac{R_{\mathrm{ub}}}{N}\,.\end{aligned}$$

Hence $\chi_w > \underline{\chi}$ by Lemma 8.62. The proof for w' such that $\nu_{w'} = \mathbf{b}(\nu)$ is analogous. □

[2]The proof requires to dualize the notion of a dominion, as discussed in Remark 6.19.

procedure Extend(Z)
  ▷ *$Z \subset V_{\text{Min}} \uplus V_{\text{Max}}$ is a nonempty subset of states of a stochastic mean payoff game*
  while `True` **do**
    if exists $k \in V_{\text{Min}} \setminus Z$, $a \in A^{(k)}$, $z \in Z$ such that $p^a_{kz} > 0$ **then**
      $Z \coloneqq Z \cup \{k\}$
    else
      if exists $i \in V_{\text{Max}} \setminus Z$ such that for all $b \in B^{(i)}$ there is $z_b \in Z$ with $p^b_{iz_b} > 0$ **then**
        $Z \coloneqq Z \cup \{i\}$
      else
        return Z
      end
    end
  done
end

Figure 8.10: Procedure that extends the set laying outside some dominion.

Remark 8.64. It is immediate to see that the complexity of the procedure above is the same as the complexity of ApproximateValue described in Theorem 8.58. Indeed, it makes $O(KR_{\text{ub}})$ calls to the Oracle$(f, \nu, 8K^2R_{\text{ub}})$ and every such call uses $O\big(a_{\text{ct}}(n+m)\big)$ arithmetic operations.

If the game does not have a constant value, then ConstantValue outputs a state that does not attain the maximal value. We now want to show how to find all such states. To this end we introduce the procedure Extend that may be used to enlarge the set of states that do not attain the maximal value. The following lemma explains the behavior of this procedure. We denote by $V \coloneqq V_{\text{Min}} \uplus V_{\text{Max}}$ the set of states of a stochastic mean payoff game.

Lemma 8.65. *Procedure* Extend *has the following properties. If $W \subset V$ is a dominion (for Player Max) and $Z \cap W = \emptyset$, then* Extend$(Z) \cap W = \emptyset$. *Furthermore, the set $V \setminus$* Extend(Z) *is a dominion.*

Proof. To prove the first claim, suppose that $Z \cap W = \emptyset$. If $k \in V_{\text{Min}} \cap W$ is a state belonging to W then, by definition, Player Min cannot leave W, i.e., $\sum_{w \in W} p^a_{kw} = 1$ for every action $a \in A^{(k)}$. In particular, k does not satisfy the condition of the first conditional statement of Extend. Likewise, if $i \in V_{\text{Max}} \cap W$ is a state belonging W, then Player Max has a possibility to stay in W, i.e., there is an action $b \in B^{(i)}$ such that $\sum_{w \in W} p^b_{iw} = 1$. In particular, i does not satisfy the condition of the second conditional statement of Extend. Hence, we have Extend$(Z) \cap W = \emptyset$. To prove the second statement, let $\tilde{W} \coloneqq V \setminus$ Extend(Z). Note that the procedure Extend stops only when both of its conditional statements are not satisfied. Therefore, for every state $k \in V_{\text{Min}} \cap \tilde{W}$ and every action $a \in A^{(k)}$ we have $\sum_{\tilde{w} \in \tilde{W}} p^a_{k\tilde{w}} = 1$. Moreover, for every state $i \in V_{\text{Max}} \cap \tilde{W}$ there is an action $b \in B^{(i)}$ such that $\sum_{\tilde{w} \in \tilde{W}} p^b_{iw} = 1$. Hence, $\tilde{W}$ is a dominion. □

Remark 8.66. We point out that Extend can be implemented to work in $O\big(a_{\text{ct}}(n+m)^2\big)$ complexity. Indeed, checking if either of the conditional statements is satisfied can be done by listing all the actions and, for every action, checking if the states to which this action may lead belong to Z. Furthermore, the procedure stops after at most $n + m$ of these verifications.

Remark 8.67. A more abstract, but equivalent, way of thinking about the procedure Extend is the following. Given Z, we define a vector $u \in \mathbb{T}^{n+m}$ as $u_w \coloneqq -\infty$ for $w \in Z$ and $u_w = 0$

```
procedure TopClass(T)
    ▷ V is the set of states of the game with operator T
    while True do
        (ConstVal, z, z') := ConstantValue(T, 3D_ub^2)
        if ConstVal = True then
            return V    ▷ V is the set of states that have the maximal value
        end
        V := V \ Extend(z)
        let T̃ denote the Shapley operator of the game induced by V
        T := T̃
    done
end
```

Figure 8.11: Procedure that finds the set of states with maximal value.

otherwise. We compute $T(u)$ and, if there is a state $w \notin Z$ such that $(T(u))_w = -\infty$, then we add w to Z.

Having the procedure Extend, we can now obtain an algorithm that finds the set of states having the maximal value. This is done by procedure TopClass presented in Fig. 8.11.

Theorem 8.68. *The procedure* TopClass *is correct and it finds the set of states with maximal value of a stochastic mean payoff game. Moreover, it performs* $O\big(a_{\mathrm{ct}}(n+m)^6 W M^{3\min\{s,n+m-1\}}\big)$ *arithmetic operations and every number that appears during these operations can be represented using* $O\Big((n+m)\log\big((n+m)MW\big)\Big)$ *bits of memory.*

Proof. Let W denote the set of all states with the maximal value. By Lemma 8.63, the procedure ConstantValue$(T, 3D_{\mathrm{ub}}^2)$ correctly decides if the game has constant value and, if not, it outputs a state $z \in V$ such that $z \notin W$. Furthermore, by Lemma 6.14, W is a dominion. Hence, by Lemma 8.65, the set $\tilde{V} := V \setminus$ Extend(z) is a dominion and $W \subset \tilde{V}$. Furthermore, note that the game induced by $\tilde{V}$ has the same set of states with the maximal value as the original game. Indeed, by Lemma 6.13, the value of every state in the game induced by $\tilde{V}$ is not greater than the value of the same state in the original game. Moreover, W is still a dominion in the game induced by $\tilde{V}$, and the game induced by W does not change when passing from the original game to the game induced by $\tilde{V}$. Hence, by Lemma 6.14, the value of every state in W is the same in all of these three games. Therefore, the problem of finding W reduces to the problem of finding the states with maximal value in the game induced by $\tilde{V}$. This shows that TopClass is correct. To show the complexity bound, note that the calls for ConstantValue are the most expensive operations in TopClass because a call for Extend and reducing the game can be done in $O\big(a_{\mathrm{ct}}(n+m)^2\big)$ complexity. Moreover, as noted in Remark 8.64, ConstantValue$(T, 3D_{\mathrm{ub}}^2)$ has the same asymptotic complexity as the problem of solving a stochastic mean payoff game presented in Theorem 8.58. Finally, TopClass does at most $n+m$ calls to ConstantValue$(T, 3D_{\mathrm{ub}}^2)$, hence the claimed bound. □

Remark 8.69. Given the procedure TopClass, one can easily find the maximal value of the game. Indeed, it is enough to first use TopClass to determine the set of states with maximal value, then restrict the game to this set of states (using the fact that this set is a dominion) and solve the remaining game by Theorem 8.58. The most expensive operation is TopClass, hence the asymptotic complexity of finding the maximal value is the same as the complexity of TopClass indicated above. Furthermore, one can dualize the procedure Extend to the dominions of Player Min. This leads to the procedure BottomClass that finds the minimal value and set of states that attain this value in the same complexity as TopClass.

Remark 8.70. In [BEGM15], it is shown that given an oracle solving TopClass, and another oracle that solves deterministic mean payoff games, one can solve arbitrary stochastic mean payoff games with pseudopolynomial number of calls to these oracles provided that the number of nondeterministic actions of the game is fixed. The authors of [BEGM15] present an oracle solving TopClass that is based on the pumping algorithm. Our analysis shows that this oracle can be replaced by an oracle based on value iteration.

Remark 8.71. As in the previous section (Remark 8.60), the results presented here extend to the case of bipartite games described by bipartite Shapley operators. Indeed, if $F\colon \mathbb{T}^V \to \mathbb{T}^V$ is such an operator, and the quantities $D_{\mathrm{ub}}, R_{\mathrm{ub}}$ are chosen accordingly to Example 8.38, then for any $N \geqslant 0$ and any state $k \in V_{\mathrm{Min}}$ with maximal value $\overline{\chi}$ we have

$$\frac{(F^N(0))_k}{N} \geqslant 2\overline{\chi} - \frac{R_{\mathrm{ub}}}{N} \, .$$

Similarly, for any state $l \in V_{\mathrm{Min}}$ with minimal value $\underline{\chi}$ we have

$$\frac{(F^N(0))_l}{N} \leqslant 2\underline{\chi} - \frac{R_{\mathrm{ub}}}{N} \, .$$

The proof of these inequalities proceeds in the same way as the proof Lemma 8.62. More precisely, if W is the set of states of game with maximal value, then $\tilde{W} \coloneqq V(W \cap V_{\mathrm{Min}})$ is a dominion and every state in the game induced by $\tilde{W}$ has value $\overline{\chi}$.[3] Therefore, if $\tilde{F}\colon \mathbb{T}^{W \cap V_{\mathrm{Min}}} \to \mathbb{T}^{W \cap V_{\mathrm{Min}}}$ is the bipartite Shapley operator of the induced game, then Lemmas 6.22 and 8.55 show that there is a bias vector $\tilde{u} \in \mathbb{R}^{W \cap V_{\mathrm{Min}}}$ such that $\tilde{F}(\tilde{u}) = 2\overline{\chi} + \tilde{u}$. Hence, for every $\varepsilon > 0$ we can take a bias vector $\tilde{u}$ such that $\|\tilde{u}\|_{\mathrm{H}} \leqslant R_{\mathrm{ub}} + \varepsilon$. As in the proof of Lemma 8.62, we define $u \in V_{\mathrm{Min}}$ as $u_k \coloneqq \tilde{u}_k$ if $k \in W \cap V_{\mathrm{Min}}$ and $u_k = -\infty$ otherwise, and we observe that $2\overline{\chi} + u \leqslant F(u)$. This gives the desired inequality. Furthermore, by applying Theorem 8.37 to $\tilde{F}$ we get that the denominator of $2\overline{\chi}$ is bounded by D_{ub} and, similarly, the denominator of $2\underline{\chi}$ is bounded by D_{ub}. Hence, we can use the procedure ConstantValue for F to decide if the game has constant value. We can also easily adapt the procedure Extend to F (e.g., by using the observation given in Remark 8.67). As a result, we can find the maximal (or minimal) value of the game and all the states controlled by Player Min that attain it in $O(nD_{\mathrm{ub}}^2 R_{\mathrm{ub}} c_{\mathrm{eval}})$ arithmetic operations, where c_{eval} and $D_{\mathrm{ub}}, R_{\mathrm{ub}}$ are as in Remark 8.60, and all the numbers that occur during these operations can be represented using $O(\log(MD_{\mathrm{ub}}R_{\mathrm{ub}}W))$ bits of memory.

8.5 Application to nonarchimedean semidefinite programming

Let us now apply the complexity results of the previous section to the problem of tropical Metzler semidefinite feasibility Tmsdfp considered in Chapter 7. For simplicity, we will apply our results to the case of matrices that define a well-formed linear matrix inequality. This gives the following theorem.

[3] This observation already appeared in the proof of Corollary 6.25.

Theorem 8.72. *Suppose that we are given symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$ such that $|Q^{(k)}| \in (\mathbb{Q} \cup \{-\infty\})^{m\times m}$ for all $k \in [n]$. Furthermore, suppose that these matrices define a well-formed linear matrix inequality. Let L denote the common denominator of all the finite entries of $|Q^{(k)}|_{k\in[n]}$ and let W denote the maximal absolute value of the entries of $|Q^{(k)}|_{k\in[n]}$. Furthermore, let $\mathcal{S}_\lambda \coloneqq \mathcal{S}_\lambda(Q^{(1)}, \dots, Q^{(n)})$ be as in Definition 4.16. Then, the highest value of $\lambda \in \mathbb{R}$ such that $\mathcal{S}_\lambda$ is nontrivial (resp. contains a real point) can be found in*

$$O(8^s n^6 m^2 LW)$$

arithmetic operations, where $s \leqslant n$ is the number of matrices $Q^{(k)}$ that are not (tropically) diagonal, i.e., have a finite off-diagonal entry. Moreover, another algorithms solves the same problem in

$$O(8^{\min\{s',m-1\}} nm^7 LW)$$

arithmetic operations, where s' denotes the number of pairs (i,j) with $i < j$ such that the tropical polynomial Q_{ij} is nonzero (i.e., the pairs such that $|Q_{ij}^{(k)}|$ is finite for at least one $k \in [n]$).

Proof. Let us start by showing the first algorithm. By Theorem 7.21, the problem of finding the valued of λ reduces to the problem of finding the maximal and minimal value of a stochastic mean payoff game. Furthermore, the game is constructed as given in Section 7.2, and its bipartite Shapley operator $F \colon \mathbb{T}^n \to \mathbb{T}^n$ is given in (7.2). The payoffs of this game correspond to the finite entries of $|Q^{(k)}|_{k\in[n]}$. Therefore, if we multiply these payoffs by L, then the maximal and minimal value of the game is multiplied by L, and we obtain a game with integer payoffs as considered in the earlier sections. Let $\tilde{W} \coloneqq LW$ denote the maximal absolute value of a payoff in this game. Therefore, as noted in Remark 8.71 the task of finding the maximal (or minimal) value of the game can be solved in $O(nD_{\mathrm{ub}}^2 R_{\mathrm{ub}} c_{\mathrm{eval}})$ arithmetic operations. In order to make this quantities explicit, first note (as in Remark 8.60) that evaluating the operator F of the form (7.2) can be done in $O(nm^2)$ arithmetic operations. Moreover, since the common denominator of every probability of this game is equal to 2, as in Example 8.38 we obtain from Theorem 8.37 that $D_{\mathrm{ub}} = O(n2^{\min\{s+1,n\}}) = O(n2^s)$ and $R_{\mathrm{ub}} = O(n^2\tilde{W}2^{\min\{n_r,n-1\}}) = O(n^2\tilde{W}2^s)$, where s is the number of deterministic states of Player Min. Moreover, note that a state $k \in [n]$ is deterministic if and only if the matrix $Q^{(k)}$ is (tropically) diagonal. This gives the first complexity bound. To obtain the second bound, note that the game can be reversed in the following way. We multiply all the payoffs by (-1) and we exchange the roles of Players Min and Max (so that Player Min in the original game becomes Player Max in the reversed game and vice versa). As a result, the value vector of the game gets multiplied by -1. Moreover, the game is still bipartite, and its bipartite Shapley operator $\tilde{F} \colon \mathbb{T}^m \to \mathbb{T}^m$ is given by

$$\forall i \in [m], (\tilde{F}(y))_i \coloneqq \min_{Q_{ii}^{(k)} \in \mathbb{T}_+} \Big(-Q_{ii}^{(k)} + \max_{Q_{i'j'}^{(k)} \in \mathbb{T}_-} \big(|Q_{i'j'}^{(k)}| + \frac{1}{2}y_{i'} + \frac{1}{2}y_{j'}\big)\Big) .$$

This operator can also can be evaluated in $c_{\mathrm{eval}} = O(nm^2)$ arithmetic operations. Furthermore, using the last part of Example 8.38, we obtain that the bounds for this operator read $D'_{\mathrm{ub}} = O(m2^{\min\{s',m-1\}})$ and $R'_{\mathrm{ub}} = O(m^2\tilde{W}2^{\min\{s',m-1\}})$, where s' is the number of nonidentical probability distributions of the game, which correspond to the number of pairs (i,j) with $i < j$ such that $|Q_{ij}^{(k)}|$ is finite for at least one $k \in [n]$. $\square$

8.5.1 A class of ergodic problems

In this section, we present a class of tropical Metzler semidefinite feasibility problems that gives rise to ergodic games. As a result, we obtain a class of feasibility problems that can be solved in a complexity depending on the condition number of the problem. We suppose that we are given tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ that fulfill the following assumption.

Assumption H. We have $Q_{ii}^{(k)} \in \mathbb{T}_{+}$ and $Q_{ij}^{(k)} \in \mathbb{T}_{-}$ for all $k \in [n]$ and $i, j \in [m]$, $i \neq j$. In other words, the matrices $Q^{(k)}$ do not have $-\infty$ entries, they have only tropically positive entries on the diagonals, and only tropically negative off-diagonal entries.

We note that if $m \geqslant 2$, then Assumption H implies Assumptions D and E.

Remark 8.73. The assumption that all diagonal entries of $Q^{(k)}$ are positive can be relaxed—it is enough to suppose that at least one entry is positive (and none of them is equal to $-\infty$). However, the class of matrices that satisfy Assumption H has an interesting symmetry shown in our experimental results presented in Section 8.5.2.

Lemma 8.74. *Suppose that $m \geqslant 2$ and that the matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ satisfy Assumption H. Then, the associated stochastic mean payoff game is ergodic.*

Proof. As noted above, the matrices $Q^{(1)}, \dots, Q^{(n)}$ satisfy Assumptions D and E. By construction, the associated stochastic mean game has n states controlled by Player Min, $V_{\mathrm{Min}} := [n]$, and$m$ states controlled by Player Max, $V_{\mathrm{Max}} := [m]$. Furthermore, at every state $k \in [n]$, Player Min has $m(m-1)/2$ actions of the form $a = (k, i, j)$ with $i, j \in [m]$, $i \neq j$. Moreover, for every such action, the associated probability distribution is given by $p_{ki}^{a} = p_{kj}^{a} = 1/2$ and the payoff is given by $r_a = -|Q_{ij}^{(k)}|$. Similarly, at every state $i \in [m]$, Player Max has n deterministic actions of the form $b = (i, k)$, $p_{ik}^{b} = 1$, and the associated payoff is given by $Q_{ii}^{(k)}$. Hence, to prove that this game is ergodic, it is enough to prove that it has a constant value for all choices of the matrices $Q^{(1)}, \dots, Q^{(n)}$. Suppose that this is not the case and let $\underline{\chi}, \bar{\chi}$ denote, respectively, the minimal and maximal value of this game, $\underline{\chi} \neq \bar{\chi}$. Let (σ, τ) denote a couple of optimal policies in this game. By Theorem 2.137, the Markov chain induced by (σ, τ) has two recurrent classes C_1, C_2 such that every state belonging to C_1 has value $\underline{\chi}$ and every state belonging to C_2 has value $\bar{\chi}$. Since the game is bipartite, C_1 contains a state $i \in [m]$ controlled by Player Max and C_2 contains a state $k \in [n]$ controlled by Player Min. Moreover, by construction, Player Max has an action $b = (i, k)$. Let $\tilde{\tau}$ denote the modification of τ defined as follows. We put $\tilde{\tau}(j) := \tau(j)$ for every $j \in [m]$ such that $j \neq i$ and $\tilde{\tau}(i) := k$. Consider the Markov chain induced by $(\sigma, \tilde{\tau})$. Note that C_2 is a recurrent class of this chain (because $i \notin C_2$). Moreover, if this chain starts at i, then it reaches C_2 is one step. Therefore, by Theorem 2.137, the average payoff $(g(\sigma, \tilde{\tau}))_i$ of Player Max in the game starting from i is equal to $\bar{\chi}$. In particular, we have $\bar{\chi} = (g(\sigma, \tilde{\tau}))_i > \underline{\chi} = (g(\sigma, \tau))_i$, which gives a contradiction with the fact that σ is an optimal policy of Player Min. $\square$

As a corollary, we get the following complexity result about solving tropical Metzler semidefinite feasibility problems.

Proposition 8.75. *Suppose that the symmetric tropical Metzler matrices $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m \times m}$ are such that $|Q^{(k)}| \in (\mathbb{Q} \cup \{-\infty\})^{m \times m}$ for all $k \in [n]$. Furthermore, suppose that $m \geqslant 2$ and that the matrices satisfy Assumption H. Let*

$$\bar{\lambda} := \sup\{\lambda \in \mathbb{R} \colon \mathcal{S}_{\lambda}(Q^{(1)}, \dots, Q^{(n)}) \text{ is nontrivial}\}$$

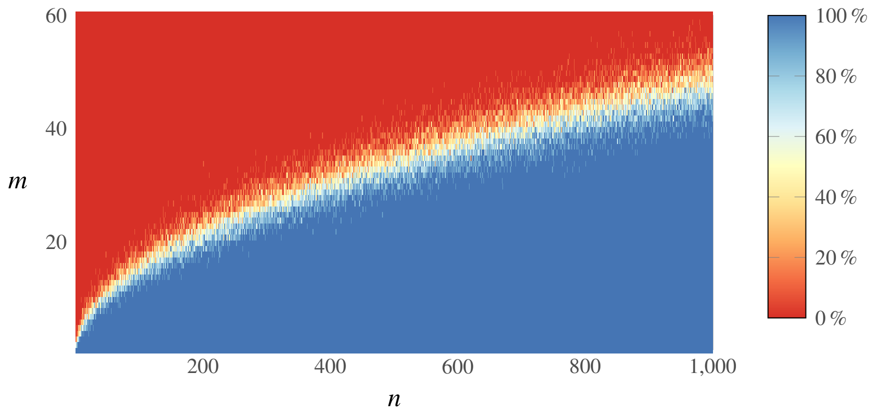

Figure 8.12: Phase transition between feasibility and infeasibility of spectrahedra. For each (n, m), the color scheme reports the ratio of feasible instances among 10 samples.

and let $F\colon \mathbb{T}^n \to \mathbb{T}^n$ *denote the Shapley operator of the associated stochastic mean payoff game. If* $\bar{\lambda} \neq 0$*, then the problem of finding the sign of* $\bar{\lambda}$ *can be solved in*

$$O((1 + \frac{R}{|\bar{\lambda}|})nm^2)$$

arithmetic operations, where $R = \inf\{\|u\|_{\mathrm{H}} \colon F(u) = \bar{\lambda} + u\}$*.*

Proof. By Lemma 8.74, the stochastic mean payoff game associated with $Q^{(1)}, \dots, Q^{(n)}$ is ergodic. Denote this value by χ. Then, by Lemmas 6.22 and 8.55 there is a vector $u \in \mathbb{R}^n$ such that $F(u) = 2\chi + u$. Moreover, we have $\bar{\lambda} = 2\chi$ by Theorem 7.21. As in the proof of Theorem 8.72, we observe that the cost of evaluating F is bounded by $O(nm^2)$. Hence, the claim follows from Remark 8.60. □

8.5.2 Experimental results

To finish, we present some numerical results for solving the class of tropical Metzler semidefinite problems defined by matrices that satisfy Assumption H. We chose all the $|Q_{ij}^{(k)}|$, for $i \leqslant j$, to be independent random variables uniformly distributed on $[0, 1]$. We created a simple floating point implementation of the procedure ValueIteration(F) given in Fig. 8.4, replacing the termination criterion $\mathbf{b}(\nu) > 0$ or $\mathbf{t}(\nu) < 0$ by $\mathbf{b}(\nu) > \varepsilon$ and $\mathbf{t}(\nu) < -\varepsilon$, where the precision was chosen as $\varepsilon := 10^{-8}$ (the performance was similar for $\varepsilon = 10^{-6}$ or $\varepsilon = 10^{-10}$). Even though we have not used the rounding techniques established in Section 8.2.1, our algorithm performs well on random instances. Furthermore, the validity of the floating point certificate of (in)feasibility provided by our program can be checked a posteriori by computing one evaluation of the operator F is exact arithmetic. We report in Table 8.1 experimental results for different values of (n, m). Our experiments were obtained using a C program, distributed as [AGS16a]. This program was compiled under Linux with gcc -O3, and executed on a single core of an Intel(R) i7-4600U CPU at 2.10 GHz with 16 GB RAM. We report the average execution time over 10 samples for every value of (n, m). The number of iterations did not exceed 731 on this benchmark, and, for most (n, m), it was limited to a few units. Furthermore, we observed that random instances exhibit experimentally a phase transition, as shown in Fig. 8.12: for a given (n, m), the system

(n,m)	$(50,10)$	$(50,40)$	$(50,50)$	$(50,100)$	$(50,1000)$
time	0.000065	0.000049	0.000077	0.000279	0.026802
(n,m)	$(100,10)$	$(100,15)$	$(100,80)$	$(100,100)$	$(100,1000)$
time	0.000025	0.000270	0.000366	0.000656	0.053944
(n,m)	$(1000,10)$	$(1000,50)$	$(1000,100)$	$(1000,200)$	$(1000,500)$
time	0.000233	0.073544	0.015305	0.027762	0.148714
(n,m)	$(2000,10)$	$(2000,70)$	$(2000,100)$	$(10000,150)$	$(10000,400)$
time	0.000487	1.852221	0.087536	19.919844	2.309174

Table 8.1: Execution time (in sec.) of Procedure ValueIteration on random instances.

is either feasible with overwhelming probability, or infeasible with overwhelming probability, unless (n,m) lies in a tiny region of the parameter space. Value iteration quickly decides feasibility, except in regions close to the phase transition. This explains why the execution time does not increase monotonically with (n,m) in our experiments (we included both easy and hard values of (n,m)).

Example 8.76. When applied to applied to the matrices given in Example 7.22, our program terminates in 20 iterations. It returns a vector having floating point entries

$$(1.05612\dots, 0.0204082\dots, 1.12755\dots)\,.$$

When converted into a vector with rational entries, it reads

$$x := \Big(1107425 \times 2^{-20}, 42799 \times 2^{-21}, 4729289 \times 2^{-22}\Big)\,.$$

We have checked using exact precision arithmetic over rationals (provided by the GNU multiple precision arithmetic library, `https://gmplib.org/`) that this vector lies in the interior of the tropical spectrahedron shown in Fig. 7.2, i.e., $x_k < (F(x))_k$ for $k \in \{1,2,3\}$. Following Lemma 4.50, the vector $\boldsymbol{x} := (t^{v_1}, t^{v_2}, t^{v_3})$ is such that $\boldsymbol{Q}(\boldsymbol{x})$ is positive definite.

CHAPTER 9

Perspectives

To finish this dissertation, let us discuss some open problems and possible directions for future research.

Tropicalization of convex semialgebraic sets The first open problem arising from of this thesis is to give a complete characterization of all tropical spectrahedra. In this work, we introduced the class of tropical Metzler spectrahedra, and used it to characterize all generic tropical spectrahedra (Chapter 4). However, we do not know if every tropical spectrahedron (even nongeneric one) can be represented as a set of the form studied in Section 4.2. A complete characterization is known for the subclass of tropical polyhedra. However, it relies on the tropical Minkowski–Weyl theorem [GK07, BSS07] and it does not carry over to spectrahedra.

Furthermore, we note that our variant of the tropical Helton–Nie conjecture (Chapter 5) is based on the simplest possible tropicalization—looking at the image of Puiseux series by their ordinary valuations. It is possible that more sophisticated tropicalizations, capturing also the sign, or higher order approximations of Puiseux series (spaces of jets) may be exploited in order to obtain a better understanding of projected spectrahedra or to construct counterexamples to the Helton–Nie conjecture.

A natural extension of this thesis would be to study the tropicalization of hyperbolicity cones. As indicated in Section 1.1, hyperbolicity cones generalize the spectrahedral cones (in the sense that every full-dimensional spectrahedral cone is a hyperbolicity cone [Ren06]). The generalized Lax conjecture asks if the converse is true, i.e., if every hyperbolicity cone is a spectrahedron. Helton and Vinnikov [HV07] proved that this conjecture is true in dimension

$n = 3$ (see also [LPR05, PV13]). Furthermore, the conjecture is true for some special classes of hyperbolic cones [AB18, Kum17]. On the other hand, some stronger versions of this conjecture are known to be false. A counterexample to one such generalization was constructed by Brändén [Brä11], and more counterexamples were given in [BVY14, AB18]. One could try to tropicalize the hyperbolicity cones over Puiseux series in order to get more insight into their structure. An important ingredient would be to study the Netwon polytopes of hyperbolic polynomials. Some properties of these polytopes have already been used in the aforementioned works (see, e.g., [Brä11]), but this study was limited to the case of stable hyperbolic polynomials. This approach to hyperbolicity cones could also shed some light on the structure of tropical spectrahedra. A drawback of the techniques that we used to study the tropicalization of spectrahedra is that they do not give much insight into the algebraic structure of the boundaries of spectrahedra. This boundary occurs more naturally when one considers a hyperbolicity cone and its defining polynomial.

A more abstract question can be asked about the tropicalizations of arbitrary convex semialgebraic sets. More precisely, we do not know if there is an analogue of the tropical basis theorem for convex semialgebraic sets. The tropical basis theorem (see, e.g., [MS15, Section 3.2]) states that if $S \subset (\mathscr{K}^*)^n$ is an algebraic variety (over the field of Puiseux series with complex coefficients) defined by some ideal I, then there exists a finite collection of polynomials belonging to I such that $\mathsf{val}(S)$ is the intersection of the tropical hypersurfaces associated with these polynomials. A natural extension to semialgebraic sets would be to replace the ideal with the cone of polynomials that are nonnegative on a given set. We do not know if one can choose a finite collection of these polynomials and describe the tropicalization of a semialgebraic set using only this collection (even if we assume that the semialgebraic set under question is convex). If one restricts attention to convex sets, and is allowed to choose infinite collections, then the answer is positive by the hyperplane separation theorem. More precisely, we have the following proposition.

Proposition 9.1. *Suppose that $\boldsymbol{S} \subset \mathbb{K}^n_{>0}$ is a convex semialgebraic set. Let $\mathcal{P}$ denote the set of all affine polynomials $\boldsymbol{P} \in \mathbb{K}[X_1, \dots, X_n]$ that are nonnegative on $\boldsymbol{S}$. Then*

$$\mathsf{val}(\boldsymbol{S}) = \bigcap_{\boldsymbol{P} \in \mathcal{P}} \{x \in \mathbb{R}^n \colon \operatorname{trop}(\boldsymbol{P})^+(x) \geqslant \operatorname{trop}(\boldsymbol{P})^-(x)\} .$$

However, it is obvious that the collection on the right-hand side has to be infinite in general (otherwise, it would describe a tropical polyhedron). Therefore, in order to find a finite collection, one has to add polynomials of higher degrees to $\mathcal{P}$.

Realizability of oriented matroids by tropical matrices Another idea would be to study the combinatorial types of polyhedra using tropical geometry. It has been shown in [ABGJ15] that, under a genericity assumption, the combinatorial type of a nonarchimedean polyhedron can be deduced from the valuation of its defining matrix. It is not known which types of polyhedra arise in this way. If this class of polyhedra is much smaller that the class of all polyhedra, then one may hope to improve the algorithms for solving deterministic mean payoff games using the correspondence with linear programming over the Puiseux series [ABGJ15, ABGJ14]. On the other hand, if this class of polyhedra is not easier to handle than the general case, then one may obtain a tighter connection between Smale's ninth problem [Sma98] and mean payoff games than the one presented in [ABGJ14]. A particular question that can be asked about such polyhedra is whether they satisfy the polynomial Hirsch conjecture (see [San12] for more information about this conjecture). As a first step towards the study of these polyhedra, one can study the realizable uniform oriented matroids that arise from generic signed tropical matrices. Currently, we have the following proposition.

Proposition 9.2. *All uniform oriented matroids of rank* 3 *with at most* 7 *elements are realizable by generic signed tropical matrices.*

We note that there are only 11 oriented matroids of rank 3 with 7 elements. We have proven this proposition by enumerating them by hand—a more systematic approach is needed to handle the larger cases (there are 135 uniform matroids of rank 3 with 8 elements).

Homotopy methods for feasibility problems In Chapter 7, we have shown that generic nonarchimedean semidefinite feasibility problems are solvable using combinatorial algorithms for stochastic mean payoff games. Moreover, as discussed in Section 8.1, a spectrahedron over Puiseux series is in fact a one-parameter family of real spectrahedra, and the nonarchimedean problem corresponds to the study of the limiting behavior of this family. Since we understand the limiting behavior of this problem, it is natural to ask what happens along the way (as the formal parameter increases), i.e, how the semidefinite feasibility problem degenerates into a combinatorial problem. This could lead to a "homotopy-like" method for solving some class of real feasibility problems (such a method would first solve a combinatorial problem and then try to decrease the value of the parameter). This problem may be interesting even in the case of linear programming, giving more insight into the combinatorial structure of polyhedra mentioned in the previous paragraph.

Value iteration In Chapter 8, we studied value iteration and its condition number. There are a few directions in which this results can be extended. First, as already noted in Section 8.2, our analysis presented in that section applies to Shapley operators of other classes of games, such as the entropy games of [ACD$^+$16, AGGCG17]. One may study the complexity bounds that can be obtained for this class of games using our approach. Second, as discussed in Remark 8.70, Boros et al. [BEGM15] gave an algorithm that can solve general stochastic mean payoff games in pseudopolynomial time when the number of randomized actions is fixes. However, this algorithm is quite complicated. One can ask if there is a simpler algorithm that obtains the same complexity bounds—for instance, if one can modify value iteration in a way that gives such a general algorithm. (A naive extension of value iteration does not have the desired properties by the counterexample presented in [BEGM13].) Third, the value iteration algorithm that we presented here converges to the value of the game at rate $1/N$. By analogy with the accelerated gradient descent, one can ask if there exists an accelerated value iteration that converges at rate $1/N^2$. Such acceleration would only slightly improve the asymptotic complexity bounds, but it may be interesting to study because we do not have information-theoretic lower bounds for the rate of convergence of such an algorithm. Finally, in Section 8.5.2 we have seen that value iteration is fast on random instances. Nevertheless, the behavior of such instances is not well understood from the theoretical perspective. For instance, it is an open question to formally prove that random instances of stochastic mean payoff games are well conditioned. Furthermore, it is clear from Fig. 8.12 that the random instances studied in Section 8.5.2 exhibit a phase transition. It would be interesting to prove this statement formally and explain the shape of the boundary between the two regions visible in Fig. 8.12.

Games with nonarchimedean payoffs and/or probabilities One could also try to study the stochastic mean payoff games using nonarchimedean perspective. More precisely, one can replace the numerical data associated with such a game (i.e., the payoffs and/or probabilities) by Puiseux series. The idea of considering nonarchimedean payoffs is already present in the literature. Indeed, parity games can be interpreted as deterministic mean payoff games with nonarchimedean payoffs [Jur98]. However, parity games form a very special subclass of games with nonarchimedean payoffs. The payoffs in these games have the property that any two payoffs with the same valuation are equal. As mentioned in Section 1.1, after the breakthrough of Calude et al. [CJK$^+$17], we now have a few different algorithms that solve parity games in quasipolynomial time (see [CJK$^+$17, JL17, Leh18, FJS$^+$17]). It is natural to ask if the quasipolynomial-time result can be extended to a wider class of games with nonarchimedean payoffs. Furthermore, the algorithm of Calude et al. relies on the fact that, given a play of a parity game, the winner of this play can be decided by an algorithm that uses only polylogarithmic memory. It would also be interesting to obtain a negative result in this direction, i.e., to show that deciding a winner of a more general game cannot be done using so little memory.

The games with nonarchimedean probabilities have not yet been considered. The simplest idea in this direction is to consider the class of games without payoffs (reachability games) with nonarchimedean probabilities, and such that one is given only the valuations of the input probabilities. The aim would be to compute the valuation of the probability of winning this game. It can be proved that this is a meaningful model, i.e., that the valuation of the probability of winning is determined by the valuations of input probabilities (this uses an extension of the Freidlin–Wentzell formula discussed in Section 8.3 to the case of absorption probabilities). Furthermore, a 0-player variant of this game (a game obtained if the strategies of both players are fixed) is solvable in polynomial time. This case is already nontrivial—it turns out that solving the 0-player case reduces to the Optimal Arborescence Problem [Edm67, Kar71] with an additional constraint on the existence of some directed path in the arborescence. Fortunately, the techniques used to solve the usual optimal arborescence problem extend to this case. It is an open question to obtain a complexity result for the 1-player case of these games.

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APPENDIX $\mathcal{A}$

Fields of convergent power series

In this appendix, we discuss different fields of convergent power series such as the Puiseux series and Hahn series. We also present the proof of van den Dries and Speissegger [vdDS98], who showed that different fields of generalized power series are real closed (or algebraically closed in the case of complex coefficients).

A.1 Basic definitions

Before presenting the definitions Puiseux and Hahn series, let us recall the definition of a well-ordered set.

Definition A.1. A set $\Gamma \subset \mathbb{R}$ is *well ordered* if every nonempty subset of Γ has a maximal element (in the usual order of the reals).

Lemma A.2. *If Γ is well ordered, then it is finite or countable.*

Sketch of the proof. Suppose that Γ is well ordered. For each $\gamma \in \Gamma$ that is not the smallest element of Γ, let $\bar{\gamma} \in \Gamma$ denote the maximal element of the set $\Gamma \cap]\gamma, -\infty[$. Then, the interval $]\gamma, \bar{\gamma}[$ contains a rational number. This gives an injection from Γ to the rational numbers. $\square$

Definition A.3. We say that $\Gamma \subset \mathbb{R}$ *has no limit points* if, for any $a, b \in \mathbb{R}$, $a < b$, the set $[a, b] \cap \Gamma$ is either empty or finite.

Lemma A.4. *Suppose that a set $\Gamma \subset \mathbb{R}$ has a maximal element and has no limit points. Then, this set is well ordered and can be written as $\Gamma = \{\gamma_0, \gamma_1, \dots\}$, where the sequence $(\gamma_0, \gamma_1, \dots)$ is strictly decreasing and either finite or unbounded.*

Sketch of the proof. Suppose that Γ is infinite. Since Γ has a maximal element and has no limit points, any strictly increasing sequence $(\gamma_0, \gamma_1, \dots)$ of elements of Γ is finite. In particular, Γ is well ordered. Let γ_0 denote the maximal element of Γ. The set $\Gamma \setminus \{\gamma_0\}$ has a maximal element. Denote it by γ_1. By repeating this procedure, we obtain an infinite strictly decreasing sequence $(\gamma_0, \gamma_1, \dots)$ of elements in Γ. Since Γ has no limit points, the sequence $(\gamma_0, \gamma_1, \dots)$ is unbounded and we have the equality $\Gamma = \{\gamma_0, \gamma_1, \dots\}$. ☐

Definition A.5. Consider a formal series

$$\boldsymbol{x} = \sum_{\gamma \in \Gamma} c_\gamma t^\gamma \tag{A.1}$$

where t is a formal parameter, the coefficients c_γ are nonzero and real, $c_\gamma \in \mathbb{R} \setminus \{0\}$, and the set $\Gamma \subset \mathbb{R}$ has a maximal element. Then, such a series is called:

- a *Puiseux series* if Γ is a subset of rational numbers, $\Gamma \subset \mathbb{Q}$, and every element of Γ has the same denominator;
- a *generalized Puiseux series* if Γ has no limit points;
- a *Hahn series* if Γ is well ordered.

By definition and Lemma A.4, the set of Hahn series contains the set of generalized Puiseux series, and the set of generalized Puiseux series contains the set of Puiseux series.

The *ring of Hahn series* is defined as the set of all Hahn series together with special empty series 0, equipped with the natural definition of addition and multiplication (i.e., we define the multiplication of two series by their Cauchy product). We analogously define the rings of Puiseux series and generalized Puiseux series. It can be checked that these three objects are, indeed, rings.

Definition A.6. Suppose that $\boldsymbol{x} = \sum_{\gamma \in \Gamma_1} c_\gamma t^\gamma$ and $\boldsymbol{y} = \sum_{\gamma \in \Gamma_2} d_\gamma t^\gamma$ are Hahn series. We define their sum as follows. For every $\gamma \in \mathbb{R}$ we define $f_\gamma \in \mathbb{R}$ as $f_\gamma \coloneqq c_\gamma + d_\gamma$ with the convention that $c_\gamma = 0$ if $\gamma \notin \Gamma_1$ and $d_\gamma = 0$ if $\gamma \notin \Gamma_2$. We take $\Gamma_+ \coloneqq \{\gamma \in \mathbb{R} \colon f_\gamma \neq 0\}$ and we put $\boldsymbol{x} + \boldsymbol{y} \coloneqq \sum_{\gamma \in \Gamma_+} f_\gamma t^\gamma$. Furthermore, we define the product of $\boldsymbol{x}$ and $\boldsymbol{y}$ in the following way. For every $\gamma \in \mathbb{R}$ we define

$$g_\gamma \coloneqq \sum_{\substack{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \\ \gamma_1 + \gamma_2 = \gamma}} c_{\gamma_1} d_{\gamma_2} \,,$$

with the convention that an empty sum is equal to 0. We take $\Gamma_\times \coloneqq \{\gamma \in \mathbb{R} \colon g_\gamma \neq 0\}$ and we put $\boldsymbol{x}\boldsymbol{y} \coloneqq \sum_{\gamma \in \Gamma_\times} g_\gamma t^\gamma$.

Lemma A.7. *The addition and multiplication of Hahn series (resp. Puiseux series, generalized Puiseux series) is well defined.*

Lemma A.8. *The set of Hahn series (resp. Puiseux series, generalized Puiseux series) forms a commutative ring with unity.*

We leave the proof of this lemmas as a (tedious but easy) exercise.[1]

[1]For instance, proving this lemmas requires to check that the sums in the definition of g_γ are finite, that the sets of exponents $\Gamma_+, \Gamma_\times$ are well ordered, that the sum/product of two (generalized) Puiseux series is again a (generalized) Puiseux series and so on.

Remark A.9. One can consider a slightly more general situation that the one presented in Definition A.5. More precisely, one can assume that the coefficients c_γ as in (A.1) belong to some fixed real closed subfield of $\mathbb{R}$ (such as the field of algebraic numbers). Furthermore, for generalized Puiseux series and Hahn series, one can also assume that the sets of exponents Γ belong to some fixed nontrivial divisible ordered abelian subgroup of $(\mathbb{R}, +)$ (such as the group of rational numbers or the group of algebraic numbers). All the proofs presented in this appendix are valid in this more general situation.

Definition A.10. If $\boldsymbol{x}$ is a Hahn series as in (A.1), and $t \in \mathbb{R}_{>0}$ is a positive number, then we define

$$\|\boldsymbol{x}\|_t \coloneqq \sum_{\gamma \in \Gamma} |c_\gamma| t^\gamma \,. \tag{A.2}$$

We say that a Hahn series $\boldsymbol{x}$ is *(absolutely) convergent* if $\|\boldsymbol{x}\|_t < +\infty$ for all t large enough.

Remark A.11. We recall that (see, e.g., [Bro96, Section 2.5]), since Γ is at most countable by Lemma A.2, the sum from (A.2) does not depend on the order of summation and is either finite or equal to $+\infty$. Furthermore, for any t satisfying $\|\boldsymbol{x}\|_t < +\infty$, the series $\boldsymbol{x}$ evaluated at t is convergent (i.e., the sum from (A.1) evaluated at t is well defined, finite, and does not depend on the order of summation). We denote by $\boldsymbol{x}(t) \in \mathbb{R}$ the value of this series. (With the convention that if $\boldsymbol{x} = 0$, then $\boldsymbol{x}(t) = 0$ for all $t > 0$.) We note that this evaluation is compatible with the ring structure on K, i.e., that we have $(\boldsymbol{x} + \boldsymbol{y})(t) = \boldsymbol{x}(t) + \boldsymbol{y}(t)$ and $(\boldsymbol{x}\boldsymbol{y})(t) = \boldsymbol{x}(t)\boldsymbol{y}(t)$ for all $\boldsymbol{x}, \boldsymbol{y} \in K$ and t such that $\|\boldsymbol{x}\|_t < +\infty, \|\boldsymbol{y}\|_t < +\infty$.

The following lemma shows that the set of absolutely convergent Hahn series is closed by addition and multiplication.

Lemma A.12. *Suppose that $\boldsymbol{x}, \boldsymbol{y}$ are Hahn series and that $t \in \mathbb{R}_{>0}$ is a positive number such that $\|\boldsymbol{x}\|_t, \|\boldsymbol{y}\|_t < +\infty$. Then, we have $\|\boldsymbol{x} + \boldsymbol{y}\|_t \leqslant \|\boldsymbol{x}\|_t + \|\boldsymbol{y}\|_t$ and $\|\boldsymbol{x}\boldsymbol{y}\|_t \leqslant \|\boldsymbol{x}\|_t \|\boldsymbol{y}\|_t$.*

Proof. The first part follows immediately from the triangle inequality. To prove the second one, denote $\boldsymbol{x} = \sum_{\gamma_1 \in \Gamma_1} c_{\gamma_1} t^{\gamma_1}$, $\boldsymbol{y} = \sum_{\gamma_2 \in \Gamma_2} d_{\gamma_2} t^{\gamma_2}$. Then

$$\|\boldsymbol{x}\boldsymbol{y}\|_t \leqslant \sum_{\gamma \in \mathbb{R}} \sum_{\substack{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \\ \gamma_1 + \gamma_2 = \gamma}} |c_{\gamma_1} d_{\gamma_2}| t^{\gamma_1 + \gamma_2} = \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2} |c_{\gamma_1} d_{\gamma_2}| t^{\gamma_1 + \gamma_2} = \|\boldsymbol{x}\|_t \|\boldsymbol{y}\|_t \,,$$

by the triangle inequality and the fact that the sums of nonnegative numbers can be arbitrarily rearranged (see [Bro96, Theorem 2.55]). □

Corollary A.13. *The set of absolutely convergent Hahn series (resp. Puiseux series, generalized Puiseux series) forms a commutative ring with unity.*

Proof. This is an immediate consequence of Lemmas A.8 and A.12. □

In the rest of this appendix, K denotes any of the rings mentioned in Corollary A.13. Before proving that these rings are fields, let us recall the basic definitions related to the valued structure of K.

Definition A.14. If $\boldsymbol{x} \in K$ is a series as in (A.1), then the maximal element of Γ is called the *valuation* of $\boldsymbol{x}$ and is denoted by $\mathsf{val}(\boldsymbol{x})$. Furthermore, the coefficient $c_{\mathsf{val}(\boldsymbol{x})}$ is called the *leading coefficient* of $\boldsymbol{x}$ and is denoted by $\mathsf{lc}(\boldsymbol{x})$. The *residue* of $\boldsymbol{x}$, denoted $\mathsf{res}(\boldsymbol{x})$, is defined as

$$\mathsf{res}(\boldsymbol{x}) \coloneqq \begin{cases} \mathsf{lc}(\boldsymbol{x}) & \text{if } \mathsf{val}(\boldsymbol{x}) = 0 \\ 0 & \text{otherwise} \,. \end{cases}$$

(With the convention that $\mathsf{val}(0) = -\infty$, $\mathsf{lc}(0) = \mathsf{res}(0) = 0$.)

Lemma A.15. *For all $\boldsymbol{x}, \boldsymbol{y} \in K$, the valuation map satisfies the relations*

$$\begin{aligned} \mathsf{val}(\boldsymbol{x} + \boldsymbol{y}) &\leqslant \max(\mathsf{val}(\boldsymbol{x}), \mathsf{val}(\boldsymbol{y})) \\ \mathsf{val}(\boldsymbol{x}\boldsymbol{y}) &= \mathsf{val}(\boldsymbol{x}) + \mathsf{val}(\boldsymbol{y}) \,. \end{aligned} \tag{A.3}$$

Proof. Immediate from the definition of addition and multiplication in K. □

Definition A.16. We say that $\mathscr{O} = \{\boldsymbol{x} \in K \colon \mathsf{val}(\boldsymbol{x}) \leqslant 0\}$ is a *valuation ring* of K.

Lemma A.17. *The valuation ring $\mathscr{O}$ is a subring of K.*

Proof. By (A.3), $\mathscr{O}$ is closed under addition and multiplication. □

The aim of this appendix is to prove that the rings of convergent series mentioned in Corollary A.13 are, in fact, real closed fields. The fact that the ring of convergent Puiseux series is a real closed field is classical, and the proof can be found in [BK12, Section 8.3]. An analogous statement for generalized Puiseux series and Hahn series was proven by van den Dries and Speissegger [vdDS98]. In the rest of this appendix, we present the proof given in [vdDS98]. We start by proving that K is a field that can be equipped with a natural order.

A.2 Structure of ordered field

The proof that K is a field requires a simple lemma.

Lemma A.18. *Suppose that $\boldsymbol{x} \in K$ is such that $\mathsf{val}(\boldsymbol{x}) < 0$. Then, the function $t \mapsto \|\boldsymbol{x}\|_t$ is nonincreasing and converges to 0.*

Proof. The fact that $\|\boldsymbol{x}\|_t$ is nonincreasing follows immediately from its definition and the fact that $\mathsf{val}(\boldsymbol{x}) < 0$. Moreover, since $\boldsymbol{x} \in K$, there exists $t_0 > 0$ such that $\|\boldsymbol{x}\|_t < +\infty$ for all $t > t_0$. If we denote $\delta = \mathsf{val}(\boldsymbol{x})$, then for every $t > t_0$ we have $\|\boldsymbol{x}\|_{2t} \leqslant 2^{\delta}\|\boldsymbol{x}\|_t$ and the claim follows. □

Lemma A.19. *The ring K is a field.*

Proof. Our proof follows [vdDS98, Lemma 4.7 and Corollary 5.6]. Let $\boldsymbol{x} = \sum_{\gamma \in \Gamma} c_\gamma t^\gamma$ be a nonzero element of K. We want to show that $\boldsymbol{x}$ in invertible. If $|\Gamma| = 1$, then the claim is trivial. Otherwise, we may suppose (by multiplying by an element of the form ct^γ) that $\boldsymbol{x}$ is of the form $\boldsymbol{x} = 1 - \tilde{\boldsymbol{x}}$, where $\mathsf{val}(\tilde{\boldsymbol{x}})$ is strictly smaller than zero, $\mathsf{val}(\tilde{\boldsymbol{x}}) = \gamma_0 < 0$. Denote $\tilde{\boldsymbol{x}} = \sum_{\gamma \in \Gamma'} c_\gamma t^\gamma$. For all $n \geqslant 0$ denote $\boldsymbol{y}_n = \sum_{k=0}^{n} \tilde{\boldsymbol{x}}^k = \sum_{\gamma \in \Gamma_n} c_\gamma^{(n)} t^\gamma$. Observe that $\mathsf{val}(\tilde{\boldsymbol{x}}^k) = k\gamma_0$. Therefore, if we let $\tilde{\Gamma}_n = \Gamma_n \cap [n\gamma_0, 0]$, then we have the inclusion $\tilde{\Gamma}_0 \subset \tilde{\Gamma}_1 \subset \dots$ and $\tilde{\Gamma}_m \cap [n\gamma_0, 0] = \tilde{\Gamma}_n$ for all $m \geqslant n$. Even more, if $\gamma \in \tilde{\Gamma}_n$ for some n, then $c_\gamma^{(m)} = c_\gamma^{(n)}$ for all $m \geqslant n$. Let $\overline{\Gamma} = \bigcup_{k=0}^{\infty} \tilde{\Gamma}_k$ and note that $\overline{\Gamma} \cap [n\gamma_0, 0] = \bigcup_{k=0}^{\infty} (\tilde{\Gamma}_k \cap [n\gamma_0, 0]) = \tilde{\Gamma}_n$ for all $n \geqslant 0$. For every $\gamma \in \overline{\Gamma}$, let n_γ denote the smallest number such that $\gamma \in \tilde{\Gamma}_{n_\gamma}$. Consider the series $\boldsymbol{y}$ defined as $\boldsymbol{y} = \sum_{\gamma \in \overline{\Gamma}} c_\gamma^{(n_\gamma)} t^\gamma$. We want to show that the series $\boldsymbol{y}$ belongs to K. To do so, we have to check that $\overline{\Gamma}$ is a correct set of exponents, and that $\boldsymbol{y}$ is absolutely convergent.

First, we show that $\overline{\Gamma}$ is well ordered. Let $\hat{\Gamma}$ be a nonempty subset of $\overline{\Gamma}$, let $\hat{\gamma}$ be the supremum of $\hat{\Gamma}$, and $\hat{\gamma}_i \to \hat{\gamma}$ be a strictly increasing sequence of elements of $\overline{\Gamma}$ that converges to $\hat{\gamma}$. Take a number n such that $\gamma \in \,]n\gamma_0, 0]$. Then, for i large enough, the sequence $\hat{\gamma}_i$ belongs to $\overline{\Gamma} \cap [n\gamma_0, 0] = \tilde{\Gamma}_n$. Since $\tilde{\Gamma}_n$ is well ordered, the sequence $\hat{\gamma}_i$ is finite and $\hat{\gamma} \in \hat{\Gamma}$. Therefore $\overline{\Gamma}$ is well ordered.

Second, we observe that if Γ' has no limit points, then the same is true for all $\tilde{\Gamma}_n$. This implies that every $\tilde{\Gamma}_n$ is finite. In this case, for any $a, b \in \mathbb{R}$, $a < b$ we can find n such that $[a, b] \cap \overline{\Gamma} \subset [n\gamma_0, 0] \cap \overline{\Gamma} = \tilde{\Gamma}_n$ and hence $\overline{\Gamma}$ has no limit points.

Third, we observe that if Γ' consists of rational numbers with the same denominator M, then the same is true for all $\tilde{\Gamma}_n$ and hence for $\overline{\Gamma}$.

To prove that $\boldsymbol{y}$ is absolutely convergent, we take $t_0 > 0$ such that $\|\tilde{\boldsymbol{x}}\|_t < 1$ for all $t > t_0$ (such t_0 exists by Lemma A.18). Then, by Lemma A.12, for all $n \geqslant 0$ we have

$$\sum_{\gamma \in \tilde{\Gamma}_n} |c_\gamma^{(n)}| t^\gamma \leqslant \sum_{\gamma \in \Gamma_n} |c_\gamma^{(n)}| t^\gamma = \| \sum_{k=0}^{n} \tilde{\boldsymbol{x}}^k \|_t \leqslant \sum_{k=0}^{n} \|\tilde{\boldsymbol{x}}\|_t^k \,.$$

Furthermore, by definition we have $\|\boldsymbol{y}\|_t = \sum_{\gamma \in \overline{\Gamma}} |c_\gamma^{(n_\gamma)}| t^\gamma$. Any finite subset of $\overline{\Gamma}$ belongs to some $\tilde{\Gamma}_n$. Therefore, any finite partial sum of $\sum_{\gamma \in \overline{\Gamma}} |c_\gamma^{(n_\gamma)}| t^\gamma$ is bounded from above by a sum of the form $\sum_{\gamma \in \tilde{\Gamma}_n} |c_\gamma^{(n)}| t^\gamma$. Hence, for all $t > t_0$ we have

$$\|\boldsymbol{y}\|_t \leqslant \sup_n \sum_{\gamma \in \tilde{\Gamma}_n} |c_\gamma^{(n)}| t^\gamma \leqslant \sum_{k=0}^{\infty} \|\tilde{\boldsymbol{x}}\|_t^k < +\infty$$

and $\boldsymbol{y}$ is absolutely convergent.

To finish the proof, we want to show that $\boldsymbol{x}\boldsymbol{y} = 1$. Note that $\mathsf{val}(\boldsymbol{y}) = 0$ and that we have $n_0 = 0$, $c_0^{(0)} = 1$. Hence $\boldsymbol{x}\boldsymbol{y}$ is of the form $\boldsymbol{x}\boldsymbol{y} = 1 + \sum_{\gamma \in \check{\Gamma}} \check{c}_\gamma t^\gamma$. Suppose that $\check{\Gamma}$ is nonempty and denote its maximal element by $\check{\gamma}_0 < 0$. Then $\boldsymbol{x}\boldsymbol{y} = 1 + \check{c}_{\check{\gamma}_0} t^{\check{\gamma}_0} +$ (terms of smaller order), where $\check{c}_{\check{\gamma}_0} \neq 0$. Take n such that $\check{\gamma}_0 > n\gamma_0$. By the definition of $\boldsymbol{y}$, the series $\boldsymbol{x}(\sum_{k=0}^n \tilde{\boldsymbol{x}}^k)$ is also of the form $\boldsymbol{x}(\sum_{k=0}^n \tilde{\boldsymbol{x}}^k) = 1 + \check{c}_{\check{\gamma}_0} +$ (terms of smaller order). However, we have $\boldsymbol{x}(\sum_{k=0}^n \tilde{\boldsymbol{x}}^k) = (1 - \tilde{\boldsymbol{x}})(\sum_{k=0}^n \tilde{\boldsymbol{x}}^k) = 1 - \tilde{\boldsymbol{x}}^{n+1}$ and the highest exponent of the series $\tilde{\boldsymbol{x}}^{n+1}$ is equal to $(n+1)\gamma_0$. This gives a contradiction and hence $\boldsymbol{x}\boldsymbol{y} = 1$. ☐

We can now equip K with an order.

Definition A.20. We say that an element $\boldsymbol{x}$ of K is *nonnegative*, what we denote by $\boldsymbol{x} \geqslant 0$, if $\mathsf{lc}(\boldsymbol{x}) \geqslant 0$. We extend this relation to arbitrary pairs $\boldsymbol{x}, \boldsymbol{y}$ of series by defining $\boldsymbol{x} \geqslant \boldsymbol{y} \iff (\boldsymbol{x} - \boldsymbol{y}) \geqslant 0$.

Lemma A.21. *The field $(K, \geqslant)$ is ordered.*

Proof. To prove the first part, we start by proving that $\geqslant$ is a total order. To this end, observe that the relations $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $\boldsymbol{y} \geqslant \boldsymbol{x}$ imply that $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) \geqslant 0$ and $\mathsf{lc}(\boldsymbol{y} - \boldsymbol{x}) \geqslant 0$. Since $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) = -\mathsf{lc}(\boldsymbol{y} - \boldsymbol{x})$, we get $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) = 0$ and $\boldsymbol{x} = \boldsymbol{y}$. Hence $\geqslant$ is antisymmetric. Furthermore, if $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $\boldsymbol{y} \geqslant \boldsymbol{z}$, then $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) \geqslant 0$ and $\mathsf{lc}(\boldsymbol{y} - \boldsymbol{z}) \geqslant 0$. This implies that $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{z}) \geqslant 0$ and $\boldsymbol{x} \geqslant \boldsymbol{z}$. Hence $\geqslant$ is transitive. Moreover, it is clear that $\geqslant$ is total. To prove that $\geqslant$ is compatible with the field operations, take $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in K$ and suppose that $\boldsymbol{x} \geqslant \boldsymbol{y}$. Then, we have $\mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) \geqslant 0$. Hence $\mathsf{lc}(\boldsymbol{x} + \boldsymbol{z} - \boldsymbol{y} - \boldsymbol{z}) \geqslant 0$ and $\boldsymbol{x} + \boldsymbol{z} \geqslant \boldsymbol{y} + \boldsymbol{z}$. Furthermore, if $\boldsymbol{x} \geqslant 0$ and $\boldsymbol{y} \geqslant 0$, then $\mathsf{lc}(\boldsymbol{x}\boldsymbol{y}) = \mathsf{lc}(\boldsymbol{x})\,\mathsf{lc}(\boldsymbol{y}) \geqslant 0$ and hence $(K, \geqslant)$ is an ordered field. ☐

The second natural definition of order over K is to say that $\boldsymbol{x} \geqslant \boldsymbol{y}$ if $\boldsymbol{x}(t) \geqslant \boldsymbol{y}(t)$ for all t large enough. Similarly, another natural definition of valuation is to put $\mathsf{val}(\boldsymbol{x}) \coloneqq \lim_{t \to \infty} \log_t |\boldsymbol{x}(t)|$. These definitions are equivalent to the previous ones. The proof requires an auxiliary lemma.

Lemma A.22. *Suppose that* $\mathsf{val}(\boldsymbol{x}) = 0$*. Then* $\boldsymbol{x}(t)$ *converges to* $\mathsf{lc}(\boldsymbol{x})$ *as* t *goes to infinity.*

Proof. Denote $\boldsymbol{x} = c + \tilde{\boldsymbol{x}}$, where $\mathsf{val}(\tilde{\boldsymbol{x}}) < 0$. We have $|\boldsymbol{x}(t) - c| = |\tilde{\boldsymbol{x}}(t)| \leqslant \|\tilde{\boldsymbol{x}}\|_t$ and the right hand side converges to zero by Lemma A.18. □

Lemma A.23. *For all* $\boldsymbol{x}, \boldsymbol{y} \in K$ *we have* $\boldsymbol{x} \geqslant \boldsymbol{y}$ *if and only if* $\boldsymbol{x}(t) \geqslant \boldsymbol{y}(t)$ *for all sufficiently large* t*. Furthermore, for every* $\boldsymbol{x} \in K$ *we have* $\mathsf{val}(\boldsymbol{x}) = \lim_{t\to\infty} \log_t |\boldsymbol{x}(t)|$ *(with the convention that* $\log_t(0) = -\infty$*).*

Proof. To prove the first claim, note that it is enough to suppose that $\boldsymbol{y} = 0$. If $\boldsymbol{x} = 0$, then the claim is trivial. Otherwise, by Lemma A.22, the function $t^{-\mathsf{val}(\boldsymbol{x})}\boldsymbol{x}(t)$ converges to $\mathsf{lc}(\boldsymbol{x}) \neq 0$. Hence $\mathsf{lc}(\boldsymbol{x}) \geqslant 0$ if and only if $\boldsymbol{x}(t) \geqslant 0$ for all t sufficiently large. The second claim is also trivial for $\boldsymbol{x} = 0$. Otherwise, since $t^{-\mathsf{val}(\boldsymbol{x})}|\boldsymbol{x}(t)|$ converges to $|\mathsf{lc}(\boldsymbol{x})|$, we have that $\log_t(t^{-\mathsf{val}(\boldsymbol{x})}|\boldsymbol{x}(t)|) = -\mathsf{val}(\boldsymbol{x}) + \log_t |\boldsymbol{x}(t)|$ converges to 0. □

To finish this section, we show that the valuation is convex with respect to the order on K.

Lemma A.24. *The valuation* val *is convex with respect to the order* $\geqslant$*. In other words, if* $\boldsymbol{y} \in \mathcal{O}$*,* $\boldsymbol{x} \leqslant \boldsymbol{y}$*, and* $-\boldsymbol{x} \leqslant \boldsymbol{y}$*, then* $\boldsymbol{x} \in \mathcal{O}$*.*

Proof. Suppose that $\mathsf{val}(\boldsymbol{x}) > 0$. Then $\mathsf{val}(\boldsymbol{x}) > \mathsf{val}(\boldsymbol{y})$ and hence $0 \leqslant \mathsf{lc}(\boldsymbol{y} - \boldsymbol{x}) = -\mathsf{lc}(\boldsymbol{x})$. Similarly $0 \leqslant \mathsf{lc}(\boldsymbol{y}+\boldsymbol{x}) = \mathsf{lc}(\boldsymbol{x})$. Therefore $\mathsf{lc}(\boldsymbol{x}) = 0$ and $\boldsymbol{x} = 0$, which gives a contradiction. □

A.3 Theorem of division, henselianity, and real closedness

We are now ready to present the proof of the main result of this appendix—the fact that K is real closed. As noted before, our proof follows [vdDS98]. The outline of the proof is as follows. First, we note that there exists a characterization of real closed valued fields, see [EP05, Theorem 4.3.7]. This characterization reduces the problem of proving that K is real closed to the problem of proving that it is henselian. The approach of van den Dries and Speisseger is to deduce henselianity from an appropriate division theorem in the ring of power series $\mathcal{O}[[\boldsymbol{x}]]$. Such division theorems are related to the Weierstrass preparation. Furthermore, it is important to note that there are proofs of Weierstrass division theorem that are well adapted to handle the questions of convergence, see [Car66].

Definition A.25. We denote by $\mathcal{O}[[\boldsymbol{x}]]$ the ring of power series over $\mathcal{O}$. More formally, this ring consists of elements of the form

$$\sum_{k=0}^{\infty} \boldsymbol{c}_k \boldsymbol{x}^k, \tag{A.4}$$

where $\boldsymbol{c}_k \in \mathcal{O}$ and $\boldsymbol{x}$ is a formal variable (with the natural definition of addition and multiplication). We say that a power series *has order* d if d is the smallest number such that $\boldsymbol{c}_d \neq 0$. We analogously define the ring of power series over $\mathbb{R}$ and the order of such a series. We denote this ring by $\mathbb{R}[[x]]$. Moreover, we denote by $\mathcal{O}[\boldsymbol{x}]$ and $\mathbb{R}[x]$ the rings of polynomials over $\mathcal{O}$ and $\mathbb{R}$ respectively, and we use the convention that zero polynomial has degree $-\infty$.

We point out that we have the following lemma.

Lemma A.26. *An element* $\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{c}_k \boldsymbol{x}^k \in \mathcal{O}[[\boldsymbol{x}]]$ *is invertible in* $\mathcal{O}[[\boldsymbol{x}]]$ *if and only if* $\mathsf{val}(\boldsymbol{c}_0) = 0$*.*

Sketch of the proof. Our proof follows [Bou03, Chapter IV, § 4.4, Proposition 6]. If $\boldsymbol{F}$ is invertible in $\mathcal{O}$, then $\boldsymbol{c}_0$ is invertible in $\mathcal{O}$, and hence $\mathsf{val}(\boldsymbol{c}_0) = 0$. Conversely, if $\mathsf{val}(\boldsymbol{c}_0) = 0$, then $\boldsymbol{c}_0$ is invertible in K by Lemma A.19, and $\mathsf{val}(\boldsymbol{c}_0^{-1}) = 0$. Write $\boldsymbol{F} = \boldsymbol{c}_0(1 + \sum_{k=1}^{\infty} \boldsymbol{c}_0^{-1}\boldsymbol{c}_k\boldsymbol{x}^k)$ and consider the series $\boldsymbol{G} = -\sum_{k=1}^{\infty} \boldsymbol{c}_0^{-1}\boldsymbol{c}_k\boldsymbol{x}^k \in \mathcal{O}[[\boldsymbol{x}]]$. For every $n \geqslant 0$ define $\boldsymbol{G}_n = \sum_{\ell=0}^{n} \boldsymbol{G}^\ell = \sum_{k=0}^{\infty} \boldsymbol{c}_k^{(n)}\boldsymbol{x}^k$ and observe that for every $m \geqslant n$ the first n coefficients of $\boldsymbol{G}_n$ and $\boldsymbol{G}_m$ are the same. We take $\boldsymbol{G}_\infty = \sum_{k=0}^{\infty} \boldsymbol{c}_k^{(k)}\boldsymbol{x}^k \in \mathcal{O}[[\boldsymbol{x}]]$. Observe that for all $n \geqslant 0$ the nth coefficient of $\boldsymbol{c}_0^{-1}\boldsymbol{F}\boldsymbol{G}_\infty$ is the same as the nth coefficient of $\boldsymbol{c}_0^{-1}\boldsymbol{F}(\sum_{\ell=0}^{n} \boldsymbol{G}^\ell)$. Furthermore, we have $\boldsymbol{c}_0^{-1}\boldsymbol{F}(\sum_{\ell=0}^{n} \boldsymbol{G}^\ell) = (1 - \boldsymbol{G})(\sum_{\ell=0}^{n} \boldsymbol{G}^\ell) = 1 - \boldsymbol{G}^{n+1}$ and hence $\boldsymbol{c}_0^{-1}\boldsymbol{F}\boldsymbol{G}_\infty = 1$. □

We now extend the definition of residue to elements of $\mathcal{O}[[\boldsymbol{x}]]$.

Definition A.27. Let $\boldsymbol{F} \in \mathcal{O}[[\boldsymbol{x}]]$ be a power series over $\mathcal{O}$ as in (A.4). We define the *residue of* $\boldsymbol{F}$ as

$$\mathsf{res}(\boldsymbol{F}) = \sum_{k=0}^{\infty} \mathsf{res}(\boldsymbol{c}_k)x^k \in \mathbb{R}[[x]] .$$

The next theorem is the main technical result of this section. We note that the statement presented in [vdDS98] is more general—we only present a particular case that is sufficient for our purposes. We also point out that our formulation is exactly the Weierstrass division theorem over the ring $\mathcal{O}$ and that the proof below can be seen as an adaptation of the proof of the Weierstrass division presented in [Car66].

Theorem A.28 (Theorem of division)**.** *Fix a series $\boldsymbol{F} \in \mathcal{O}[[\boldsymbol{x}]]$ and let d denote the order of $\mathsf{res}(\boldsymbol{F})$. Then, for any $\boldsymbol{A} \in \mathcal{O}[[\boldsymbol{x}]]$, there exists a unique $\boldsymbol{Q} \in \mathcal{O}[[\boldsymbol{x}]]$ such that $\boldsymbol{A} - \boldsymbol{F}\boldsymbol{Q} = \boldsymbol{R}$ is a polynomial over $\mathcal{O}$ of degree strictly smaller than d.*

Before giving the proof of the theorem, we will prove its special case. This case can be thought of as an algorithm of long division of power series over $\mathcal{O}$.

Lemma A.29. *Suppose that $\boldsymbol{F} \in \mathcal{O}[[\boldsymbol{x}]]$ is of the form $\boldsymbol{F} = \boldsymbol{x}^d - \boldsymbol{H}$, where every coefficient of the series $\boldsymbol{H}$ has a strictly negative valuation. Then, there exists a series $\boldsymbol{Q} \in \mathcal{O}[[\boldsymbol{x}]]$ such that $\boldsymbol{A} - \boldsymbol{F}\boldsymbol{Q} = \boldsymbol{R}$ is a polynomial over $\mathcal{O}$ of degree strictly smaller than d. Furthermore, if there exists $\delta < 0$ such that every coefficient of the series $\boldsymbol{H}$ has a valuation smaller than δ, then $\boldsymbol{Q}$ is unique.*

Proof. Our proof follows [vdDS98, Theorems 4.17 and 5.10] and [Car66]. To prove the first part, we define sequences $\boldsymbol{A}_n, \boldsymbol{Q}_n, \boldsymbol{R}_n \in \mathcal{O}[[\boldsymbol{x}]]$ by the following recurrence. We put $\boldsymbol{A}_0 = \boldsymbol{A}$. Furthermore, if $\boldsymbol{A}_n = \sum_{k=0}^{\infty} \boldsymbol{a}_k^{(n)}\boldsymbol{x}^k$, then we put $\boldsymbol{R}_n = \sum_{k=0}^{d-1} \boldsymbol{a}_k^{(n)}\boldsymbol{x}^k$, $\boldsymbol{Q}_n = \sum_{k=d}^{\infty} \boldsymbol{a}_k^{(n)}\boldsymbol{x}^{k-d}$, and $\boldsymbol{A}_{n+1} = \boldsymbol{A}_n - \boldsymbol{Q}_n\boldsymbol{F} - \boldsymbol{R}_n = \boldsymbol{Q}_n\boldsymbol{H}$. Denote $\boldsymbol{H} = \sum_{k=0}^{\infty} \boldsymbol{h}_k\boldsymbol{x}^k$, let $\delta_k = \max\{\mathsf{val}(\boldsymbol{h}_\ell) \colon \ell \leqslant k\} < 0$ and observe that

$$\boldsymbol{a}_k^{(n+1)} = \sum_{p=0}^{k} \boldsymbol{a}_p^{(n)}\boldsymbol{h}_{k-p} \tag{A.5}$$

for all $n, k \geqslant 0$. Hence, by a straightforward induction, $\mathsf{val}(\boldsymbol{a}_k^{(n)}) \leqslant n\delta_k$ for all $n, k \geqslant 0$. For any fixed $k \geqslant 0$ we define $\boldsymbol{a}_k^{(\infty)} = \sum_{\gamma \in \Gamma_k} a_{k,\gamma}^{(\infty)}t^\gamma$ as follows. We denote $\boldsymbol{a}_k^{(n)} = \sum_{\gamma \in \Gamma_{k,n}} a_{k,\gamma}^{(n)}t^\gamma$, take $\Gamma_k = \bigcup_{n=0}^{\infty} \Gamma_{k,n}$, and for all $\gamma \in \Gamma_k$ define

$$a_{k,\gamma}^{(\infty)} = \sum_{\ell=0}^{\infty} a_{k,\gamma}^{(\ell)} = \sum_{\ell=0}^{\lfloor \gamma/|\delta_k| \rfloor} a_{k,\gamma}^{(\ell)}$$

(where $a_{k,\gamma}^{(\ell)} = 0$ if $\gamma \notin \Gamma_{k,\ell}$). We will show that $\boldsymbol{a}_k^{(\infty)}$ is an element of $\mathcal{O}$. To do that, we have to check that Γ_k is a correct set of exponents and that $\boldsymbol{a}_k^{(\infty)}$ is absolutely convergent.

First, we show that Γ_k is well ordered. Let $\hat{\Gamma}$ be a nonempty subset of Γ_k, let $\hat{\gamma} \leqslant 0$ be the supremum of $\hat{\Gamma}$, and $\hat{\gamma}_i \to \hat{\gamma}$ be a strictly increasing sequence of elements of Γ_k that converges to $\hat{\gamma}$. Take a number n such that $\gamma \in]n\delta_k, 0]$. Then, for i large enough, the sequence $\hat{\gamma}_i$ belongs to $\bigcup_{\ell=0}^{n-1} \Gamma_{k,\ell}$. Hence, we can extract from $\hat{\gamma}_i$ an infinite subsequence that belongs to some $\Gamma_{k,\ell}$. Since $\Gamma_{k,\ell}$ is well ordered, this subsequence is finite and $\hat{\gamma} \in \hat{\Gamma}$. Therefore Γ_k is well ordered.

Second, if K is the field of generalized Puiseux series, then $\Gamma_{k,\ell}$ has no limit points for all $\ell \geqslant 0$. In this case, for any $a, b \in \mathbb{R}$, $a < b$ we can find n such that $[a,b] \cap \Gamma_k \subset [n\delta_k, 0] \cap \Gamma_k \subset \bigcup_{\ell=0}^{n} \Gamma_{k,\ell}$ and hence Γ_k has no limit points.

Third, if K is the field of Puiseux series and $M \in \mathbb{N}$ is the common denominator of all exponents of the series $\{\boldsymbol{a}_0^{(0)}, \dots, \boldsymbol{a}_k^{(0)}, \boldsymbol{h}_0, \dots, \boldsymbol{h}_k\}$, then (A.5) shows that the same is true for all $\boldsymbol{a}_p^{(n)}$ where $n \geqslant 0$ and $p \in \{0, \dots, k\}$. Hence, this is also true for $\boldsymbol{a}_k^{(\infty)}$.

To show that $\boldsymbol{a}_k^{(\infty)}$ is absolutely convergent, let $t_0 > 0$ be such that $\|\boldsymbol{h}_p\|_t < \frac{1}{2k}$ and $\|\boldsymbol{a}_p^{(0)}\|_t < +\infty$ for all $p \in \{0, \dots, k\}$ and $t > t_0$ (such t_0 exists by Lemma A.18). Furthermore, denote $s_k(t) = \max\{\|\boldsymbol{a}_p^{(0)}\|_t \colon p \leqslant k\}$. By Lemma A.12 and a straightforward induction using (A.5), for all $n \geqslant 0$ and $t > t_0$ we have

$$\max\{\|\boldsymbol{a}_p^{(n)}\|_t \colon p \leqslant k\} \leqslant (\frac{1}{2})^n s_k(t) \,.$$

In particular, for any such t we have

$$\sum_{\gamma \in \Gamma_k} |a_{k,\gamma}^{(\infty)}| t^\gamma \leqslant \sum_{\gamma \in \Gamma_k} \sum_{\ell=0}^{\lfloor \gamma / |\delta_k| \rfloor} |a_{k,\gamma}^{(\ell)}| t^\gamma \leqslant \sum_{\ell=0}^{\infty} \sum_{\gamma \in \Gamma_k} |a_{k,\gamma}^{(\ell)}| t^\gamma = \sum_{\ell=0}^{\infty} \|\boldsymbol{a}_k^{(\ell)}\|_t < +\infty \,.$$

Hence $\boldsymbol{a}_k^{(\infty)}$ is absolutely convergent and $\boldsymbol{a}_k^{(\infty)} \in \mathcal{O}$. Denote $\boldsymbol{Q}_\infty = \sum_{k=d}^{\infty} \boldsymbol{a}_k^{(\infty)} \boldsymbol{x}^{k-d}$ and $\boldsymbol{R}_\infty = \sum_{k=0}^{d-1} \boldsymbol{a}_k^{(\infty)} \boldsymbol{x}^k$. We want to show that $\boldsymbol{A} = \boldsymbol{F}\boldsymbol{Q}_\infty + \boldsymbol{R}_\infty$. Note that for every $n \geqslant 0$ we have

$$\boldsymbol{A} - \boldsymbol{A}_{n+1} = \boldsymbol{A}_0 - \boldsymbol{A}_{n+1} = (\boldsymbol{A}_0 - \boldsymbol{A}_1) + \dots + (\boldsymbol{A}_n - \boldsymbol{A}_{n+1}) = \boldsymbol{F}(\sum_{\ell=0}^{n} \boldsymbol{Q}_n) + (\sum_{\ell=0}^{n} \boldsymbol{R}_\ell) \,.$$

Hence $\boldsymbol{A} = \boldsymbol{A}_{n+1} + \boldsymbol{F}(\sum_{\ell=0}^{n} \boldsymbol{Q}_n) + (\sum_{\ell=0}^{n} \boldsymbol{R}_\ell)$. By definition, for every fixed $k \geqslant 0$, the kth coefficients of the series $\boldsymbol{A}_{n+1} + \boldsymbol{F}(\sum_{\ell=0}^{n} \boldsymbol{Q}_n) + (\sum_{\ell=0}^{n} \boldsymbol{R}_\ell)$ and $\boldsymbol{F}\boldsymbol{Q}_\infty + \boldsymbol{R}_\infty$ have the same expansion up to order $n\delta_k$ and the claim follows.

To show the second claim, suppose that there are two elements $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{O}[[\boldsymbol{x}]]$ such that $\boldsymbol{A} - (\boldsymbol{x}^d - \boldsymbol{H})\boldsymbol{Q}_1 = \boldsymbol{R}_1$ and $\boldsymbol{A} - (\boldsymbol{x}^d - \boldsymbol{H})\boldsymbol{Q}_2 = \boldsymbol{R}_2$ are polynomials of degree smaller than d. If we denote $\overline{\boldsymbol{Q}} = \boldsymbol{Q}_1 - \boldsymbol{Q}_2$ and $\overline{\boldsymbol{R}} = \boldsymbol{R}_2 - \boldsymbol{R}_1$, then $\overline{\boldsymbol{R}}$ is also a polynomial of degree smaller than d and we have

$$\overline{\boldsymbol{R}} = (\boldsymbol{A} - (\boldsymbol{x}^d - \boldsymbol{H})\boldsymbol{Q}_2) - (\boldsymbol{A} - (\boldsymbol{x}^d - \boldsymbol{H})\boldsymbol{Q}_1) = (\boldsymbol{x}^d - \boldsymbol{H})(\boldsymbol{Q}_1 - \boldsymbol{Q}_2) = (\boldsymbol{x}^d - \boldsymbol{H})\overline{\boldsymbol{Q}} \,.$$

We want to show that $\overline{\boldsymbol{Q}} = 0$. Suppose that $\overline{\boldsymbol{Q}} = \sum_{k=0}^{\infty} \boldsymbol{q}_k \boldsymbol{x}^k$, denote as previously $\boldsymbol{H} = \sum_{k=0}^{\infty} \boldsymbol{h}_k \boldsymbol{x}^k$ and let $\delta < 0$ be such that $\mathsf{val}(\boldsymbol{h}_\ell) < \delta < 0$ for all $\ell \geqslant 0$. Furthermore, suppose that the inequality $\mathsf{val}(\boldsymbol{q}_k) \leqslant n\delta_k$ holds for some $n \geqslant 0$ and all $k \geqslant 0$ (it trivially holds for $n = 0$). For any $\ell \geqslant 0$, the $(d+\ell)$th coefficient of $\overline{\boldsymbol{R}}$ is equal to zero and hence

$$\boldsymbol{q}_\ell - \sum_{k=0}^{d+\ell} \boldsymbol{q}_k \boldsymbol{h}_{d+\ell-k} = 0 \,.$$

Note that the valuation of $\sum_{k=0}^{d+\ell} \boldsymbol{q}_k \boldsymbol{h}_{d+\ell-k}$ is not greater than $(n+1)\delta$. Therefore, the same is true for $\boldsymbol{q}_\ell$. As a result, we have $\mathsf{val}(\boldsymbol{q}_\ell) = -\infty$ for all $\ell \geqslant 0$ and $\overline{\boldsymbol{Q}} = 0$. ☐

Theorem A.28 follows from the previous lemma.

Proof of Theorem A.28. As previously, our proof follows [vdDS98, Theorems 4.17 and 5.10] and [Car66]. Denote $\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{c}_k \boldsymbol{x}^k$ and $\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{c}_{k+d} \boldsymbol{x}^k$. Since $\mathsf{res}(\boldsymbol{F})$ has order d, the series $\boldsymbol{G}$ is invertible in $\mathscr{O}[[\boldsymbol{x}]]$ by Lemma A.26. Hence, if we denote $\boldsymbol{H} = \boldsymbol{G}^{-1}(-\sum_{k=0}^{d-1} \boldsymbol{c}_k \boldsymbol{x}^k)$, then $\boldsymbol{F} = \boldsymbol{G}(\boldsymbol{x}^d - \boldsymbol{H})$. Furthermore, note that there exists $\delta < 0$ such that all coefficients of $\boldsymbol{H}$ have valuation not greater than δ (because this is true for the coefficients of $\sum_{k=0}^{d-1} \boldsymbol{c}_k \boldsymbol{x}^k$ and $\boldsymbol{G}^{-1} \in \mathscr{O}[[\boldsymbol{x}]]$). Therefore, we can apply Lemma A.29 to $\overline{\boldsymbol{F}} = \boldsymbol{x}^d - \boldsymbol{H}$. Hence, there exists a unique series $\overline{\boldsymbol{Q}} \in \mathscr{O}[[\boldsymbol{x}]]$ such that $\boldsymbol{A} - \overline{\boldsymbol{F}}\overline{\boldsymbol{Q}}$ is a polynomial of degree smaller than d. By taking $\boldsymbol{Q} = \boldsymbol{G}^{-1}\overline{\boldsymbol{Q}}$ we have $\boldsymbol{A} - \boldsymbol{F}\boldsymbol{Q} = \boldsymbol{A} - \boldsymbol{G}\overline{\boldsymbol{F}}\boldsymbol{G}^{-1}\overline{\boldsymbol{Q}} = \boldsymbol{A} - \overline{\boldsymbol{F}}\overline{\boldsymbol{Q}}$. The uniqueness of $\boldsymbol{Q}$ follows from the uniqueness of $\overline{\boldsymbol{Q}}$. Indeed, if $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathscr{O}[[\boldsymbol{x}]]$ are two different series that satisfy the claim for $\boldsymbol{F}$, then $\boldsymbol{G}\boldsymbol{Q}_1$, $\boldsymbol{G}\boldsymbol{Q}_2 \in \mathscr{O}[[\boldsymbol{x}]]$ are two different series that satisfy the claim for $\overline{\boldsymbol{F}}$. ☐

Corollary A.30. *Fix a series $\boldsymbol{F} \in \mathscr{O}[[\boldsymbol{x}]]$ and let d denote the order of $\mathsf{res}(\boldsymbol{F})$. Then, there exists a unique pair $\boldsymbol{Q} \in \mathscr{O}[[\boldsymbol{x}]], \boldsymbol{P} \in \mathscr{O}[\boldsymbol{x}]$ such that $\boldsymbol{Q}$ is invertible, $\boldsymbol{P}$ is a monic polynomial of degree d, and $\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{P}$.*

Proof. Our proof follows [vdDS98, Theorem 4.17]. By Theorem A.28, there exists a unique $\boldsymbol{Q}$ such that $\boldsymbol{x}^d - \boldsymbol{F}\boldsymbol{Q} = \boldsymbol{R}$ is a polynomial of degree smaller than d. Hence $\boldsymbol{F}\boldsymbol{Q} = \boldsymbol{P}$, where $\boldsymbol{P}$ is a monic polynomial of degree d. We will show that $\boldsymbol{Q}$ is invertible. To do so, write $\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{c}_k \boldsymbol{x}^k$ and $\boldsymbol{Q} = \sum_{k=0}^{\infty} \boldsymbol{q}_k \boldsymbol{x}^k$. We have

$$\boldsymbol{q}_0 \boldsymbol{c}_d + \sum_{k=0}^{d-1} \boldsymbol{q}_{d-k} \boldsymbol{c}_k = 1 \,.$$

Since $\mathsf{res}(\boldsymbol{F})$ has order d, we have $\mathsf{val}(\boldsymbol{c}_k) < 0$ for all $k \in \{0, 1, \dots, d-1\}$. In particular, $\mathsf{val}(\sum_{k=0}^{d-1} \boldsymbol{q}_{d-k} \boldsymbol{c}_k) < 0$. This implies that $0 = \mathsf{val}(\boldsymbol{q}_0 \boldsymbol{c}_d) = \mathsf{val}(\boldsymbol{q}_0)$ and $\boldsymbol{Q}$ is invertible by Lemma A.26. Thus $\boldsymbol{F} = \boldsymbol{Q}^{-1}\boldsymbol{P}$. Conversely, if $\boldsymbol{F} = \boldsymbol{Q}^{-1}\boldsymbol{P}$, then $\boldsymbol{x}^d - \boldsymbol{F}\boldsymbol{Q}$ is a polynomial of degree smaller than d. Therefore, by Theorem A.28, $\boldsymbol{Q}$ is unique. ☐

Corollary A.31. *The field K is henselian. In other words, if $\boldsymbol{F} \in \mathscr{O}[\boldsymbol{x}]$ is a polynomial and $\alpha \in \mathbb{R}$ is such that $\mathsf{res}(\boldsymbol{F})(\alpha) = 0$ and $(\mathsf{res}(\boldsymbol{F}))'(\alpha) \neq 0$, then there exists $\boldsymbol{\alpha} \in \mathscr{O}$ such that $\boldsymbol{F}(\boldsymbol{\alpha}) = 0$ and $\mathsf{res}(\boldsymbol{\alpha}) = \alpha$.*

Proof. Denote $\boldsymbol{F} = \sum_{k=0}^{d} \boldsymbol{c}_k \boldsymbol{x}^k$ and consider the polynomial $\overline{\boldsymbol{F}}(\boldsymbol{x}) = \boldsymbol{F}(\alpha + \boldsymbol{x})$. We have

$$\overline{\boldsymbol{F}}(\boldsymbol{x}) = \sum_{k=0}^{d} \boldsymbol{c}_k (\alpha + \boldsymbol{x})^k = \sum_{k=0}^{d} \sum_{\ell=0}^{k} \binom{k}{\ell} \alpha^{k-\ell} \boldsymbol{c}_k \boldsymbol{x}^\ell = \sum_{\ell=0}^{d} \Big(\sum_{k=\ell}^{d} \binom{k}{\ell} \alpha^{k-\ell} \boldsymbol{c}_k \Big) \boldsymbol{x}^\ell \,.$$

Note that for $\ell = 0$ the constant coefficient of $\overline{\boldsymbol{F}}$ is equal to $\sum_{k=0}^{d} \boldsymbol{c}_k \alpha^k$. Since $\mathsf{res}(\boldsymbol{F})(\alpha) = 0$, we have $\mathsf{val}(\sum_{k=0}^{d} \boldsymbol{c}_k \alpha^k) < 0$. Furthermore, for $\ell = 1$, the linear term of $\overline{\boldsymbol{F}}$ is equal to $\sum_{k=1}^{d} \boldsymbol{c}_k k \alpha^{k-1}$. Since $(\mathsf{res}(\boldsymbol{F}))'(\alpha) \neq 0$, we have $\mathsf{val}(\sum_{k=1}^{d} \boldsymbol{c}_k k \alpha^{k-1}) = 0$. In particular, $\mathsf{res}(\overline{\boldsymbol{F}})$ is of order 1. Therefore, we can use Corollary A.30 to decompose it into $\overline{\boldsymbol{F}} = \boldsymbol{Q}(\boldsymbol{x} - \overline{\boldsymbol{\alpha}})$, where $\boldsymbol{Q} \in \mathscr{O}[[\boldsymbol{x}]]$ is an invertible series and $\overline{\boldsymbol{\alpha}} \in \mathscr{O}$ is a constant. We want to show that $\boldsymbol{Q}$ is a polynomial. To do so, denote $\overline{\boldsymbol{F}} = \sum_{k=0}^{d} \overline{\boldsymbol{c}_k} \boldsymbol{x}^k$ and $\boldsymbol{Q} = \sum_{k=0}^{\infty} \boldsymbol{q}_k \boldsymbol{x}^k$. We have $\overline{\boldsymbol{c}_0} = -\boldsymbol{\alpha}\boldsymbol{q}_0$. Moreover, since $\boldsymbol{Q}$ is invertible, we have $\mathsf{val}(\boldsymbol{q}_0) = 0$ (by Lemma A.26). Hence $\mathsf{val}(\overline{\boldsymbol{\alpha}}) = \mathsf{val}(\overline{\boldsymbol{c}_0}) < 0$. Furthermore, note that for every $\ell \geqslant 1$ we have $\overline{\boldsymbol{\alpha}}\boldsymbol{q}_{d+\ell} = \boldsymbol{q}_{d+\ell-1}$. In particular, $\mathsf{val}(\boldsymbol{q}_{d+\ell}) = \mathsf{val}(\boldsymbol{q}_d) - \ell\,\mathsf{val}(\overline{\boldsymbol{\alpha}})$. Since $\boldsymbol{Q} \in \mathscr{O}[[\boldsymbol{x}]]$ and $\mathsf{val}(\overline{\boldsymbol{\alpha}}) < 0$, this is possible only if $\boldsymbol{q}_d = 0$. Therefore, $\boldsymbol{Q}$ is a polynomial of degree at most $d-1$ and hence $\overline{\boldsymbol{F}}(\overline{\boldsymbol{\alpha}}) = 0$. To finish, observe that we have $\boldsymbol{F}(\alpha + \overline{\boldsymbol{\alpha}}) = 0$ and $\mathsf{res}(\alpha + \overline{\boldsymbol{\alpha}}) = \alpha$. ☐

Corollary A.32. *The field K is real closed.*

Proof. By Corollary A.31 and Lemma A.24 the field K is an ordered henselian valued field equipped with a nontrivial and convex valuation. Furthermore, its value group is divisible and its residue field is real closed. Hence, K is real closed by [EP05, Theorem 4.3.7]. ☐

Remark A.33. We note that if a field K is real closed, then its extension $K(\sqrt{-1})$ is algebraically closed [BPR06, Theorem 2.11]. It is easy to see that if K is any of the fields considered here (Puiseux series, generalized Puiseux series, Hahn series etc. with real coefficients), then $K(\sqrt{-1})$ is isomorphic to the analogous field with complex coefficients. Therefore, all such fields are algebraically closed.

APPENDIX $\mathcal{B}$

Additional proofs

B.1 Model theory

In this section, we give proofs of Theorem 2.106 and Lemma 2.108. We start by the following technical lemmas.

Lemma B.1. *Suppose that $\mathcal{M} = (M, 0, -\infty, +, \leqslant)$ is a model of $\mathrm{Th}_{\mathrm{doagb}}$. Then $0 \neq -\infty$ and if $x, y \in M$ are such that $x + y = -\infty$, then either x or y is equal to $-\infty$.*

Proof. Suppose that $0 = -\infty$. Then $x = x + 0 = x + (-\infty) = -\infty$ for all $x \in M$, which is a contradiction with the nontriviality axiom. Furthermore, if $x + y = -\infty$ and $y \neq -\infty$, then there exists $z \neq -\infty$ such that $x = x + 0 = x + y + z = -\infty + z = -\infty$. $\square$

Lemma B.2. *Suppose that $\mathcal{M} = (M, 0, -\infty, +, \leqslant)$ is a model of $\mathrm{Th}_{\mathrm{doagb}}$. Then $\hat{\mathcal{M}} = (M \setminus \{-\infty\}, 0, +, \leqslant)$ is a model of $\mathrm{Th}_{\mathrm{doag}}$. Moreover, if $\phi(y_1, \dots, y_m)$ is a quantifier-free $\mathcal{L}_{\mathrm{og}}$-formula and $\bar{y} \in (M \setminus \{-\infty\})^m$, then $\mathcal{M} \models \phi(\bar{y})$ if and only if $\hat{\mathcal{M}} \models \phi(\bar{y})$.*

Proof. We note that the addition in $M \setminus \{-\infty\}$ is well defined by Lemma B.1. This lemma also shows that $0 \in M \setminus \{-\infty\}$. Therefore, $\hat{M}$ is an $\mathcal{L}_{\mathrm{og}}$-structure. It is immediate to see that it verifies all the axioms of $\mathrm{Th}_{\mathrm{doag}}$. The second claim follows from induction. First, if $\psi(y_1, \dots, y_n)$ is an $\mathcal{L}_{\mathrm{og}}$-term and $\bar{y} \in (M \setminus \{-\infty\})^n$, then $\psi(\bar{y})$ is an element of $M \setminus \{-\infty\}$. Moreover, this is the same element if we treat ψ as an $\mathcal{L}_{\mathrm{og}}$-term over $\hat{\mathcal{M}}$ or if we treat it as an $\mathcal{L}_{\mathrm{ogb}}$-term over $\mathcal{M}$. (This follows by induction from Definition 2.87.) Therefore, the claim is true for atomic formulas or their negations. As noted in the proof of Lemma 2.103, every quantifier-free formula arises as a conjunction and disjunction of atomic formulas or its negations, and the claim follows. □

We now give the proof of Theorem 2.106.

Proof of Theorem 2.106. Suppose that ϕ is an $\mathcal{L}_{\mathrm{ogb}}$-formula and let $m = |\mathsf{Fvar}(\phi)|$. We will show that there exists an $\mathcal{L}_{\mathrm{ogb}}$-formula that is quantifier free and equivalent to ϕ in $\mathrm{Th}_{\mathrm{doagb}}$. Furthermore, if $m \geqslant 1$, then this formula will have the form described in the claim, while if $m = 0$, then this formula will be denoted by $\psi_\emptyset$ and will be of the form $0 = 0$ or $0 \neq 0$. This shows both quantifier elimination and completeness. Throughout the proof, $\mathcal{M} = (M, 0, -\infty, +, \leqslant)$ denotes any model of $\mathrm{Th}_{\mathrm{doagb}}$.

To begin, observe that if $\psi(y_1, \dots, y_n)$ is an $\mathcal{L}_{\mathrm{ogb}}$-term and $\bar{y} \in M^n$, then $\psi(\bar{y})$ evaluates to $-\infty$ in one of the two cases: if ψ contains a symbol $-\infty$ or if at least one coordinate of $\bar{y}$ is equal to $-\infty$. Otherwise, $\psi(\bar{y})$ evaluates to a quantity in $M \setminus \{-\infty\}$. This can be proven by an immediate induction using Lemma B.1. Therefore, if ϕ is atomic, then we can construct the desired formula by the following case-by-case analysis:

- If ϕ is of the form $\psi_1 = \psi_2$ for two $\mathcal{L}_{\mathrm{ogb}}$-terms, then we have the following subcases:
  - If both ψ_1 and ψ_2 contain the symbol $-\infty$, then we set ψ_Σ to be $0 = 0$ for all Σ.
  - If ψ_1 contains the symbol $-\infty$, but ψ_2 does not, then we set ψ_Σ to $0 \neq 0$ if $\mathsf{Fvar}(\psi_2) \subset \Sigma$ and to $0 = 0$ otherwise.
  - If neither ψ_1 nor ψ_2 contains the symbol $-\infty$, then we set ψ_Σ to be of the form $\psi_1 = \psi_2$ if $\mathsf{Fvar}(\phi) = \Sigma$, of the form $0 = 0$ if $\mathsf{Fvar}(\psi_1) \setminus \Sigma \neq \emptyset$ and $\mathsf{Fvar}(\psi_2) \setminus \Sigma \neq \emptyset$, and of the form $0 \neq 0$ otherwise.
- If ϕ is of the form $\psi_1 \leqslant \psi_2$ for two $\mathcal{L}_{\mathrm{ogb}}$-terms, then we have the following subcases:
  - If ψ_1 contains the symbol $-\infty$, then we set ψ_Σ to be $0 = 0$ for all Σ.
  - If ψ_2 contains the symbol $-\infty$, but ψ_1 does not, then we set ψ_Σ to be $0 \neq 0$ if $\mathsf{Fvar}(\psi_1) \subset \Sigma$, and to $0 = 0$ otherwise.
  - If neither ψ_1 nor ψ_2 contains the symbol $-\infty$, then we set ψ_Σ to be of the form $\psi_1 = \psi_2$ if $\mathsf{Fvar}(\phi) = \Sigma$, of the form $0 = 0$ if $\mathsf{Fvar}(\psi_1) \setminus \Sigma \neq \emptyset$, and of the form $0 \neq 0$ otherwise.

We continue the proof by induction. If ϕ is of the form $\neg\psi$ and ψ has no free variables, then the claim is trivial from induction hypothesis. Otherwise, $\psi(y_1, \dots, y_m)$ is of the form

$$\bigvee_{\Sigma \subset [m]} \big((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \land \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \ \land \ \psi_\Sigma\big) \, .$$

Then $\neg\psi$ is equivalent in $\mathrm{Th}_{\mathrm{doagb}}$ to

$$\bigvee_{\Sigma \subset [m]} \big((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \land \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \ \land \ \neg\psi_\Sigma\big) \, .$$

Indeed, if we fix $\bar{y} \in M^m$, then there exists exactly one set $\Sigma \subset [m]$ that satisfies $((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \land \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty))$, namely $\overline{\Sigma} = \{\sigma \in [m] \colon \bar{y}_\sigma \neq -\infty\}$. Hence $\psi(\bar{y})$ is true in $\mathcal{M}$ if and only if $\psi_{\overline{\Sigma}}(\bar{y})$ is true in $\mathcal{M}$. Therefore, by negating all ψ_Σ we negate ψ. If ϕ is an

$\mathcal{L}_{\mathrm{ogb}}$-formula of the form $(\phi_1 \wedge \phi_2)$, then, by induction hypothesis (and adding some disjunctions if necessary), we can assume that ϕ_k are of the form

$$\bigvee_{\Sigma \subset [m]} \left((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \ \wedge \ \psi_{k,\Sigma}\right)$$

for $k \in \{1, 2\}$. The same argument as above shows that ϕ is equivalent in $\mathrm{Th}_{\mathrm{doagb}}$ to

$$\bigvee_{\Sigma \subset [m]} \left((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \ \wedge \ (\psi_{1,\Sigma} \wedge \psi_{2,\Sigma})\right).$$

Finally, if ϕ is of the form $\exists y_{m+1}, \psi$, where ψ has $m+1$ free variables $y_1, \dots, y_m, y_{m+1}$, then by induction hypothesis we can assume that ϕ is of the form

$$\exists y_{m+1}, \bigvee_{\Sigma \subset [m+1]} \left((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \ \wedge \ \psi_\Sigma\right).$$

We divide the sets $\Sigma \subset [m+1]$ into two groups. First, if $y_{m+1} \notin \mathsf{Fvar}(\psi_\Sigma)$, then we define $\hat{\psi}_\Sigma$ as ψ_Σ. Second, if Σ is such that $y_{m+1} \in \mathsf{Fvar}(\psi_\Sigma)$, then we use Theorem 2.101 to find a quantifier-free $\mathcal{L}_{\mathrm{og}}$-formula $\hat{\psi}_\Sigma$ that is equivalent in $\mathrm{Th}_{\mathrm{doag}}$ to the formula $\exists y_{m+1}, \psi_\Sigma$. In both cases, the formulas $\exists y_{m+1}, ((\forall \sigma \in \Sigma, y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin \Sigma, y_\sigma = -\infty) \wedge \psi_\Sigma)$ and $(\forall \sigma \in ([m] \cap \Sigma), y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \in ([m] \setminus \Sigma), y_\sigma = -\infty) \wedge \hat{\psi}_\Sigma$ are equivalent in $\mathrm{Th}_{\mathrm{doagb}}$.[1] Indeed, this is trivial in the first case. In the second case, the fact that $y_{m+1} \in \mathsf{Fvar}(\psi_\Sigma)$ implies, by induction hypothesis, that $m+1 \in \Sigma$. Therefore, the claim follows from Lemma B.2. Thus, ϕ is equivalent in $\mathrm{Th}_{\mathrm{doagb}}$ to

$$\bigvee_{\Sigma \subset [m+1]} \left((\forall \sigma \in ([m] \cap \Sigma), y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \in ([m] \setminus \Sigma), y_\sigma = -\infty) \ \wedge \ \hat{\psi}_\Sigma\right),$$

that can be rewritten as

$$\bigvee_{\Xi \subset [m]} \left((\forall \sigma \in \Xi, y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin \Xi, y_\sigma = -\infty) \ \wedge \ (\hat{\psi}_\Xi \vee \hat{\psi}_{\Xi \cup \{m+1\}})\right). \qquad \square$$

We now present the proof of Lemma 2.108.

Proof of Lemma 2.108. Let $\hat{\mathcal{M}} = (\Gamma, 0, +, \leqslant)$ denote the $\mathcal{L}_{\mathrm{og}}$-structure of Γ. If all strata of $\mathcal{S}$ are semilinear, then, by Lemma 2.103, for every stratum $K = \{k_1, \dots, k_p\} \subset [n]$ there exists an $\mathcal{L}_{\mathrm{og}}$-formula $\psi_K(x_{k_1}, \dots, x_{k_p}, y_1^{(K)}, \dots, y_{\ell_K}^{(K)})$ and a vector $\bar{y}^{(K)} \in \Gamma^{\ell_K}$ such that $\mathcal{S}_K = \{\bar{x} \in \Gamma^K \colon \hat{\mathcal{M}} \models \psi_K(\bar{x}, \bar{y}^{(K)})\}$. Therefore, $\mathcal{S}$ can be written down as an union of the sets of the form

$$\{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \forall k \in K, \bar{x}_k \neq -\infty \wedge \forall k \notin K, \bar{x}_k = -\infty \wedge \hat{\mathcal{M}} \models \psi_K((\bar{x}_k)_{k \in K}, \bar{y}^{(K)})\} \quad \text{(B.1)}$$

(and, possibly, the point $(-\infty, \dots, -\infty)$). Let $\mathcal{M} = (\Gamma \cup \{-\infty\}, 0, -\infty, \leqslant)$ be the $\mathcal{L}_{\mathrm{ogb}}$-structure of $\Gamma \cup \{-\infty\}$. Then, by Lemma B.2, the set (B.1) can be rewritten as

$$\{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \forall k \in K, \bar{x}_k \neq -\infty \ \wedge \ \forall k \notin K, \bar{x}_k = -\infty \ \wedge \ \mathcal{M} \models \psi_K((\bar{x}_k)_{k \in K}, \bar{y}^{(K)})\}$$

[1]With the convention that we replace $(\forall \sigma \in ([m] \cap \Sigma), y_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \in ([m] \setminus \Sigma), y_\sigma = -\infty) \wedge \hat{\psi}_\Sigma$ by $\hat{\psi}_\Sigma$ if $m = 0$.

and $\{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \mathcal{M} \models \tilde{\psi}_K(\bar{x}, \bar{y}^{(K)})\}$, where $\tilde{\psi}_K$ is defined as

$$(\forall k \in K, x_k \neq -\infty) \ \wedge \ (\forall k \notin K, x_k = -\infty) \ \wedge \psi_K \,.$$

This means that $\mathcal{S}$ is definable in $\mathcal{L}_{\mathrm{ogb}}$. To prove the opposite implication, note that if $\mathcal{S}$ is definable in $\mathcal{L}_{\mathrm{ogb}}$, then, by Theorem 2.106, there exists $\ell \geqslant 0$, a formula $\phi(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+\ell})$ of the form

$$\bigvee_{\Sigma \subset [n+\ell]} \big((\forall \sigma \in \Sigma, x_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin \Sigma, x_\sigma = -\infty) \ \wedge \ \psi_\Sigma\big) \,,$$

and a vector $\bar{y} \in (\Gamma \cup \{-\infty\})^\ell$ such that $\mathcal{S} = \{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \mathcal{M} \models \phi(\bar{x}, \bar{y})\}$. Let $L = \{l \in \{n+1, \dots, n+\ell\} \colon \bar{y}_l \neq -\infty\}$ denote the support of $\bar{y}$, take a nonempty set $K \subset [n]$, and consider the formula ϕ_K given by

$$(\forall \sigma \in (K \cup L), x_\sigma \neq -\infty) \ \wedge \ (\forall \sigma \notin (K \cup L), x_\sigma = -\infty) \ \wedge \ \psi_\Sigma \,.$$

Note that, by Lemma B.2, we have

$$\begin{aligned}
&\{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \forall k \in K, \bar{x}_k \neq -\infty \ \wedge \ \forall k \notin K, \bar{x}_k = -\infty \ \wedge \ \mathcal{M} \models \phi(\bar{x}, \bar{y})\} \\
&= \{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \forall k \in K, \bar{x}_k \neq -\infty \ \wedge \ \forall k \notin K, \bar{x}_k = -\infty \ \wedge \mathcal{M} \models \phi_K(\bar{x}, \bar{y})\} \\
&= \{\bar{x} \in (\Gamma \cup \{-\infty\})^n \colon \forall k \in K, \bar{x}_k \neq -\infty \wedge \forall k \notin K, \bar{x}_k = -\infty \wedge \hat{\mathcal{M}} \models \phi_K((\bar{x}_k)_{k \in K}, (\bar{y}_l)_{l \in L})\} \,.
\end{aligned}$$

Therefore, the stratum $\mathcal{S}_K$ is definable in $\mathcal{L}_{\mathrm{og}}$ as $\mathcal{S}_K = \{\bar{x} \in \Gamma^K \colon \hat{\mathcal{M}} \models \phi_K((\bar{x}_k)_{k \in K}, (\bar{y}_l)_{l \in L})\}$. By Lemma 2.103, $\mathcal{S}_K$ is semilinear. □

B.2 Ergodic theorem for Markov chains with rewards

In this section we give the proof of Theorem 2.137 based on the analysis given in the textbook of Chung [Chu67]. We use the same notation as in Section 2.7. Let μ_{uw} denote the probability that the Markov chain starting from $u \in V$ will reach $w \in V$ at least once, $\mu_{uw} = \mathbb{P}(\exists s \geqslant 1, X_s = w | X_0 = u)$. By definition, the state u is recurrent if $\mu_{uu} = 1$, and it is transient otherwise. By ζ_{uw} we denote the expected number of visits in w before returning to u, i.e.,

$$\zeta_{uw} = \mathbb{E}\Big(\sum_{s=0}^{T_u - 1} \mathbf{1}_{\{X_s = w\}} \Big| X_0 = u\Big) \,.$$

The following theorem describes the ergodic behavior of any finite (or countable) Markov chain.

Theorem B.3 ([Chu67, Part I, §6, Theorem 4 and its Corollary])**.** *The Cesaro limit*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{s=0}^{N} P^s$$

is well defined (we will denote it by M). Moreover, the entries of M are given as follows: if w is a transient state, then $M_{uw} = 0$ for all u. If w is recurrent, then $M_{uw} = \frac{\mu_{uw}}{\theta_w}$ for all u.

Remark B.4. Note that the theorem above does not state that $\theta_w < \infty$ if the state w is recurrent. (We work under the convention that $\frac{a}{+\infty} = 0$ for all finite a.) Nevertheless, if the chain is finite, then we have $\theta_w < \infty$ for all recurrent states w, and this can be deduced as a corollary of the theorem above, as discussed below.

Observe that M is a stochastic matrix (as a limit of stochastic matrices). Moreover, we have $MP = PM = M$. This leads to the following corollary.

Corollary B.5. *If $C \subset V$ is a recurrent class, then $M_{uw} = \frac{1}{\theta_w}$ for all $u, w \in C$. Furthermore, $(\pi_u)_{u \in C}$ defined as $\pi_u = \frac{1}{\theta_u}$ is the unique stationary distribution on C. In particular, if $u \in V$ is a recurrent state, then $\theta_u < \infty$.*

Proof. Let first claim follows immediately from Theorem B.3. We will prove that $(\pi_u)_{u \in C}$ is a stationary distribution on C. Let P_C denote the square submatrix of P formed by the rows and columns of P with indices in C. We define M_C analogously. The first claim implies that M_C has identical rows. Since C is a recurrent class, we have $P_{uw'} = 0$ for all $u \in C, w' \notin C$. Hence, for all s we have $[P^s]_C = [P_C]^s$. Therefore $M_C = \lim_{N \to \infty} \frac{1}{N} \sum_{s=0}^{N} P_C^s$. Hence, M_C is stochastic. In other words, every row of M_C is a probability distribution on C and, since $M_C P_C = M_C$, this distribution is a stationary distribution on C. Since C is a recurrent class, stationary distribution on C has only strictly positive values. Hence we have $\theta_u < \infty$. The fact that the stationary distribution is unique follows from [Chu67, Part I, §7, Theorem 1]. □

The next theorem characterizes the relationship between entries of M and the values ζ_{uw}.

Theorem B.6. *If (u, w) belong to the same recurrent class, then $0 < \zeta_{uw} < \infty$. Moreover, if (f, g, h, u, w) are (not necessarily distinct) states belonging to the same recurrent class, then $M_{fg}/M_{hu} = \zeta_{wg}/\zeta_{wu}$.*

Proof. The fact that $0 < \zeta_{uw} < \infty$ follows from [Chu67, Part I, §9, Theorem 2 and 3]. Moreover, by Corollary B.5 we have $M_{uw} = \frac{1}{\theta_w} > 0$. Hence the claim follows from [Chu67, Part I, §9, Theorem 5 and remarks that precede it]. □

Corollary B.7. *If (u, w) belong to the same recurrent class, then*

$$M_{uw} = \frac{1}{\theta_w} = \frac{\zeta_{uw}}{\theta_u} \, .$$

Proof. By Theorems B.3 and B.6 we have $\theta_u/\theta_w = M_{ww}/M_{uu} = \zeta_{uw}/\zeta_{uu}$. By definition $\zeta_{uu} = 1$ and hence $\theta_u/\theta_w = \zeta_{uw}$. □

Proof of Theorem 2.137. Fix $u \in V$. Observe that for all $N \geqslant 1$ we have

$$\mathbb{E}\Big(\sum_{s=0}^{N} r_{X_s} \Big| X_0 = u\Big) = [r + Pr + \dots + P^N r]_u \, .$$

Therefore $g_u = [Mr]_u$. In particular, g_u is well defined.

Let us suppose that the initial state u is recurrent and denote its recurrent class by C. In this case, Corollary B.5 and Theorem B.3 imply that $g_u = \sum_{w \in C} r_w \pi_w$, where $(\pi_w)_{w \in C}$ is the stationary distribution on C. Moreover, Corollary B.7 gives the identity

$$\begin{aligned} g_u &= \frac{1}{\theta_u} \sum_{w \in C} \zeta_{uw} r_w \\ &= \frac{1}{\theta_u} \mathbb{E}\Big(\sum_{w \in C} \sum_{s=0}^{T_u - 1} r_w \mathbf{1}_{\{X_s = w\}} \Big| X_0 = u \Big) \\ &= \frac{1}{\theta_u} \mathbb{E}\Big(\sum_{s=0}^{T_u - 1} \sum_{w \in V} r_w \mathbf{1}_{\{X_s = w\}} \Big| X_0 = u \Big) \\ &= \frac{1}{\theta_u} \mathbb{E}\Big(\sum_{s=0}^{T_u - 1} r_{X_s} \Big| X_0 = u \Big) = \frac{\xi_u}{\theta_u} \,. \end{aligned}$$

Now, suppose that the initial state u is transient. Let $C_1, \dots, C_p$ denote the recurrent classes in our Markov chain. In this case, Theorem B.3 gives the identity

$$g_u = \sum_{s=1}^{p} \sum_{w \in C_s} \frac{\psi_s}{\theta_w} r_w = \sum_{s=1}^{p} \psi_s g_{u_s} \,. \qquad \square$$

B.3 Kohlberg's theorem

In this appendix, we give a proof of the Kohlberg's theorem (Theorem 6.6) and Proposition 8.39. The original proof presented in [Koh80] uses Farkas' lemma in the ordered field of rational functions. We present a more abstract variant of this proof, replacing Farkas' lemma by a variant of quantifier elimination in ordered fields. It is easy to see that the theory of ordered fields (without the assumption of real closedness) does not admit quantifier elimination in the sense considered in Section 2.6. Indeed, this theory is not model complete, because the statement $\exists x,\, 2 = x \cdot x$ is true in $\mathbb{R}$ but false in $\mathbb{Q}$. However, there exists a weaker variant of quantifier elimination, called linear quantifier elimination (see, e.g, [Wei88, LW93, ER92]), that is true over ordered fields. This requires to introduce the definition of a linear formula in the language $\mathcal{L}_{\mathrm{or}}$. This requires to make a slight adaptation of the construction of formulas presented in Section 2.6.

Definition B.8. Let $I \subset \mathbb{N}$. The set of *I-linear terms* of $\mathcal{L}_{\mathrm{or}}$ is constructed in the following way:

- constant symbols 0 and 1 are an *I-linear* terms;
- every variable symbol x_k is an *I-linear* term;
- if ϕ, ψ are *I-linear* terms, then $\phi + \psi$ is an *I-linear* term;
- if ϕ, ψ are *I-linear* terms and the free variables of ψ do not contain a symbol from $\{x_i\}_{i \in I}$, $\mathsf{Fvar}(\psi) \cap \{x_i\}_{i \in I} = \emptyset$, then $\phi \cdot \psi$ is an *I-linear* term.

In other words, an I-linear term is a polynomial of the form $P(x) = P_0(x) + \sum_{i \in I} P_i(x) x_i$, where the polynomials $P_0(x), P_i(x)$ have natural coefficients and do not depend on the variables from $\{x_i\}_{i \in I}$.

Definition B.9. The set of *I-linear formulas* of $\mathcal{L}_{\mathrm{or}}$ is constructed from the *I-linear* terms in the same way as the set of $\mathcal{L}_{\mathrm{or}}$-formulas is constructed from $\mathcal{L}_{\mathrm{or}}$-terms (see Section 2.6). An $\mathcal{L}_{\mathrm{or}}$-formula ϕ is *linear* if it is linear with respect to its bound variables, i.e., if it is I-linear, where $\{x_i\}_{i \in I} = \mathsf{Bvar}(\phi)$.

The following result is the linear quantifier elimination.

Theorem B.10 ([Wei88, LW93, ER92])**.** *If $\phi(x_1, \dots, x_p)$ is a linear $\mathcal{L}_{\mathrm{or}}$-formula, then there exists a quantifier-free $\mathcal{L}_{\mathrm{or}}$-formula $\psi(x_1, \dots, x_p)$ such that ϕ is equivalent to ψ in the theory of ordered fields.*

Moreover, given the quantifier elimination result above, as a corollary we obtain a model completeness result.

Corollary B.11 ([ER92, Corollary 2])**.** *Suppose that $\mathscr{K}_1, \mathscr{K}_2$ are ordered fields and that $\mathscr{K}_1$ is a substructure of $\mathscr{K}_2$ with an embedding $\eta\colon \mathscr{K}_1 \to \mathscr{K}_2$. Moreover, suppose that $\phi(x_1, \dots, x_p)$ is a linear $\mathcal{L}_{\mathrm{or}}$-formula. Then, for any $\bar{x}_1, \dots, \bar{x}_p \in \mathscr{K}_1$ we have $\mathscr{K}_1 \models \phi(\bar{x}_1, \dots, \bar{x}_p)$ if and only if $\mathscr{K}_2 \models \phi(\eta(\bar{x}_1), \dots, \eta(\bar{x}_p))$.*

We will apply Corollary B.11 to the field of Laurent series. Let us recall the definition of this field.

Definition B.12. We define the set of *Laurent series*, denoted $\mathbb{R}\{t\}$, in the following way. Let $\boldsymbol{x} = \sum_{i=1}^{\infty} c_{\lambda_i} t^{\lambda_i}$ be a Puiseux series as in (2.2). Then, $\boldsymbol{x}$ belongs to $\mathbb{R}\{t\}$ if the sequence $(\lambda_i) \geqslant 1$ consists of integer numbers. Moreover, the empty series 0 belongs to $\mathbb{R}\{t\}$.

One can easily adapt the proof of Lemma A.19 to show that $\mathbb{R}\{t\}$ is a subfield of $\mathbb{K}$, and the embedding is given by the identity map. We can now prove Kohlberg's theorem. To this end, fix a piecewise-affine function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ be its piecewise description. Furthermore, for all $s \in [p]$, let $C^{(s)} \in \mathbb{R}^{m_s\times n}, d^{(s)} \in \mathbb{R}^{m_s}$ be such that $\mathcal{W}^{(s)} = \{x \in \mathbb{R}^n \colon C^{(s)}x \leqslant d^{(s)}\}$.

Theorem B.13 ([Koh80])**.** *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is piecewise affine and nonexpansive in any norm. Then, there exist $u, \eta \in \mathbb{R}^n$, $s \in [p]$, and $\gamma_0 \geqslant 0$ such that $(A^{(s)} - I)\eta = 0$, $(A^{(s)} - I)u = \eta - b^{(s)}$, and $f(u + \gamma\eta) = u + (\gamma + 1)\eta$ for all $\gamma \geqslant \gamma_0$. Furthermore, for every $\pi \in \mathbb{R}^n$ such that $\pi^\intercal A^{(s)} = \pi$ we have $\pi^\intercal u = 0$.*

Remark B.14. We note that the theorem above immediately implies the claims of Theorem 6.6 and Proposition 8.39.

Proof of Theorem B.13. Let $\|\cdot\|$ denote any norm for which f is nonexpansive. Then, for any fixed $0 < \alpha < 0$, the function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as $g(x) \coloneqq f(\alpha x)$ is a contraction in $\|\cdot\|$ and hence it has a unique fixed point. Consider the statement "for every $\alpha \in]0, 1[$, there exist $s \in [p]$ and x such that $\alpha A^{(s)}x + b^s = x$ and $\alpha C^{(s)}x \leqslant d^{(s)}$." This statement is true in $\mathbb{R}$ and, by the model completeness of real closed fields (Theorem 2.110 and Proposition 2.124), it is true in $\mathbb{K}$. In particular, this statement is true in $\mathbb{K}$ for $\alpha \coloneqq 1 - t^{-1}$. Moreover, after fixing α, the statement becomes linear. More precisely, if we consider $(\tilde{\alpha}, (\tilde{A}^{(s)}, \tilde{b}^{(s)}, \tilde{C}^{(s)}, \tilde{d}^{(s)})_{s\in[p]})$ to be variables (without fixing their value), then the $\mathcal{L}_{\mathrm{or}}$-formula "there exist $s \in [p]$ and x such that $\tilde{\alpha}\tilde{A}^{(s)}x + \tilde{b}^s = x$ and $\tilde{\alpha}\tilde{C}^{(s)}x \leqslant \tilde{d}^{(s)}$" is linear. Since this formula is true in $\mathbb{K}$ after fixing $(\tilde{\alpha}, (\tilde{A}^{(s)}, \tilde{b}^{(s)}, \tilde{C}^{(s)}, \tilde{d}^{(s)})_{s\in[p]}) = (1-t^{-1}, (A^{(s)}, b^{(s)}, C^{(s)}, d^{(s)})_{s\in[p]})$, by Corollary B.11, it is true in $\mathbb{R}\{t\}$. In other words, there exists a vector $\boldsymbol{x} \in \mathbb{R}\{t\}^n$ and $s \in [p]$ such that $(1 - t^{-1})A^{(s)}\boldsymbol{x} + b^{(s)} = \boldsymbol{x}$ and $(1 - t^{-1})C^{(s)}\boldsymbol{x} \leqslant d^{(s)}$.

First, we will show that $\mathsf{val}(\boldsymbol{x}_k) \leqslant 1$ for all $k \in [n]$. By Proposition 8.1, there exists $t_0 > 0$ such that $(1-t^{-1})A^{(s)}\boldsymbol{x}(t)+b^{(s)} = \boldsymbol{x}(t)$ and $(1-t^{-1})C^{(s)}\boldsymbol{x}(t) \leqslant d^{(s)}$ for all $t > t_0$. In particular, for all such t we have

$$f\big((1 - t^{-1})\boldsymbol{x}(t)\big) = (1 - t^{-1})A^{(s)}\boldsymbol{x}(t) + b^{(s)} = \boldsymbol{x}(t)\,. \tag{B.2}$$

Moreover, since f is nonexpansive, we have

$$\|f(0) - \boldsymbol{x}(t)\| = \|f(0) - f\big((1 - t^{-1})\boldsymbol{x}(t)\big)\| \leqslant \|(1 - t^{-1})\boldsymbol{x}(t)\| = (1 - t^{-1})\|\boldsymbol{x}(t)\|\,.$$

Hence $\|\boldsymbol{x}(t)\| \leqslant (1-t^{-1})\|\boldsymbol{x}(t)\| + \|f(0)\|$ and $\|\boldsymbol{x}(t)\| \leqslant t\|f(0)\|$ for all $t > t_0$. By the equivalence of norms in $\mathbb{R}^n$, there exists a constant $c > 0$ such that $|\boldsymbol{x}_k(t)| \leqslant ct\|f(0)\|_\infty$ for all $k \in [n]$. In particular, we have $\log_t |\boldsymbol{x}_k(t)| \leqslant 1 + \log_t(c\|f(0)\|_\infty)$ and hence $\mathsf{val}(\boldsymbol{x}_k) \leqslant 1$.

Second, denote $\boldsymbol{x}_k = c^{(k)}_1 t + c^{(k)}_0 + c^{(k)}_{-1}t^{-1} + \dots$ for every $k \in [n]$, where we allow $c^{(k)}_i = 0$. We put $\eta \coloneqq (c^{(1)}_1, \dots, c^{(n)}_1)$ and $u \coloneqq (c^{(1)}_0, \dots, c^{(n)}_0)$. We will show that (η, u) satisfy the claim. By comparing the first two terms in the series development of the equality $(1-t^{-1})A^{(s)}\boldsymbol{x} + b^{(s)} = \boldsymbol{x}$ we get $A^{(s)}\eta = \eta$ and $A^{(s)}u - A^{(s)}\eta + b^{(s)} = u$, as claimed. Furthermore, if $\pi \in \mathbb{R}^n$ is such that $\pi^\intercal A^{(s)} = \pi$, then $(1 - t^{-1})\pi^\intercal\boldsymbol{x} + \pi^\intercal b^{(s)} = \pi^\intercal\boldsymbol{x}$. Hence $\pi^\intercal b^{(s)} = t^{-1}\pi^\intercal\boldsymbol{x}$. In particular, we have $\pi^\intercal u = 0$. It remains to prove that there exists $\gamma_0 \geqslant 0$ such that $f(u + \gamma\eta) = u + (\gamma + 1)\eta$ for all $\gamma \geqslant \gamma_0$. By the fact that f is nonexpansive, and the equivalence of norms in $\mathbb{R}^n$, there exists a constant $c' > 0$ such that for all $t > t_0$ we have

$$\begin{aligned}\|f\big((1 - t^{-1})\boldsymbol{x}(t)\big) - f\big(t\eta + (u - \eta)\big)\| &\leqslant \|(1 - t^{-1})\boldsymbol{x}(t) - t\eta - u + \eta\| \\ &\leqslant c' \sum_{k\in[n]} |(1 - t^{-1})\boldsymbol{x}_k(t) - t\eta_k - u_k + \eta_k|\,.\end{aligned} \tag{B.3}$$

Since $\mathsf{val}\big((1-t^{-1})\boldsymbol{x}_k(t) - t\eta_k - u_k + \eta_k\big) < 0$, the last expression in (B.3) goes to 0 as t goes to $+\infty$ (by the triangle inequality and Lemma A.18). In a similar way, the expression

$$\begin{aligned}&\|((1-t^{-1})A^{(s)}\boldsymbol{x}(t) + b^{(s)}) - (A^{(s)}(t\eta + u - \eta) + b^{(s)})\| \\ &= \|(1-t^{-1})A^{(s)}\boldsymbol{x}(t) - t\eta - u + b^{(s)}\| = \|\boldsymbol{x}(t) - t\eta - u\|\end{aligned}$$

goes to zero as t goes to $+\infty$. Therefore, by (B.2), the function $g(t) \colon \mathbb{R}_{>0} \to \mathbb{R}^n$

$$g(t) \coloneqq f(t\eta + (u-\eta)) - A^{(s)}(t\eta + (u-\eta)) + b^{(s)}$$

satisfies $\lim_{t\to+\infty} \|g(t)\| = 0$. Moreover, note that the points $t\eta + (u-\eta)$ lie on the same half-line. Since f is piecewise affine, for sufficiently large t, these points belong to the same polyhedron of the piecewise description of f. Thus, $g(t)$ is affine for t large enough. Hence, by $\lim_{t\to+\infty} \|g(t)\| = 0$, the function $g(t)$ is equal to 0 for t large enough and we the claim follows. $\square$

ANNEXE C

Résumé en français

C.1 Contexte et motivations

La programmation semi-définie est un outil fondamental d'optimisation convexe. Elle revient à optimiser une fonction linéaire sur un spectraèdre, qui est un ensemble défini par une inégalité matricielle linéaire

$$\mathcal{S} \coloneqq \{x \in \mathbb{R}^n \colon Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succcurlyeq 0\} \, ,$$

où $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{R}^{m \times m}$ est une suite des matrices symétriques et $\succcurlyeq$ est l'ordre de Loewner sur l'espace des matrices symétriques. Par définition, $A \succcurlyeq B$ si et seulement si $A - B$ est semi-définie positive. À cause de sa puissance expressive, la programmation semi-définie a trouvé des applications nombreuses. Par exemple, les relaxations semi-définies sont utilisées pour obtenir des approximations polynomiales aux problèmes NP-difficiles d'optimisation combinatoire, tels que le problème de Max-Cut [GW95]. La fonction thêta de Lovász [Lov79] est une autre application classique de la programmation semi-définie dans le domaine d'optimisation combinatoire. On peut calculer cette fonction en temps polynomial en résolvant un programme semi-défini. De plus, elle est comprise entre le nombre de cliques et le nombre chromatique, qui sont NP-difficiles à calculer. Nous renvoyons à [GM12, LR05] pour plus d'informations sur les applications de la programmation semi-définie à l'optimisation combinatoire. La programmation semi-définie est aussi importante dans la domaine d'optimisation polynomiale. Même si les problèmes d'optimisation polynomiale ne sont pas convexes en général, Lasserre [Las01, Las02] et Parrilo [Par03] ont montré qu'une grande partie de ces problèmes peut être résolue avec une précision arbitraire en utilisant une hiérarchie de programmes semi-définis. On trouve plus d'informations sur les applications de la programmation semi-définie en optimisation polynomiale dans les livres [Las15, Las09b]. Grâce à sa puissance expressive, la programmation semi-définie a des applications dans la théorie du contrôle [BEGFB94], en information quantique [Wat18, FSP18, GdLL18], en preuve formelle [MC11, MAGW15], en vérification de programmes [RVS16], pour les plans d'expérience [VBW98], en traitement du signal [PE10] et dans d'autres domaines. Nous renvoyons à [WSV00, AL12, dK02, BPT13, BTN01, BV04] pour plus d'informations sur la programmation semi-définie et ses applications.

En pratique, on résout des programmes semi-définis en utilisant des méthodes de points intérieurs. Ces méthodes ont été généralisées de la programmation linéaire à la programmation semi-définie par Alizadeh [Ali95] et à la programmation convexe par Nesterov et Nemirovski [NN94]. Nous renvoyons à [dK02, Ren01, GM12] pour plus d'informations sur les méthodes de points intérieurs.

Il y a beaucoup des questions ouvertes sur les spectraèdres et la programmation semi-définie. Par exemple, Nemirovski [Nem07] a demandé de caractériser les spectraèdres projetés. Helton et Nie [HN09] ont conjecturé que tout ensemble semi-algébrique convexe est une projection d'un spectraèdre. La conjecture a été confirmée pour plusieurs classes d'ensembles [HN09, HV07, HN10, Las09a, GPT10, GN11, NPS08]. De plus, on sait qu'elle est vraie en dimension 2 [Sch18a]. Cependant, la conjecture à été récemment réfutée par Scheiderer, qui a montré que le cône des formes semi-définies positives n'est pas une projection d'un spectraèdre, sauf pour quelques cas particuliers [Sch18b]. Son article contient une liste exhaustive de références.

La conjecture généralisée de Lax est une autre question ouverte. Elle demande si tout cône d'hyperbolicité est un spectraèdre. La réponse est positive pour plusieurs classes de ces cônes [HV07, LPR05, PV13, AB18, Kum17]. Néanmoins, quelques généralisations de la conjecture sont fausses [Brä11, AB18, BVY14]. Nous renvoyons aux travaux cités pour plus d'informations.

La géométrie des spectraèdres a été étudié, sous des perspectives différentes, dans [RG95, DI10, ORSV15, FSED18]. Cependant, il y a plusieurs questions ouvertes dans cette domaine. Par exemple, on ne comprend pas bien la structure faciale des spectraèdres et des cônes d'hyperbolicité.

Dans cette thèse, nous nous intéressons à la complexité théorique de la programmation semi-définie. Dans le modèle de machine du Turing, on peut obtenir des solutions approchées aux programmes semi-définis bien structurés en utilisant l'algorithme de l'ellipsoïde [GLS93, Ram93] (cependant, cette méthode n'est pas efficace en pratique). Le même résultat pour les méthodes de points intérieurs a été montré très récemment [dKV16]. Ces deux méthodes fournissent uniquement des solutions approchées. Une solution exacte au programme semi-défini est un nombre algébrique et le degré de son polynôme minimal peut être élevé [NRS10]. De plus, les méthodes mentionnées ci-dessus dépendent d'hypothèses supplémentaires sur la structure d'un spectraèdre sous-jacent. Nous renvoyons à [Ram97, dKV16, LMT15] pour plus d'informations et à [LP18] pour une classe de programmes semi-définis de petite taille qui sont difficiles pour les logiciels contemporains. Les méthodes de pointe qui sont capables de résoudre les programmes semi-définis généraux sont fondées sur les méthodes de points critiques [HNSED16, Nal18, HNSED18] (en outre, on peut décider de la vacuité d'un spectraèdre par l'elimination des quantificateurs [PK97]). De point de vue théorique, Ramana [Ram97] a montré que le problème de la vacuité semi-définie (étant données des matrices $Q^{(0)}, \dots, Q^{(1)}$, decider si le spectraèdre associé est non vide) appartient à la classe $\mathrm{NP}_{\mathbb{R}} \cap \mathrm{coNP}_{\mathbb{R}}$, où l'indice $\mathbb{R}$ dénote le modèle de calcul BSS. On ne sait pas si ce problème appartient au NP dans le modèle de machine de Turing. La difficulté de ce problème est liée au fait que tout point inclus dans un spectraèdre peut avoir des coordonnées de taille doublement exponentielle par rapport à la taille des données. En outre, il est possible qu'un spectraèdre soit non vide sans contenir des points rationnels [Sch16]. Encore plus concrètement, Tarasov et Vyalyi [TV08] ont montré que le problème de comparaison des nombres définis par des circuits arithmétiques se réduit au problème de la vacuité semi-définie. Ce problème est plus difficile que le problème Sum of Square Roots. La complexité de ce dernier problème est ouverte depuis (au moins) 1976, cf. [GGJ76, Pap77, ABKPM09, EY10, JT18] pour plus d'informations. En outre, certains problèmes de complétion de matrices se réduisent au problème de la vacuité semi-définie [Lau01, LV14]. Différents certificats de (non-)vacuité, du caractère borné, et d'inclusion de spectraèdres ont été étudiés dans [Ram97, KS13, LP18, KTT13, The17, KPT18].

Il y a plusieurs caractérisations équivalentes du fait qu'une matrice symétrique est semi-définie positive. Par exemple, on peut définir cette notion en supposant que les valeurs propres d'une matrice sont positives, que ses mineurs principaux sont positifs, que la forme quadratique associée à cette matrice est positive ou par le fait que cette matrice admet une décomposition de Cholesky. On peut montrer que ces définitions sont équivalentes dans tout corps réel clos (et pas seulement dans le corps de nombres réels). Cela est une conséquence de la complétude de la théorie des corps réels clos. En particulier, la notion d'une matrice semi-définie positive est valable dans tout corps réel clos. Par conséquent, on peut étudier les spectraèdres et la programmation semi-définie dans chaque corps réel clos, même si ce corps est non-archimédien. C'est l'objet de cette thèse. L'exemple le plus important pour nous est le corps des séries de Puiseux. Classiquement, le corps des séries de Puiseux (qui avait été déjà considéré par Newton [BK12, Chapter 8.3]) est défini comme un corps des séries formelles ayant des exposants rationnelles et telles que chaque exposant dans une série fixée a le même dénominateur. Néanmoins, du point de vue de la géométrie tropicale, il est utile de considérer un corps encore plus grand, composé de séries ayant des exposants réels [Mar10]. Pour cette raison, nous considérons ici le corps des séries de Puiseux généralisées qui a été proposé dans [Mar10]. Ce corps, a un changement de variable près, est identique au corps des séries de Dirichlet généralisées qui a été étudiée par Hardy et Riesz [HR15]. Une série de Puiseux généralisée est une série de la forme

$$\boldsymbol{x} = \sum_{i=1}^{\infty} c_{\lambda_i} t^{\lambda_i} \, ,$$

où t est un paramètre formel et $(\lambda_i)_{i \geqslant 1}$ est une suite strictement décroissante et soit finie soit non bornée. Si nous supposons que les coefficients $c_{\lambda_i} \in \mathbb{C} \setminus \{0\}$ sont complexes, alors l'ensemble des séries de Puiseux généralisées est un corps algébriquement clos. Si nous supposons que c_{λ_i} sont réels, nous obtenons un sous-corps qui est réel clos. De plus, van den Dries et Speissegger [vdDS98] ont montré que l'ensemble de séries absolument convergentes (pour t suffisamment grand) est un corps réel clos aussi. Ce corps est le corps principal considéré dans cette thèse. Nous le notons par $\mathbb{K}$. Cependant, on remarque qu'on a choisi $\mathbb{K}$ pour des raisons de simplicité et pour rendre la présentation plus concrète. Dans les chapitres ultérieurs de la thèse, nous montrerons que les résultats que nous obtenons pour $\mathbb{K}$ peuvent être transférés (par une élimination des quantificateurs) aux autres corps réels clos non-archimédiens. Pour la brièveté, nous omettons l'adjectif "généralisées" quand nous discutons les séries de Puiseux.

D'après la discussion ci-dessus, la définition d'un spectraèdre dans les séries de Puiseux est la même que la définition dans les nombres réels. Autrement dit, un spectraèdre dans $\mathbb{K}$ est un ensemble défini par

$$\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \cdots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} \succcurlyeq 0\} \, ,$$

où $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ est une suite de matrices symétriques. Un spectraèdre dans les séries de Puiseux fournit une famille des spectraèdres réels, obtenus en remplaçant le paramètre formel t par une valeur réelle et grande :

$$\boldsymbol{\mathcal{S}}(t) \coloneqq \{x \in \mathbb{R}^n \colon \boldsymbol{Q}^{(0)}(t) + x_1\boldsymbol{Q}^{(1)}(t) + \dots + x_n\boldsymbol{Q}^{(n)}(t) \succcurlyeq 0\} \,.$$

L'élimination des quantificateurs dans les corps réels clos assure que la programmation semi-définie dans les séries de Puiseux a les mêmes propriétés fondamentales que la programmation semi-définie dans les nombres réels. De plus, elle implique que, pour t suffisamment grand, les spectraèdres réels $\boldsymbol{\mathcal{S}}(t)$ héritent les propriétés du spectraèdre non-archimédien $\boldsymbol{\mathcal{S}}$. Il y a plusieurs motivations pour étudier ces types des programmes semi-définis. On peut obtenir ces programmes en considérant des programmes paramétriques dans les réels ou des programmes structurés ayant des entrées d'ordres de grandeur différents. Ils ont aussi leur propre intérêt car, par l'analogie avec la programmation linéaire [Meg89], on présume que le cas non-archimédien peut nous aider à comprendre la complexité du problème dans les nombres réels. En particulier, les polyèdres sur les séries de Puiseux encodent une classe de polyèdres réels définis par des matrices à grands coefficients. La complexité des programmes linéaires définis par ces matrices est un cas particulier du neuvième problème de Smale [Sma98], qui demande si on peut résoudre la programmation linéaire en temps fortement polynomial dans le modèle de machine de Turing et en temps polynomial dans le modèle BSS. En plus, les spectraèdres dans les séries de Puiseux peuvent fournir de nouvelles classes d'exemples des spectraèdres ayant des propriétés géométriques inattendues. Nos méthodes d'étudier les spectraèdres non-archimédiens sont basés sur la géométrie tropicale.

Une question générale en géométrie tropicale consiste à fournir des caractérisations combinatoires des amibes non-archimédiennes, définies comme des images par la valuation non-archimédienne des ensembles algébriques dans les corps non-archimédiens algébriquement clos. C'était le sujet du travail de Bieri et Groves [BG84] qui ont généralisé les résultats précédents obtenus par Bergman [Ber71]. Le théorème de Kapranov [EKL06] donne une caractérisation des images des hypersurfaces en utilisant la notion d'une hypersurface tropicale. La généralisation de ce théorème au cas des variétés algébriques est connue sous le nom de “théorème fondamental de la géométrie tropicale”. Une démonstration constructive de ce théorème et une discussion historique se trouvent dans [JMM08]. La géométrie tropicale a attiré beaucoup d'attention à cause de ses liens avec la géométrie algébrique énumérative [Mik05]. On trouve plus d'informations sur ces aspects de la géométrie tropicale dans [RGST05, IMS09, MS15].

L'étude des objets provenant de la géométrie algébrique réel en utilisant des outils tropicaux a été initié par Viro [Vir89, Vir08], qui a utilisé des méthodes combinatoires pour construire des courbes planaires avec la topologie prescrite, en relation avec le seizième problème d'Hilbert. Ultérieurement, cette méthode a été utilisée par Itenberg et Viro pour construire un contre-exemple à la conjecture de Ragsdale [IV96]. De plus, elle a été généralisée au cas des intersections completes [Stu94, Bih02]. Cependant, en toute généralité, il n'y a pas d'analogue du théorème fondamental pour les tropicalisations des variétés réelles. Nous renvoyons à [SW05, Ale13, Vin12] pour une discussion. Les images par valuation des ensembles semi-algébriques ont été étudiées par Alessandrini [Ale13], qui a montré un analogue réel du théorème de Bieri et Groves.

Indépendamment des travaux cités ci-dessus, la convexité et séparation dans les semi-modules idempotents ont été étudiées par plusieurs auteurs [Zim77, Hel88, SS92, LMS01, CGQ04, BH04, CGQS05]. Le lien entre ces deux domaines a été observé par Develin et Sturmfels [DS04], qui ont introduit le terme “convexité tropicale”. Develin et Yu [DY07] ont caractérisé des polyèdres tropicaux comme des images par valuation des polyèdres dans les séries de Puiseux. La géométrie des polyèdres tropicaux à été étudié dans plusieurs travaux, par exemple [Jos05, GK07, BSS07, AGK11, AK17]. Dans une série d'articles étroitement liés à cette thèse, Allamigeon, Benchimol, Gaubert et Joswig ont étudié la tropicalisation des polytopes définis par des matrices génériques, développé l'algorithme du simplexe tropical et décrit l'équivalent tropical du chemin central [ABGJ15, ABGJ14, ABGJ18]. Yu [Yu15] a tropicalisé le cône des matrices semi-définies positives et montré que cette tropicalisation est donnée en utilisant uniquement des mineurs de taille 2. Akian, Gaubert et Guterman [AGG12] ont montré que le problème de la vacuité pour les polyèdres tropicaux est équivalent au problème des jeux déterministes à paiement moyen. Grigoriev et Podolskii [GP15] ont montré le même résultat pour le problème de pré-variétés tropicales linéaires. Bodirsky et Mamino [BM16] ont étudié les liens entre la convexité tropicale, les jeux stochastiques à paiement moyen et les problèmes de la satisfiabilité des contraintes.

Les jeux répétés à somme nulle ont été introduits par Shapley [Sha53], qui a étudié l'existence des stratégies optimales en temps fini et infini avec des paiements escomptés. Gillette [Gil57] a proposé d'étudier les jeux à information parfaite et paiement moyen. Liggett et Lippman [LL69] ont montré que ces jeux possèdent des stratégies optimales stationnaires. Le cas déterministe de ces jeux a été étudié par Ehrenfeucht et Mycielski [EM79]. L'existence de stratégies stationnaires optimales n'est plus vraie pour les jeux à information imparfaite. Néanmoins, Bewley et Kohlberg [BK76] ont montré que la valeur limite des jeux stochastiques à information imparfaite est bien définie et Mertens et Neyman [MN81] ont montré que cette limite est la valeur du jeu avec un paiement moyen. Il existe de nombreux autres travaux sur la théorie des jeux répétés à somme nulle. Nous nous référons à [MSZ15, LS15] pour plus d'information. Lune des manières possibles détudier ces jeux est dutiliser les propriétés de lopérateur de Shapley associé. Cette approche a été présentée dans l'article original par Shapley [Sha53] et a ensuite été étendue à d'autres classes de jeux [RS01, Ney03]. De plus, cette approche a été utilisée dans [AGG12] et nous faisons la même chose dans cette thèse.

Indépendamment, des cas particuliers de jeux stochastiques à information parfaite et paiement moyen ont été découverts dans la communauté informatique. Les jeux de parité ont été introduits par Emerson et Jutla [EJ91] et Mostowski [Mos91]. Ils sont équivalents à la vérification de modèle dans la logique du μ-calcul modal [EJS93, Jur98]. De plus, ils ont été utilisés par Friedmann [Fri11, DH17] pour construire une borne inférieure sous-exponentielle pour la règle de pivotage de Zadeh pour l'algorithme du simplexe (voir aussi [Fea10, FHZ14]). Les jeux à paiement moyen déterministes ont été étudiés par Gurvich, Karzanov et Khachiyan [GKK88] et les jeux stochastiques simples ont été introduits par Condon [Con92]. Les réductions étudiées dans [Jur98, ZP96, AM09] et l'analyse de [Con92] donnent les faits suivants. Premièrement, les jeux de parité sont réductibles en temps polynomial à des jeux à paiement moyen déterministes, qui forment un sous-ensemble de jeux à paiement moyen stochastiques. Deuxièmement, les jeux à paiement moyen stochastiques sont équivalents en temps polynomial à des jeux stochastiques simples et à des jeux à paiement escompté. Troisièmement, le problème de décision associé (étant donné létat dans un jeu, décider si sa valeur est positive) appartient à NP $\cap$ coNP (même à UP $\cap$ coUP). D'un autre coté, le problème de calcul de la valeur de jeux stochastiques simples appartient à la classe de complexité CLS, qui est une sous-classe de PPAD et PLS [DP11, EY10]. Malgré tous ces résultats, il n'existe aucun algorithme polynomial connu pour ces classes de jeux. Halman [Hal07] a montré (en généralisant les résultats de [Lud95, BSV03, BV07]) que les jeux à paiement moyen stochastiques peuvent être décrits comme un problème de type LP[1]. Cela implique quils peuvent être résolus en temps fortement

[1]Halman considérait uniquement une sous-classe des jeux à paiement moyen stochastiques, mais chaque jeu

sous-exponentiel en espérance, en utilisant la règle randomisée de pivotage proposée par Matoušek, Sharir et Welzl [MSW96] (voir aussi [HZ15] pour des améliorations récentes). On ne sait pas si les jeux à paiement moyen stochastiques peuvent être résolus en temps sous-exponentiel par un algorithme déterministe (cette question est ouverte même si nous limitons l'attention aux jeux déterministes). Par conséquent, divers auteurs ont étudié la complexité paramétrée de ces jeux. Par exemple, Gimbert et Horn [GH08] ont montré que des jeux stochastiques simples peuvent être résolus en temps polynomial lorsque le nombre de positions aléatoires est fixé. Zwick et Paterson [ZP96] ont donné un algorithme pseudo-polynomial pour les jeux à paiement moyen déterministes. Nous nous référons à [IJM12, CR17] pour les améliorations de ces résultats et plus d'informations. De plus, Hansen, Miltersen et Zwick [HMZ13] ont fourni un algorithme fortement polynomial pour les jeux à paiement escompté et taux d'escompte fixé. Jurdziński, Paterson et Zwick [JPZ08] ont fourni un algorithme déterministe sous-exponentiel pour les jeux de parité. Dans une découverte plus récente, Calude et al. [CJK+17] ont montré que les jeux de parité peuvent être résolus en temps quasi-polynomial. Ce résultat a déjà fourni de nombreux travaux de suivi [GIJ17, JL17, FJS+17, Leh18, CDF+18]. Nous renvoyons également à [Mam17, BEGM13, AGH15, ACS14, HK66, ACTDG12, vD18, Zie98, VJ00] pour plus d'informations sur ces jeux et leurs algorithmes et à [CJH04, DJL18, CDGO14, SWZ18, CD12] pour les classes de jeux associées. Cependant, notons quelques articles dans ce domaine qui sont étroitement liés à nos travaux. Boros, Elbassioni, Gurvich et Makino [BEGM15] ont utilisé une généralisation de l'algorithme du "pompage", initialement introduit par Gurvich, Karzanov et Khachiyan [GKK88] dans le cas déterministe, pour montrer que les jeux à paiement moyen stochastiques peuvent être résolus en temps pseudo-polynomial si le nombre de positions aléatoires est fixé. (Nous notons que cela ne découle pas des résultats analogues obtenus pour les jeux stochastiques simples cités plus haut, car la réduction des jeux stochastiques généraux à des jeux stochastiques simples ajoute trop de positions aléatoires au jeu, voir [BEGM13] pour une discussion détaillée.) Dans cette thèse, nous proposons d'utiliser l'itération sur les valeurs comme une alternative aux algorithmes de pompage et nous analysons ses bornes de complexité. L'itération sur les valeurs a été étudiée par Zwick et Paterson [ZP96] pour les jeux déterministes. Ibsen-Jensen et Miltersen [IJM12] ont montré que l'itération sur les valeurs modifiée peut résoudre des jeux stochastiques simples en temps polynomial lorsque le nombre de positions aléatoires est fixe. Chatterjee et Ibsen-Jensen [CIJ14] ont étudié la complexité de l'itération sur les valeurs pour les jeux à informations imparfaite. L'itération sur les valeurs pour les jeux à un joueur a été également étudiée, par exemple dans [Put05, ACD+17]. Allamigeon, Benchimol, Gaubert et Joswig [ABGJ14] ont utilisé la relation entre les jeux à paiement moyen déterministes et la programmation linéaire dans les séries de Puiseux pour montrer qu'une règle de pivotage polynomiale et semi-algébrique pour un algorithme du simplexe fournissait un algorithme polynomial pour les jeux à paiement moyen déterministes. Par des techniques similaires, Allamigeon, Benchimol et Gaubert [ABG14] ont montré que les jeux à paiement moyen déterministes peuvent être résolus en temps polynomial en moyenne.

C.2 Contributions

La thèse est divisée en deux parties. Dans la première partie, nous présentons des résultats structurels concernant les tropicalisations des ensembles semi-algébriques. Dans la seconde partie,

de ce type peut être réduit à la forme de Halman dans un temps fortement polynomial en utilisant les résultats de [AM09].

nous discutons des conséquences algorithmiques de notre approche.

Dans le Chapitre 3, nous commençons par étudier les tropicalisations des ensembles semi-algébriques généraux. Dans le cas de variétés algébriques (complexes), un résultat fondamental de Bieri et Groves [BG84, EKL06] montre que si $(\mathscr{K}, \mathsf{val})$ est un corps valué algébriquement clos dont le groupe de valeur est égal à $\mathbb{R}$ et $S \subset (\mathscr{K}^*)^n$ est une variété algébrique dans le tore, alors $\mathsf{val}(S) \subset \mathbb{R}^n$ est une union de polyèdres. Alessandrini [Ale13] a montré un analogue de ce théorème pour les sous-ensembles définissables de corps de Hardy des structures o-minimales à croissance polynomiale. Son analyse implique que si $(\mathscr{K}, \mathsf{val})$ est un corps réel clos avec la valuation convexe dont le groupe de valeur est égal à $\mathbb{R}$ et $S \subset (\mathscr{K}^*)^n$ est un ensemble semi-algébrique, alors $\mathsf{val}(S)$ est une union de polyèdres. Les résultats d'Alessandrini s'appliquent également aux corps avec des groupes de valeurs contenus dans $\mathbb{R}$. Dans le Chapitre 3, nous donnons une preuve constructive de ce résultat. La preuve est basée sur l'élimination des quantificateurs de Denef et Pas [Pas89] et s'applique aux corps avec des groupes de valeurs arbitraires (pas uniquement les sous-groupes de $\mathbb{R}$). L'élimination des quantificateurs de Denef et Pas donne également un principe de transfert qui est utilisé plus tard pour prouver l'analogue tropical de la conjecture de Helton et Nie pour les corps valués réel clos avec des groupes de valeurs arbitraires.

Théorème B (Theorem 3.1)**.** *Supposons que $(\mathscr{K}, \mathsf{val})$ soit un corp valué réel clos dont la valuation $\mathsf{val}\colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ est convexe et non triviale. Soit $S \subset \mathscr{K}^n$ un ensemble semi-algébrique. Alors, l'ensemble $\mathsf{val}(S) \subset (\Gamma \cup \{-\infty\})^n$ est semi-linéaire et ses strates sont fermées. Réciproquement, tout sous-ensemble semi-linéaire de $(\Gamma \cup \{-\infty\})^n$ qui a des strates fermées est une image par valuation d'un ensemble semi-algébrique.*

Étant donné un ensemble S, il est parfois possible de trouver une description simple de $\mathsf{val}(S)$. Un résultat important dans cette direction est le théorème de Kapranov [EKL06, Theorem 2.1.1], qui montre que si $\mathscr{K}$ est le corps des séries de Puiseux avec des coefficients complexes et $S := \{x \in (\mathscr{K}^*)^n \colon P(x) = 0\}$ est une hypersurface donnée par un polynôme $P \in \mathscr{K}[X_1, \dots, X_n]$, alors $\mathsf{val}(S)$ est entièrement décrit par la tropicalisation formelle du polynôme P. Cette affirmation n'est pas vraie pour les ensembles semi-algébriques réels, même s'ils sont décrits par une (in)égalité polynomiale. Cependant, grâce au théorème mentionné ci-dessus, nous pouvons donner une description explicite des images par valuation des ensembles semi-algébriques définis par des systèmes d'inégalités, sous l'hypothèse qu'une certaine condition de régularité soit satisfaite.

Théorème C (Theorem 3.4)**.** *Soit $\mathbb{K}$ le corps des séries de Puiseux avec des coefficients réels et soit $\boldsymbol{S} \subset \mathbb{K}^n_{>0}$ un ensemble semi-algébrique défini par*

$$\boldsymbol{S} := \{\boldsymbol{x} \in \mathbb{K}^n_{>0} \colon \boldsymbol{P}_1(\boldsymbol{x}) \,\square_1\, 0, \dots, \boldsymbol{P}_m(\boldsymbol{x}) \,\square_m\, 0\} \, ,$$

où $\boldsymbol{P}_i \in \mathbb{K}[X_1, \dots, X_n]$ sont des polynômes non-nuls et $\square \in \{\geqslant, >\}^m$. Soit $P_i := \mathrm{trop}(\boldsymbol{P}_i)$ pour tout i et supposons que $\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ ait un support régulier. Alors

$$\mathsf{val}(\boldsymbol{S}) = \{x \in \mathbb{R}^n \colon \forall i, P_i^+(x) \geqslant P_i^-(x)\} \, .$$

Le théorème ci-dessus est une généralisation d'un résultat de Develin et Yu [DY07] qui ont montré le Théorème C dans le cas où les polynômes $\boldsymbol{P}_i$ sont affines. De plus, nous notons que la condition "$\mathcal{C}^{\geqslant}(P_1, \dots, P_m)$ a un support régulier" est générique – elle est satisfaite si les coefficients des polynômes tropicaux $P_1, \dots, P_m$ se trouvent en dehors d'un ensemble de mesure nulle.

Ensuite, nous analysons les tropicalisations d'ensembles semi-algébriques convexes définis dans les séries de Puiseux. Dans le Chapitre 4, nous introduisons la notion des *spectraèdres tropicaux* – des ensembles définis comme des images par valuation des spectraèdres dans les séries de Puiseux. Nous montrons qu'une classe de spectraèdres tropicaux définis par les matrices de Metzler peut être décrite explicitement à l'aide d'inégalités polynomiales de degré deux dans le demi-corps tropical. Puis, nous utilisons cette classe pour décrire explicitement les spectraèdres tropicaux génériques. Il se trouve que les mineurs de taille 2×2 suffisent pour décrire un spectraèdre tropical générique. Ceci étend le résultat de Yu [Yu15] qui a tropicalisé le cône semi-défini positif.

Théorème D (Theorem 4.28)**.** *Supposons que* $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ *soient des matrices symétriques. Soit* $\boldsymbol{Q}(\boldsymbol{x}) \coloneqq \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)}$ *et notons* $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$ *le spectraèdre associé. De plus, supposons que les matrices* $\mathsf{val}(\boldsymbol{Q}^{(0)}), \dots, \mathsf{val}(\boldsymbol{Q}^{(n)}) \in \mathbb{T}^{m \times m}$ *soient génériques. Alors, l'ensemble* $\mathsf{val}(\boldsymbol{\mathcal{S}})$ *est décrit par des inégalités polynomiales tropicales données par une variante tropicale des mineurs du faisceau affine* $\boldsymbol{Q}(\boldsymbol{x})$ *de taille* 2×2.

Dans le Chapitre 5, nous étudions les ensembles semi-algébriques convexes généraux et nous donnons de multiples caractérisations équivalentes de leurs images par valuation. En particulier, nous montrons que l'analogue tropical de la conjecture de Helton et Nie est vrai.

Théorème E (Theorem 5.5)**.** *Soit* $\mathcal{S} \subset \mathbb{T}^n$*. Alors les conditions suivantes sont équivalentes :*
(a) $\mathcal{S}$ *est une tropicalisation d'un ensemble semi-algébrique convexe ;*
(b) $\mathcal{S}$ *est tropicalement convexe et a des strates semi-linéaires fermées ;*
(c) $\mathcal{S}$ *est une projection dun spectraèdre tropical de Metzler ;*
(d) il existe un spectraèdre projeté $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ *tel que* $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$.

Ce théorème est le premier endroit où la connexion avec les jeux à paiement moyen stochastiques joue un rôle. L'un des ingrédients de la démonstration est un lemme de Zwick et Paterson [ZP96] qui était à l'origine utilisé pour montrer une reduction des jeux à paiement escompté à des jeux stochastiques simples. En utilisant le principe de transfert donné par l'élimination des quantificateurs de Denef et Pas, on peut ensuite montrer que la conjecture classique de Helton et Nie est vraie “à valuation près”.

Théorème F (Theorem 5.2)**.** *Soit* $\mathscr{K}$ *un corps valué réel clos dont la valuation* $\mathsf{val} \colon \mathscr{K} \to \Gamma \cup \{-\infty\}$ *est convexe et non triviale. Supposons que* $S \subset \mathscr{K}^n$ *soit un ensemble semi-algébrique convexe. Alors, il existe un spectraèdre projeté* $S' \subset \mathscr{K}^n$ *tel que* $\mathsf{val}(S) = \mathsf{val}(S')$.

Dans la seconde partie de la thèse, nous étudions la relation entre les tropicalisations d'ensembles convexes et les jeux à paiement moyen stochastiques. Dans le Chapitre 6, nous présentons cette classe de jeux et l'analysons à l'aide des opérateurs de Shapley [Sha53], du théorème de Kohlberg [Koh80] et de la propriété de Collatz et Wielandt [Nus86]. Notre présentation est basée sur [AGG12]. Nous donnons un nouveau résultat dans ce domaine – nous généralisons la caractérisation tropicale des états gagnants de jeux déterministes fourni dans [AGG12] au cas des jeux stochastiques. En combinant ce résultat avec les résultats des chapitres précédents, nous obtenons la correspondance suivante entre la tropicalisation des cônes et les jeux à paiement moyen stochastiques.

Théorème G (Theorem 6.30)**.** *Soit* $\mathcal{S} \subset \mathbb{T}^n$*. Alors,* $\mathcal{S}$ *est une tropicalisation d'un cône semi-algébrique convexe fermé si et seulement si il existe un jeu à paiement moyen stochastique tel que son opérateur de Shapley* $F \colon \mathbb{T}^n \to \mathbb{T}^n$ *satisfait* $\mathcal{S} = \{x \in \mathbb{T}^n \colon x \leqslant F(x)\}$*. De plus, le* *support de $\mathcal{S}$ est donné par le plus grand dominion gagnant de ce jeu. En particulier, $\mathcal{S}$ n'est pas trivial si et seulement si le jeu a au moins un état gagnant.*

En conséquence du théorème ci-dessus, le problème de vacuité d'un cône semi-algébrique dans les séries de Puiseux peut, en théorie, être réduit à la résolution d'un jeu à paiement moyen. Néanmoins, même si nos démonstrations sont constructives, il ne pas facile de trouver un jeu adapté en général. En effet, comme indiqué précédemment, bien que les jeux à paiement moyen stochastiques appartiennent à la classe NP $\cap$ coNP, il existe des problèmes d'une complexité inconnue qui peuvent être réduits aux problèmes de vacuité semi-définie conique. Cependant, les systèmes polynomiaux tropicaux mentionnés dans le Théorème D peuvent être convertis en opérateurs de Shapley. Cela implique que les problèmes de vacuité semi-définie non-archimédienne pour les cônes génériques dans les séries de Puiseux peuvent être résolus par une réduction à des jeux à paiement moyen. De plus, notre approche peut résoudre certains problèmes de vacuité semi-définie qui ne satisfont pas la condition de généricité. Ceci est montré dans le Chapitre 7 et résumé ci-dessous.

Théorème H (Theorem 7.6)**.** *Supposons que $\boldsymbol{Q}^{(1)},\dots,\boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ soient des matrices symétriques. Soit $\boldsymbol{Q}(\boldsymbol{x}) := \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)}$ et $\boldsymbol{\mathcal{S}} := \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{Q}(\boldsymbol{x}) \succcurlyeq 0\}$. De plus, supposons que les matrices $\boldsymbol{Q}^{(k)}$ aient des valuations rationnelles, $\mathsf{val}(\boldsymbol{Q}^{(1)}),\dots,\mathsf{val}(\boldsymbol{Q}^{(k)}) \in (\mathbb{Q}\cup\{-\infty\})^{m\times m}$. Alors, étant donné seulement les valuations signées $\mathsf{sval}(\boldsymbol{Q}^{(k)})$ de ces matrices, nous pouvons construire (en temps polynomial) un jeu à paiement moyen stochastique qui a les propriétés suivantes. Si la valeur maximale du jeu est strictement positive, alors $\boldsymbol{\mathcal{S}}$ n'est pas trivial. Si la valeur maximale est strictement négative, alors $\boldsymbol{\mathcal{S}}$ est trivial. De plus, si la valeur maximale est égale à 0 et que les matrices $\mathsf{val}(\boldsymbol{Q}^{(1)}),\dots,\mathsf{val}(\boldsymbol{Q}^{(n)})$ sont génériques, alors $\boldsymbol{\mathcal{S}}$ n'est pas trivial. Réciproquement, la résolution des jeux à paiement moyens stochastiques peut être réduite au problème de la vacuité des cônes spectraèdrales $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ décrits ci-dessus.*

En plus, nous montrons que le problème de la vacuité dun spectraèdre tropical de Metzler (défini uniquement par les inégalités polynomiales tropicales, sans référence aux séries de Puiseux) est équivalent à la résolution de jeux à paiement moyen stochastiques.

Théorème I (Theorem 7.4)**.** *Le problème de la vacuité dun spectraèdre tropical de Metzler est équivalent en temps polynomial au problème de la résolution de jeux à paiement moyen stochastiques.*

Dans le dernier chapitre de la thèse, nous étudions l'itération sur les valeurs, qui est un algorithme simple qui peut être utilisé pour résoudre des jeux à paiement moyens stochastiques. Cet algorithme est basé sur le fait que si $F\colon \mathbb{T}^n \to \mathbb{T}^n$ est l'opérateur de Shapley d'un jeu, alors la limite $\lim_{N\to\infty} F^N(0)/N$ (où $F^N = F \circ \dots \circ F$) existe et est égale au vecteur de valeur de ce jeu. L'algorithme d'itération sur les valeurs calcule les valeurs successives de $F^N(0)$ et en déduit les propriétés de la limite. Comme indiqué ci-dessus, le problème de la vacuité des spectraèdres tropicaux correspond à la détermination du signe de la valeur. L'observation suivante fournit un nombre de conditionnement pour ce problème.

Théorème J (Theorem 8.25)**.** *Soit $f\colon \mathbb{R}^n \to \mathbb{R}^n$ une fonction monotone et additivement homogène. De plus, supposons que l'équation $f(u) = \eta + u$ ait une solution $(\eta, u) \in \mathbb{R}\times\mathbb{R}^n$. Alors, nous avons $\lim_{N\to\infty} f^N(0)/N = \eta(1,1,\dots,1)$. De plus, supposons que $\eta \neq 0$ et notons $R := \inf\{\|u\|_{\mathrm{H}} \in \mathbb{R}^n \colon f(u) = \eta + u\}$, où $\|\cdot\|_{\mathrm{H}}$ est la semi-norme d'Hilbert. Alors, pour tout*

$$N \geqslant \lceil 1 + \frac{R}{|\eta|} \rceil$$

les entrées de $f^N(0)$ ont le même signe et ce signe est identique au signe de η.

Les hypothèses de ce théorème sont satisfaites, par exemple, si f est l'opérateur de Shapley d'un jeu à paiement moyen stochastique à valeur constante. Dans ce cas, η est la valeur du jeu. Cependant, il existe d'autres classes de jeux pour lesquels ce résultat peut être intéressant, telles que les jeux d'entropie de [ACD+16, AGGCG17]. Le nombre de conditionnement possède une interprétation géométrique.

Proposition K (Proposition 8.16)**.** *Supposons que $F\colon \mathbb{T}^n \to \mathbb{T}^n$ soit un opérateur de Shapley associé à un cône spectraèdrale tropical de Metzler $\mathcal{S} \subset \mathbb{T}^n$. De plus, supposons que F satisfasse les conditions du Théorème J et que $\eta > 0$. Alors, il existe un cône spectraèdrale tropical de Metzler $\tilde{\mathcal{S}} \subset \mathbb{T}^{2n}$ tel que $\mathcal{S}$ est la projection de $\tilde{\mathcal{S}}$ et que $\eta/2$ est le rayon d'une boule la plus grosse (dans la semi-norme d'Hilbert) inclus dans $\tilde{\mathcal{S}}$. De plus, si $u \in \mathbb{R}^n$ est tel que $F(u) = \eta + u$, alors u est la projection du centre d'une telle boule.*

De cette manière, la quantité $|\eta|$ mesure la largeur du cône spectraèdrale tropical de Metzler, tandis que la quantité R mesure la distance de ce cône à l'origine. Intuitivement, cela devrait être comparé aux quantités qui déterminent la complexité de la méthode d'ellipsoïde [GLS93]. La complexité de cette méthode dépend de façon polynomiale de $\log(R/r)$, où R est le rayon dune boule contenant un corps convexe donné et r est le rayon d'une boule incluse dans ce corps. Ainsi, R mesure la distance qui sépare le corps de l'origine, tandis que r mesure sa taille. Le fait que l'itération sur les valeurs dépende de manière polynomiale de $R/|\eta|$ au lieu de $\log(R/|\eta|)$ est intuitivement justifié par le fait que la valuation applique le logarithme aux données initiales. L'analogie avec l'ellipsoïde est renforcée par le fait que l'itération sur des valeurs est basée sur un oracle – pour utiliser cette méthode, il suffit d'avoir un oracle qui évalue f approximativement.

Nous donnons également une autre application du nombre de conditionnement, reliant les problèmes de vacuité archimédiens et non-archimédiens. Comme discuté précédemment, un spectraèdre $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ dans les séries de Puiseux peut être vu comme une famille de spectraèdres réels. Par un argument d'o-minimalité, il existe $t_0 > 0$ tel que pour tout $t > t_0$ les problèmes de vacuité pour $\boldsymbol{\mathcal{S}}$ et $\boldsymbol{\mathcal{S}}(t)$ coïncident (c'est-à-dire, $\boldsymbol{\mathcal{S}}(t)$ est non vide si et seulement si $\boldsymbol{\mathcal{S}}$ est non vide). Cette correspondance peut potentiellement conduire à un algorithme de type homotopique pour décider de la vacuité de certains spectraèdres. Pour créer un tel algorithme, il est souhaitable d'avoir des bornes sur la quantité t_0. Le théorème suivant montre que t_0 n'est pas grand si le jeu à paiement moyen stochastique associé est bien conditionné.

Théorème L (Theorem 8.4)**.** *Supposons que $Q^{(1)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$ soient les matrices symétriques tropicales de Metzler qui créent une inégalité matricielle linéaire tropicale bien formée. Notons $\bar{\eta} \in \mathbb{R}$ la valeur maximale du jeu associé. Soit $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ un soulèvement monomial de $Q^{(1)}, \dots, Q^{(n)}$ défini par $\boldsymbol{Q}^{(k)}_{ij} \coloneqq \delta_{ij} t^{|Q^{(k)}_{ij}|}$, où $\delta_{ij} \coloneqq 1$ si $i = j$ et $\delta_{ij} \coloneqq -1$ sinon. Alors, pour tout*

$$t > (2(m-1)n)^{1/(2\bar{\eta})}$$

le cône spectraèdrale réel $\boldsymbol{\mathcal{S}}(t) \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n_{\geqslant 0} \colon x_1\boldsymbol{Q}^{(1)}(t) + \dots + x_n\boldsymbol{Q}^{(n)}(t)\}$ est non triviale si et seulement si le cône spectraèdrale non-archimédien $\boldsymbol{\mathcal{S}} \coloneqq \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \colon \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)}\}$ est non triviale.

La contribution principale du Chapitre 8 consiste à donner des bornes explicites pour le nombre de conditionnement $R/|\eta|$ dans le cas où f est un opérateur de Shapley d'un jeu à paiement moyen stochastique. Ceci est basé sur le théorème suivant qui estime la taille en bits de la mesure invariante d'une chaîne de Markov finie.

Théorème M (Theorem 8.44)**.** *Supposons que $P \in [0,1]^{n\times n}$ soit une matrice stochastique irréductible avec des entrées rationnelles et notons $\pi \in]0,1]^n$ la distribution stationnaire de P. Soit $M \in \mathbb{N}^*$ le dénominateur commun de toutes les entrées de P. Alors, le plus petit dénominateur commun des entrées de π n'est pas supérieur à $nM^{\min\{n_r,n-1\}}$, où $n_r \leqslant n$ est le nombre de lignes de P qui ne sont pas déterministes (c'est-à-dire qui ont une entrée dans l'intervalle ouvert $]0,1[$). De plus, la borne $nM^{\min\{n_r,n-1\}}$ est optimale.*

Des bornes similaires ont déjà été montrées dans la littérature concernant les jeux à paiement moyen stochastiques [BEGM15, AGH18, Con92, ACS14]. Cependant, les démonstrations dans ces travaux sont basées sur l'inégalité de Hadamard qui conduit à des résultats sous-optimaux. Pour obtenir la borne optimale, nous remplaçons l'inégalité de Hadamard par la formule combinatoire de Freidlin et Wentzell [FW12]. Comme corollaire, nous estimons le nombre maximal ditérations nécessaires pour résoudre un jeu à paiement moyen stochastique. Cela donne un résultat de complexité paramétrée pour cette classe de jeux.

Théorème N (Theorem 8.58 et Theorem 8.68)**.** *L'algorithme d'itération sur les valeurs résout les jeux à valeur constante en temps pseudo-polynomial et mémoire polynomiale lorsque le nombre d'actions aléatoires est fixé. De plus, pour les jeux généraux (avec une valeur arbitraire), une modification de l'itération sur les valeurs peut trouver la valeur maximale (ou minimale) du jeu, ainsi que l'ensemble des états qui l'atteignent, en temps pseudo-polynomial et mémoire polynomiale lorsque le nombre d'actions aléatoires est fixé.*

Nous notons que la seconde partie du théorème utilise la caractérisation des dominions gagnants établis dans le Chapitre 6. On connait seulement deux autres algorithmes qui atteignent la complexité donnée dans le Théorème N : l'algorithme de pompage étudié dans [BEGM15] et une variante de l'algorithme d'ellipsoïde donnée dans [BEGM17]. Cependant, l'itération sur les valeurs est plus simple et nos bornes de complexité sont meilleures que celles obtenues pour les algorithmes de pompage et d'ellipsoïde. Si on spécifie nos bornes pour le cas des spectraèdres tropicaux de Metzler, nous obtenons des bornes qui dépendent de la densité des matrices définissant ce spectraèdre tropical. Ces bornes sont données dans la Section 8.5, où nous introduisons également une classe explicite de spectraèdres tropicaux qui fournit des jeux à valeur constante. Enfin, nous expérimentons notre approche à la résolution de programmes semi-définis non-archimédiens aléatoires de grande taille sur cette classe d'entrées.

Titre : Spectraèdres tropicaux: application à la programmation semi-définie et aux jeux à paiement moyen

Mots clés : géométrie tropicale, programmation semi-définie, jeux à paiement moyen, ensembles semi-algébriques

Résumé : La programmation semi-définie est un outil fondamental d'optimisation convexe et polynomiale. Elle revient à optimiser une fonction linéaire sur un spectraèdre (un ensemble défini par des inégalités matricielles linéaires). En particulier, la programmation semi-définie est une généralisation de la programmation linéaire.
Nous étudions l'analogue non-archimédien de la programmation semi-définie, en remplaçant le corps des nombres réels par le corps des séries de Puiseux. Notre approche est fondée sur des méthodes issues de la géométrie tropicale et, en particulier, sur l'étude de la tropicalisation des spectraèdres.
En première partie de la thèse, nous analysons les images par la valuation des ensembles semi-algébriques généraux définis dans le corps des séries de Puiseux. Nous montrons que ces images ont une structure polyédrale, ce qui fournit un analogue réel du théorème de Bieri et Groves. Ensuite, nous introduisons la notion de spectraèdres tropicaux et nous montrons que, sous une hypothèse de généricité, ces objets sont décrits par des systèmes d'inégalités polynomiales de degré 2 sur le semi-corps tropical. Cela généralise un résultat de Yu sur la tropicalisation du cône des matrices positives.
Une question importante relative à la programmation semi-définie sur les réels consiste à caractériser des projections de spectraèdres. Dans ce cadre, Helton et Nie ont conjecturé que tout ensemble semi-algébrique convexe est la projection d'un spectraèdre. La conjecture a été réfutée par Scheiderer. Néanmoins, nous montrons qu'elle est vraie “à valuation près”: dans le corps réel clos des séries de Puiseux, les ensembles semi-algébriques convexes et les spectraèdres projetés ont exactement les mêmes images par la valuation non-archimédienne.
En seconde partie de la thèse, nous étudions des questions algorithmiques liées à la programmation semi-définie. Le problème algorithmique de base consiste à décider si un spectraèdre est vide. On ne sait pas si ce problème appartient à NP dans le modèle de la machine de Turing, et les algorithmes fondés sur la décomposition cylindrique algébrique ou la méthode de points critiques constituent l'état de l'art dans ce domaine. Nous montrons que, dans le cadre non-archimédien, les spectraèdres tropicaux génériques sont décrits par des opérateurs de Shapley associés aux jeux à paiement moyen stochastiques. Cela donne une méthode pour résoudre des problèmes de réalisabilité en programmation semi-définie non-archimédienne en utilisant les algorithmes combinatoires conçus pour les jeux stochastiques.
Dans les chapitres finaux de la thèse, nous établissons des bornes de complexité pour l'algorithme d'itération sur les valeurs qui exploitent la correspondance entre les jeux stochastiques et la convexité tropicale. Nous montrons que le nombre d'itérations est contrôlé par un nombre de conditionnement relié au diamètre intérieur du spectraèdre tropical associé. Nous fournissons des bornes supérieures générales sur le nombre de conditionnement. Pour cela, nous établissons des bornes optimales sur la taille en bits des mesures invariantes de chaînes de Markov. Comme corollaire, notre estimation montre que l'itération sur la valeur résout les jeux ergodiques à paiement moyen en temps pseudo-polynomial si le nombre de positions aléatoires est fixé. Enfin, nous expérimentons notre approche à la résolution de programmes semi-définis non-archimédiens aléatoires de grande taille.

Title: Tropical spectrahedra: Application to semidefinite programming and mean payoff games

Keywords: tropical geometry, semidefinite programming, mean payoff games, semialgebraic sets

Abstract: Semidefinite programming (SDP) is a fundamental tool in convex and polynomial optimization. It consists in minimizing the linear functions over the spectrahedra (sets defined by linear matrix inequalities). In particular, SDP is a generalization of linear programming.
The purpose of this thesis is to study the nonarchimedean analogue of SDP, replacing the field of real numbers by the field of Puiseux series. Our methods rely on tropical geometry and, in particular, on the study of tropicalization of spectrahedra.
In the first part of the thesis, we analyze the images by valuation of general semialgebraic sets defined over the Puiseux series. We show that these images have a polyhedral structure, giving the real analogue of the Bieri–Groves theorem. Subsequently, we introduce the notion of tropical spectrahedra and show that, under genericity conditions, these objects can be described explicitly by systems of polynomial inequalities of degree 2 in the tropical semifield. This generalizes the result of Yu on the tropicalization of the SDP cone.
One of the most important questions about real SDPs is to characterize the sets that arise as projections of spectrahedra. In this context, Helton and Nie conjectured that every semialgebraic convex set is a projected spectrahedron. This conjecture was disproved by Scheiderer. However, we show that the conjecture is true “up to taking the valuation”: over a real closed nonarchimedean field of Puiseux series, the convex semialgebraic sets and the projections of spectrahedra have precisely the same images by the nonarchimedean valuation.
In the second part of the thesis, we study the algorithmic questions related to SDP. The basic computational problem associated with SDP over real numbers is to decide whether a spectrahedron is nonempty. It is unknown whether this problem belongs to NP in the Turing machine model, and the state-of-the-art algorithms that certify the (in)feasibility of spectrahedra are based on cylindrical decomposition or the critical points method. We show that, in the nonarchimedean setting, generic tropical spectrahedra can be described by Shapley operators associated with stochastic mean payoff games. This provides a tool to solve nonarchimedean semidefinite feasibility problems using combinatorial algorithms designed for stochastic games.
In the final chapters of the thesis, we provide new complexity bounds for the value iteration algorithm, exploiting the correspondence between stochastic games and tropical convexity. We show that the number of iterations needed to solve a game is controlled by a condition number, which is related to the inner radius of the associated tropical spectrahedron. We provide general upper bounds on the condition number. To this end, we establish optimal bounds on the bit-length of stationary distributions of Markov chains. As a corollary, our estimates show that value iteration can solve ergodic mean payoff games in pseudopolynomial time, provided that the number of random positions of the game is fixed. Finally, we apply our approach to large scale random nonarchimedean SDPs.

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