Programmation dynamique tropicale en optimisation stochastique multi-étapes

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Programmation dynamique tropicale en optimisation stochastique multi-étapes

Tropical dynamic programming in multistage stochastic optimization

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Part I

Introduction

Chapter 1

Version Française

Problèmes d'optimisation stochastique multi-étapes

Dans cette thèse nous étudions les problèmes d'optimisation stochastique multi-étapes dans le cadre hasard-décision (le risque vient en premier, la décision en seconde). En partant d'un état donné x_0, un décideur observe le résultat w_1 d'une variable aléatoire $\mathbf{W_1}$, puis décide d'un contrôle u_0 qui induit un coût $c_0^{w_1}(x_0, u_0)$ *connu* et le système évolue vers un état futur x_1 à partir d'une dynamique *connue* : $x_1 = f_0^{w_1}(x_0, u_0)$. Après avoir observé un nouveau résultat aléatoire, le décideur prend une nouvelle décision basée sur ce constat qui induit un coût connu, le système évolue alors vers un état futur, et ainsi de suite jusqu'à ce que T décisions aient été prises. A l'étape finale, on impose des contraintes sur l'état final x_T, modélisées par une fonction de coût final ψ. Le décideur vise à minimiser le coût moyen de ses décisions.

Les problèmes d'optimisation stochastique multi-étapes (MSP) peuvent être décrits formellement par le problème d'optimisation suivant

$$
\begin{aligned}
&\min_{(\mathbf{X},\mathbf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t^{\mathbf{W_{t+1}}}(\mathbf{X_t}, \mathbf{U_t}) + \psi(\mathbf{X_T})\right], \\
&\text{s.t. } \mathbf{X_0} = x_0 \text{ donné}, \forall t \in [\![0, T-1]\!], \\
&\mathbf{X_{t+1}} = f_t^{\mathbf{W_{t+1}}}(\mathbf{X_t}, \mathbf{U_t}), \\
&\sigma(\mathbf{U_t}) \subset \sigma(\mathbf{X_0}, \mathbf{W_1}, \dots, \mathbf{W_{t+1}}),
\end{aligned}
\tag{1.1}
$$

où $(\mathbf{W_t})_{t\in[\![1,T]\!]}$ est une suite donnée de variables aléatoires indépendantes ayant chacune des valeurs dans un ensemble mesurable $(\mathbb{W}_t, \mathcal{W}_t)$.

Résolution par programmation dynamique

Une approche pour résoudre les problèmes MSP est la programmation dynamique, voir par exemple [Bel54, Ber16, CCCDL15, PP14, SDR09]. On note $\mathbb{X} = \mathbb{R}^n$ *l'espace d'état* et $\mathbb{U} = \mathbb{R}^m$ *l'espace des commandes* pour certains entiers $n, m \in \mathbb{N}$. Les espaces $\mathbb{X}$ et les $\mathbb{U}$ sont tous deux dotés de leurs structures euclidienne et borélienne. Nous définissons les opérateurs de Bellman ponctuels $\mathcal{B}_t^w$ et les opérateurs de Bellman moyennés $\mathfrak{B}_t$ pour chaque $t \in [\![0, T-1]\!]$. Pour chaque réalisation possible $w \in \mathbb{W}_{t+1}$ du bruit $\mathbf{W_{t+1}}$, pour chaque fonction $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ prenant des valeurs réelles étendues dans $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, la fonction $\mathcal{B}_t^w(\phi)(\cdot) : \mathbb{X} \to \overline{\mathbb{R}}$ est définie par

$$
\forall x \in \mathbb{X}, \ , \ \ \mathcal{B}_t^w(\phi)(x) = \min_{u\in\mathbb{U}} \Big(c_t^w(x, u) + \phi\big(f_t^w(x, u)\big)\Big) .
$$

L'opérateur de Bellman moyenné $\mathfrak{B}_t$ est la moyenne de tous les opérateurs de Bellman ponctuels par rapport à la loi de probabilité de $\mathbf{W_{t+1}}$. C'est-à-dire que pour chaque fonction $\phi : \mathbb{X} \to \overline{\mathbb{R}}$, on pose

$$
\forall x \in \mathbb{X} \ , \ \ \mathfrak{B}_t(\phi)(x) = \mathbb{E}\big[\mathcal{B}_t^{\mathbf{W_{t+1}}}(\phi)(x)\big] = \mathbb{E}\Big[\min_{u\in\mathbb{U}} \Big(c_t^{\mathbf{W_{t+1}}}(x, u) + \phi\big(f_t^{\mathbf{W_{t+1}}}(x, u)\big)\Big)\Big] .
$$

L'opérateur Bellman moyenné peut être vu comme un opérateur à une étape qui calcule en un état donné x, le coût de la meilleure (en moyenne) commande. Notez que dans le cadre hasard-décision, la commande est prise après observation du bruit. La programmation dynamique stipule que pour résoudre les problèmes MSP (4.1), il suffit de résoudre le système suivant d'*équations de Bellman*,

$$V_T = \psi \quad \text{et} \quad \forall t \in [\![0, T-1]\!], V_t = \mathfrak{B}_t \left(V_{t+1} \right) \ . \tag{1.2}$$

Pour résoudre les équations de Bellman, il faut calculer récursivement en remontant dans le temps les *fonctions valeur (de Bellman)* V_t. Enfin, la valeur $V_0(x_0)$ est la solution du problème à plusieurs étapes (1.1).

Vers une atténuation du fléau de la dimension

Une limitation de la programmation dynamique pour résoudre les problèmes d'optimisation en plusieurs étapes est ce qu'on appelle le "fléau de la dimension" [Bel54]. C'est-à-dire que lorsque l'espace d'état $\mathbb{X}$ est un espace vectoriel, toute méthode basée sur une grille pour calculer les fonctions valeur a une complexité qui est exponentielle en la dimension de l'espace d'état $\mathbb{X}$. Un algorithme populaire (voir [GLP15, Gui14, GR12, PP91, Sha11, ZAS18]) qui vise à atténuer le fléau de la dimension est l'algorithme Stochastic Dual Dynamic Programming (ou SDDP en abrégé) introduit par Pereira et Pinto en 1991. En supposant que les fonctions de coût c_t^w sont convexes et les dynamiques f_t^w sont linéaires, les fonctions valeur définies par la Programmation Dynamique (3.3) sont convexes [GLP15]. Sous ces hypothèses, l'algorithme SDDP vise à construire des les approximations des fonctions valeur en tant que suprema de fonctions affines et ne repose pas sur une discrétisation de l'espace d'état. L'un des principaux inconvénients de l'algorithme SDDP est l'absence d'un critère d'arrêt efficace : il construit des sur-approximations des fonctions valeur mais les approximations par le dessus (ou internes) sont obtenues par à un schéma de Monte-Carlo coûteux et les critères arrêts associés ne sont pas déterministes. Nous suivons une autre voie pour fournir des sur approximations comme expliqué maintenant.

Dans [Qu13, Ch. 8] et [Qu14], Qu a conçu un algorithme qui construit des approximations par le dessus de la fonction valeur survenant dans le cadre de problèmes de commande optimale en horizon infini et continu où l'ensemble des commandes est à la fois discret et continu. Ce travail a été inspiré par celui de McEneaney [McE07] en utilisant des techniques issues de l'algèbre tropicale, aussi appelées techniques max-plus ou min-plus. Supposons que pour chaque contrôle discret, les fonctions de coût sont quadratiques convexes et la dynamique est linéaire en l'état et le contrôle continu. Si l'ensemble des contrôles discrets est fini, alors en exploitant la linéarité min-plus des opérateurs de Bellman $\mathcal{B}_t$, on peut montrer que les fonctions valeur peuvent être calculées comme un infimum ponctuel fini de fonctions quadratiques convexes :

$$V_t = \inf_{\phi_t \in F_t} \phi_t \ ,$$

où F_t est un ensemble fini de formes quadratiques convexes. De plus, dans ce cadre les éléments de F_t peuvent être explicitement calculés par le biais de l'Équation Algébrique Discrète de Riccati (DARE [LR95]). Ainsi, une schéma d'approximation qui calcule une suite croissante de sous-ensembles $\left(F_t^k\right)_{k\in\mathbb{N}}$ de F_t donne un algorithme qui converge après un nombre fini d'itérations

$$V_t^k = \inf_{\phi_t \in F_t^k} \phi_t \approx \inf_{\phi_t \in F_t} \phi_t = V_t.$$

Cependant, la taille de l'ensemble des fonctions F_t qui doivent être calculées a une croissance exponentielle en $T - t$. Dans [Qu14], face à la croissance exponentielle de F_t, Qu a introduit un schéma aléatoire qui ajoute à F_t^k “le meilleur” élément (étant donné les approximations actuelles) de F_t en un point tiré uniformément sur la sphère de l'unité.

Aperçu de la thèse

Chaque chapitre de cette thèse a été écrit comme un chapitre indépendant. Le lecteur peut les lire indépendamment des autres.

1. Aux chapitres 3 et 4, nous construisons un algorithme général qui englobe à la fois l'algorithme SDDP et une adaptation du travail de Qu à un cadre stochastique, à temps discret et à horizon fini. Notre algorithme construit itérativement des approximations des fonctions valeur sous forme de combinaisons max-plus ou min-plus linéaires. Ces deux chapitres constituent le cœur de la thèse.

2. Le chapitre 5 est une contribution sur le calcul d'une distance entre processus stochastiques appelée la Distance Imbriquée. Nous présentons une relaxation entropique de la Distance Imbriquée qui peut être calculée efficacement.

3. Dans le chapitre 6, nous donnons un théorème général sur l'échange entre intégration et minimisation. Il généralise notamment celui de Giner [Gin09] et celui de Rockafellar and Wets [RW09].

Dans chacune des sections suivantes, nous présentons plus en détail chaque chapitre et sa contribution.

Programmation dynamique tropicale : le cas déterministe

On s'est d'abord intéressé au cadre simplifié des problèmes déterministes d'optimisation multi-étapes,

$$\begin{aligned} \min_{\substack{x=(x_0,\dots,x_T)\in\mathbb{X}^{T+1}\\ u=(u_0,\dots u_{T-1})\in\mathbb{U}^T}} & \sum_{t=0}^{T-1} c_t(x_t,u_t)+\psi(x_T) \\ \text{s.t. } & \forall t\in [\![0,T-1]\!]\ , \quad x_{t+1}=f_t(x_t,u_t) \text{ et } x_0\in\mathbb{X} \ \text{ donné}\ . \end{aligned}$$

Dans ce contexte déterministe, on a cherché à comprendre comment, à chaque pas de temps, une combinaison linéaire max-plus ou min-plus de fonctions élémentaires pourrait converger vers la fonction valeur. On présente un algorithme itératif qui ajoute à chaque itération une fonction élémentaire à la combinaison linéaire max-plus ou min-plus courante. Cet algorithme, appelé Programmation Dynamique Tropicale (TDP), peut être vu comme une variante tropicale des approximations parametriques utilisées en Programmation Dynamique Approximative (voir [Ber19, Pow11]) où les fonctions valeurs sont approchées par des combinaisons linéaires de fonctions élémentaires.

L'algorithme TDP détermine une fonction élémentaire à rajouter en tirant aléatoirement un *point*, dit *de raffinement*. De plus, sachant que les fonctions valeurs vérifient le système d'équations de Bellman, on souhaiterait que les approximations générées par TDP vérifient aussi ce système d'équations.

TDP exige deux propriétés sur les fonctions élémentaires ajoutées itérativement aux approximations courantes de V_t, $t < T$. La première propriété, locale, est appelée *exactitude* de la fonction élémentaire. Elle demande à ce que la fonction élémentaire ϕ_t ajoutée vérifie la t-ème équation de Bellman au point de raffinement $x \in \mathbb{X}$:

$$\phi_t(x) = \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(x) .$$

La seconde propriété, globale, est appelée *validité* de la fonction élémentaire. Celle-ci exige que la fonction élémentaire ϕ_t ajoutée au temps t soit toujours en dessous ou au dessus de l'image par le t-ème opérateur de Bellman $\mathcal{B}_t$ de l'approximation courante au temps $t+1$, *i.e.* en notant $\mathcal{V}_{F_{t+1}}$ une combinaison max-plus or min-plus linéaire de fonctions élémentaires (approchant V_{t+1}),

$$\phi(\cdot) \geq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(\cdot) , \qquad \text{(pour des combinaisons min-plus linéaires)}$$
$$\phi(\cdot) \leq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(\cdot) . \qquad \text{(pour des combinaisons max-plus linéaires)}$$

Ainsi, ne disposant que d'information exacte qu'en les points de raffinement, il est en général impossible de garantir que de telles combinaisons linéaires min-plus ou max-plus vérifient le système d'équations de Bellman. Toutefois, nous avons établi une condition suffisante portant sur la richesse des points de raffinement afin que, presque sûrement, asymptotiquement les approximations générées par TDP vérifient un système d'équations de Bellman restreintes. Si les restrictions ne sont à leur tour pas trop contraignantes, alors il est possible de conclure que toute suite de fonctions vérifiant un tel système d'équations est égale à la suite de fonctions valeurs, en des points d'intérêts.

On obtient de cette façon le premier résultat de cette thèse qui est une condition suffisante portant sur la richesse des points de raffinement, asymétrique entre combinaison linéaire min-plus et max-plus, afin d'obtenir convergence presque sûre asymptotique des approximations générées par TDP vers la fonction valeur en des points d'intérêts.

On a ensuite illustré (voir Figure 1.1) ce résultat de convergence en appliquant l'algorithme TDP pour générer des approximations min-plus des fonctions valeurs (au dessus des fonctions valeurs), comme infima de formes quadratiques convexes. Parallèlement on a aussi généré des sous-approximations des fonctions valeurs comme suprema de fonctions affines. Les sur-approximations adaptent aux problèmes multi-étapes (discrets en temps) un algorithme dû à Zheng Qu ([Qu14]). Les sous-approximations sont générées par l'algorithme Dual Dynamic Programming (DDP), version déterministe d'un algorithme introduit par Pereira et Pinto en 1991, une fois retranscrit dans le cadre de TDP. Sur la Figure 1.1, on a représenté l'écart entre sur approximations et sous approximations le long des trajectoires optimales des sous approximations courantes.

Programmation dynamique tropicale : vers le cas stochastique

Dans [BDZ18, PdF13], on étudie des schémas d'approximation où les sous-approximations sont données sous forme de suprema de fonctions affines et les sur-approximations sont des fonctions polyhédrales. Dans ce chapitre 4, nous cherchons à étendre, avec TDP, l'approche de [BDZ18, PdF13] en considérant plus généralement que les sous-approximations sont des combinaisons linéaires max-plus de certaines *fonction de base* et les sur-approximations sont des combinaisons linéaires min-plus de certaines autres fonctions de base. On va :

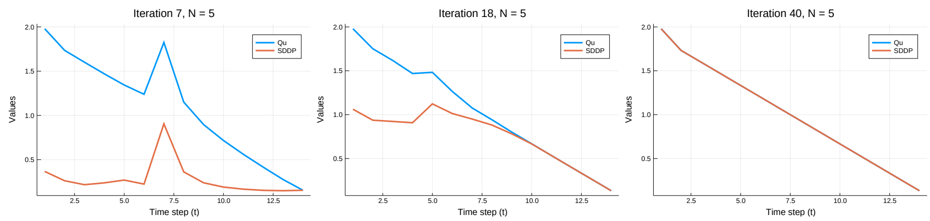

Figure 1.1: Écart en fonction du temps t entre une adaptation de la méthode de Qu (infima de quadratiques convexes) et une variante déterministe de SDDP (suprema de fonctions affines). Cet écart est évalué le long des trajectoires optimales courantes de la variante déterministe de SDDP.

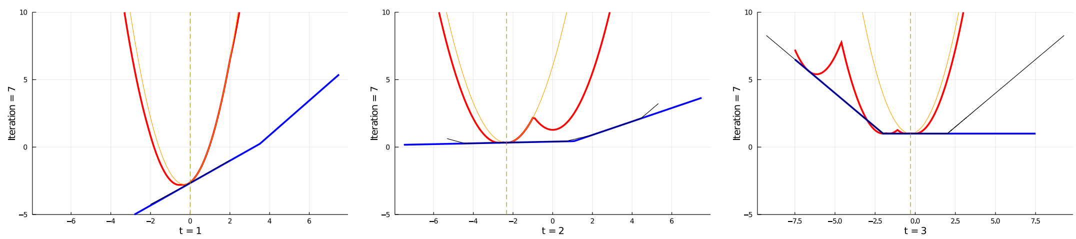

Figure 1.2: Approximations U-SDDP des fonctions valeurs. L'écart entre sur-approximation (infimum de quadratiques, en rouge) et sous-approximations (supremum de droites, en bleu) tend vers 0 le long d'une trajectoire spécifique d'états (en pointillées).

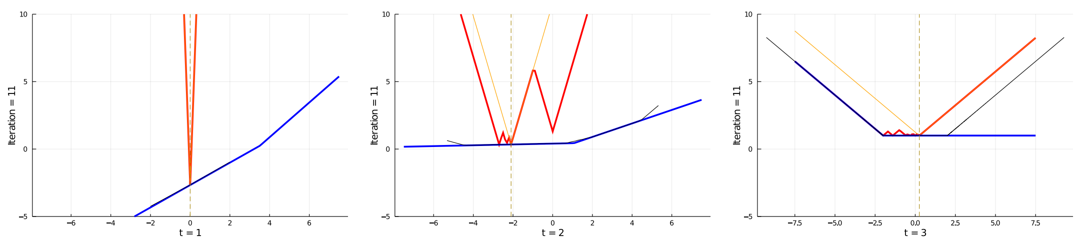

Figure 1.3: Approximations V-SDDP des fonctions valeurs. L'écart entre sur-approximation (infimum de fonctions “en forme de V”, en rouge) et sous-approximations (supremum de droites, en bleu) tend vers 0 le long d'une trajectoire spécifique d'états (en pointillées).

1. Élargir le cadre déterministe du Chapitre 3 au cadre des MSP Lipschitz (voir la Section 4.2).

2. S'assurer que les sur et sous approximations convergent vers la véritable fonction valeur V_t sur un ensemble de points communs, (voir la Section 4.3). Le résultat de la Section 4.3 se généralise à tout schéma d'approximation min-plus/max-plus le résultat de [BDZ18] qui a été donné pour un variante de SDDP.

3. Donner explicitement plusieurs moyens numériquement efficaces de construire des sous approximations inférieures des fonctions valeur V_t, comme combinaisons min-plus et max-plus linéaires de fonctions simples, (voir la Section 4.4).

Relaxation entropique de la Distance Imbriquée

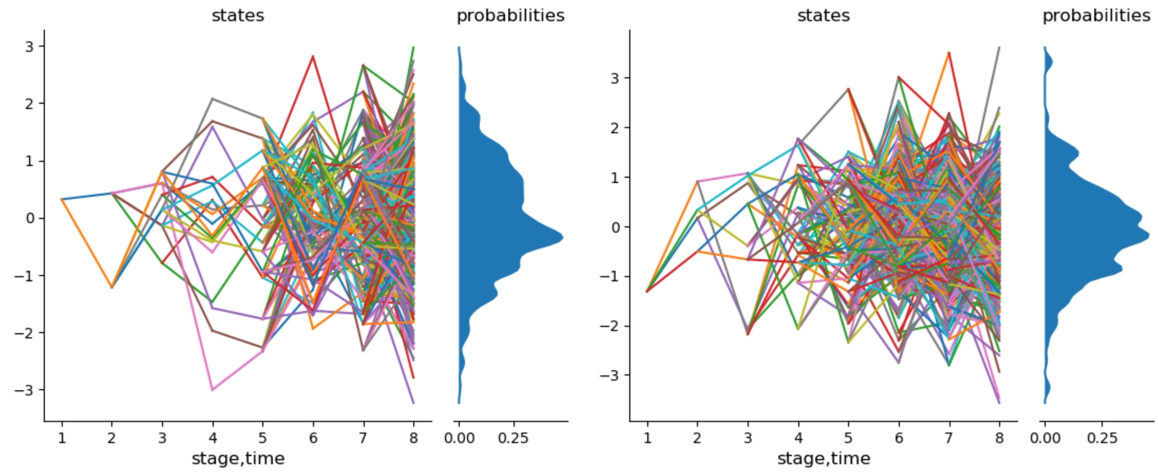

Figure 1.4: Deux arbres de scenarios X et Y avec (en bleu à droite) une approximation continue de l'histogramme des feuilles. Leur Distance Imbriquée (ND) est de $\mathrm{ND}_2(X, Y) = 1.009$ et sa relaxation entropique (END) est de $\mathrm{END}_2(X, Y) = 1.011$, voir Section 5.3. Ces arbres ont été générés via le package Julia ScenTrees.jl [KPP20].

Dans le cadre de la programmation stochastique multi-étapes (MSP), Georg Pflug a introduit en 2009 [Pfl09] la *Distance Imbriquée*, qui est un raffinement de la distance de Wasserstein pour tenir compte de la proximité dans les filtrations entre deux processus stochastiques à temps discret. Selon la dénomination habituelle dans la communauté de la programmation stochastique (voir [HR09, PP14, SDR09]), nous désignons également par *arbre de scénarios* un processus stochastique à temps discret qui est également discret et fini en espace.

Il existe de nombreuses distances différentes entre les arbres de scénarios. Cependant, peu d'entre elles sont adaptées aux besoins des MSP : on voudrait garantir la continuité de la fonction de valeur d'une MSP par rapport aux arbres de scénarios, *i.e.* si deux arbres de scénarios sont arbitrairement proches l'un de l'autre, alors la valeur de la MSP associée (avec la même structure sauf pour les arbres de scénarios) est également arbitrairement proche.

Une distance entre les arbres de scénarios est la distance de Wasserstein. Intuitivement, la distance de Wasserstein entre deux probabilités p et q (pour l'arbre de scénarios $(X_t)_{t\in[\![1,T]\!]}$, considérer la loi de probabilité du T-uplet $(X_1, \dots, X_T)$) correspond au coût optimal de la division et du transport de la masse de l'un à l'autre. On écrit $\mathbf{1}_k$, $k \in \mathbb{N}$, pour le vecteur $(1; \dots; 1)$ de $\mathbb{R}^k$.

Définition 1 (Transport optimal discret et distances de Wasserstein)**.** *Soit n, m deux entiers et $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$ et $\mathbb{Y} = \{y_1, \dots, y_m\}$ deux ensembles finis inclus dans $\mathbb{R}^t$, $t \geq 1$. Notons $c = (c_{ij})_{i,j}$ une matrice $n \times m$ positive appelée* matrice de coût*. Le* coût de transport optimal *entre deux mesures de probabilité p et q sur respectivement $\mathbb{X}$ et $\mathbb{Y}$, est la valeur du problème d'optimisation suivant*

$$\mathrm{OT}(p, q; c) = \min_{\pi \in \mathbb{R}_+^{n \times m}} \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} c_{ij} \pi_{ij} \ \text{s.t.}\ \pi \mathbf{1}_m = p \ \text{et}\ \pi^T \mathbf{1}_n = q. \tag{1.4}$$

De plus, on définit la fonction de coût *c par $c(x_i, y_j) = c_{ij}$ pour chaque indice i, j. Lorsque pour un certain réel $r \geq 1$, la fonction de coût c est égale à d^r avec d une métrique sur $\mathbb{X} \times \mathbb{Y}$, alors $\mathrm{OT}(p, q; d^r)^{1/r}$ est la r-*ème distance de Wasserstein *entre p et q, notée $\mathrm{W_r}(p, q)$.*

Nous nous référons aux ouvrages [PC19, Vil09] pour une présentation du transport optimal.

Dans les problèmes d'optimisation à deux étapes, sous certaines hypothèses de régularité, la fonction valeur d'une MSP à deux niveaux est lipschitzienne par rapport à la distance de Wasserstein, voir [PP14, Chapitre 6]. Cependant, la fonction valeur d'un MSP avec plus de 2 d'étapes n'est pas lipschitzienne par rapport à la distance de Wasserstein, comme le montre l'Exemple 1, où nous montrons que pour un MSP à 3 étapes, deux arbres de scénarios peuvent être arbitrairement proches l'un de l'autre dans la métrique de Wasserstein, mais l'écart entre les valeurs des MSPs associés est arbitrairement grand.

Exemple 1 (La distance de Wasserstein n'est pas adaptée aux MSPs)**.** *Dans cet exemple, nous montrons que la distance 1-Wasserstein n'est pas pertinente pour évaluer la distance entre les arbres de scénarios impliqués dans une MSP : une petite distance de Wasserstein arbitraire entre deux arbres de scénarios peut être associé à un écart arbitrairement grand entre les valeurs des MSPs associés.*

Étant donné un arbre de scénario Z (processus stochastique discret en temps et en espace) équipé de sa filtration naturelle $(\mathcal{F}_t)_{t \in [\![0,2]\!]}$[1]*, on voudrait acheter au coût moyen minimal un unique objet*

$$v(Z) = \min_{\mathbf{u}} \left\{ \mathbb{E}\left[\sum_{t=0}^{2} Z_t \mathbf{u_t}\right] \;\middle|\; \begin{array}{c} \mathbf{u_t} \in \{0, 1\}, \\ \mathbf{u_t} \ \text{est } \mathcal{F}_t \ \text{-mesurable}, \\ \sum_{t=0}^{T} \mathbf{u_t} = 1, \end{array} \right\}.$$

Soit $A \gg \epsilon > 0$, sur la Figure 1.5 on a représenté deux arbres de scénarios modélisant le prix d'un objet pendant 3 pas de temps. Leurs filtrations naturelles sont différentes. Intuitivement, sur l'arbre de scénario de gauche, le décideur observe une variation de ϵ du prix à $t = 1$ et sait qu'elle entraînera une forte hausse ou une forte baisse du prix à $t = 2$. Alors que dans l'arbre de scénario de droite, le décideur ne reconnaît pas cette information en $t = 1$. Exemple inspiré de [HRS06].

D'une part, nous avons la proximité dans la métrique de 1-Wasserstein W comme

$$\mathrm{W}(X, Y) = 2\epsilon.$$

[1] $\forall t \in [\![0, 2]\!]$, $\mathcal{F}_t = \sigma(Z_0, \dots, Z_t)$.

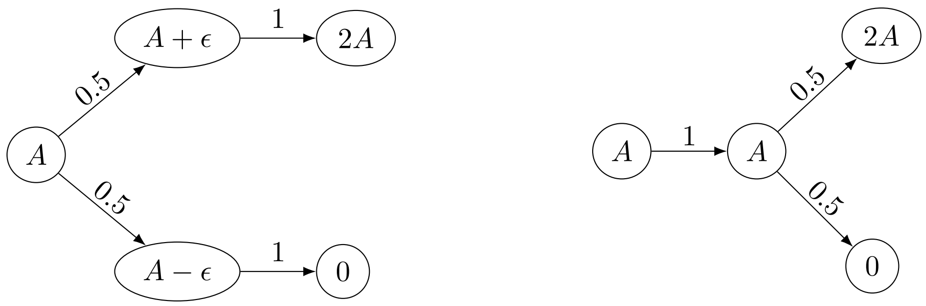

Figure 1.5: Gauche : arbre de scénarios $X := (X_0, X_1, X_2)$. Droite : arbre de scénarios $Y = (Y_0, Y_1, Y_2)$.

En revanche, les valeurs optimales sont $v(X) = \frac{A+\epsilon}{2}$ et $v(Y) = A$. Nous avons donc un écart de valeurs arbitrairement important

$$|v(X) - v(Y)| = \frac{A - \epsilon}{2} \underset{A \to +\infty}{\to} +\infty.$$

En 2012, Pflug et Pichler ont prouvé dans [PP12] que la Distance Imbriquée introduite précédemment par Pflug est l'adaptation correcte de la distance de Wasserstein pour la programmation stochastique multi-étapes : sous des hypothèses de régularité, la fonction valeur d'un MSP est continue de Lipschitz par rapport à la Distance Imbriquée entre les arbres de scénarios. Depuis lors, elle a été utilisée comme outil pour quantifier la qualité des arbres d'approximation : étant donné un arbre de scénario initial, on aimerait avoir un bon arbre d'approximation avec moins de nœuds. La Distance Imbriquée quantifie la qualité d'un arbre approximatif et le plan de transport optimal associé permet également de réduire les arbres de scénario, voir par exemple [KP15, HVKM20].

La Distance Imbriquée est généralement calculée par un algorithme rétrograde dans le temps (introduit dans [PP12], voir aussi [PS19, Definition 15]) qui revient à résoudre un nombre exponentiel (en T) de problèmes de transport optimal en l'absence de structure (indépendence) sur le processus de bruit. Il décompose dans le temps le calcul de la Distance Imbriquée comme le calcul dynamique d'un nombre fini de problèmes de transport optimaux entre des probabilités conditionnelles avec des coûts mis à jour en amont.

Le transport optimal entre des probabilités discrètes de taille n peut être résolu par l'algorithme Hongrois avec une complexité de $O(n^3)$ (voir [EK72]) ou avec l'algorithme d'enchères avec une complexité d'environ $O(n^3 \log n)$, voir [BC89].

En ajoutant un terme entropique à la formulation primale du problème de transport optimal associé au calcul d'un coût de Wasserstein, un schéma de projection alternatif donne l'algorithme de Sinkhorn, introduit en Transport Optimal dans [Cut13]. En sélectionnant soigneusement le terme de relaxation entropique, la complexité de l'algorithme de Sinkhorn est d'environ $O(n^2)$.

En relaxant chaque problème de transport optimal impliqué dans le calcul récursif de la Distance Imbriquée, on obtient une relaxation entropique de la Distance Imbriquée.

Le chapitre est organisé comme suit :

- Dans la section 5.2, nous définissons d'abord formellement la Distance Imbriquée comme la valeur d'un système dynamique de problèmes de transport optimal entre des probabilités conditionnelles avec des coûts variables. Ensuite, nous présentons une relaxation entropique du problème de transport optimal discret (1.4) et comment ce problème de transport optimal relaché peut être résolu efficacement par l'algorithme de Sinkhorn. Enfin, nous définissons une régularisation entropique naturelle de la Distance Imbriquée en relaxant chaque problème de transport optimal impliqué dans sa formulation dynamique.

- Dans la section 5.3, nous terminons ce chapitre 5 par une expérience numérique montrant à la fois l'accélération de notre approche du calcul des Distance Imbriquées et sa précision relative.

Échange entre intégration et minimisation

La question de l'échange entre intégration et minimisation est une question importante en optimisation stochastique (où l'intégration correspond à une espérance mathématique). Étant donné un espace de mesure $(\Omega, \mathcal{F}, \mu)$ et un sous-ensemble $U \subset \overline{\mathbb{R}}^{\Omega}$ de fonctions, on se demande quand l'égalité suivante est vraie

$$\inf_{u \in U} \int_{\Omega} u \, \mathrm{d}\mu = \int_{\Omega} \inf_{u \in U} u \, \mathrm{d}\mu \; . \tag{1.5}$$

Le cadre mathématique et les conditions pour obtenir l'équation (1.5) peuvent être trouvés dans [BG01, EKT13, Gin09, RW09, SDR09]. Nous nous concentrons sur [Gin09] et [RW09].

Pour commencer, dans l'équation (1.5), il convient de préciser dans quel sens l'intégrale $\int$ doit être comprise et dans quel sens les infima $\inf_{u \in U} u$ ou $\inf_{u \in U} \int u \, \mathrm{d}\mu$ sont définis. Ensuite, lorsque le sous-ensemble U sur lequel la minimisation est effectuée est un sous-ensemble de $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ et quand l'intégrale $\int$ est l'intégrale habituelle de Lebesgue, Giner a obtenu dans [Gin09] une condition nécessaire et suffisante pour obtenir l'Equation (1.5). Dans ce cadre, l'espace $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ est doté de l'ordre habituel μ-presque partout, et l'infimum est $\inf_{u \in U} u = \operatorname{ess\,inf}_{u \in U} u$, qui est bien défini par [Nev70, Proposition II.4.1]. Compte tenu d'un sous-ensemble $U \subset L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ de fonctions, Giner établit que l'Équation (1.5) est vraie si et seulement si, pour chaque famille finie $u_1, \ldots, u_n$ de U, nous avons

$$\inf_{u \in U} \int_{\Omega} (u - \inf_{1 \leq i \leq n} u_i) \, \mathrm{d}\mu \leq 0 \; .$$

Cependant, la vérification de la condition ci-dessus n'est pas une tâche facile, car elle dépend conjointement de l'intégrale $\int$ et du sous-ensemble U. De plus, on peut se demander si on peut encore avoir l'Equation (1.5) pour des sous-ensembles U qui sont intégrables dans un sens plus faible que Lebesgue intégrable.

Lorsqu'un sous-ensemble de fonctions $U \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ est l'image d'un ensemble X par une application $f : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, *i.e.* $U = f(X)$, un célèbre théorème de Rockafellar et Wets ([RW09, Theorem 14.60]) donne une condition sur l'application f et une condition sur l'ensemble X de sorte que l'équation (1.5) soit vérifiée. Dans ce cas, on s'intéresse à l'échange entre minimisation sur des sous-ensembles U de $L^0(\Omega, \mathcal{F}, \mu; \mathbb{R})$ avec l'intégrale externe, une généralisation de l'intégrale de Lebesgue à $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. Nous étudions l'intégrale externe et ses propriétés dans l'Annexe 6.5.

Le chapitre est organisé comme suit. La section 6.2 est consacré à un théorème d'échange énoncé sur des ensembles partiellement ordonnés. Plus précisément, nous présentons un théorème d'échange abstrait de la forme

$$\bigwedge_{x \in X} \Phi(x) = \Phi\big(\bigwedge_{x \in X} x\big) \ . \tag{1.6}$$

Une fois supposé des conditions sur l'application $\Phi : \mathbb{X} \to \mathbb{Y}$ et des propriétés structurelles sur les ensembles $\mathbb{X}$, $\mathbb{Y}$, nous fournissons une condition nécessaire et suffisante pour que l'Équation (1.6) soit vraie. Notre résultat s'inscrit dans la lignée de celui de Giner, car notre condition nécessaire et suffisante utilise à la fois la cartographie Φ et l'ensemble X.

La section 6.3 aborde la question d'origine sur l'échange entre minimisation et intégration en utilisant les résultats de la section 6.2.

Nous espérons que notre théorème d'échange abstrait ou que son application à l'intégrale de Lebesgue étendue donne un aperçu de la manière dont on peut obtenir l'échange entre l'intégration et minimisation comme dans l'équation (1.5), ainsi que sur la manière dont nous pouvons aller au-delà du cas intégral (mesures de risque en optimisation stochastique).

Chapter 2

English version

Multistage Stochastic optimization Problems

In this thesis we study multistage stochastic optimal control problems in the hazard-decision framework (hazard comes first, decision second). Starting from a given state x_0, a decision maker observes the outcome w_1 of a random variable $\mathbf{W_1}$, then decides on a control u_0 which induces a *known* cost $c_0^{w_1}(x_0, u_0)$ and the system evolves to a future state x_1 from a *known* dynamic: $x_1 = f_0^{w_1}(x_0, u_0)$. Having observed a new random outcome, the decision maker makes a new decision based on this observation which induces a known cost, then the system evolves to a known future state, and so on until T decisions have been made. At the last step, there are constraints on the final state x_T which are modeled by a final cost function ψ. The decision maker aims to minimize the average cost of her decisions.

Multistage Stochastic optimization Problems (MSP) can be formally described by the following optimization problem

$$\begin{aligned}
&\min_{(\mathbf{X},\mathbf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t^{\mathbf{W_{t+1}}}(\mathbf{X_t}, \mathbf{U_t}) + \psi(\mathbf{X_T})\right], \\
&\text{s.t. } \mathbf{X_0} = x_0 \text{ given}, \forall t \in [\![0, T-1]\!], \\
&\mathbf{X_{t+1}} = f_t^{\mathbf{W_{t+1}}}(\mathbf{X_t}, \mathbf{U_t}), \\
&\sigma(\mathbf{U_t}) \subset \sigma(\mathbf{X_0}, \mathbf{W_1}, \dots, \mathbf{W_{t+1}}),
\end{aligned} \tag{2.1}$$

where $(\mathbf{W_t})_{t\in[\![1,T]\!]}$ is a given sequence of independent random variables each with values in some measurable set $(\mathbb{W}_t, \mathcal{W}_t)$.

Dynamic Programming

One approach to solving MSP problems is by dynamic programming, see for example [Bel54, Ber16, CCCDL15, PP14, SDR09]. For some integers $n, m \in \mathbb{N}$, denote by $\mathbb{X} = \mathbb{R}^n$ the *state space* and $\mathbb{U} = \mathbb{R}^m$ the *control space*. Both $\mathbb{X}$ and $\mathbb{U}$ are endowed with their euclidean structure and borelian structure. We define the pointwise Bellman operators $\mathcal{B}_t^w$ and the average Bellman operators $\mathfrak{B}_t$ for every $t \in [\![0, T-1]\!]$. For each possible realization $w \in \mathbb{W}_{t+1}$ of the noise $\mathbf{W_{t+1}}$, for every function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ taking extended real values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, the function $\mathcal{B}_t^w(\phi)(\cdot) : \mathbb{X} \to \overline{\mathbb{R}}$ is defined by

$$\forall x \in \mathbb{X}, \ , \ \ \mathcal{B}_t^w(\phi)(x) = \min_{u\in\mathbb{U}} \Big(c_t^w(x, u) + \phi\big(f_t^w(x, u)\big)\Big) \ .$$

Now, the average Bellman operator $\mathfrak{B}_t$ is the mean of all the pointwise Bellman operators with respect to the probability law of $\mathbf{W_{t+1}}$. That is, for every $\phi : \mathbb{X} \to \overline{\mathbb{R}}$, we have that

$$\forall x \in \mathbb{X} \ , \ \ \mathfrak{B}_t(\phi)(x) = \mathbb{E}\big[\mathcal{B}_t^{\mathbf{W_{t+1}}}(\phi)(x)\big] = \mathbb{E}\Big[\min_{u\in\mathbb{U}} \Big(c_t^{\mathbf{W_{t+1}}}(x, u) + \phi\big(f_t^{\mathbf{W_{t+1}}}(x, u)\big)\Big)\Big] \ .$$

The average Bellman operator can be seen as a one stage operator which computes the value of applying the best (average) control at a given state x. Note that in the hazard-decision framework assumed here, the control is taken after observing the noise. Now, the Dynamic Programming approach states that in order to solve MSP Problems (2.1), it suffices to solve the following system of *Bellman equations* (2.2),

$$V_T = \psi \quad \text{and} \quad \forall t \in [\![0, T-1]\!], V_t = \mathfrak{B}_t(V_{t+1}) \ . \tag{2.2}$$

Solving the Bellman equations means computing recursively backward in time the *(Bellman) value functions* V_t. Finally, the value $V_0(x_0)$ is the solution of the multistage Problem (2.1).

Damping the curse of dimensionnality

One issue of using Dynamic Programming to solve multistage optimization problems is the so-called *curse of dimensionality* [Bel54]. That is, when the state space $\mathbb{X}$ is a vector space, grid-based methods to compute the value functions have a complexity which is exponential in the dimension of the state space $\mathbb{X}$. One popular algorithm (see [GLP15, Gui14, GR12, PP91, Sha11, ZAS18]) that aims to dampen the curse of dimensionality is the Stochastic Dual Dynamic Programming algorithm (or SDDP for short) introduced by Pereira and Pinto in 1991. Assuming that the cost functions c_t^w are convex and the dynamics f_t^w are linear, the value functions defined in the Dynamic Programming formulation (3.3) are convex [GLP15]. Under these assumptions, the SDDP algorithm aims to build lower (or outer) approximations of the value functions as suprema of affine functions and does not rely on a discretization of the state space. One of the main drawback of the SDDP algorithm is the lack of an efficient stopping criterion: it builds lower approximations of the value functions but upper (or inner) approximations are built through a Monte-Carlo scheme that is costly and the associated stopping criteria is not deterministic. We follow another path to provide upper approximations as explained now.

In [Qu13, Ch. 8] and [Qu14], Qu devised an algorithm which builds upper approximations of a Bellman value function arising in an infinite horizon and continuous time framework where the set of controls is both discrete and continuous. This work was inspired by the work of McEneaney [McE07] using techniques coming from tropical algebra, also called max-plus or min-plus techniques. Assume that for each fixed discrete control the cost functions are convex quadratic and the dynamics are linear in both the state and the continuous control. If the set of discrete controls is finite, then exploiting the min-plus linearity of the Bellman operators $\mathcal{B}_t$, one can show that the value functions can be computed as a finite pointwise infimum of convex quadratic functions:

$$V_t = \inf_{\phi_t \in F_t} \phi_t \ ,$$

where F_t is a finite set of convex quadratic forms. Moreover, in this framework, the elements of F_t can be explicitly computed through the Discrete Algebraic Riccati Equation (DARE [LR95]). Thus, an approximation scheme that computes an increasing sequence of subsets $\left(F_t^k\right)_{k\in\mathbb{N}}$ of F_t yields an algorithm that converges after a finite number of improvements

$$V_t^k := \inf_{\phi_t \in F_t^k} \phi_t \approx \inf_{\phi_t \in F_t} \phi_t = V_t.$$

However, the size of the set of functions F_t that need to be computed is growing exponentially with $T - t$. In [Qu14], in order to address the exponential growth of F_t, Qu introduced a probabilistic scheme that adds to F_t^k the “best” (given the current approximations) element of F_t at some point drawn on the unit sphere.

Outline of the thesis

Every chapter of this thesis was written as an independent chapter. The reader may read them independently of the others.

1. In Chapter 3 and Chapter 4, we build a general algorithm that encompasses both SDDP algorithm and an adaptation of Qu's work to a stochastic, discrete time and finite horizon framework. TDP iteratively builds approximations of the value functions as max-plus or min-plus linear combinations. These two chapters form the core of the thesis.

2. In the second part Chapter 5 is a contribution about the Nested Distance: we present an entropic relaxation of the Nested Distance which can be computed efficiently.

3. In the third Chapter 6 we give a general theorem about the interchange between integration and minimization. It notably generalizes the one of Giner [Gin09] and the one from Rockafellar and Wets [RW09].

In each of the following sections, we give a presentation of each chapter and its contribution.

Tropical Dynamic Programming: the deterministic case

First, we focus on the simplified framework of deterministic multistage optimisation problems,

$$\min_{\substack{x=(x_0,\dots,x_T)\in\mathbb{X}^{T+1}\\ u=(u_0,\dots u_{T-1})\in\mathbb{U}^T}} \sum_{t=0}^{T-1} c_t(x_t,u_t) + \psi(x_T)$$
$$\text{s.t. } \forall t \in [\![0, T-1]\!] \ , \quad x_{t+1} = f_t(x_t,u_t) \text{ and } x_0 \in \mathbb{X} \ \text{ given} \ .$$

In this deterministic context, we tried to understand how, at each time step, a linear max-plus or min-plus combination of elementary functions could converge towards the value function. An iterative algorithm is presented which adds at each iteration an elementary function to the current max-plus or min-plus linear combination. This algorithm, called Tropical Dynamic Programming (TDP), can be seen as a tropical variant of parametric approximations used in Adaptive Dynamic Programming (see [Ber19, Pow11]) where the value functions are approximated by linear combinations of basis functions.

TDP determines an elementary function to be added by randomly drawing a point, called a *trial point*. Moreover, since the value functions verify Bellman's system of equations, we would like that the approximations generated by TDP verify this system of equations as well.

TDP requires two properties on the elementary function ϕ_t added iteratively to the current approximation of V_t, $t < T$. The first property, local, is called *tightness* of the elementary function. It requires that the added elementary function verifies the t-th Bellman equation at the trial point $x \in \mathbb{X}$:

$$\phi_t(x) = \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(x) \, .$$

The second property, global, is called the *validity* of the elementary function. It requires that the elementary function ϕ_t added at time t is always below or above the image by the t-th

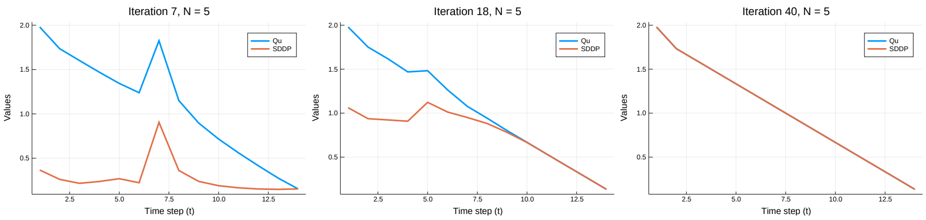

Figure 2.1: Gap w.r.t. the time t between an adaptation of Qu's algorithm (infima of convex quadratics) and a deterministic variant of SDDP (suprema of affine cuts). The gap is evaluated along current optimal trajectories of deterministic SDDP.

Bellman operator $\mathcal{B}_t$ of the current approximation at time $t+1$, *i.e.* denoting by $\mathcal{V}_{F_{t+1}}$ a max-plus or min-plus linear combinations of basic functions (approximating V_{t+1}),

$$\phi(\cdot) \geq \mathfrak{B}_t \left(\mathcal{V}_{F_{t+1}}\right)(\cdot)\,, \qquad \text{(for min-plus linear combinations)}$$
$$\phi(\cdot) \leq \mathfrak{B}_t \left(\mathcal{V}_{F_{t+1}}\right)(\cdot)\,. \qquad \text{(for max-plus linear combinations)}$$

Thus, having exact information about this system only at the trial points, it is generally impossible to guarantee that such min-plus or max-plus linear combinations verify the Bellman system of equations. However, we have established a sufficient condition regarding the richness of the trial points such that, almost surely, asymptotically the approximations generated by TDP verify a restricted Bellman system of equations. If the restrictions are in turn not too restrictive, then it is possible to conclude that any sequence of functions verifying such a system of equations is equal to the sequence of value functions at points of interest.

This gives us the first result of this thesis, Theorem 15 which is a sufficient condition concerning the richness of the trial points, asymmetric between linear min-plus and max-plus combinations, in order to obtain an almost sure asymptotic convergence of the approximations generated by TDP towards the value function at points of interest.

This convergence result was then illustrated (see Figure 2.1) by applying the TDP algorithm to generate min-plus approximations of the value functions (above the value functions), as infima of convex quadratic shapes. At the same time, sub-approximations of the value functions were also generated as suprema of affine functions. The upper approximations adapt to multistage problems an algorithm from Zheng Qu ([Qu14]). Lower approximations are generated by the Dual Dynamic Programming (DDP) algorithm, deterministic version of an algorithm introduced by Pereira and Pinto in 1991, once transcribed into TDP. In Figure 1.1, the difference between upper and lower approximations along the optimal trajectories of the current lower approximations has been represented.

Tropical Dynamic Programming: toward the stochastic case

In [BDZ18, PdF13], is studied approximation schemes where lower approximations are given as a suprema of affine functions and upper approximations are given as a polyhedral function. We aim in this chapter 4 to extend, with TDP, the approach of [BDZ18, PdF13] considering more generally that lower approximations are max-plus linear combinations of some *basic functions* and upper approximations are min-plus linear combinations of some other basic functions. In this chapter, we will:

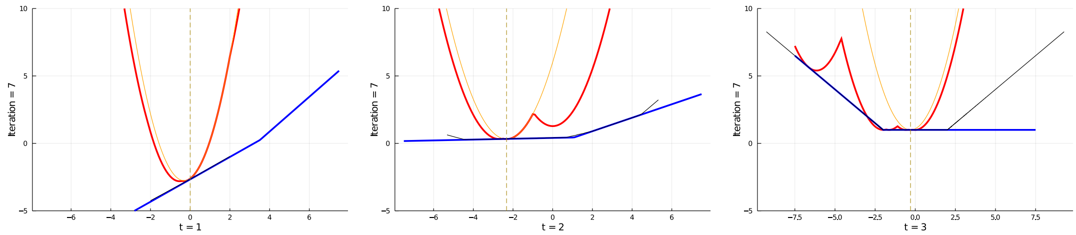

Figure 2.2: U-SDDP approximations of the value functions. We observe that the gap between upper approximations (infimum of quadratics, in red) and lower approximations (supremum of cuts, in red) vanishes along a specific trajectory of states (in dashed lines).

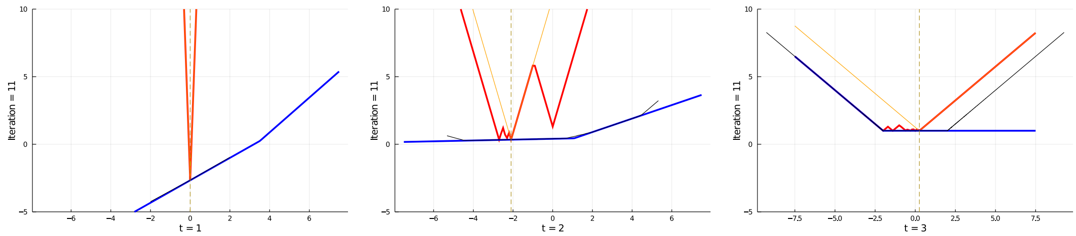

Figure 2.3: V-SDDP approximations of the value functions. We observe that the gap between upper approximations (infimum of "V-shaped functions", in red) and lower approximations (supremum of cuts, in blue) vanishes along a specific trajectory of states (in dashed lines).

1. Extend the deterministic framework of Chapter 3 to Lipschitz MSP defined in Equation (4.1) and introduce TDP, (see Section 4.2). The noises are independent and each with finite support.

2. Ensure that upper and lower approximations converge to the true value functions on a common set of points, see Section 4.3. The main result of (Section 4.3) generalizes to any min-plus/max-plus approximation scheme the result of [BDZ18] which was stated for a variant of SDDP.

3. Explicitly give several numerically efficient ways to build upper and lower approximations of the value functions, as min-plus and max-plus linear combinations of some simple functions, see (Section 4.4).

Entropic relaxation of the Nested Distance

In Multistage Stochastic Programming (MSP), Georg Pflug introduced in 2009 [Pfl09] the *Nested Distance*, which is a refinement of the Wasserstein distance to account proximity in the filtrations between two discrete time stochastic processes. Following usual denomination in the Stochastic Programming community (see [HR09, PP14, SDR09]), we also denote by *scenario tree* a discrete time stochastic process which is also discrete and finite in space.

There are many different distances between scenario trees. However, few are suited for MSP purposes: one would like to guarantee continuity of the value function of a MSP with

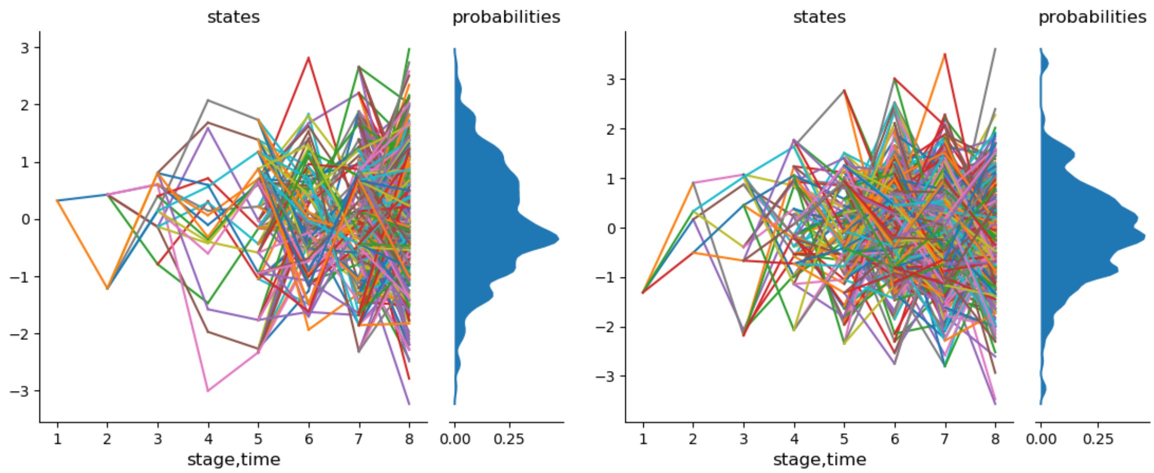

Figure 2.4: Two scenario trees X and Y with (in blue, right) a continuous probability approximation of the histogram the leaves. Their Nested Distance is $\mathrm{ND}_2(X, Y) = 1.009$ and its entropic relaxation is $\mathrm{END}_2(X, Y) = 1.011$, see Section 5.3. The trees were generated using the ScenTrees.jl package [KPP20].

respect to scenario trees, *i.e.* if two scenario trees are arbitrarily close to each other, then the value of the associated MSP (with the same structure except for the scenario trees) is arbitrarily close as well.

One distance between scenario tree is the Wasserstein distance. Intuitively, the Wasserstein distance between two probabilities p and q (for scenario tree $(X_t)_{t\in[\![1,T]\!]}$, consider the probability law of the tuple $(X_1, \dots, X_T)$) corresponds to the optimal cost of splitting and transporting the mass from one to the other. We write $\mathbf{1}_k$, $k \in \mathbb{N}$, for the vector $(1; \dots; 1)$ of $\mathbb{R}^k$.

Definition 2 (Discrete optimal transport and Wasserstein distances)**.** *Let n, m be two integers and $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$ and $\mathbb{Y} = \{y_1, \dots, y_m\}$ be two finite sets included in $\mathbb{R}^t$, $t \geq 1$. Denote by $c = (c_{ij})_{i,j}$ a $n \times m$ positive matrix called* cost matrix. *The* optimal transport cost *between two probability measures p and q on respectively $\mathbb{X}$ and $\mathbb{Y}$, is the value of the following optimization problem*

$$\mathrm{OT}(p, q; c) = \min_{\pi \in \mathbb{R}_+^{n\times m}} \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} c_{ij}\pi_{ij} \ \ s.t.\ \ \pi\mathbf{1}_m = p \ \ and\ \ \pi^T\mathbf{1}_n = q. \tag{2.4}$$

Now, define the cost function *by $c(x_i, y_j) = c_{ij}$ for every indexes i, j. When for some real $r \geq 1$, the cost function c is equal to d^r with d a metric on $\mathbb{X} \times \mathbb{Y}$, then $\mathrm{OT}(p, q; d^r)^{1/r}$ is the* r-th Wasserstein distance *between p and q, denoted $\mathrm{W_r}(p, q)$.*

We refer to the textbooks [PC19, Vil09] for a presentation and references on optimal transport.

In two stage multistage optimization problems, under some regularity assumptions, the value function of a bilevel MSP is Lipschitz continuous with respect to the Wasserstein distances, see [PP14, Chapter 6]. However the value function of MSP with more than 2 stages is not continuous with respect to the Wasserstein distances, as seen in Example 1, where we show that for a 3 stage MSP, two scenario trees can be arbitrarily close to each other in the 1-Wasserstein metric but the gap in the values of the associated MSPs is arbitrarily large.

Example 1 (The Wasserstein distance is not suited for MSP)**.** *In this example we illustrate that the* 1*-Wasserstein is not an interesting metric to evaluate distance between scenario trees involved in a MSP: an arbitrary small Wasserstein distance between two scenario trees may yield an arbitrary large gap in values of the same MSP.*

Given a scenario tree Z (see Definition 53 for a formal definition) with natural filtration $(\mathcal{F}_t)_{t\in[\![0,2]\!]}$[1]*, we want to buy a single object at the minimal average cost*

$$v(Z) = \min_{\mathbf{u}} \left\{ \mathbb{E}\left[\sum_{t=0}^{2} Z_t \mathbf{u_t}\right] \;\middle|\; \begin{array}{c} \mathbf{u_t} \in \{0,1\}\,, \\ \mathbf{u_t} \textit{ is } \mathcal{F}_t \textit{ -measurable,} \\ \sum_{t=0}^{T} \mathbf{u_t} = 1, \end{array} \right\}.$$

Fix $A \gg \epsilon > 0$, here are two scenario tree modeling the price of an object during 3 *time steps. Their natural filtrations are different. Intuitively, on the left scenario tree, the decision maker observes that an ϵ variation of the price at $t = 1$ and knows that it will yield an explosion (upward or downward) of the price at $t = 2$. Whereas on the right scenario tree, the decision maker does not recognize such information at time $t = 1$. Example adapted but inspired from [HRS06].*

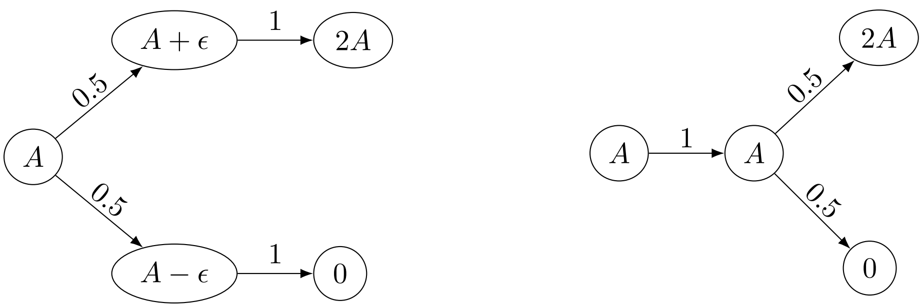

Figure 2.5: Left: scenario tree $X := (X_0, X_1, X_2)$. Right: scenario tree $Y = (Y_0, Y_1, Y_2)$.

On the one hand we have proximity in the 1*-Wasserstein metric* W *as*

$$\mathrm{W}(X, Y) = 2\epsilon.$$

On the other hand, the optimal values are $v(X) = \frac{A+\epsilon}{2}$ and $v(Y) = A$. Thus, we have an arbitrarily large gap in values

$$|v(X) - v(Y)| = \frac{A-\epsilon}{2} \underset{A\to+\infty}{\to} +\infty.$$

In 2012, Pflug and Pichler proved in [PP12] that the Nested Distance previously introduced by Pflug, is the correct adaptation of the Wasserstein distance for multistage stochastic programming: under regularity assumptions, the value function of MSPs is Lipschitz continuous with respect to the Nested Distance between scenario trees. Since then, it has been used as a tool to quantify the quality of approximating trees: given an initial scenario tree, one

[1] For every $t \in [\![0, 2]\!]$, $\mathcal{F}_t = \sigma(Z_0, \ldots, Z_t)$.

would like to have a good approximating tree with fewer nodes. The Nested Distance both quantifies the quality of an approximating tree and the associated optimal transport plan also allows for reduction of scenario trees, see for example [KP15, HVKM20].

The Nested Distance is usually computed via a backward recursive algorithm (introduced in [PP12], see also [PS19, Definition 15]) which amounts to solve an exponential number (in T) number of optimal transportation problems if there is no further assumption (like independence) on the noise process. It decomposes over the time the computation of the Nested Distance as the dynamic computation of a finite number of optimal transport problems between conditional probabilities with costs updated backward.

Optimal transport between discrete probabilities of size n can be solved by the Hungarian algorithm with complexity $O(n^3)$ (see [EK72]) or with the auction algorithm with complexity roughly $O(n^3 \log n)$, see [BC89].

By adding an entropic term to the primal of the optimal transport problem associated with the computation of a Wasserstein cost, an alternating projection scheme yield Sinkhorn's algorithm, introduced in Optimal Transport in [Cut13] to compute Wasserstein distances. By carefully selecting the entropic regularization term, Sinkhorn's algorithm computes an ϵ-overestimation of the Wasserstein distance in $O(n^2 \log(n)\epsilon^{-3})$ operations.

Relaxing each optimal transport problem involved in the recursive computation of the Nested Distance, we end up with an entropic regularization of the Nested Distance.

The chapter is organized as follows:

- In Section 5.2, we first formally define the Nested Distance as the value of a dynamic system of optimal transport problems between conditional probabilites and varying costs. Then, we present an entropic relaxation of the discrete optimal transport Problem (2.4) and how this relaxed OT problem can be solved efficiently by Sinkhorn's algorithm. Lastly, we define a natural entropic regularization of the Nested Distance by relaxing each OT problem involved in its dynamic formulation.

- In Section 5.3, we end this chapter 5 with a numerical experiment showing both the speedup of our approach to compute Nested Distances and also its relative preciseness.

Interchange between integration and minimization

The question of interchanging integration and minimization is an important issue in stochastic optimization (where integration corresponds to mathematical expectation). Loosely stated, given a measure space $(\Omega, \mathcal{F}, \mu)$ and a subset $U \subset \overline{\mathbb{R}}^\Omega$ of functions, we wonder when does the following equality hold

$$\inf_{u \in U} \int_\Omega u \, \mathrm{d}\mu = \int_\Omega \inf_{u \in U} u \, \mathrm{d}\mu \; . \tag{2.5}$$

Mathematical framework and conditions to get Equation (2.5) can be found in [BG01, EKT13, Gin09, RW09, SDR09]. We focus on [Gin09] and [RW09].

To begin with, in Equation (2.5) one needs to clarify in which sense the integral $\int$ is to be understood and in which sense the infima $\inf_{u \in U} u$ or $\inf_{u \in U} \int u \, \mathrm{d}\mu$ are defined. Then, when the subset U, over which minimization is performed, is a subset of $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and when the integral $\int$ is the usual Lebesgue integral, Giner obtained in [Gin09] a necessary and sufficient condition for (2.5) as follows. In this case, the space $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ is endowed with the usual μ-pointwise order, and the infimum is $\inf_{u\in U} u = \operatorname{ess\,inf}_{u\in U} u$, which is well-defined by [Nev70, Proposition II.4.1]. Given a subset $U \subset L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ of functions, Giner establishes that Equation (2.5) holds true if and only if, for every finite family $u_1, \ldots, u_n$ in U, we have

$$\inf_{u\in U} \int_{\Omega} (u - \inf_{1\leq i\leq n} u_i)\, \mathrm{d}\mu \leq 0 \;.$$

However, checking the above condition is not an easy task, as it depends jointly on the integral $\int$ and on the subset U. Moreover, one may wonder if we can still have Equation (2.5) for more general subsets U which are integrable in a weaker sense than Lebesgue integrable.

When a subset of functions $U \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ is the image of a set X by a mapping $f : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, *i.e.* $U = f(X)$, a celebrated theorem of Rockafellar and Wets ([RW09, Theorem 14.60]) gives a condition on the mapping f and a condition on the set X so that Equation (2.5) holds. In this case, we deal with minimization over subsets U of $L^0(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and interchange with the outer integral, a generalization of the Lebesgue integral to $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. We study the outer integral and its properties in Appendix 6.5.

The Chapter is organized as follows. Sect. 6.2 is devoted to a minimization interchange theorem on posets. More precisely, we provide an abstract interchange theorem of the form

$$\bigwedge_{x\in X} \Phi(x) = \Phi\big(\bigwedge_{x\in X} x\big) \;. \tag{2.6}$$

Once assumed conditions on the mapping $\Phi : \mathbb{X} \to \mathbb{Y}$ and structural properties of the sets $\mathbb{X}$, $\mathbb{Y}$, we provide a necessary and sufficient condition so that Equation (2.6) holds true. Our result is in the lineage of Giner's, as our necessary and sufficient condition involves both the mapping Φ and the set X.

Sect. 6.3 then tackles the original question of interchange between minimization and integration by specifying the results of Section 6.2.

We hope that either our abstract interchange theorem or its application to the extended Lebesgue integral provide insight as to how one may obtain the interchange between integration and minimization as in Equation (2.5), and as to how we can go beyond the integral case (risk measures in stochastic optimization).

Part II

Tropical Dynamic Programming

Chapter 3

Tropical Dynamic Programming: the deterministic case

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3.1 Introduction

Throughout this chapter, we aim to study a deterministic optimal control problem with discrete time. Informally, given a time t and a state $x_t \in \mathbb{X}$, one can apply a control $u_t \in \mathbb{U}$ and the next state is given by the dynamic f_t, that is $x_{t+1} = f_t(x_t, u_t)$. Then, one wants to minimize the sum of costs $c_t(x_t, u_t)$ induced by the controls starting from a given state x_0 and during a given time horizon T. Furthermore, one can add some final restrictions on the states at time T which will be modeled by an additional cost function ψ depending only on the final state x_T. We will call such optimal control problems, *multistage optimization problems* and *switched multistage optimization problems* if the controls are both continuous and discrete:

$$\min_{\substack{x=(x_0,\dots,x_T)\in\mathbb{X}^{T+1}\\ u=(u_0,\dots u_{T-1})\in\mathbb{U}^T}} \sum_{t=0}^{T-1} c_t(x_t,u_t) + \psi(x_T) \tag{3.1a}$$

$$\text{s.t. } \forall t \in [\![0, T-1]\!] \ , \quad x_{t+1} = f_t(x_t,u_t) \text{ and } x_0 \in \mathbb{X} \ \text{ given} \ . \tag{3.1b}$$

One can solve the multistage Problem (3.1) by Dynamic Programming as introduced by Richard Bellman around 1950 [Bel54, Dre02]. This method breaks the multistage Problem (3.1) into T sub-problems that one can solve by backward recursion over time. More precisely, denoting by $\mathcal{B}_t : \overline{\mathbb{R}}^{\mathbb{X}} \to \overline{\mathbb{R}}^{\mathbb{X}}$ the operator from the set of functions over $\mathbb{X}$ that may take infinite values to itself, defined by

$$\mathcal{B}_t(\phi) : x \mapsto \min_{u\in\mathbb{U}} \Big(c_t(x,u) + \phi\big(f_t(x,u)\big)\Big) \ , \tag{3.2}$$

one can show (see for example [Ber16]) that solving Problem (3.1) amounts to solve the following sequence of sub-problems:

$$V_T = \psi \quad \text{ and } \quad \forall t \in [\![0, T-1]\!] \quad V_t = \mathcal{B}_t(V_{t+1}) \ . \tag{3.3}$$

We will call each operator $\mathcal{B}_t$ the *Bellman operator* at time t and each equation in (3.3) will be called the *Bellman equation* at time t. Lastly, the function V_t defined in Equation (3.3) will be called the (Bellman) *value function* at time t. Note that the value of Problem (3.1) is equal to the value function V_0 at point x_0, that is $V_0(x_0)$, whereas solving the sequence of sub-problems given by Equation (3.3) means to compute the value functions V_t at each point $x \in \mathbb{X}$ and time $t \in [\![0, T-1]\!]$.

We will state several assumptions on these operators in Section 3.2 under which we will devise an algorithm to solve the system of Bellman Equation (3.3), also called the Dynamic Programming formulation of the multistage problem. Let us stress on the fact that although we want to solve the multistage Problem (3.1), we will mostly focus on its (equivalent) Dynamic Programming formulation given by Equation (3.3).

One issue of using Dynamic Programming to solve multistage optimization problems is the so-called *curse of dimensionality* [Bel54]. That is, when the state space $\mathbb{X}$ is a vector space, grid-based methods to compute the value functions have a complexity which is exponential in the dimension of the state space $\mathbb{X}$. One popular algorithm (see [GLP15, Gui14, GR12, PP91, Sha11, ZAS18]) that aims to dampen the curse of dimensionality is the Stochastic Dual Dynamic Programming algorithm (or SDDP for short) introduced by Pereira and Pinto in 1991. Assuming that the cost functions c_t are convex and the dynamics f_t are linear, the value functions defined in the Dynamic Programming formulation (3.3) are convex [GLP15]. Under these assumptions, the SDDP algorithm aims to build lower (or outer) approximations of the value functions as suprema of affine functions and thus, does not rely on a discretization of the state space. In the aforementioned references, this approach is used to solve stochastic multistage convex optimization problems, however in this chapter we will restrict our study to deterministic multistage convex optimization problems as formulated in Problem (3.1). In our deterministic framework, the SDDP algorithm boils down to the classical Nested Benders decomposition and can be applied to our framework. One of the main drawback of the SDDP algorithm (in the stochastic case) is the lack of an efficient stopping criterion: it builds lower approximations of the value functions but upper (or inner) approximations are built through a Monte-Carlo scheme that is costly and the associated stopping criteria is not deterministic. We follow another path to provide upper approximations as explained now.

In [Qu13, Ch. 8] and [Qu14], Qu devised an algorithm which builds upper approximations of a Bellman value function arising in an infinite horizon and continuous time framework where the set of controls is both discrete and continuous. Qu's work was inspired by the work of McEneaney [McE07] using techniques coming from tropical algebra, also called max-plus or min-plus techniques. Assume that $\mathbb{X} = \mathbb{R}^n$ and that for each fixed discrete control the cost functions are convex quadratic and the dynamics are linear in both the state and the continuous control. If the set of discrete controls is finite, then exploiting the min-plus linearity of the Bellman operators $\mathcal{B}_t$, one can show that the value functions can be computed as a finite pointwise infimum of convex quadratic functions:

$$V_t = \inf_{\phi_t \in F_t} \phi_t \ ,$$

where F_t is a finite set of convex quadratic forms. Moreover, in this framework, the elements of F_t can be explicitly computed through the Discrete Algebraic Riccati Equation (DARE [LR95]). Thus, an approximation scheme that computes an increasing sequence of subsets $\left(F_t^k\right)_{k\in\mathbb{N}}$ of F_t yields an algorithm that converges after a finite number of improvements

$$V_t^k := \inf_{\phi_t \in F_t^k} \phi_t \approx \inf_{\phi_t \in F_t} \phi_t = V_t.$$

However, the size of the set of functions F_t that need to be computed is growing exponentially with $T - t$. In [Qu14], in order to address the exponential growth of F_t, Qu introduced a probabilistic scheme that adds to F_t^k the "best" (given the current approximations) element of F_t at some point drawn on the unit sphere.

Our work aims to build a general algorithm that encompasses both a deterministic version of the SDDP algorithm and an adaptation of Qu's work to a discrete time and finite horizon framework.

The remainder of this chapter is structured as follows. In Section 3.2, we make several assumptions on the Bellman operators $\mathcal{B}_t$ and define an algorithm which builds approximations of the value functions as a pointwise optimum (*i.e.* either a pointwise infimum or a pointwise supremum) of basic functions in order to solve the associated Dynamic Programming formulation (3.3) of the multistage Problem (3.1). At each iteration, the so-called basic function that is added to the current approximation will have to satisfy two key properties at a randomly drawn point, namely, *tightness* and *validity*. A key feature of the proposed algorithm is that it can yield either upper or lower approximations. More precisely,

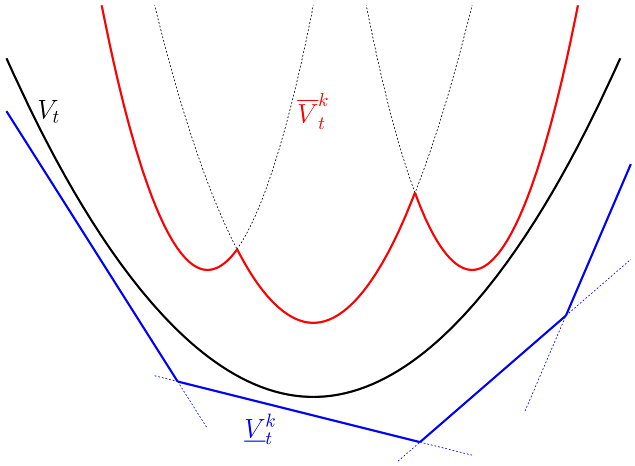

Figure 3.1: The lower approximations $\underline{V}_t^k$ will be built as a supremum of basic functions (here affine functions) that will always be below V_t. Likewise, the upper approximations $\overline{V}_t^k$ will be built as an infimum of some other basic functions (here quadratic functions) that will always be above V_t.

• if the basic functions are affine, then approximating the value functions by a pointwise supremum of affine functions will yield the SDDP algorithm;
• if the basic functions are quadratic convex, then approximating the value functions by a pointwise infimum of convex quadratic functions will yield an adaptation of Qu's min-plus algorithm.

In Section 3.3, we study the convergence of the approximations of the value functions generated by our algorithm at a given time $t \in [\![0, T]\!]$. We use an additional assumption on the random points on which current approximations are improved, which state that they need to cover a "rich enough set" and show that the approximating sequence converges almost surely (over the draws) to the Bellman value function on a set of interest.

In the last sections, we will specify our algorithm to three special cases. In Section 3.4, we prove that when building lower approximations as a supremum of affine cuts, the condition on the draws is satisfied on the optimal current trajectory, as done in SDDP. Thus, we get another point of view on the usual (see [GLP15, Sha11]) asymptotic convergence of SDDP, in the deterministic case. In Section 3.5, we describe an algorithm which builds upper approximations as an infimum of quadratic forms. It will be a step toward addressing the issue of computing efficient upper approximations for the SDDP algorithm. In Section 3.6, we present on a toy example some numerical experiments where we simultaneously compute lower approximations of the value functions by a deterministic version of SDDP of the value functions and upper approximations of the value functions by a discrete time version of Qu's min-plus algorithm.

3.2 Notations and definitions

In the sequel, we will use the following notations
• $\mathbb{X} := \mathbb{R}^n$, endowed with its Euclidean structure and its Borel σ-algebra denotes the *set of states*.

• T, a finite integer that we will call the time *horizon.*
• opt, denotes a generic operation that is either the *pointwise infimum* or the *pointwise supremum* of functions which we will call the *pointwise optimum.*
• $\overline{\mathbb{R}}$, denotes the extended real line endowed with the operations $+\infty + (-\infty) = -\infty + \infty = +\infty$.
• $\mathrm{dom}\phi$, denotes the *domain* of $\phi \in \left(\overline{\mathbb{R}}\right)^{\mathbb{X}}$ defined as the subset of $\mathbb{X}$ in which $\phi(x) \in \mathbb{R}$.
• $\mathbf{F}_t$ and $\mathbb{F}_t$, denote for every $t \in [\![0, T]\!]$, two subsets of the set $\left(\overline{\mathbb{R}}\right)^{\mathbb{X}}$ such that $\mathbf{F}_t \subset \mathbb{F}_t$.
• ϕ is said to be a *basic function* if it is an element of $\mathbf{F}_t$ for some $t \in [\![0, T]\!]$.
• δ_X denotes, for every set $X \subset \mathbb{X}$, the function equal to 0 on X and $+\infty$ elsewhere.
• For every $t \in [\![0, T]\!]$ and every set of basic functions $F_t \subset \mathbf{F}_t$, we denote by $\mathcal{V}_{F_t}$ its pointwise optimum, $\mathcal{V}_{F_t} := \mathrm{opt}_{\phi \in F_t} \phi$, that is

$$\begin{array}{rccl} \mathcal{V}_{F_t} : & \mathbb{X} & \longrightarrow & \overline{\mathbb{R}} \\ & x & \longmapsto & \mathrm{opt}\left\{\phi(x) \mid \phi \in F_t\right\}. \end{array} \tag{3.4}$$

• $(\mathcal{B}_t)_{t \in [\![0, T-1]\!]}$ denotes a sequence of T operators from $\overline{\mathbb{R}}^{\mathbb{X}}$ to $\overline{\mathbb{R}}^{\mathbb{X}}$, called the *Bellman operators.*
• $(V_t)_{t \in [\![0, T]\!]}$, denotes, for a fixed function $\psi : \mathbb{X} \to \overline{\mathbb{R}}$, a sequence of *value functions* given by the system of Bellman Equations (3.3).

Now, we make several assumptions on the structure of Problem (3.3). These assumptions will be satisfied in the examples developed in Sections 3.4 to 3.6. These assumptions will make it possible to propagate backward in time, regularity of the value function at the final time $t = T$ to the value function at the initial time $t = 0$.

Assumption 1 (Structural assumptions)**.**

–*(a)* ***Stability by pointwise optimum:*** *for every $t \in [\![0, T]\!]$, if $F_t \subset \mathbf{F}_t$ then $\mathcal{V}_{F_t} \in \mathbb{F}_t$.*

–*(b)* ***Stability by pointwise convergence:*** *for every $t \in [\![0, T]\!]$ if a sequence of functions $(\phi^k)_{k \in \mathbb{N}} \subset \mathbb{F}_t$ converges pointwise to ϕ on the domain of V_t, then $\phi \in \mathbb{F}_t$.*

–*(c)* ***Common regularity:*** *for every $t \in [\![0, T]\!]$, there exists a common (local) modulus of continuity of all $\phi \in \mathbb{F}_t$,* i.e. *for every $x \in \mathrm{dom}(V_t)$, there exist $\omega_{t,x} : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ which is increasing, equal to 0 in 0, continuous at 0 and such that for every $\phi \in \mathbb{F}_t$ and for every $x' \in \mathrm{dom}(V_t)$, we have that $|\phi(x) - \phi(x')| \leq \omega_{t,x}(\|x - x'\|)$.*

–*(d)* ***Final condition:*** *the value function V_T at time T is a pointwise optimum for some given subset F_T of $\mathbf{F}_T$, that is $\psi := \mathcal{V}_{F_T}$.*

–*(e)* ***Stability by the Bellman operators:*** *for every $t \in [\![0, T-1]\!]$, if $\phi \in \mathbb{F}_{t+1}$, then $\mathcal{B}_t(\phi)$ belongs to $\mathbb{F}_t$.*

–*(f)* ***Order preserving operators:*** *for every $t \in [\![0, T-1]\!]$, the operators $\mathcal{B}_t$ are* order preserving, i.e. *if $\phi, \varphi \in \mathbb{F}_{t+1}$ are such that $\phi \leq \varphi$, then $\mathcal{B}_t(\phi) \leq \mathcal{B}_t(\varphi)$.*

–*(g)* ***Additively subhomogeneous operators:*** *for every time step $t \in [\![0, T-1]\!]$, and every given compact set K_t, there exists $M_t > 0$ such that the operator $\mathcal{B}_t$ restricted to K_t is* additively subhomogeneous *over $\mathbb{F}_{t+1}$, meaning that for every constant function $\lambda \geq 0$ and every function $\phi \in \mathbb{F}_{t+1}$, we have*

$$\mathcal{B}_t(\phi + \lambda) + \delta_{K_t} \leq \mathcal{B}_t(\phi) + \lambda M_t + \delta_{K_t}.$$

–(h) ***Proper value functions:*** *the solution* $(V_t)_{t\in[\![0,T]\!]}$ *to the Bellman equations* (3.3) *never takes the value* $-\infty$ *and is not identically equal to* $+\infty$*.*

–(i) ***Compactness condition:*** *for every* $t \in [\![0, T-1]\!]$ *and every compact set* $K_t \subset \mathrm{dom}(V_t)$*, there exists a compact set* $K_{t+1} \subset \mathrm{dom}(V_{t+1})$ *such that, for every function* $\phi \in \mathbb{F}_{t+1}$ *and constant* $\lambda \geq 0$*, we have*

$$\mathcal{B}_t \left(\phi + \lambda + \delta_{K_{t+1}}\right) \leq \mathcal{B}_t \left(\phi + \lambda\right) + \delta_{K_t}.$$

Remark 3. *Assumption 1-*(c) *ensures that the domain of each function of* $\mathbb{F}_t$ *includes the domain of* V_t*. Note that if* $\mathbb{F}_t$ *is the set of all functions satisfying Assumption 1-*(c)*, then Assumption 1-*(b) *is trivially satisfied. Also note that the domain of* V_t *is known as in [GLP15].*

Remark 4. *Note that Assumption 1-*(h) *and 1-*(i) *do not change whether* $\mathrm{opt} = \inf$ *or* $\mathrm{opt} = \sup$ *as the optimal control problem that we consider is formulated as a minimization problem.*

Lemma 5. *For every* $t \in [\![0, T]\!]$ *we have that* $V_t \in \mathbb{F}_t$*.*

Proof. By Assumption 1-(d) and Assumption 1-(a), V_T is in $\mathbb{F}_T$. Now, assume that for some $t \in [\![0, T-1]\!]$ we have that $V_{t+1} \in \mathbb{F}_{t+1}$. By Assumption 1-(e), we have that $V_t = \mathcal{B}_t(V_{t+1}) \in \mathbb{F}_t$ which ends the proof by backward induction. □

From a set of basic functions $F_t \subset \mathbf{F}_t$, one can build its pointwise optimum $\mathcal{V}_{F_t} = \mathrm{opt}_{\phi\in F_t} \phi$. We build a monotone sequence of approximations of the value functions as optima of basic functions which will be computed through *compatible selection functions* as defined below. We illustrate this definition in Figure 3.2.

If $\mathrm{opt} = \inf$, then we will build upper approximations of the value function V_t as a min-plus linear combinations of basic functions. If $\mathrm{opt} = \sup$, we will build lower approximations as a max-plus linear combinations of basic functions.

Definition 6 (Compatible selection function)**.** *Let a time step* $t \in [\![0, T-1]\!]$ *be fixed. A* compatible selection function*, or simply* selection function*, is a function* $\mathcal{S}_t$ *from* $2^{\mathbf{F}_{t+1}} \times \mathbb{X}$ *to* $\mathbf{F}_t$ *satisfying the two following properties*
– ***Validity****: for every set of basic functions* $F_{t+1} \subset \mathbf{F}_{t+1}$ *and every* $x \in \mathbb{X}$*, we have* $\mathcal{S}_t\left[F_{t+1}, x\right] \leq \mathcal{B}_t\left(\mathcal{V}_{F_{t+1}}\right)$ *(resp.* $\mathcal{S}_t\left[F_{t+1}, x\right] \geq \mathcal{B}_t\left(\mathcal{V}_{F_{t+1}}\right)$*) when* $\mathrm{opt} = \sup$ *(resp.* $\mathrm{opt} = \inf$*).*
– ***Tightness****: for every set of basic functions* $F_{t+1} \subset \mathbf{F}_{t+1}$ *and every* $x \in \mathbb{X}$ *the functions* $\mathcal{S}_t\left[F_{t+1}, x\right]$ *and* $\mathcal{B}_t\left(\mathcal{V}_{F_{t+1}}\right)$ *coincide at point* x*, that is* $\mathcal{S}_t\left[F_{t+1}, x\right](x) = \mathcal{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(x)$*.*

For $t = T$*, we say that* $\mathcal{S}_T : \mathbb{X} \to \mathbf{F}_T$ *is a* compatible selection function *if it is* valid *and* tight*. There,* $\mathcal{S}_T$ *is* valid *if, for every* $x \in \mathbb{X}$*, the function* $\mathcal{S}_T[x]$ *remains below (resp. above) the value function at time* T *when* $\mathrm{opt} = \sup$ *(resp.* $\mathrm{opt} = \inf$*). Moreover, the function* $\mathcal{S}_T$ *is* tight *if it coincides with the value function at point* x*, that is for every* $x \in \mathbb{X}$*, we have* $\mathcal{S}_T[x](x) = V_T(x)$*.*

Note that the Tightness assumption only asks for equality at the point x between the functions $\mathcal{S}_t\left[F_{t+1}, x\right]$ and $\mathcal{B}_t\left(\mathcal{V}_{F_{t+1}}\right)$ and not necessarily everywhere. The only global property between the functions $\mathcal{S}_t\left[F_{t+1}, x\right]$ and $\mathcal{B}_t\left(\mathcal{V}_{F_{t+1}}\right)$ is an inequality given by the validity assumption.

In Algorithm 1 we will generate, for every time t, a sequence of random points of crucial importance that we will call *trial points*. They will be the ones where the selection functions

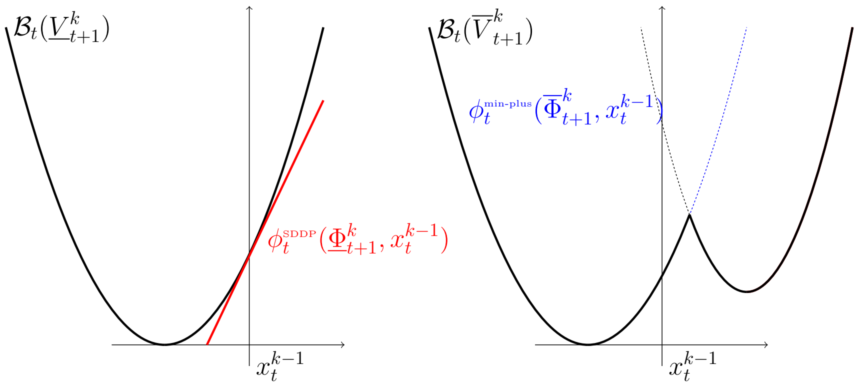

Figure 3.2: In Sections 3.4 and 3.5, we will specify two selection functions, ϕ_t^{SDDP} and $\phi_t^{\text{min-plus}}$, respectively, that will respectively yield upper and lower approximations of the value functions. In both cases, the selection function computes a basic function (in red or blue) which is equal, at the point x_t^{k-1}, to the image by the Bellman operator of the current approximation (in black), that is the tightness assumption. Moreover it remains above (or below) the image by the Bellman operator of the current approximation, that is the validity assumption.

will be evaluated, given the set F_t^k which characterizes the current approximation. In order to generate those points, we will assume that we have at our disposition an Oracle which, given $T{+}1$ sets of functions (characterizing the current approximations), computes $T{+}1$ compact sets and a probability law.

Definition 7 (Oracle)**.** *The Oracle takes as input $T{+}1$ sets of functions $F = (F_0, \dots, F_T)$ included in $\mathbf{F}_0, \dots, \mathbf{F}_T$ respectively. Its output consists of $T{+}1$ compact sets $K_0, \dots, K_T$, each included in $\mathbb{X}$, and of a probability measure $\mathbb{P}_F$ on the space $\mathbb{X}^{T+1}$ which are such that*
– Initialization. If for every $t \in [\![0, T]\!]$, $F_t = \emptyset$, then return $T{+}1$ given compact sets and a given probability measure.
– For every $t \in [\![0, T]\!]$, $K_t \subset \mathrm{dom}\,(V_t)$.
– The support of $\mathbb{P}_F$ is included in $K_0 \times \dots \times K_T$.

For every time $t \in [\![0, T]\!]$, we construct a sequence of functions $\big(V_t^k\big)_{k\in\mathbb{N}}$ belonging to $\mathbb{F}_t$ as follows. For every time $t \in [\![0, T]\!]$ and for every $k \geq 0$, we build a subset F_t^k of the set $\mathbf{F}_t$ and define the sequence of functions by pointwise optimum

$$V_t^k := \mathcal{V}_{F_t^k} = \operatorname*{opt}_{\phi \in F_t^k} \phi \; . \tag{3.5}$$

As described here, the functions are just byproducts of Algorithm 1, which only describes the way the sets F_t^k are computed.

As the following algorithm was inspired by Qu's work which uses tropical algebra techniques, we will call this algorithm "Tropical Dynamic Programming".

At each iteration, Algorithm 1 generates a trial point x_t^k which only depends on the data available at the current iteration. We loosely explain this point. Define for every $k \in \mathbb{N}$, $F^k = (F_t^k)_{t\in[\![0,T]\!]}$ and $x^k = (x_t^k)_{t\in[\![0,T]\!]}$. Then, there exists a deterministic function ξ and a sequence of independent random variables $(W^k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $x^k = \xi(F^k, W^k)$ where $(W^k)_{k\in\mathbb{N}}$ is furthermore independent from F^0. Throughout the remainder of the chapter,

Algorithm 1 Tropical Dynamic Programming (TDP)

Input: For every $t \in [\![0, T]\!]$, $\mathcal{S}_t$ a compatible selection function and a Trial point Oracle satisfying Definition 7.

Output: For every $t \in [\![0, T]\!]$, a sequence of sets $\left(F_t^k\right)_{k\in\mathbb{N}}$ and the associated sequence $V_t^k = \operatorname{opt}_{\phi\in F_t^k} \phi$.

Define for every $t \in [\![0, T]\!]$, $F_t^0 := \emptyset$.

for $k \geq 0$ **do**

Forward phase

Compute $\left(K_0^k, \ldots K_T^k, \mathbb{P}^k\right) = \text{Oracle}\left(F_0^k, \ldots, F_T^k\right)$.

Draw trial points $\left(x_t^k\right)_{t\in[\![0,T]\!]}$ over $K_0^k \times K_1^k \times \ldots \times K_T^k$ according to $\mathbb{P}^k$ knowing the past iterations.

Backward phase

Compute $\phi_T^{k+1} := \mathcal{S}_T\left[F_T^k, x_T^k\right]$.

Define $F_T^{k+1} := F_T^k \cup \left\{\phi_T^{k+1}\right\}$.

for t from $T-1$ to 0 **do**

Compute $\phi_t^{k+1} := \mathcal{S}_t\left[F_{t+1}^{k+1}, x_t^k\right]$.

Define $F_t^{k+1} := F_t^k \cup \left\{\phi_t^{k+1}\right\}$.

end for

end for

denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space on which the random variables $(W^k)_{k\in\mathbb{N}}$ are defined and independent.

We will denote by $+$ the Minkowski sum between sets, by $\mathbb{B}$ the unit closed Euclidean ball of $\mathbb{X}^{T+1}$ and for every $x \in \mathbb{X}^{T+1}$ and radius $r > 0$, $B(x, r)$ is the Euclidean open ball of radius r centered at x. Furthermore, we define for every t, $K_t^* := \overline{\lim}_{k\in\mathbb{N}} K_t^k$ the set of all possible limit points of K_t^k. We make the following assumption on the Oracle which, loosely stated, ensures that if a state x_t is close to K_t^*, then x_t is almost a limit point of the sequence of trial points $(x_t^k)_{k\in\mathbb{N}}$.

Assumption 2 (Trial point assumption)**.** *For every radius $r' > 0$, there exists $r > 0$ such that*

$$\forall x \in \mathbb{X}\ ,\quad \mathbb{P}\Big[\big(x \in \overline{\lim}_{k\in\mathbb{N}}{}^{1} K^k + r\mathbb{B}\big) \Rightarrow x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r')\Big] = 1\ . \tag{3.6}$$

Remark that $(\overline{\lim}_{k\in\mathbb{N}} K^k) + r\mathbb{B} = \overline{\lim}_{k\in\mathbb{N}}(K^k + r\mathbb{B})$, hence the lack of parenthesis. The following lemma gives some more insight on the Trial point assumption.

Lemma 8. *Consider the sequence of trial points $(x^k)_{k\in\mathbb{N}}$ generated by Algorithm 1 with an Oracle satisfying Assumption 2. Given $r' > 0$ and $x \in \mathbb{X}$, for every $r'' > r'$, $\mathbb{P}$-a.s.,*

$$x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r') \Rightarrow x^k \in B(x, r'') \textit{ for infinitely many indices } k \in \mathbb{N}\ . \tag{3.7}$$

Conversely, given $r'' > 0$ and $x \in \mathbb{X}$, for every $r' > r''$, $\mathbb{P}$-a.s.

$$x^k \in B(x, r'') \textit{ for infinitely many indices } k \in \mathbb{N} \Rightarrow x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r')\ . \tag{3.8}$$

[1] See [RW09, Definition 4.1 p. 109].

Proof. First, we prove Equation (3.7). Fix $r'' > r' > 0$ and assume that $x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r')$, $\mathbb{P}$-a.s.. Then, there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence $(y^{\sigma(k)})_{k\in\mathbb{N}} \subset r'\mathbb{B}$ such that $x^{\sigma(k)} + y^{\sigma(k)} \underset{k\to+\infty}{\longrightarrow} x$. As $r'' - r' > 0$, there exists a rank $k_0 \in \mathbb{N}$ such that when $k \geq k_0$ we have $\|x - x^{\sigma(k)} + y^{\sigma(k)}\| \leq r'' - r'$. By triangle inequality, we have

$$\|x - x^{\sigma(k)}\| \leq \|x - x^{\sigma(k)} + y^{\sigma(k)}\| + \|-y^{\sigma(k)}\| \leq (r'' - r') + r' = r'',$$

i.e. $\mathbb{P}$-a.s., for every $k \geq k_0$, $x \in B(x^k, r'')$, which yields Equation (3.7).

Second, we prove Equation (3.8). Fix $r' > r'' > 0$ and assume that $x^k \in B(x, r'')$ for infinitely many indices $k \in \mathbb{N}$. Thus, $\mathbb{P}$-a.s, there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence $(y^{\sigma(k)})_{k\in\mathbb{N}} \subset r''\mathbb{B}$ such that $x^{\sigma(k)} - x = y^{\sigma(k)}$. As $r' > r''$, $\mathbb{P}$-a.s. $x \in B(x^k, r')$ and $x = x^{\sigma(k)} - y^{\sigma(k)}$. Hence, we obtain Equation (3.8). □

Now, we give two examples of Oracles that satisfy the Trial point assumption 2. They are used respectively in Section 3.4 and 3.5.

Example 2 (Independent uniform draws over the unit sphere)**.** *Consider the Oracle which constantly outputs $T+1$ times the unit Euclidean sphere S of $\mathbb{X}$ and the uniform probability measure $\mathbb{P}^k := \sigma_U$ of S^{T+1} on $\mathbb{X}^{T+1}$.*[2] *Here, we have for every $k \in \mathbb{N}$, $K^k = S^{T+1}$. Fix an arbitrary $r' > 0$ and set $r = r'/2$, we prove that*

$$\forall x \in \mathbb{X}, \mathbb{P}\Big[x \in (S^{T+1} + r\mathbb{B}) \Rightarrow x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r')\Big] = 1\ .$$

Proof. Fix $x \in S^{T+1} + r\mathbb{B}$, we need to show that we have $\mathbb{P}\big[x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r')\big] = 1$. Now, fix $r'' > 0$ such that $r < r'' < r'$. Using Lemma 8-(3.8), it is enough to show that

$$\mathbb{P}[x^k \in B(x, r'') \text{ for infinitely many indices } k \in \mathbb{N}] = 1. \tag{3.9}$$

As $r'' > r$ and x is distant from S^{T+1} by less than r, the quantity $\mathbb{P}[x^k \in B(x, r'')] = \sigma_U[B(x, r'') \cap S^{T+1}]$ is a positive constant in k. Thus, we have that $\sum_{k\in\mathbb{N}} \mathbb{P}\big[x^k \in B(x, r'')\big] = +\infty$. Moreover, the sequence of events $(x^k \in B(x, r))_{k\in\mathbb{N}}$ is independent, thus by Borel-Cantelli's Lemma, Equation (3.9) holds. □

Example 3 (Dirac on the current optimal trajectory)**.** *The sequence of probability measures $(\mathbb{P}^k)_{k\in\mathbb{N}}$ is recursively build as follows:*
– Set $\mathbb{P}^0 := (\delta_{x_0^0}, \dots \delta_{x_T^0})$ where $x_t^0 \in K_t^0$ for every $t \in [\![0, T]\!]$.
– Given sets of functions $F_0^k, \dots, F_T^k$. Start, by fixing $x_0^k = x_0$ and compute forward in time, for $t \in [\![0, T-1]\!]$, optimal controls by $u_t^k \in \arg\min_u \mathcal{B}_t^u(\mathcal{V}_{F_{t+1}^k})(x_t^k)$, and successive states by $x_{t+1}^k = f_t(x_t^k, u_t^k)$.
– Define a probability measures $\mathbb{P}^k := (\delta_{x_0^k}, \dots, \delta_{x_T^k})$.

Consider the Oracle which, given sets of functions $F_0^k, \dots, F_T^k$, outputs the singleton $\{x^k\} = \big\{(x_t^k)_{t\in[\![0,T]\!]}\big\}$ and the probability measure $\mathbb{P}^k := (\delta_{x^k})$ defined at previous step. Fix $r > 0$, take $r' = r > 0$ and $x \in \mathbb{X}^{T+1}$. We obtain that

$$\mathbb{P}\Big[x \in \big(\overline{\lim}_{k\in\mathbb{N}} \underbrace{\{x^k\} + r\mathbb{B}}_{=B(x^k,r)}\big)^c \text{ or } x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, r)\Big] = 1\ ,$$

which is equivalent to the Trial point assumption with $K^k = \{x^k\}$.

[2]For every $A \in \mathcal{B}(\mathbb{X}^{T+1})$, $\sigma_U(A) = C\,\mathrm{Leb}(\pi_{S^{T+1}}^{-1}(A \cap S^{T+1}))$, where Leb is the Lebesgue measure on $\mathbb{X}^{T+1}$, $\pi_{S^{T+1}}$ is the Euclidean projector on S^{T+1} restricted to the ball $B(0,1)^{T+1}$ without 0 and C a normalization constant.

3.3 Almost sure convergence on the set of accumulation points

In this section, we will prove the convergence result stated in Theorem 15. For this purpose, we state several crucial properties of the approximation functions $\left(V_t^k\right)_{k\in\mathbb{N}}$ generated by Algorithm 1. They are direct consequences of the facts that the Bellman operators are order preserving and that the basic functions building our approximations are computed through compatible selection functions. Algorithm 1 is stochastic as trial points are drawn at each iteration from $\mathbb{P}^k$. Therefore, equalities, inequalities and statements where the functions V_t^k are involved hold $\mathbb{P}$-almost surely. However, for the sake of simplicity, we will refrain from always adding $\mathbb{P}$-almost surely in equalities, inequalities and some statements.

Lemma 9. *The sequence of functions $\left(V_t^k\right)_{k\in\mathbb{N}}$, for every $t \in [\![0,T]\!]$, given by Equation (3.5) and produced by Algorithm 1 satisfy the following properties.*

1. ***Monotone approximations:*** *for every indices $k < k'$ and every $t \in [\![0,T]\!]$, we have that $V_t^k \geq V_t^{k'} \geq V_t$ when $\mathrm{opt} = \inf$ and $V_t^k \leq V_t^{k'} \leq V_t$ when $\mathrm{opt} = \sup$.*

2. *For every $k \in \mathbb{N}$ and every $t \in [\![0,T-1]\!]$, we have that $\mathcal{B}_t\left(V_{t+1}^k\right) \leq V_t^k$ when $\mathrm{opt} = \inf$ and $\mathcal{B}_t\left(V_{t+1}^k\right) \geq V_t^k$ when $\mathrm{opt} = \sup$.*

3. *For every $k \geq 1$ and every $t \in [\![0,T-1]\!]$, we have $\mathcal{B}_t\left(V_{t+1}^k\right)\left(x_t^{k-1}\right) = V_t^k\left(x_t^{k-1}\right)$.*

4. *For every $k \geq 1$, we have $V_T^k\left(x_T^{k-1}\right) = V_T\left(x_T^{k-1}\right)$.*

Proof. We prove each point when $\mathrm{opt} = \inf$. The case $\mathrm{opt} = \sup$ is similar and left to the reader.
• (1) (left inequality). Let $t \in [\![0,T]\!]$ be fixed. By construction of Algorithm 1, the sequence of sets $\left(F_t^k\right)_{k\in\mathbb{N}}$ is non-decreasing. Now, using the definition of the sequence $\left(V_t^k\right)_{k\in\mathbb{N}}$ given by Equation (3.5) we have that $V_t^{k+1} = \mathcal{V}_{F_t^{k+1}}(x) = \inf_{\phi\in F_t^{k+1}} \phi(x) \leq \inf_{\phi\in F_t^k} \phi(x) = \mathcal{V}_{F_t^k}(x) = V_t^k$ and thus the sequence $\left(V_t^k\right)_{k\in\mathbb{N}}$ is non-increasing.
• (2). We prove the assertion by induction on $k \in \mathbb{N}$. For $k = 0$, as $F_t^0 = \emptyset$, we have $V_t^0 = +\infty$ for all $t \in [\![0,T-1]\!]$ and thus the assertion is true. Now, assume that for some $k \in \mathbb{N}$, we have for all $t \in [\![0,T-1]\!]$

$$\mathcal{B}_t\left(V_{t+1}^k\right) \leq V_t^k \ . \tag{3.10}$$

Since $\left(V_{t+1}^{k'}\right)_{k'\in\mathbb{N}}$ is non-increasing by already proved Item (1) and $\mathcal{B}_t$ is order preserving using Assumption 1-(f), we have that $\mathcal{B}_t\left(V_{t+1}^{k+1}\right) \leq \mathcal{B}_t(V_{t+1}^k)$. This last inequality combined with induction assumption given by Equation (3.10) gives the inequality

$$\mathcal{B}_t\left(V_{t+1}^{k+1}\right) \leq V_t^k \ . \tag{3.11}$$

Moreover, we also have that

$$\mathcal{B}_t\left(V_{t+1}^{k+1}\right) \underset{\text{(by (3.5))}}{=} \mathcal{B}_t\left(\mathcal{V}_{F_{t+1}^{k+1}}\right) \underset{\text{(by } \mathcal{S}_t \text{ validity at } x_t^k)}{\leq} \mathcal{S}_t\left[F_{t+1}^{k+1}, x_t^k\right] = \phi_t^{k+1} \ , \tag{3.12}$$

where the last equality is obtained by definition of function ϕ_t^{k+1} in Algorithm 1. Thus, combining Equation (3.11) and (3.12) we have that $\mathcal{B}_t(V_{t+1}^{k+1}) \leq \inf\big(V_t^k, \phi_t^{k+1}\big)$. Finally, using Equation (3.5) and Algorithm 1, we have that

$$\inf\big(V_t^k, \phi_t^{k+1}\big) = \inf\Big(\inf_{\phi \in F_t^k} \phi, \phi_t^{k+1}\Big) = \inf_{\phi \in F_t^k \cup \{\phi_t^{k+1}\}} \phi = \inf_{\phi \in F_t^{k+1}} \phi = V_t^{k+1} .$$

Thus, we obtain that $\mathcal{B}_t\left(V_{t+1}^{k+1}\right) \leq \inf\big(V_t^k, \phi_t^{k+1}\big) = V_t^{k+1}$, which gives the induction assumption for $k+1$ and concludes the proof of (2).

• (3). As the selection function $\mathcal{S}_t$ is tight in the sense of Definition 6, we have by construction of Algorithm 1 that $\mathcal{B}_t(V_{t+1}^k)(x_t^{k-1}) = \phi_t^k(x_t^{k-1})$. Combining this equation with Item (2) and the definition of V_t^k, one gets Lemma 9-(3).

• (4). Similarly, we have that $V_T(x_T^{k-1}) = \phi_T^k(x_T^{k-1})$, which combined with the inequality given in Item (1) and the definition of V_T^k gives Lemma 9-(4).

• (1) (right inequality). We prove that $V_t^k \geq V_t$ for all for all $k \in \mathbb{N}$ and all $t \in [\![0, T]\!]$. Fix $k \in \mathbb{N}$, we show that $V_t^k \geq V_t$ for all $t \in [\![0, T]\!]$ by backward recursion on time t. For $t = T$, by validity of the selection functions given in Definition 6, for every $\phi \in F_T^k$, we have that $\phi \geq V_T$. Thus $V_T^k = \mathcal{V}_{F_T^k} = \inf_{\phi \in F_T^k} \phi \geq V_T$. Now, suppose that for some $t \in [\![0, T-1]\!]$, we have that $V_{t+1} \leq V_{t+1}^k$. Then, using the definition of the value function in Equation (3.3), the fact that the Bellman operators are order preserving and the inequality already proved in Item (2) we obtain that: $V_t = \mathcal{B}_t\big(V_{t+1}\big) \leq \mathcal{B}_t\big(V_{t+1}^k\big) \leq V_t^k$, which gives the assertion for time t. This ends the proof. □

In the following two propositions, we state that the sequences $\big(V_t^k\big)_{k\in\mathbb{N}}$ and $(\mathcal{B}_t\left(V_{t+1}^k\right))_{k\in\mathbb{N}}$ converge uniformly on any compact included in the domain of V_t. The limit function V_t^* of $\big(V_t^k\big)_{k\in\mathbb{N}}$ will be a natural candidate to be the value function V_t.

Lemma 10. *Fix $t \in [\![0, T]\!]$. Let $(\phi^k)_{k\in\mathbb{N}}$ be a monotonic sequence in $\mathbb{F}_t$ such that there exists $\phi_1, \phi_2 \in \mathbb{F}_t$ satisfying for every $k \in \mathbb{N}$ $\phi_1 \leq \phi^k \leq \phi_2$. Then, the sequence $(\phi^k)_{k\in\mathbb{N}}$ converges uniformly on every compact set included in* $\mathrm{dom}(V_t)$ *to a function $\phi^* \in \mathbb{F}_t$.*

Proof. The proof relies on the Arzela-Ascoli theorem [Sch95, Theorem 2.13.30 p.347]. Since ϕ_1 and ϕ_2 belong to $\mathbb{F}_t$, they are finite on $\mathrm{dom}(V_t)$. Then, the sequence of functions $(\phi^k)_{k\in\mathbb{N}}$ is monotonic and bounded, so it converges pointwise on $\mathrm{dom}(V_t)$ to a limit function ϕ^*. By Assumption 1-(b), this implies that $\phi^* \in \mathbb{F}_t$.

Now, fix a compact set $K \subset \mathrm{dom}(V_t)$. First, since $(\phi^k)_{k\in\mathbb{N}} \subset \mathbb{F}_t$, we have that for every $k \in \mathbb{N}$, $\mathrm{dom}(\phi^k)$ contains $\mathrm{dom}(V_t)$ and the sequence of functions $(\phi^k)_{k\in\mathbb{N}}$ share a common modulus of continuity. Second, $\sup_{k\in\mathbb{N}} \sup_{x\in K} |\phi^k(x)|$ is finite as $|\phi_1|$ and $|\phi_2|$ are continuous functions on the compact K. Hence, by Arzela-Ascoli theorem the monotonic sequence $(\phi^k)_{k\in\mathbb{N}}$ converges uniformly on the compact K towards the continuous function ϕ^*. □

Proposition 11 (Existence of an approximating limit)**.** *Let $t \in [\![0, T]\!]$ be fixed. The sequence of functions $\big(V_t^k\big)_{k\in\mathbb{N}}$ defined by Equation (3.5) and Algorithm 1 $\mathbb{P}$-a.s. converges uniformly on every compact set included in the domain of V_t (solution of Equation (3.3)) to a function $V_t^* \in \mathbb{F}_t$.*

Proof. By Lemma 9-(1), for every $k \geq 1$ we have that $V_t^1 \leq V_t^k \leq V_t$, when $\mathrm{opt} = \sup$ (and the inequalities are reversed when $\mathrm{opt} = \inf$). Now, we have that $V_t^1 \in \mathbb{F}_t$ and by Lemma 5, the mapping V_t is also in $\mathbb{F}_t$. Moreover, by Lemma 9-(1), the sequence $(V_t^k)_{k\geq 1}$ is monotonic. Thus, by Lemma 10, we have that $(V_t^k)_{k\geq 1}$ converges uniformly on every compact set included in $\mathrm{dom}(V_t)$ towards a function $V_t^* \in \mathbb{F}_t$.

This ends the proof. $\square$

Proposition 12. *Let $t \in [\![0, T-1]\!]$ be fixed and V_{t+1}^* be the function defined in Proposition 11. The sequence $\mathcal{B}_t\left(V_{t+1}^k\right)$ $\mathbb{P}$-a.s. converges uniformly to the continuous function $\mathcal{B}_t\left(V_{t+1}^*\right)$ on every compact sets included in the domain of V_t.*

Proof. First we consider the case opt $=$ inf. As the sequence $\left(V_{t+1}^k\right)_{k\in\mathbb{N}}$ is non-increasing and using the fact that the operator $\mathcal{B}_t$ is order preserving, the sequence $(\mathcal{B}_t(V_{t+1}^k))_{k\in\mathbb{N}}$ is also non-increasing. Moreover, we have that

$$
\begin{aligned}
V_t^1 &\geq V_t^k && \text{(Lemma 9-(1))}\\
&\geq \mathcal{B}_t(V_{t+1}^k) && \text{(Lemma 9-(2))}\\
&\geq \mathcal{B}_t(V_{t+1}) && \text{(Lemma 9-(1))}\\
&= V_t .
\end{aligned}
$$

Thus, by Lemma 10, the sequence of functions $(\mathcal{B}_t(V_{t+1}^k))_{k\geq 1}$ converges uniformly on every compact set included in $\mathrm{dom}(V_t)$ to a function $\phi \in \mathbb{F}_t$. Let K_t be a given compact set included in $\mathrm{dom}(V_t)$. We now show that the function ϕ is equal to $\mathcal{B}_t\left(V_{t+1}^*\right)$ on the given compact K_t or equivalently we show that $\phi + \delta_{K_t} = \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{K_t}$. As already shown in Proposition 11, we have that $V_{t+1}^k \geq V_{t+1}^*$, which combined with the fact that the operator $\mathcal{B}_t$ is order preserving, gives, for every $k \geq 1$, that $\mathcal{B}_t(V_{t+1}^k) \geq \mathcal{B}_t(V_{t+1}^*)$. Now, adding on both side of the previous inequality the mapping δ_{K_t} and taking the limit as k goes to infinity, we have that

$$\phi + \delta_{K_t} \geq \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{K_t} .$$

For the converse inequality, start by recalling that, by the compactness condition (see Assumption 1-(i)), there exists a compact set $K_{t+1} \subset \mathrm{dom}(V_{t+1})$ such that, for every $\phi \in \mathbb{F}_{t+1}$ and every $\lambda \geq 0$, we have that

$$\mathcal{B}_t\left(\phi + \lambda + \delta_{K_{t+1}}\right) \leq \mathcal{B}_t\left(\phi + \lambda\right) + \delta_{K_t} . \tag{3.13}$$

Now, by Proposition 11, the non-increasing sequence $\left(V_{t+1}^k\right)_{k\in\mathbb{N}}$ converges uniformly to $V_{t+1}^* \in \mathbb{F}_{t+1}$ on the compact set K_{t+1}. Thus, for any fixed $\epsilon > 0$, there exists an integer $k_0 \in \mathbb{N}$, such that we have

$$V_{t+1}^k \leq V_{t+1}^k + \delta_{K_{t+1}} \leq V_{t+1}^* + \epsilon + \delta_{K_{t+1}} ,$$

for all $k \geq k_0$. By Assumption 1-(f) and Assumption 1-(g), the operator $\mathcal{B}_t$ is order preserving and additively M_t-subhomogeneous, thus we get using Equation (3.13) that

$$
\begin{aligned}
\mathcal{B}_t\left(V_{t+1}^k\right) \leq \mathcal{B}_t\left(V_{t+1}^k + \delta_{K_{t+1}}\right) &\leq \mathcal{B}_t(V_{t+1}^* + \epsilon + \delta_{K_{t+1}}), && \text{(by Assumption 1-(f))}\\
&\leq \mathcal{B}_t(V_{t+1}^* + \epsilon) + \delta_{K_t}, && \text{(by Equation (3.13))}\\
&\leq \mathcal{B}_t(V_{t+1}^*) + M_t\epsilon + \delta_{K_t} . && \text{(by Assumption 1-(g))}
\end{aligned}
$$

Adding δ_{K_t} on the left hand side, we have for every $k \geq k_0$ that $\mathcal{B}_t\left(V_{t+1}^k\right) + \delta_{K_t} \leq \mathcal{B}_t\left(V_{t+1}^*\right) + M_t\epsilon + \delta_{K_t}$. Thus, taking the limit when k goes to infinity we obtain that

$$\phi + \delta_{K_t} \leq \mathcal{B}_t\left(V_{t+1}^*\right) + M_t\epsilon + \delta_{K_t} .$$

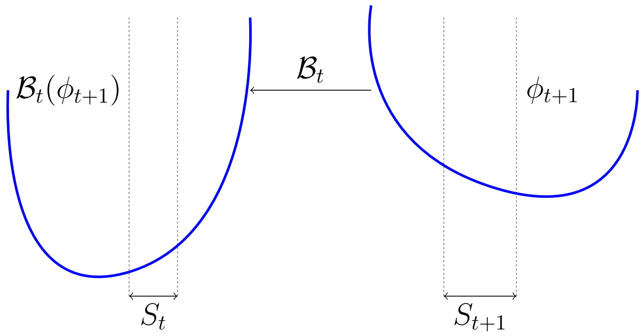

Figure 3.3: The optimality of the sets $(S_t)_{t\in[\![0,T]\!]}$ means that in order to compute the restriction of $\mathcal{B}_t(\phi_{t+1})$ on S_t, one only needs to know the values of ϕ_{t+1} on the set S_{t+1}.

The result has been proved for all $\epsilon > 0$ and we have thus shown that $\phi = \mathcal{B}_t\left(V_{t+1}^*\right)$ on the compact set K_t. We conclude that $\left(\mathcal{B}_t\left(V_{t+1}^k\right)\right)_{k\in\mathbb{N}}$ converges uniformly to the function $\mathcal{B}_t\left(V_{t+1}^*\right)$ on the compact set K_t. For the case opt $=$ sup, *mutatis mutantis* we have that $\mathcal{B}_t(V_{t+1}^k) \leq \mathcal{B}_t(V_{t+1}^*)$. Similarly, as the sequence (V_{t+1}^k) is non-decreasing and $\mathcal{B}_t$ is order preserving, one gets that for every k large enough

$$\mathcal{B}_t(V_{t+1}^*) \geq \mathcal{B}_t(V_{t+1}^*) + \delta_{K_{t+1}} \geq \mathcal{B}_t(V_{t+1}^* + \epsilon + \delta_{K_{t+1}}).$$

Thus, by Equation (3.13) and M_t-subhomogeneity we have that $\mathcal{B}_t(V_{t+1}^*) + \delta_{K_t} \leq \mathcal{B}_t(V_{t+1}^k) + M_t\epsilon + \delta_{K_t}$, which yields the result when k goes to infinity. This ends the proof. □

We want to exploit the fact that our approximations of the final cost function are exact in the sense that we have equality between V_T^k and V_T at the points drawn in Algorithm 1, that is, the tightness assumption of the selection function is much stronger at time T than for times $t < T$. Thus we want to propagate the information backward in time: starting from time $t = T$ we want to deduce information on the approximations for times $t < T$.

In order to show that $V_t = V_t^*$ on some set S_t, a dissymmetry between upper and lower approximations is emphasized. We introduce the notion of optimal sets $(S_t)_{t\in[\![0,T]\!]}$ with respect to a sequence of functions $(\phi_t)_{t\in[\![0,T]\!]}$ as a condition on the sets $(S_t)_{t\in[\![0,T]\!]}$ such that in order to compute the restriction of $\mathcal{B}_t(\phi_{t+1})$ to S_t, one only needs to know ϕ_{t+1} on the set S_{t+1}. The Figure 3.3 illustrates this notion.

Definition 13 (Optimal sets)**.** *Let $(\phi_t)_{t\in[\![0,T]\!]}$ be $T{+}1$ functions on $\mathbb{X}$. A sequence of sets $(S_t)_{t\in[\![0,T]\!]}$ is said to be* (ϕ_t)-optimal *if for every $t \in [\![0, T-1]\!]$, we have*

$$\mathcal{B}_t\left(\phi_{t+1} + \delta_{S_{t+1}}\right) + \delta_{S_t} = \mathcal{B}_t\left(\phi_{t+1}\right) + \delta_{S_t}. \tag{3.14}$$

When approximating from below, the optimality of sets is only needed for the limit functions $(V_t^*)_{t\in[\![0,T]\!]}$, whereas when approximating from above, one needs the optimality of sets with respect to the value functions $(V_t)_{t\in[\![0,T]\!]}$. It seems easier to ensure the (V_t^*)-optimality of sets than (V_t)-optimality as the function V_t^* is known through the sequence $\left(V_t^k\right)_{k\in\mathbb{N}}$, whereas the function V_t is, *a priori*, unknown. This fact is discussed in Sections 3.4 and 3.5.

Lemma 14 (Uniqueness in restricted Bellman Equations)**.** *Let $(X_t)_{t\in[\![0,T]\!]}$ be a sequence of sets such that for every $t \in [\![0, T]\!]$, $X_t \subset \mathrm{dom}(V_t)$ and which is*

– (V_t)-*optimal when* $\mathrm{opt} = \inf$,

– (V_t^*)-*optimal when* $\mathrm{opt} = \sup$.

If the sequence of functions $(V_t^*)_{t\in[\![0,T]\!]}$ *satisfies the following restricted Bellman Equations:*

$$V_T^* + \delta_{X_T} = \psi + \delta_{X_T} \quad \textit{and} \quad \forall t \in [\![0, T-1]\!],\ \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{X_t} = V_t^* + \delta_{X_t}\ . \tag{3.15}$$

Then, for every $t \in [\![0,T]\!]$ *and every* $x \in X_t$, *we have that* $V_t^*(x) = V_t(x)$.

Proof. We prove the lemma by backward induction on time $t \in [\![0,T]\!]$. We first treat the case $\mathrm{opt} = \inf$. At time $t = T$, since V_T is given by Equation (3.3), we have $V_T = \psi$. We therefore have by Equation (3.15) that $V_T^* + \delta_{X_T} = \psi + \delta_{X_T} = V_T + \delta_{X_T}$, which gives the fact that functions V_T^* and V_T coincide on the set X_T. Now, let time $t \in [\![0, T-1]\!]$ be fixed and assume that we have $V_{t+1}^*(x) = V_{t+1}(x)$ for every $x \in X_{t+1}$, or equivalently:

$$V_{t+1}^* + \delta_{X_{t+1}} = V_{t+1} + \delta_{X_{t+1}}\ . \tag{3.16}$$

Using Lemma 9-(1), the sequence of functions $(V_t^k)_{k\in\mathbb{N}}$ is lower bounded by V_t. Taking the limit in k, we obtain that $V_t^* \geq V_t$, thus we only have to prove that $V_t^* \leq V_t$ on X_t, that is $V_t^* + \delta_{X_t} \leq V_t + \delta_{X_t}$. We successively have:

$$\begin{aligned}
V_t^* + \delta_{X_t} &= \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{X_t} && \text{(by (3.15))}\\
&\leq \mathcal{B}_t\left(V_{t+1}^* + \delta_{X_{t+1}}\right) + \delta_{X_t} && \text{($\mathcal{B}_t$ is order preserving)}\\
&= \mathcal{B}_t\left(V_{t+1} + \delta_{X_{t+1}}\right) + \delta_{X_t} && \text{(by induction assumption (3.16))}\\
&= \mathcal{B}_t\left(V_{t+1}\right) + \delta_{X_t} && \text{(by (3.14), $(X_t)_{t\in[\![0,T]\!]}$ is (V_t)-optimal)}\\
&= V_t + \delta_{X_t}, && \text{(by (3.3))}
\end{aligned}$$

which concludes the proof in the case of $\mathrm{opt} = \inf$.

Now we prove the case $\mathrm{opt} = \sup$ in a similar way by backward induction on time $t \in [\![0,T]\!]$. As for the case $\mathrm{opt} = \inf$, at time $t = T$, one has $V_T^* + \delta_{X_T} = V_T + \delta_{X_T}$. Now assume that for some $t \in [\![0, T-1]\!]$ one has $V_{t+1}^* + \delta_{X_{t+1}} = V_{t+1} + \delta_{X_{t+1}}$. By Lemma 9-(1), the sequence of functions (V_t^k) is now upper bounded by V_t. Thus, taking the limit in k we obtain that $V_t^* \leq V_t$ and we only need to prove that $V_t^* + \delta_{X_t} \geq V_t + \delta_{X_t}$. We successively have:

$$\begin{aligned}
V_t + \delta_{X_t} &= \mathcal{B}_t\left(V_{t+1}\right) + \delta_{X_t} && \text{(by (3.3))}\\
&\leq \mathcal{B}_t\left(V_{t+1} + \delta_{X_{t+1}}\right) + \delta_{X_t} && \text{($\mathcal{B}_t$ is order preserving)}\\
&= \mathcal{B}_t\left(V_{t+1}^* + \delta_{X_{t+1}}\right) + \delta_{X_t} && \text{(by induction assumption (3.16))}\\
&= \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{X_t} && \text{($(X_t)_{t\in[\![0,T]\!]}$ is (V_t^*)-optimal)}\\
&= V_t^* + \delta_{X_t}, && \text{(by (3.15))}
\end{aligned}$$

This ends the proof. □

One cannot expect the limit function, V_t^*, to be equal everywhere to the value function, V_t, given by Equation (3.3). However, one can expect an (almost sure over the draws) equality between the two functions V_t and V_t^* on all possible cluster points of sequences $(y_k)_{k\in\mathbb{N}}$ with $y_k \in K_t^k$ for all $k \in \mathbb{N}$, that is, on the set $\overline{\lim} K_t^k$.

Theorem 15 (Convergence of Tropical Dynamic Programming)**.** *Define* $K_t^* := \overline{\lim}_k K_t^k$, *for every time* $t \in [\![0,T]\!]$. *Assume that,* $\mathbb{P}$*-a.s. the sets* $(K_t^*)_{t\in[\![0,T]\!]}$ *are* (V_t)*-optimal when* $\mathrm{opt} = \inf$ *(resp.* (V_t^*)*-optimal when* $\mathrm{opt} = \sup$*). Then,* $\mathbb{P}$*-a.s. for every* $t \in [\![0,T]\!]$ *the function* V_t^* *defined in Proposition 11 is equal to the value function* V_t *on* K_t^*.

Proof. We will only consider the case opt $=$ inf as the proof for the case opt $=$ sup is analogous. We will show that Equation (3.15) holds $\mathbb{P}$-almost surely with $X_t = K_t^*$, $t \in [\![0, T]\!]$. The proof is decomposed in several steps

• Reformulation using the separability of $\mathbb{X}$. Let $C := (C_t)_t \subset \mathbb{X}^{T+1}$ be compact in $\mathrm{dom}(V_0) \times \ldots \times \mathrm{dom}(V_T)$. For every $t \in [\![0, T-1]\!]$, set $\Delta_t : x_t \in \mathbb{X} \to V_t^*(x_t) - \mathcal{B}_t(V_{t+1}^*)(x_t) \in \overline{\mathbb{R}}$, $\Delta_T : x_T \in \mathbb{X} \to V_T^*(x_T) - V_T(x_T) \in \overline{\mathbb{R}}$ and $\Delta := (\Delta)_{t \in [\![0,T]\!]}$. Also write $K^* := (K_t^*)_{t \in [\![0,T]\!]}$. We want to show that

$$\mathbb{P}\Big[\forall x \in C, \big(x \in K^* \Rightarrow \Delta(x) = 0\big)\Big] = 1 \; . \tag{3.17}$$

By continuity of $V_t^* - \mathcal{B}_t(V_{t+1}^*)$ (resp. $V_T^* - V_T$) for every $t \in [\![0, T-1]\!]$ (resp. $t = T$) and compactness of K, Equation (3.17) is equivalent to

$$\mathbb{P}\Big[\forall \epsilon > 0, \exists r > 0, \forall x \in K, \big(x \in (K^* + r\mathbb{B}) \Rightarrow \Delta(x) \leq \epsilon\big)\Big] = 1 \; . \tag{3.18}$$

Without loss of generality, by density, one may restrict ϵ and r to the countable set $\mathbb{Q}_+^*$ and the set C to the set $C \cap (\mathbb{Q}^n)^{T+1}$, that is, Equation (3.18) is equivalent to

$$\forall \epsilon \in \mathbb{Q}_+^*, \exists r \in \mathbb{Q}_+^*, \forall x \in C \cap (\mathbb{Q}^n)^{T+1}, \mathbb{P}\big[x \in (K^* + r\mathbb{B}) \Rightarrow \Delta(x) \leq \epsilon\big] = 1 \; . \tag{3.19}$$

For the remainder of the proof, we fix $\epsilon \in \mathbb{Q}_+^*$. Now, we exploit the equicontinuity of the sequence of functions $(V_t^k)_{k \in \mathbb{N}}$ and $\big(\mathcal{B}_t(V_{t+1}^k)\big)_{k \in \mathbb{N}}$ in order to compute a suitable radius $r' \in \mathbb{Q}_+^*$ so as to satisfy Equation (3.19). We separate the cases $t = T$ and $t < T$.

• Equicontinuity and uniform convergence, case $\mathbf{t} = \mathbf{T}$. As the functions V_T and V_t^k, for $k \in \mathbb{N}$ are in $\mathbb{F}_T$, they share a common modulus of continuity on the compact C_T. Thus, they share a common uniform modulus of continuity. Hence, there exists a radius $r_T \in \mathbb{Q}_+^*$, such that for every $x_T \in C_T \cap \mathbb{Q}^n$, if $y_T \in B(x_T, r_T) \cap \mathrm{dom} V_T$, then

$$\big|V_T^{k+1}(x_T) - V_T^{k+1}(y_T)| \leq \frac{\epsilon}{3} \text{ and } |V_T(y_T) - V_T(x_T)| \leq \frac{\epsilon}{3} \; . \tag{3.20}$$

Now, as $(V_T^k)_{k \in \mathbb{N}}$ converges uniformly to V_T^* on the compact $C_T \subset \mathrm{dom} V_T$, there exists a rank $k_T \in \mathbb{N}$ such that, if $k \geq k_T$, then for all $x_T \in K_T$,

$$|V_T^{k+1}(x_T) - V_T^*(x_T)| \leq \frac{\epsilon}{3}. \tag{3.21}$$

• Equicontinuity and uniform convergence, case $\mathbf{t} \in [\![\mathbf{0}, \mathbf{T} - \mathbf{1}]\!]$. The sequences $(V_t^k)_{k \in \mathbb{N}}$, resp. $(\mathcal{B}_t(V_{t+1}^k))_{k \in \mathbb{N}}$ are uniformly equicontinuous on the compact $C_t \subset \mathrm{dom}(V_t)$. There exists a radius $r_t \in \mathbb{Q}_+^*$ such that for every $x_t \in C_t$, if $y_t \in B(x_t, r_t) \cap \mathrm{dom} V_t$, then for every $k \in \mathbb{N}$,

$$|V_t^{k+1}(x_t) - V_t^{k+1}(y_t)| \leq \frac{\epsilon}{4} \quad \text{and} \quad |\mathcal{B}_t(V_{t+1}^{k+1}(y_t)) - \mathcal{B}_t(V_{t+1}^{k+1}(x_t))| \leq \frac{\epsilon}{4} \; . \tag{3.22}$$

By uniform convergence of the sequence $(V_t^k)_{k \in \mathbb{N}}$ (resp. $\mathcal{B}_t(V_{t+1}^k)_{k \in \mathbb{N}}$) to V_t^* (resp. to $\mathcal{B}_t(V_{t+1}^*)$) on the compact $C_t \subset \mathrm{dom}(V_t)$, there exists a rank $k_t \in \mathbb{N}$ such that, if $k \geq k_t$, then for every $x_t \in K$

$$|V_t^*(x_t) - V_t^{k+1}(x_t)| \leq \frac{\epsilon}{4} \text{ and } |\mathcal{B}_t(V_{t+1}^{k+1})(x_t) - \mathcal{B}_t(V_{t+1}^*)(x_t)| \leq \frac{\epsilon}{4} \tag{3.23}$$

• There exists a draw $x_t^{k^*}$ of the sequence of trial points $(x_t^k)_{k\in\mathbb{N}}$ arbitrarily close to any given point of K^*. Throughout the remainder of the proof, we fix ranks $k_t \in \mathbb{N}$ and radii $r_t \in \mathbb{Q}_+^*$ defined in Step 2 and set

$$\overline{k} := \max_{t\in[\![0,T]\!]} k_t \in \mathbb{N} \text{ and } \underline{r} := \min_{t\in[\![0,T]\!]} r_t \ .$$

By the Trial point oracle assumption, there exists $r \in \mathbb{Q}_+^*$ such that, for every $x \in C$,

$$\mathbb{P}[x \in K^* + r\mathbb{B} \Rightarrow x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, \underline{r}/2)] = 1 \ . \tag{3.24}$$

Now, fix $x \in C \cap (\mathbb{Q}^n)^{T+1}$. By Equation (3.24), $\mathbb{P}$-a.s., if $x \in K^* + r\mathbb{B}$ then $x \in \overline{\lim}_{k\in\mathbb{N}} B(x^k, \underline{r}/2)$, so by Lemma 8, $(x^k)_k \in B(x, \underline{r})$ infinitely often. Hence, $\mathbb{P}$-a.s., if $x \in K^* + r\mathbb{B}$, then there exists $k^* \geq \overline{k}$ such that

$$x^{k^*} \in B(x, \underline{r}). \tag{3.25}$$

• Conclusion. When $t = T$, by triangle inequality, $\mathbb{P}$-a.s. we have that

$$\begin{aligned}
\Delta(x_T) \leq & \underbrace{|V_T^*(x_T) - V_T^{k^*+1}(x_T)|}_{\leq\epsilon/3 \text{ by } (3.21) \text{ and } (3.25)} + \underbrace{|V_T^{k^*+1}(x_T) - V_T^{k^*+1}(x_T^{k^*})|}_{\leq\epsilon/3 \text{ by } (3.20)} \\
& + \underbrace{|V_T^{k^*+1}(x_T^{k^*}) - V_T(x_T^{k^*})|}_{=0 \text{ by Tightness Lemma } 9\text{-}(4)} + \underbrace{|V_T(x_T^{k^*}) - V_T(x_T)|}_{\leq\epsilon/3 \text{ by } (3.21) \text{ and } (3.25)} \\
\leq & \ \epsilon \ .
\end{aligned}$$

When $t \in [\![0, T-1]\!]$, by triangle inequality, $\mathbb{P}$-a.s. we have that

$$\begin{aligned}
\Delta(x_t) \leq & \underbrace{|V_t^*(x_t) - V_t^{k+1}(x_t)|}_{\leq\epsilon/4 \text{ by } (3.23)} + \underbrace{|V_t^{k+1}(x_t) - V_t^{k+1}(x_t^k)|}_{\leq\epsilon/4 \text{ by } (3.22) \text{ and } (3.25)} \\
& + \underbrace{|V_t^{k+1}(x_t^k) - \mathcal{B}_t(V_{t+1}^{k+1})(x_t^k)|}_{= 0 \text{ by Lemma } 9\text{-}(3)} \\
& + \underbrace{|\mathcal{B}_t(V_{t+1}^{k+1})(x_t^k) - \mathcal{B}_t(V_{t+1}^{k+1})(x_t)|}_{\leq\epsilon/4 \text{ by } (3.22) \text{ and } (3.25)} + \underbrace{|\mathcal{B}_t(V_{t+1}^{k+1})(x_t) - \mathcal{B}_t(V_{t+1}^*)(x_t)|}_{\leq\epsilon/4 \text{ by } (3.23)} \\
\leq & \ \epsilon \ .
\end{aligned}$$

Thus, we have shown Equation (3.19), *i.e.* $\mathbb{P}$-a.s., for every $t \in [\![0, T]\!]$ we have $V_t^* = \mathcal{B}_t(V_{t+1}^*)$ on K_t^*. The sequence $(V_t^*)_{k\in\mathbb{N}}$ satisfies the restricted Bellman Equation (3.15) with the sequence $(K_t^*)_{k\in\mathbb{N}}$. The conclusion follows from the Uniqueness lemma (Lemma 14). □

3.4 SDDP selection function: lower approximations in the linear-convex framework

We will show that our setting contains a similar framework of (the deterministic version of) the SDDP algorithm as described in [GLP15] and yields the same result of convergence. Let $\mathbb{X} = \mathbb{R}^n$ be a continuous state space and $\mathbb{U} = \mathbb{R}^m$ a continuous control space. We want to solve the following problem

$$
\begin{aligned}
\min_{\substack{x=(x_0,\dots,x_T)\\ u=(u_0,\dots u_{T-1})}} \ & \sum_{t=0}^{T-1} c_t(x_t,u_t) + \psi(x_T) \\
\text{s.t. } & x_0 \in \mathbb{X} \text{ is given}, \\
& \forall t \in [\![0,T]\!],\ x_t \in \mathbb{X}, \\
& \forall t \in [\![0,T-1]\!],\ u_t \in \mathbb{U},\ x_{t+1} = f_t(x_t,u_t).
\end{aligned}
\tag{3.26}
$$

We make similar assumptions as in the literature of SDDP (*e.g.* [GLP15]), note that in our formulation, we have put the constraints on the states and controls on the cost functions. We refer to [AE84] and [RW09] for results on set-valued mappings.

Assumption 3. *For all $t \in [\![0,T-1]\!]$ we assume that:*

1. *The dynamic $f_t : \mathbb{X}\times\mathbb{U} \longrightarrow \mathbb{X}$ is linear, $f_t(x,u) = A_t x + B_t u$, for some given matrices A_t and B_t of compatible dimensions.*
2. *The cost function $c_t : \mathbb{X}\times\mathbb{U} \longrightarrow \overline{\mathbb{R}}$ is a proper lower semicontinuous (l.s.c.) convex function which is L_{c_t}-Lipschitz continuous on its (convex) domain, $\mathrm{dom}(c_t)$.*
3. *The projection on $\mathbb{X}$ of $\mathrm{dom}(c_t)$, denoted X_t, is a convex polytope with non-empty interior.*
4. *Define the set-valued mapping $U_t : \mathbb{X} \rightrightarrows \mathbb{U}$, for every $x \in \mathbb{X}$*

$$
U_t(x) := \left\{ u \in \mathbb{U} \mid (x,u) \in \mathrm{dom}(c_t) \ \textit{and}\ f_t(x,u) \in X_{t+1} \right\},
$$

where we assume that

- *For every $x \in X_t$, $U_t(x)$ is compact.*
- *The graph of the set-valued mapping U_t has a non-empty interior.*
- *For every $x \in X_t$, there exists $u \in U_t(x)$.*[3]
- *The set-valued mapping U_t is L_{U_t}-*Lipschitz continuous[4] *(hence, both upper and lower semicontinuous).*

Moreover, at time $t = T$, we assume that $X_T := \mathrm{dom}(V_T) \subset \mathbb{X}$ is convex and compact with non-empty interior, the final cost function $\psi : \mathbb{X} \longrightarrow \overline{\mathbb{R}}$ is a proper convex l.s.c. function with known compact convex domain and ψ is C_T-Lipschitz continuous on its domain.

Remark 16. *Under Assumption 3, the graph of the set-valued mapping U_t is convex, and its domain is X_t.*

Remark 17. *A sufficient condition to ensure that the set-valued mapping U_t is Lipschitz continuous is given in [RW09, Example 9.35]: U_t is Lipschitz when its graph is convex polyhedral, which is the classical framework of SDDP. Moreover a Lipschitz constant can be explicitly computed.*

[3] known as a *Relatively Complete Recourse* assumption.

[4] For all $x, x' \in \mathbb{X}$, $U_t(x') \subset U_t(x) + L_{U_t}\|x'-x\|$.

For every time step $t \in [\![0, T-1]\!]$, recall the *Bellman operator* $\mathcal{B}_t$ for every function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ by:

$$\mathcal{B}_t(\phi) := \inf_{u \in \mathbb{U}} \Big(c_t(\cdot, u) + \phi\big(f_t(\cdot, u)\big) \Big) . \tag{3.27}$$

Moreover, for every function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ and every $(x, u) \in \mathbb{X} \times \mathbb{U}$ we define

$$\mathcal{B}_t^u (\phi) (x) := c_t (x, u) + \phi\big(f_t(x, u)\big) \in \overline{\mathbb{R}} . \tag{3.28}$$

The Bellman equations of Problem (3.26) can be written using the Bellman operators $\mathcal{B}_t$ given by Equation (3.27):

$$V_T = \psi \quad \text{and} \quad \forall t \in [\![0, T-1]\!], V_t : x \in \mathbb{X} \mapsto \mathcal{B}_t(V_{t+1})(x) \in \overline{\mathbb{R}} . \tag{3.29}$$

In Proposition 18, we establish a stability property of the Bellman operators given by Equation (3.27). The image of a Lipschitz continuous function by the operator $\mathcal{B}_t$ will also be Lipschitz continuous and we give an explicit (conservative) Lipschitz constant.

Proposition 18. *Under Assumption 3, for every $t \in [\![0, T-1]\!]$, given a constant $L_{t+1} > 0$, there exist a constant $L_t > 0$ such that if $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ is convex l.s.c. proper with domain X_{t+1} and L_{t+1}-Lipschitz continuous on X_{t+1} then $\mathcal{B}_t(\phi)$ is convex l.s.c. proper with domain X_t and L_t-Lipschitz continuous on X_t.*

Proof of Proposition 18. Fix $t \in [\![0, T-1]\!]$ and let $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ be a convex l.s.c. proper function with domain X_{t+1} and L_{t+1}-Lipschitz continuous function on X_{t+1}. We show that $\mathrm{dom}\big(\mathcal{B}_t(\phi)\big) = X_t$. Let $x_t \in X_t$ be arbitrary. By the RCR Assumption, there exist $u_t \in U_t(x_t)$ such that $f_t(x_t, u_t) \in X_{t+1}$ and $(x_t, u_t) \in \mathrm{dom}(c_t)$. As the domain of ϕ is X_{t+1}, we have that

$$\inf_{u \in \mathbb{U}} \Big(c_t(x_t, u) + \phi\big(f_t(x_t, u)\big) \Big) \leq c_t(x_t, u_t) + \phi\big(f_t(x_t, u_t)\big) < +\infty .$$

Thus, we have shown that $\mathrm{dom}\big(\mathcal{B}_t(\phi)\big)$ includes X_t. Conversely, if $x \notin X_t$, then for every $u \in \mathbb{U}$, we have $c_t(x, u) = +\infty$, hence $x \notin \mathrm{dom}(\mathcal{B}_t(\phi))$. This implies that $\mathrm{dom}\big(\mathcal{B}_t(\phi)\big) \subset X_t$ and the equality follows.

Moreover, the above infimum can be restricted to $U_t(x)$, which is compact. As the function $x \mapsto \mathcal{B}_t(\phi)(x)$ is convex (resp. l.s.c.) on X_t as $(x, u) \mapsto c_t(x, u) + \phi(f_t(x, u))$ is jointly convex (resp. l.s.c. and $U_t(x)$ is compact).

Since $c_t(x, \cdot)$, ϕ are l.s.c. and $f_t(x, \cdot)$ is continuous, the above infimum is attained. We will denote by $u_x \in U_t(x)$ a minimizer, note that $f_t(x, u_x) \in X_{t+1}$.

We finally show that the function $\mathcal{B}_t(\phi)$ is Lipschitz on X_t with a constant $L_t > 0$ that only depends on the data of Problem (3.26). Fix $x, x' \in X_t$ and denote by $u_{x'} \in U_t(x')$ an optimal control at x', *i.e.* $\mathcal{B}_t^{u_{x'}}(\phi)(x') = \mathcal{B}_t(\phi)(x')$. For every $u \in U_t(x)$, we have that

$$\begin{aligned}
\mathcal{B}_t(\phi)(x) &\leq \mathcal{B}_t(\phi)(x') + \mathcal{B}_t^u(\phi)(x) - \mathcal{B}_t(\phi)(x') \\
&= \mathcal{B}_t(\phi)(x') + \big(c_t(x, u) - c_t(x', u_{x'})\big) + \Big(\phi\big(f_t(x, u)\big) - \phi\big(f_t(x', u_{x'})\big)\Big) \\
&\leq \mathcal{B}_t(\phi)(x') + L_{c_t}(\|x - x'\| + \|u - u_{x'}\|) \\
&\quad + L_{t+1}\Big(\lambda_{\max}(A_t^T A_t)^{1/2}\|x - x'\| + \lambda_{\max}(B_t^T B_t)^{1/2}\|u - u_{x'}\|\Big).
\end{aligned} \tag{3.30}$$

Indeed, as the domain of $c_t(x, \cdot)$ is $U_t(x)$, the domain of ϕ is X_{t+1} and that for every $u \in U_t(x)$, we have $f_t(x, u) \in X_{t+1}$, Equation (3.30) holds for every $u \in U_t(x)$.

Now, we will bound from above $\|u - u_{x'}\|$ by $\|x - x'\|$ multiplied by a constant. By Assumption 3-(4) the set-valued mapping U_t is L_{U_t}-Lipschitz on its domain X_t. Hence, by definition, there exists $\tilde{u} \in U_t(x)$ such that:

$$\|\tilde{u} - u_{x'}\| \leq L_{U_t}\|x - x'\|. \tag{3.31}$$

Replacing u by $\tilde{u}$ in Equation (3.30), by Equation (3.31) we deduce that $\mathcal{B}_t(\phi)(x) - \mathcal{B}_t(\phi)(x') \leq L_t\|x - x'\|$, where the Lipschitz constant $L_t > 0$ only depends on the data of Problem (3.26). *Mutatis mutandis*, we have that $\mathcal{B}_t(\phi)(x') - \mathcal{B}_t(\phi)(x) \leq L_t\|x - x'\|$, and the result follows. □

Remark 19. *Knowing the value function at time $t = T$, by Proposition 18 we can compute recursively backward in time the domain of V_t for each $t < T$: it is equal to the projection on $\mathbb{X}$ of the domain of the cost function, which is X_t and known to the decision maker. Moreover using Proposition 18 we have that, for every t, the value function V_t is convex l.s.c. proper and Lipschitz continuous on its domain, with a computable constant.*

As lower semicontinuous proper convex functions can be approximated by a supremum of affine function, for every $t \in [\![0, T]\!]$ we define $\mathbf{F}_t^{\mathrm{SDDP}}$ to be the set of affine functions $\phi : x \in \mathbb{X} \mapsto \langle a, x\rangle + b \in \mathbb{R}$, $a \in \mathbb{X}$, $b \in \mathbb{R}$ with $\|a\|_2 \leq L_t$ if $x \in X_t$ and $+\infty$ otherwise. Moreover, we shall denote by $\mathbb{F}_t^{\mathrm{SDDP}}$ the set of convex functions $\phi : \mathbb{X} \mapsto \overline{\mathbb{R}}$ which are L_t-Lipschitz continuous on X_t, of domain X_t and proper.

Proposition 20. *Under Assumption 3, the Problem 3.26 and the Bellman operators defined in Equation* (3.29) *satisfy the structural assumptions given in Assumption 1.*

Proof. We prove successively each assumption listed in Assumption 1.
•1-(a). Recall that we are here on the case opt = sup. Fix $t \in [\![0, T]\!]$ and let $F \subset \mathbf{F}_t^{\mathrm{SDDP}}$ be a set of affine L_t-Lipschitz continuous functions with domain X_t. For every $x, x' \in X_t$, we have that

$$|\mathcal{V}_F(x) - \mathcal{V}_F\left(x'\right)| = |\sup_{\phi \in F} \phi(x) - \sup_{\phi \in F} \phi(x')| \leq \sup_{\phi \in F} |\phi(x) - \phi(x')| \leq L_t\|x - x'\|.$$

Thus, the function $\mathcal{V}_F$ is L_t-Lipschitz continuous. As a supremum of affine functions is convex and l.s.c., $\mathcal{V}_F$ is also convex and l.s.c., we have thus shown that $\mathcal{V}_F \in \mathbb{F}_t^{\mathrm{SDDP}}$.
•1-(b) and 1-(c). By construction, for all $t \in [\![0, T]\!]$, every element of $\mathbb{F}_t^{\mathrm{SDDP}}$ is L_t-Lipschitz continuous. Thus, by the previous point, $\mathbb{F}_t^{\mathrm{SDDP}}$ is also stable by pointwise convergence.
•1-(d). As ψ is convex proper and L_T-Lipschitz continuous on X_T, it is a countable (as $\mathbb{R}^n$ is separable) supremum of L_T-Lipschitz affine functions.
•1-(e). This has been shown in Proposition 18.
•1-(f). Let ϕ_1 and ϕ_2 be two functions over $\mathbb{X}$ such that $\phi_1 \leq \phi_2$ *i.e.* for every $x \in \mathbb{X}$, we have $\phi_1(x) \leq \phi_2(x)$. We want to show that $\mathcal{B}_t(\phi_1) \leq \mathcal{B}_t(\phi_2)$. Let $x \in \mathbb{X}$, we have:

$$\begin{aligned}
\mathcal{B}_t(\phi_1)(x) &= \inf_{u \in \mathbb{U}} c_t(x, u) + \phi_1(f_t(x, u)) \\
&\leq \inf_{u \in \mathbb{U}} c_t(x, u) + \phi_2(f_t(x, u)) \\
&= \mathcal{B}_t(\phi_2)(x).
\end{aligned}$$

•1-(g). We will show that $\mathcal{B}_t$ is additively homogeneous, hence one can choose $M_t = 1$ in Assumption 1-(g). Let $\lambda \in \mathbb{R}$ be a given constant and ϕ a given function in $\mathbb{F}_{t+1}$. We identify the constant λ with the constant function $\lambda : x \mapsto \lambda$ and we have for all $x \in \mathbb{X}$:

$$\begin{aligned}\mathcal{B}_t\left(\lambda+\phi\right)(x) &= \inf_{u\in\mathbb{U}}\Big(c_t(x,u)+(\lambda+\phi)\big(f_t(x,u)\big)\Big)\\ &= \inf_{u\in\mathbb{U}}\Big(c_t(x,u)+\lambda+\phi\big(f_t(x,u)\big)\Big)\\ &= \lambda+\inf_{u\in\mathbb{U}}\Big(c_t(x,u)+\phi\big(f_t(x,u)\big)\Big)\\ &= \lambda+\mathcal{B}_t(\phi)(x).\end{aligned}$$

• 1-(h). By backward recursion on time step $t \in [\![0, T]\!]$ and by Proposition 18, for every time step $t \in [\![0, T]\!]$ the function V_t given by the Dynamic Programming Equation (3.29) is convex and L_t-Lipschitz continuous on X_t.

•1-(i). Fix $t \in [\![0, T-1]\!]$, an arbitrary element $\phi \in \mathbb{F}_t^{\mathrm{SDDP}}$, a constant $\lambda \geq 0$ and set $\tilde{\phi} := \phi+\lambda$. We will show that for every compact set $K_t \subset X_t$, there exist a compact set $K_{t+1} \subset X_{t+1}$ such that

$$\mathcal{B}_t\left(\tilde{\phi}+\delta_{K_{t+1}}\right)+\delta_{K_t} = \mathcal{B}_t\left(\tilde{\phi}\right)+\delta_{K_t}, \tag{3.32}$$

which will imply the desired result. Now, Equation (3.32) is equivalent to the fact that for every state $x_t \in K_t$, there exist a control $u_t \in U_t(x)$ such that

$$f_t(x_t,u_t) \in K_{t+1} \quad \text{where} \quad u_t \in \operatorname*{arg\,min}_{u\in U_t(x)} \mathcal{B}_t^u(\tilde{\phi})(x_t) = c_t(x_t,u)+\tilde{\phi}\big(f_t(x_t,u)\big)\ .$$

Set $K_{t+1} := f_t(X_t, U_t(X_t))$, it satisfies Equation (3.32), we show that it is compact. As X_t is compact and f_t is continuous, it is sufficient to prove that $U_t(X_t)$ is compact, which is true as U_t is upper semicontinuous (u.s.c.) and non-empty compact valued, see [AE84, Proposition 11 p.112]. This ends the proof □

Now, we define a compatible selection function for opt = sup. Let $t \in [\![0, T-1]\!]$ be fixed, for any $F \subset \mathbf{F}_t^{\mathrm{SDDP}}$ and $x \in \mathbb{X}$, we define the following optimization problem

$$\min_{(x',u,\lambda)\in X_t\times U_t(x')\times\mathbb{R}} c_t(x',u)+\lambda \tag{3.33a}$$

$$\text{s.t.} \quad x'=x \quad \text{and} \quad \phi\big(f_t(x',u)\big)\leq\lambda \quad \forall\phi\in F\ . \tag{3.33b}$$

If we denote by b its optimal value and by a a Lagrange multiplier associated to the constraint $x'-x=0$ at the optimum, that is such that $(x', u; \lambda, a)$ is a stationary point of the Lagrangian $c_t(x', u)+\lambda-\langle a, x'-x\rangle$, then we define

$$\phi_t^{\mathrm{SDDP}}(F,x) := x' \mapsto \langle a, x'-x\rangle + b + \delta_{X_t}(x')\ .$$

Finally, at time $t = T$, for any $F \subset \mathbf{F}_T^{\mathrm{SDDP}}$ and $x \in \mathbb{X}$, fix $a \in \partial V_T(x)$ and define

$$\phi_T^{\mathrm{SDDP}}(F,x) := x' \mapsto \langle a, x'-x\rangle + V_T(x)\,.$$

Proposition 21. *For every time $t \in [\![0, T]\!]$, the function ϕ_t^{SDDP} is a compatible selection function for* opt = sup *in the sense of Definition 6.*

Proof. Fix $t \in [\![0, T-1]\!]$, $F \subset \mathbf{F}^{\text{SDDP}}_{t+1}$ and $x \in \mathbb{X}$. Using Equation (3.27) we obtain that $\mathcal{B}_t(\mathcal{V}_F)(x)$ is equal to b the optimal value of optimization problem (3.33a). Thus, since $\phi^{\text{SDDP}}_t(F, x)(x) = b$ we obtain that the selection function is tight. It is also valid as a is a subgradient of the convex function $\mathcal{B}_t(V_F)$ at x. For $t = T$, the selection function ϕ^{SDDP}_T is tight and valid by convexity of V_T. □

If we want to apply the convergence result from Theorem 15, as we approximate from below the value functions (opt = sup) then one has to make the draws according to some sets K^k_t such that the sets $K^*_t := \overline{\lim}_{k\in\mathbb{N}} K^k_t$ are V^*_t optimal. As done in the literature of the Stochastic Dual Dynamic Programming algorithm (see for example [GLP15] and [ZAS18] or [PP91]), one can study the case when the draws are made along the optimal trajectories of the current approximations.

More precisely, fix $k \in \mathbb{N}$ we define a sequence $(x^k_0, x^k_1, \dots, x^k_T)$ by

$$x^k_0 := x_0 \quad \text{and} \quad \forall t \in [\![0, T-1]\!],\ x^k_{t+1} := f_t(x^k_t, u^k_t)\ ,$$

where $u^k_t \in \arg\min_u \mathcal{B}^u_t\left(V^k_t\right)\left(x^k_t\right)$. We say that such a sequence $(x^k_0, x^k_1, \dots, x^k_T)$ is an *optimal trajectory for the k-th approximations starting from x_0*. We show that optimal trajectories for the current approximations become (V^*_t)-optimal when k goes to infinity, using a result of convergence in minimization by Rockafellar and Wets [RW09, Theorem 7.33].

Proposition 22. *For every $k \in \mathbb{N}$, let $(x^k_0, x^k_1, \dots, x^k_T)$ be an optimal trajectory for the k-th approximations starting from x_0 and define a sequence of singletons for every $t \in [\![0, T]\!]$, $K^k_t := \{x^k_t\}$. Then the sets $(K^*_t)_{t\in[\![0,T]\!]}$ defined by $K^*_t := \overline{\lim}_k K^k_t$ are (V^*_t)-optimal.*

Proof. Fix $t \in [\![0, T-1]\!]$, we want to show that Equation (3.14) is satisfied for K^*_t which is equivalent to prove that for every $x^*_t \in K^*_t$, we have that

$$\mathcal{B}_t\left(V^*_{t+1} + \delta_{K^*_{t+1}}\right)(x^*_t) = \mathcal{B}_t\left(V^*_{t+1}\right)(x^*_t). \tag{3.34}$$

Now, using the definition of the Bellman operators in Equation (3.27) and Equation (3.34) we have to prove that there exists a control $u^*_t \in U_t(x^*_t)$ such that

$$u^*_t \in \left\{u \,\middle|\, f_t(x^*_t, u) \in K^*_{t+1}\right\} \cap \operatorname*{arg\,min}_{u\in U_t(x^*_t)} \Big(c_t(x^*_t, u) + V^*_{t+1}\big(f_t(x^*_t, u)\big)\Big)\ . \tag{3.35}$$

Fix $x^*_t \in K^*_t$ and extracting if needed a subsequence, without loss of generality, assume that $(x^k_t)_{k\in\mathbb{N}}$ converges to x^*_t. Fix $k \in \mathbb{N}$ and the sequence of controls $(u^k_0, \dots, u^k_{T-1})$ associated with the optimal trajectory for the k-th approximations $(x^k_0, \dots, x^k_T)$. We have that

$$u^k_t \in \left\{u \,\middle|\, f_t(x^k_t, u) \in K^k_{t+1}\right\} \cap \operatorname*{arg\,min}_{u\in U_t(x^k_t)} \Big(c_t(x^k_t, u) + V^k_{t+1}\big(f_t(x^k_t, u)\big)\Big)\ . \tag{3.36}$$

Extracting, if needed, a subsequence $(u^n_t)_{n\in\mathbb{N}}$ of $(u^k_t)_{n\in\mathbb{N}}$, we will show that the sequence $(u^n_t)_{n\in\mathbb{N}}$ converges to some $u^*_t \in \arg\min_{u\in U_t(x^*_t)} c_t(x^*_t, u) + V^*_{t+1}(f_t(x^*_t, u))$. Equation (3.35) will be satisfied as for every $n \in \mathbb{N}$, $f_t(x^n_t, u^n_t) \in K^n_{t+1}$, the continuity of f_t and definition of K^*_{t+1} will ensure that $u^*_t \in \left\{u \mid f_t(x^*_t, u) \in K^*_{t+1}\right\}$.

We will use the result of convergence in minimization [RW09, Theorem 7.33]. We define

$$B^k : \mathbb{U} \to \overline{\mathbb{R}},\ u \mapsto c_t(x^k_t, u) + V^k_{t+1}\big(f_t(x^k_t, u)\big)\ ,$$
$$B^* : \mathbb{U} \to \overline{\mathbb{R}},\ u \mapsto c_t(x^*_t, u) + V^*_{t+1}\big(f_t(x^*_t, u)\big)\ .$$

Recall that, under Assumption 3-(4), the set-valued mapping U_t has compact values with non-empty interior and is L_{U_t}-Lipschitz continuous for some constant $L_F > 0$. Moreover, the functions B^* and every B^k, $k \in \mathbb{N}$ are convex, l.s.c., proper, inf-compact, with compact domains $U_t(x_t^*)$ and $U_t(x_t^k)$, respectively. As U_t is Lipschitz continuous, the sequence of functions $(B^k)_{k\in\mathbb{N}}$ converges uniformly to B^* on every compact K included in the interior of $\mathrm{dom}(B^*) = U_t(x_t^*)$. Thus, by [RW09, Theorem 7.17.c], $(B_t^k)_{k\in\mathbb{N}}$ epiconverges to B^*. Finally, $(u_t^k)_{k\in\mathbb{N}} \subset f_t(X_t, U_t(X_t))$ which is compact as U_t is u.s.c. and f_t is continuous. We conclude that we can extract a converging subsequence out of $(u_t^k)_k$. Denoting by $u_t^* \in U_t(x_t^*)$ its limit, by [RW09, Theorem 7.33] we finally have that $u_t^* \in \arg\min_{u\in\mathbb{U}} B^*(u)$. This ends the proof. □

Hence, when applying TDP with the SDDP selection function, we will refine the approximations along the current optimal trajectories, *i.e.* we use Oracle defined in Example 3. We conclude this section by proving the convergence of TDP algorithm in the SDDP case.

Theorem 23 (Lower (outer) approximations of the value functions)**.** *Under Assumption 3, for every $t \in [\![0, T]\!]$, denote by $\left(V_t^k\right)_{k\in\mathbb{N}}$ the sequence of functions generated by Tropical Dynamic Programming with the selection function ϕ_t^{SDDP} and the draws made uniformly over the sets K_t^k defined in Proposition 22. Then, the sequence $\left(V_t^k\right)_{k\in\mathbb{N}}$ is non-decreasing, bounded from above by V_t, and converges uniformly to V_t^* on every compact set included in $\mathrm{dom}\,(V_t)$. Moreover, almost surely over the draws, $V_t^* = V_t$ on $\overline{\lim}_{k\in\mathbb{N}} K_t^k$.*

Proof. As the structural assumptions Assumption 1 are satisfied, as the functions ϕ_t^{SDDP}, $0 \leq t \leq T$, are compatible selections and the sets $(K_t^*)_{t\in[\![0,T]\!]}$ are (V_t^*)-optimal (case opt = sup) by Theorem 15, we have the result. □

3.5 A min-plus selection function: upper approximations in the linear-quadratic framework with both continuous and discrete controls

In §3.5.1, we study the case where the cost functions and dynamics are homogeneous. We conclude this section in §3.5.2 with an example which shows that using optimal trajectories of the best current approximations as trial points in Tropical Dynamic Programming may generate functions $(V_t^k)_{k\in\mathbb{N}}$ which do not converge to the value function V_t. In the appendix 3.9, we show how one can use the homogeneous case to solve the non-homogeneous case by augmenting the state dimension by one.

3.5.1 The pure homogeneous case

We will denote by $\mathbb{M}_n$ the set of $n\times n$ real matrices and by $\mathbb{S}_n \subset \mathbb{M}_n$ the subset of symmetric matrices.

Definition 24 (Pure quadratic form)**.** *We say that a function $q : \mathbb{X} \to \mathbb{R}$ is a pure quadratic form if there exist a symmetric real matrix $M \in \mathbb{S}_n$ such that for every $x \in \mathbb{X}$, we have $q(x) = x^T M x$.*

Similarly, a function $q : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is a pure quadratic form if there exist two symmetric real matrices $M_1 \in \mathbb{S}_n$ and $M_2 \in \mathbb{S}_m$ such that for every $x \in \mathbb{X}$, we have $q(x, u) = x^T M_1 x + u^T M_2 u$.

Let us insist that pure quadratic forms are not general 2-homogeneous quadratic forms in the sense that they lack a mixing term of the form $x^T Mu$. In 3.9 we show why we do not lose generality by studying this case instead of general polynomials of degree 2. Let $\mathbb{X} = \mathbb{R}^n$ be a continuous state space (endowed with its Euclidean and Borel structure), $\mathbb{U} = \mathbb{R}^m$ a continuous control space and $\mathbb{V}$ a finite set of discrete (or switching) controls. We want to solve the following optimization problem

$$\min_{(x,u,v)\in\mathbb{X}^{T+1}\times\mathbb{U}^T\times\mathbb{V}^T} \sum_{t=0}^{T-1} c_t^{v_t}(x_t,u_t) + \psi(x_T) \tag{3.37a}$$

$$\text{s.t.} \quad x_0 \in \mathbb{X} \text{ given, and} \quad \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t^{v_t}(x_t,u_t)\ . \tag{3.37b}$$

Assumption 4. *Let $t \in [\![0, T-1]\!]$ and $v \in \mathbb{V}$ be arbitrary.*
– The dynamic $f_t^v : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{X}$ is linear. That is, $f_t^v(x,u) = A_t^v x + B_t^v u$, for some given matrices A_t^v and B_t^v of compatible dimensions.
– The cost function $c_t^v : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{R}$ is a pure convex quadratic form, $c_t^v(x,u) = x^T C_t^v x + u^T D_t^v u$, where the matrix C_t^v is symmetric semidefinite positive and the matrix D_t^v is symmetric definite positive.
– The final cost function $\psi := \inf_{i\in I_T} \psi_i$ is a finite infimum of pure convex quadratic form, of matrix $M_i \in \mathbb{S}_n$ with $i \in I_T$ a finite set, such that there exists a constant $\alpha_T \geq 0$ satisfying for every $i \in I_T$ $\quad 0 \preceq M_i \preceq \alpha_T \operatorname{Id}$.

One can write the Dynamic Programming equation for Problem 3.37 as follows

$$V_T = \psi \quad \text{ and } \forall t \in [\![0, T-1]\!], \forall x \in \mathbb{X}, V_t(x) = \inf_{v\in\mathbb{V}} \inf_{u\in\mathbb{U}} c_t^v(x,u) + V_{t+1}\big(f_t^v(x,u)\big)\ . \tag{3.38}$$

The following result is crucial in order to study this example: the value functions are 2-homogeneous, allowing us to restrict their study to the unit sphere.

Proposition 25. *For every time step $t \in [\![0, T]\!]$, the value function V_t, solution of Equation (3.38) is 2-homogeneous, that is, for every $x \in \mathbb{X}$ and every $\lambda \in \mathbb{R}$, we have $V_t(\lambda x) = \lambda^2 V_t(x)$.*

Proof. We proceed by backward recursion on time step $t \in [\![0, T]\!]$. For $t = T$ it is true by Assumption 4. Assume that it is true for some $t \in [\![1, T]\!]$. Fix $\lambda \in \mathbb{R}$, then by definition of V_{t-1}, for every $x \in \mathbb{X}$, we have

$$\begin{aligned} V_{t-1}(\lambda x) &= \min_{v\in\mathbb{V}} \min_{u\in U} c_{t-1}^v(\lambda x, u) + V_t\big(f_{t-1}^v(\lambda x, u)\big) \\ &= \min_{v\in\mathbb{V}} \min_{u'=u/\lambda\in U} c_{t-1}^v(\lambda x, \lambda u') + V_t\big(f_{t-1}^v(\lambda x, \lambda u')\big)\ , \end{aligned}$$

which yields the result by 2-homogeneity of $x \mapsto c_{t-1}^v(x,u)$, linearity of f_{t-1}^v and 2-homogeneity of V_t. $\square$

Thus, in order to compute V_t, one only needs to know its values on the unit (Euclidean) sphere S as for every non-zero $x \in \mathbb{X}$, $V_t(x) = \|x\|^2\ V_t\big(\frac{x}{\|x\|}\big)$. Hence, we will refine our approximations only on the sphere, that is we will draw trial points uniformly on the sphere and use the Oracle defined in Example 2. Now, for every time $t \in [\![0, T-1]\!]$ and every switching control $v \in \mathbb{V}$ we define the *Bellman operator with fixed switching control* $\mathcal{B}_t^v$ for every function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ by:

$$\mathcal{B}_t^v(\phi) := \inf_{u \in \mathbb{U}} c_t^v(\cdot, u) + \|f_t^v(\cdot, u)\|^2 \phi\left(\frac{f_t^v(\cdot, u)}{\|f_t^v(\cdot, u)\|}\right).$$

For every time $t \in [\![0, T-1]\!]$ we define the *Bellman operator* $\mathcal{B}_t$ for every function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ by:

$$\mathcal{B}_t\left(\phi\right) := \inf_{v \in \mathbb{V}} \mathcal{B}_t^v\left(\phi\right). \tag{3.39}$$

This definition of the Bellman operator emphasizes that the unit sphere S is (V_t)-optimal in the sense of Definition 13. Note that for 2-homogeneous functions, we have that $\mathcal{B}_t^v(\phi) = \inf_{u \in \mathbb{U}} c_t^v(\cdot, u) + \phi(f_t^v(\cdot, u))$. Using Equation (3.39), one can rewrite the Dynamic Programming Equation (3.38) as

$$V_T = \psi \quad \text{ and } \quad \forall t \in [\![0, T-1]\!], V_t = \mathcal{B}_t(V_{t+1})\ . \tag{3.40}$$

Now, in order to apply the Tropical Dynamic Programming algorithm to Equation (3.40), we need to check Assumption 1. Under Assumption 4, there exist an interval in the cone of symmetric semidefinite matrices which is stable by every Bellman operator $\mathcal{B}_t$ in the sense of the proposition below. We will consider the Loewner order on the cone of (real) symmetric semidefinite matrices, *i.e.* for every couple of matrices of symmetric matrices (M_1, M_2) we say that $M_1 \preceq M_2$ if, and only if, $M_2 - M_1$ is semidefinite positive. Moreover we will identify a pure quadratic form with its symmetric matrix, thus when we write an infimum over symmetric matrices, we mean the pointwise infimum over their associated pure quadratic forms.

Proposition 26 (Existence of a stable interval)**.** *Under Assumption 4, we define a sequence of positive reals $(\alpha_t)_{t \in [\![0,T]\!]}$ by backward recursion on $t \in [\![0, T-1]\!]$ such that we have:*

$$0 \preceq M \preceq \alpha_{t+1} \operatorname{Id} \Rightarrow 0 \preceq \mathcal{B}_t(M) \preceq \alpha_t \operatorname{Id}, \tag{3.41}$$

where α_T is a given constant by Assumption 4.

Proof. First, given an arbitrary $t \in [\![0, T]\!]$, we want to show that if $M \succeq 0$ then $\mathcal{B}_t(M) \succeq 0$. As in Proposition 20 one can show that the Bellman operator $\mathcal{B}_t$ is order preserving. Therefore, if $M \succeq 0$ then $\mathcal{B}_t(M) \succeq \mathcal{B}_t(0)$. Hence it is enough to show that $\mathcal{B}_t(0) \succeq 0$. But by Formula (3.53), we have that $B_t(0) = \min_{v \in \mathbb{V}} C_t^v \succeq 0$ (by Assumption 4) hence the result follows.

Second, let $t \in [\![0, T-1]\!]$ and $\alpha_{t+1} > 0$ be fixed. We consider $\alpha_t > 0$ defined by

$$\alpha_t := \max_{v \in \mathbb{V}} \alpha_{t+1} \lambda_{\max}\big(A_t^v (A_t^v)^T\big) + \lambda_{\max}(C_t^v) > 0\ , \tag{3.42}$$

and we prove that if $M \preceq \alpha_{t+1} \operatorname{Id}$ then we have that $\mathcal{B}_t(M) \preceq \alpha_t \operatorname{Id}$. For that purpose, consider M such that $M \preceq \alpha_{t+1} \operatorname{Id}$. Then, denoting by $\overline{B}_t^v$ the matrix $\operatorname{Id} + \alpha_{t+1} B_t^v (D_t^v)^{-1} (B_t^v)^T$, we have that $\lambda_{\min}(\overline{B}_t^v) = 1 + \alpha_{t+1} \lambda_{\min}(B_t^v (D_t^v)^{-1} (B_t^v)^T) \geq 1$ using the fact that the matrix $B_t^v (D_t^v)^{-1} (B_t^v)^T$ is positive semi-definite by Assumptions 4. Now, we successively have for any $v \in \mathbb{V}$

$$\begin{aligned}
\lambda_{\max}\big(\mathcal{B}_t^v(M)\big) &\leq \lambda_{\max}\big(\mathcal{B}_t^v(\alpha_{t+1}\,\mathrm{Id})\big) && (\mathcal{B}_t^v \text{ is order preserving})\\
&= \lambda_{\max}\Big(\alpha_{t+1}(A_t^v)^T(\overline{B}_t^v)^{-1}A_t^v + C_t^v\Big) && (\text{using } (3.53))\\
&\leq \alpha_{t+1}\lambda_{\max}\Big((A_t^v)^T(\overline{B}_t^v)^{-1}A_t^v\Big) + \lambda_{\max}(C_t^v) && (\text{by Proposition } 35)\\
&\leq \alpha_{t+1}\lambda_{\max}\big(A_t^v A_t^{vT}\big)\lambda_{\max}\big((\overline{B}_t^v)^{-1}\big) + \lambda_{\max}(C_t^v) && (\text{by Proposition } 35)\\
&\leq \alpha_{t+1}\lambda_{\max}\big(A_t^v A_t^{vT}\big) + \lambda_{\max}(C_t^v) && \Big(\text{as } \lambda_{\max}\big((\overline{B}_t^v)\big) = \lambda_{\min}\big((\overline{B}_t^v)\big)^{-1} \leq 1\Big)\\
&\leq \alpha_t\ , && (\text{using } (3.42))
\end{aligned}$$

which gives that $\mathcal{B}_t^v(M) \preceq \alpha_t\,\mathrm{Id}$. Then, the same result follows for the operator $\mathcal{B}_t$ using Equation (3.39). This ends the proof. □

Using Proposition 26, one can deduce by backward recursion on $t \in [\![0, T-1]\!]$ the existence of intervals of matrices, in the Loewner order, which are stable by the Bellman operators.

Corollary 27. *Under Assumption 4, using the sequence of positive reals $(\alpha_t)_{t\in[\![0,T]\!]}$ defined in Proposition 26, we define a sequence of positive reals $(\beta_t)_{t\in[\![0,T]\!]}$ by $\beta_T := \alpha_T$ and $\forall t \in [\![0, T-1]\!]$, $\beta_t := \max(\alpha_t, \beta_{t+1})$. Then, one has that*

$$0 \preceq M \preceq \beta_T\,\mathrm{Id} \Rightarrow \forall t \in [\![0, T-1]\!],\ 0 \preceq \mathcal{B}_t(\dots\mathcal{B}_{T-2}(\mathcal{B}_{T-1}(M))) \preceq \beta_t\,\mathrm{Id}.$$

The basic functions $\mathbf{F}_t^{\text{min-plus}}$ will be pure quadratic convex forms bounded in the Loewner sense by 0 and $\beta_t I$,

$$\mathbf{F}_t^{\text{min-plus}} := \big\{\phi : x \in \mathbb{X} \mapsto x^T M x \in \mathbb{R} \,\big|\, M \in \mathbb{S}_n,\ 0 \preceq M \preceq \beta_t\,\mathrm{Id}\big\}\ ,$$

and we define the following class of functions which will be stable by pointwise infimum of elements in $\mathbf{F}_t^{\text{min-plus}}$,

$$\mathbb{F}_t^{\text{min-plus}} := \big\{\mathcal{V}_F \,\big|\, F \subset \mathbf{F}_t^{\text{min-plus}}\big\}\ . \tag{3.43}$$

Exploiting the min additivity of the Bellman operator, which gives that $\mathcal{B}_t\big(\inf(\phi_1, \phi_2)\big) = \inf\big(\mathcal{B}_t(\phi_1), \mathcal{B}_t(\phi_2)\big)$, and the fact that the final cost $\widetilde{\psi}$ is a finite infima of basic functions, one deduces by backward induction on $t \in [\![0, T]\!]$ that the value functions are finite infima of basic functions.

Lemma 28. *For every time $t \in [\![0, T]\!]$, there exists a finite set F_t of convex pure quadratic forms such that*

$$V_t = \inf_{\phi \in F_t} \phi.$$

Proof. For $t = T$, set $F_T := \{\psi_i\}_{i \in I_T}$. Now, assume that for some $t \in [\![0, T-1]\!]$, we have that $V_{t+1} = \inf_{\phi \in F_{t+1}} \phi$, where F_{t+1} is a finite set of convex pure quadratic functions. Then, by definition of the Bellman operators $\mathcal{B}_t$ (see Equation (3.39)), we have that

$$V_t = \mathcal{B}_t(V_{t+1}) = \inf_{v \in \mathbb{V}} \mathcal{B}_t^v\big(\inf_{\phi \in F_{t+1}} \phi\big) = \inf_{\phi \in F_{t+1}} \inf_{v \in \mathbb{V}} \mathcal{B}_t^v(\phi) = \inf_{\phi \in F_{t+1}, v \in \mathbb{V}} \big(\mathcal{B}_t^v(\phi)\big)$$

As the image by $\mathcal{B}_t$ of a convex pure quadratic function is still a convex pure quadratic function (see Appendix 3.7), setting $F_t := \big\{\mathcal{B}_t^v(\phi) \,|\, \phi \in F_{t+1} \text{ and } v \in \mathbb{V}\big\}$, we obtain that $V_t = \inf_{\phi \in F_t} \phi$, where F_t is a finite set of convex pure quadratic functions. Backward induction on time $t \in [\![0, T]\!]$ ends the proof. □

Proposition 29. *Under Assumption 4, the Problem 3.37 and the Bellman operators defined in Equation* (3.38) *satisfy the structural assumptions given in Assumption 1.*

Proof. We prove successively each assumption listed in Assumption 1.

• 1-(a). By construction, $\mathbb{F}_t^{\text{min-plus}}$ in Equation (3.43) is stable by pointwise infimum.

• 1-(b) and 1-(c). We will show that every element of $\mathbb{F}_t^{\text{min-plus}}$ is 2β-Lipschitz continuous on S. Let $F = \{\phi_i\}_{i\in I} \subset \mathbb{F}_t^{\text{min-plus}}$ with $I \subset \mathbb{N}$ and $\phi_i \in \mathbf{F}_t^{\text{min-plus}}$ with associated symmetric matrix M_i. Fix $x, y \in S$, we have successively

$$
\begin{aligned}
|V_F(x) - V_F(y)| &= |\inf_{i\in I} x^T M_i x - \inf_{i\in I} y^T M_i y| \\
&\leq \max_{i\in I} |x^T M_i x - y^T M_i y| \\
&\leq \max_{i\in I} |x^T M_i\,(x-y) + y^T M_i\,(x-y)\,| \\
&\leq \max_{i\in I} |\langle x+y, M_i(x-y)\rangle| && (M^T = M) \\
&\leq \|x+y\| \cdot \max_{i\in I} \|M_i\,(x-y)\,\| && \text{(Cauchy-Schwarz)} \\
&\leq \|x+y\| \cdot \max_{i\in I} \|M_i\| \|x-y\| \\
&\leq \beta_t \|x+y\| \cdot \|x-y\| && (\|M_i\| \leq \beta_t) \\
&\leq 2\beta_t \|x-y\|\ , && (3.44)
\end{aligned}
$$

since $\|x+y\| \leq 2$. Thus, every element of $\mathbb{F}_t^{\text{min-plus}}$ is $2\beta_t$-Lipschitz on S and by stability by pointwise infimum, $\mathbb{F}_t^{\text{min-plus}}$ is stable by pointwise convergence.

• 1-(d). By Assumption 4, the final cost function ψ is an element of $\mathbb{F}_T^{\text{min-plus}}$.

• 1-(e). This is given by Corollary 27.

• 1-(f). Proceed as in Proposition 20.

• 1-(g). Fix a time step $t \in [\![0, T-1]\!]$, a compact $K_t \subset \mathrm{dom}(V_t)$ $(= \mathbb{X})$, a function $\phi \in \mathbb{F}_{t+1}^{\text{min-plus}}$ and a constant $\lambda \geq 0$. By definition of $\mathbb{F}_{t+1}^{\text{min-plus}}$, there exists a finite set $F := \{\phi_i\}_{i\in I} \subset \mathbf{F}_{t+1}^{\text{min-plus}}$ such that $\phi = \inf_{i\in I} \phi_i$. By Equation (3.56), for each $i \in I$ and $v \in \mathbb{V}$, there exists a linear map L_i^v such that

$$
\min_{u\in\mathbb{U}} c_t^v(x,u) + \phi_i(f_t^v(x,u)) = c_t^v(x, L_i^v(x)) + \phi_i(f_t^v(x, L_i^v(x)))\ , \tag{3.45}
$$

with $\|L_i^v\| \leq \alpha_{t+1} C_t$, where C_t is a constant depending on the parameters of the control problem only. Hence the maps $x \mapsto f_t^v(x, L_i^v(x))$ are linear and their norm are bounded by $(\alpha_{t+1}+1)C_t'$ for some constant C_t' depending on the parameters of the control problem only. Set $M_t := ((\alpha_{t+1}+1)C_t'\|K_t\|)^2$, where $\|K_t\|$ is the radius of a ball centered in 0 including K_t. Therefore, for $x \in K_t$, we have $\|f_t^v(x,u)\|^2 \leq M_t$. Now, for $x \in K_t$, using the bound on f_t we have

$$
\begin{aligned}
\mathcal{B}_t\left(\phi+\lambda\right)(x) &= \min_{\substack{i\in I\\ u\in\mathbb{U}\\ v\in\mathbb{V}}} c_t^v(x,u) + \|f_t^v(x,u)\|^2(\phi_i+\lambda)(\frac{f_t^v(x,u)}{\|f_t^v(x,u)\|}) \\
&\leq \min_{\substack{i\in I\\ v\in\mathbb{V}}} c_t^v(x,L_i^v(x)) + \|f_t^v(x,L_i^v(x))\|^2(\phi_i+\lambda)(\frac{f_t^v(x,L_i^v(x))}{\|f_t^v(x,L_i^v(x))\|}) \\
&\leq \min_{\substack{i\in I\\ v\in\mathbb{V}}} c_t^v(x,L_i^v(x)) + \|f_t^v(x,L_i^v(x))\|^2\phi_i(\frac{f_t^v(x,L_i^v(x))}{\|f_t^v(x,L_i^v(x))\|}) + M_t\lambda \\
&= \min_{\substack{i\in I\\ v\in\mathbb{V}}} \Big(c_t^v(x,L_i^v(x)) + \phi_i(f_t^v(x,L_i^v(x)))\Big) + M_t\lambda \\
&= \min_{\substack{i\in I\\ u\in\mathbb{U}\\ v\in\mathbb{V}}} \Big(c_t^v(x,u) + \phi_i\big(f_t^v(x,u)\big)\Big) + M_t\lambda \qquad \text{(by (3.45))} \\
&= \mathcal{B}_t(\phi)(x) + M_t\lambda \ .
\end{aligned}
$$

hence the desired result $\mathcal{B}_t\left(\phi+\lambda\right)(x) \leq M_t\lambda + \mathcal{B}_t(\phi)(x)$.

- 1-(h). This is a consequence of Lemma 28.
- 1-(i) Fix $\phi\in\mathbb{F}_t$ and $\lambda\geq 0$. Denote by $\tilde{\phi}=\phi+\lambda$. For every $x\in\mathbb{X}$, we have that

$$
\begin{aligned}
\mathcal{B}_t(\tilde{\phi})(x) &= \min_{(u,v)\in\mathbb{U}\times\mathbb{V}} c_t^v(x,u) + \|f_t^v(x,u)\|^2\tilde{\phi}(\frac{f_t^v(x,u)}{\|f_t^v(x,u)\|}) \\
&= \min_{(u,v)\in\mathbb{U}\times\mathbb{V}} c_t^v(x,u) + \|f_t^v(x,u)\|^2(\tilde{\phi}+\delta_S)(\frac{f_t^v(x,u)}{\|f_t^v(x,u)\|}) \\
&= \mathcal{B}_t(\tilde{\phi}+\delta_S)(x),
\end{aligned}
$$

which implies the desired result.

□

Remark 30. *We have shown that $\mathcal{B}_t$ is additively subhomogeneous with constant M_t. An upper bound of M_t can be computed as in the proof of Proposition 26, by bounding the greatest eigenvalue of each matrices L_i^v.*

We now define, for any $t\in[\![0,T]\!]$, a selection functions $\phi_t^{\text{min-plus}}$ and prove that it is a compatible selection function. As each arg min mentioned below involves a finite set, selecting an element in the arg min raises no issue.

Proposition 31. *For every time $t\in[\![0,T]\!]$, any $F\subset\mathbf{F}_t^{\text{min-plus}}$ and any $x\in\mathbb{X}$, define a function $\phi_t^{\text{min-plus}}$ as follows*

$$
\phi_t^{\text{min-plus}}(F,x) \in \begin{cases} \mathcal{B}_t\Big(\arg\min_{\phi\in F}\big(\mathcal{B}_t(\phi)(x)\big)\Big) & \text{for} \quad t\neq T\ , \\ \arg\min_{\psi_i\in F}\psi_i(x) & \text{for} \quad t=T\ , \end{cases} \tag{3.46}
$$

is a compatible selection *function as defined in Definition 6.*

Proof. Fix $t = T$. The function $\phi_t^{\text{min-plus}}$ is tight and valid as $V_T = \psi$. Now fix $t \in [\![0, T-1]\!]$. Let $F \subset \mathbb{F}_{t+1}^{\text{min-plus}}$ and $x \in \mathbb{X}$ be arbitrary. We have

$$\begin{aligned}
\mathcal{B}_t(\mathcal{V}_F)(x) &= \mathcal{B}_t\big(\inf_{\phi \in F} \phi\big)(x) \\
&= \inf_{(u,v) \in \mathbb{U} \times \mathbb{V}} \Big(c_t^v(x,u) + \inf_{\phi \in F} \phi\big(f_t^v(x,u)\big)\Big) \\
&= \inf_{\phi \in F} \inf_{(u,v) \in \mathbb{U} \times \mathbb{V}} \Big(c_t^v(x,u) + \phi\big(f_t^v(x,u)\big)\Big) \\
&= \inf_{\phi \in F} \big(\mathcal{B}_t\left(\phi\right)(x)\big) \\
&= \phi_t^{\text{min-plus}}\left(F, x\right)(x) \ .
\end{aligned}$$

Thus, $\phi_t^{\text{min-plus}}$ is tight. By similar arguments, we have for every $x' \in \mathbb{X}$ that

$$\mathcal{B}_t(\mathcal{V}_F)(x') = \big(\inf_{\phi \in F} \mathcal{B}_t(\phi)\big)(x') \leq \phi_t^{\text{min-plus}}(F, x)(x') \ .$$

This shows that $\phi_t^{\text{min-plus}}(F, x)$ is valid and ends the proof. □

We conclude this section by proving the convergence of TDP algorithm in the Min-plus case.

Theorem 32 (Upper (inner) approximations of the value functions)**.** *For every $t \in [\![0, T]\!]$, denote by $\left(V_t^k\right)_{k \in \mathbb{N}}$ the sequence of functions generated by Tropical Dynamic Programming with the selection function $\phi_t^{\text{min-plus}}$ and the draws made uniformly over the sphere $K_t := S$. Under Assumption 4, the sequence $\left(V_t^k\right)_{k \in \mathbb{N}}$ is non increasing, bounded from below by V_t and converges uniformly to V_t^* on S. Moreover, almost surely over the draws, $V_t^* = V_t$ on S.*

Proof. As the structural Assumption 1 are satisfied, as the functions $\phi_t^{\text{min-plus}}$, $0 \leq t \leq T$ are compatible selections and the unit sphere S is V_t-optimal (case opt $=$ inf), we can apply Theorem 15. □

3.5.2 Optimal trajectories for upper approximations may not converge

We now give an example showing that approximating from above may fail to converge when the points are drawn along optimal trajectories for the current upper approximations of V_t (in contrast with Section 3.4 where we approximate from below V_t). As shown by Proposition 36 there is no loss of generality in considering the framework of §3.5.1 but with non-homogeneous functions.

We consider a (non-homogeneous) problem with only two time steps, that is $T = 1$ and $t \in \{0, 1\}$ such that

- The state space $\mathbb{X}$ and the control space $\mathbb{U}$ are equal to $\mathbb{R}$.
- The linear dynamic is $f(x, u) = x + u$.
- The quadratic cost is $c(x, u) = x^2 + u^2$.
- The final cost function is the infimum, $\psi = \inf(\psi_1, \psi_2)$, of two given quadratic mappings, $\psi_1(x) = (x+2)^2 + 1$ and $\psi_2(x) = x^2$.

The Bellman operator $\mathcal{B}$, associated to this multistage optimization problem is defined for every $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ and every $x \in \mathbb{X}$ by

$$\mathcal{B}(\phi)(x) = \min_{u \in U} \left(x^2 + u^2 + \phi(x+u)\right) = x^2 + \min_{u \in U} \left(u^2 + \phi(x+u)\right) \ .$$

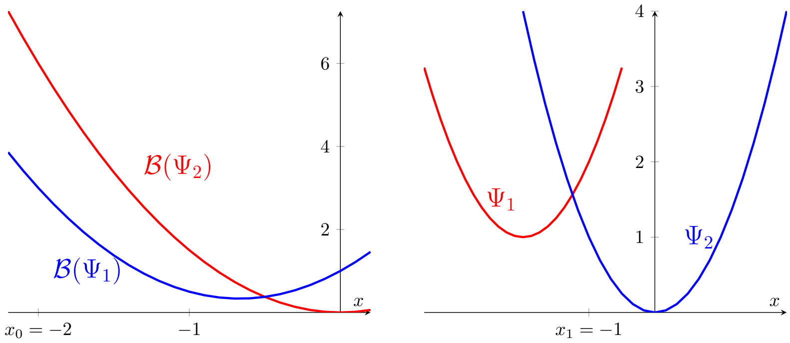

Figure 3.4: Illustration of the multistage (two stages here) optimization problem studied in §3.5.2. The final cost function, $\psi = \inf(\psi_1, \psi_2)$ is shown in the right subfigure and displays of the images, by the Bellman operator $\mathcal{B}$, of the functions ψ_1 and ψ_2 are drawn in the left subfigure. We have that $\mathcal{B}(\psi_2)(x_0) > \mathcal{B}(\psi_1)(x_0)$. At the final time $t = 1$, the "best function" at the point -1 is ψ_2. The image by the k-th optimal dynamic of $x_0 = -2$ is $x_1 = -1$.

For the case where $\phi_{a,b}(\cdot) = (\cdot + a)^2 + b$ with $a, b \in \mathbb{R}$ one has for every $x \in \mathbb{R}$

$$\mathcal{B}(\phi_{a,b})(x) = \frac{3}{2}x^2 + ax + b. \tag{3.47}$$

Fix $x_0 = x_0^k = -2$ for every $k \in \mathbb{N}$. As described in Algorithm 1, the approximations of the value functions V_1 and V_0 are initialized to $+\infty$. Thus every control $u \in \mathbb{U}$ is optimal in the sense that $u \in \arg\min_{u' \in \mathbb{U}} x^2 + (u')^2 + \phi(x + u')$. Hence if we set $x_1^0 := -1 = f(x_0, 1)$ then (x_0, x_1^0) is an optimal trajectory as described in Proposition 22.

We deduce from Equation (3.47) the following facts, illustrated in Figure 3.4.

1. The image of ψ_2 is strictly greater than the image of ψ_1 by the Bellman operator $\mathcal{B}$, *i.e.*

$$\mathcal{B}(\psi_2)(-2) > \mathcal{B}(\psi_1)(-2).$$

2. The image by the k-th optimal dynamic of -2 is -1, *i.e.* setting $u_0^k := \arg\min_{u' \in \mathbb{U}} (-2)^2 + (u')^2 + V_1^k(-2 + u')$ (the arg min is here a singleton) one has

$$f(-2, u_0^k) = -1.$$

3. At the final step $t = 1$, the best function at the point -1 is ψ_2, *i.e.*

$$\psi(-1) = \inf(\psi_1(-1), \psi_2(-2)) = \psi_2(-2).$$

From those three facts, one can deduce starting $x_0 = -2$ and $x_1 = -1$, the optimal trajectory for the current approximations will always be sent to $x_1 = -1$. But, as shown in the proof of Proposition 31 one can show that the image by $\mathcal{B}$ of an infimum is the infimum of the images by $\mathcal{B}$:

$$V_0(-2) = \mathcal{B}(\inf(\psi_1, \psi_2))(-2) = \inf(\mathcal{B}(\psi_1)(-2), \mathcal{B}(\psi_1)(-2)).$$

Thus for every $k \in \mathbb{N}$, we have $V_0(-2) = \mathcal{B}(\psi_1)(-2) < \mathcal{B}(\psi_1)(-2) = V_0^k(-2)$. The constant sequence $V_0^k(-2)$ fails to converge to $V_0(-2)$.

3.6 Numerical experiments on a toy example

In §3.6.1, we propose a toy optimization Problem (3.48) on which we run TDP-SDDP and TDP-Minplus. Problem (3.48) falls in the framework described in Section 3.4. Thus, we are able to obtain lower approximations of V_t using TDP-SDDP. TDP-Minplus cannot be applied directly. We apply a "discretization" step to Problem (3.48) (see §3.6.2) which yields Problem (3.49) parameterized by an integer $N > 0$. Then we apply to Problem (3.49) an "homogenization" step (see §3.6.3) to obtain Problem (3.50). The value functions V_t of the original Problem (3.48) are bounded from above by $\widetilde{V}_{t,N}$, the value functions of Problem (3.50). We apply TDP-Minplus (described in Section 3.5) to Problem (3.50) which gives upper approximations of $\widetilde{V}_{t,N}$ and *a fortiori*, of V_t. In §3.6.6, we show numerical experiments which show the convergence of this approximation scheme to V_t.

3.6.1 A toy example: constrained linear-quadratic framework

Let β, γ be two given reals such that $\beta < \gamma$, we study the following multistage linear quadratic problem involving a constraint on one of the controls:

$$\min_{(x,u,v)\in\mathbb{X}^{T+1}\times\mathbb{U}^T\times[\beta,\gamma]^T} \sum_{t=0}^{T-1} c_t(x_t,u_t,v_t) + \psi(x_T) \tag{3.48a}$$

$$\text{s.t.} \quad x_0 \in \mathbb{X} \text{ given, and} \quad \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t(x_t,u_t,v_t)\ , \tag{3.48b}$$

where $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$, with quadratic convex costs functions of the form

$$c_t(x,u,v) = x^T C_t x + u^T D_t u + v^2 d_t,$$

where $C_t \in \mathbb{S}_n^+$, $D_t \in \mathbb{S}_m^{++}$ and $d_t > 0$, linear dynamics $f_t(x,u,v) = A_t x + B_t u + v b_t$, where A_t (resp. B_t) is a $n \times n$ (resp. $n \times m$) matrix, $b_t \in \mathbb{X}$, and final cost function $\psi := x^T M x$ with $M \in \mathbb{S}_n^{++}$.

For every $t \in [\![0,T]\!]$, the value function V_t is L_t-Lipschitz continuous and convex. Moreover the Lipschitz constant $L_t > 0$ can be explicitly computed. As done in Section 3.4 we will generate lower approximations of the value functions V_t through compatible selection functions $(\phi_t^{\text{SDDP}})_{t\in[\![0,T]\!]}$. In this example, the structural Assumptions 1 are not satisfied as the sets of states and controls are not compacts. As we will still observe convergence of the lower approximations $(\phi_t^{\text{SDDP}})_{t\in[\![0,T]\!]}$ generated by TDP to the value functions, this suggests that the (classical) framework presented in Section 3.4 can be extended. This will be the object of a future work and here we would like to stress on the numerical scheme and results.

3.6.2 Discretization of the constrained control

We approximate Problem (3.48) by discretizing the constrained control in order to obtain an unconstrained switched multistage linear quadratic problem. More precisely, we fix an integer $N \geq 2$, set $v_i = \beta + i\frac{\gamma-\beta}{N-1}$ for every $i \in [\![0, N-1]\!]$ and set $\mathbb{V} := \{v_0, v_1, \dots v_{N-1}\}$. Then, we define the following unconstrained switched multistage linear quadratic problem:

$$\min_{(x,u,v)\in\mathbb{X}^{T+1}\times\mathbb{U}^T\times\mathbb{V}^T} \sum_{t=0}^{T-1} c_t^{v_t}(x_t,u_t) + \psi(x_T) \tag{3.49a}$$

$$\text{s.t.} \quad x_0 \in \mathbb{X} \text{ given, and} \quad \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t^{v_t}(x_t,u_t)\ , \tag{3.49b}$$

where for every $v \in \mathbb{V}$, $f_t^v = f_t(\cdot, \cdot, v)$ and $c_t^v = c_t(\cdot, \cdot, v)$. As the set of controls of Problem (3.48) contains the set of controls of Problem (3.49), upper approximations of the value functions of Problem (3.49) yield upper approximations of the value functions of Problem (3.48). Thus we will construct upper approximations for Problem (3.49).

3.6.3 Homogenization of the costs and dynamics

We add a dimension to the state space in order to homogenize the costs and dynamics, when a sequence of switching controls is fixed. Define the following homogenized costs and dynamics for every $t \in [\![0, T-1]\!]$ by:

$$\tilde{f_t}^v(x, y, u) = \begin{pmatrix} A_t & vb_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} B_t \\ 0 \end{pmatrix} u,$$

$$\tilde{c_t}^v(x, y, u) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} C_t & 0 \\ 0 & v^2 d_t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + u^T D_t u,$$

And as the final cost function is already homogeneous, $\widetilde{\psi}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Using these homogenized functions we define a multistage optimization problem with one more (compared to Problem (3.49)) dimension on the state variable:

$$\begin{aligned} &\min_{(x,y,u,v) \in \mathbb{X}^{T+1} \times \mathbb{R}^{T+1} \times \mathbb{U}^T \times \mathbb{V}^T} \sum_{t=0}^{T-1} \tilde{c_t}^{v_t}(x_t, y_t, u_t) + \widetilde{\psi}(x_T, y_T) \\ &\text{s.t. } \begin{cases} (x_0, y_0) \in \mathbb{X} \times \mathbb{R} \text{ is given,} \\ \forall t \in [\![0, T-1]\!], \ (x_{t+1}, y_{t+1}) = \tilde{f_t}^{v_t}(x_t, y_t, u_t) \ . \end{cases} \end{aligned} \tag{3.50}$$

One can deduce the value functions $V_{t,N}$ of the multistage optimization problem (3.49) (with non-homogeneous costs and dynamics) from the value functions $\widetilde{V}_{t,N}$ of (3.50) (with homogeneous costs and dynamics) by Proposition 36. For every $x \in \mathbb{X}$, we have that

$$V_{t,N}(x) = \widetilde{V}_{t,N}(x, 1) \ . \tag{3.51}$$

For every time step $t \in [\![0, T]\!]$ the value function $\widetilde{V}_{t,N}$ solution of Problem (3.50) is 2-homogeneous. That is, for every $(x, y) \in \mathbb{X} \times \mathbb{R}$ and every $\lambda \in \mathbb{R}$, we have $\widetilde{V}_{t,N}(\lambda x, \lambda y) = \lambda^2 \widetilde{V}_{t,N}(x, y)$. This will allow us to restrict the study of the value functions to the unit sphere, which is compact.

3.6.4 Min-plus upper approximations of the value functions of Problem (3.50)

We apply the results of Section 3.5 as follows. Let $v \in \mathbb{V}$ be a given switching control, in this framework, the operator $\mathcal{B}_t^v$ is defined as in Section 3.5 but with an augmented state. More precisely, for every function $\phi : \mathbb{X} \times \mathbb{R} \to \overline{\mathbb{R}}$ and every point $(x, y) \in \mathbb{X} \times \mathbb{R}$:

$$\mathcal{B}_t^v(\phi)(x, y) = \inf_{u \in \mathbb{U}} \tilde{c}_t^v(x, y, u) + \|\tilde{f}_t^v(x, y, u)\|^2 \phi\left(\frac{\tilde{f}_t^v(x, y, u)}{\|\tilde{f}_t^v(x, y, u)\|}\right) .$$

Then, for every time $t \in [\![0, T-1]\!]$, the Dynamic Programming operator $\mathcal{B}_t$ associated to Problem (3.50) satisfies $\mathcal{B}_t(\phi) := \inf_{v \in \mathbb{V}} \mathcal{B}_t^v(\phi)$.

A key property of the operators $\mathcal{B}_t^v$ and $\mathcal{B}_t$ is that they are min-additive, meaning that for every functions $\phi_1, \phi_2 : \mathbb{X} \to \overline{\mathbb{R}}$ one has:

$$\mathcal{B}_t^v\big(\inf(\phi_1, \phi_2)\big) = \inf\big(\mathcal{B}_t^v(\phi_1), \mathcal{B}_t^v(\phi_2)\big) \ ,$$

and a similar equation for $\mathcal{B}_t$. Moreover, by Riccati formula (see Equation (3.52)), the image of a convex quadratic function by $\mathcal{B}_t^v$ is also a convex quadratic function.

Lemma 28 suggests to use the following set of basic functions:

$$\mathbf{F}_t^{\text{min-plus}} := F_t \ \text{ and } \ \mathbb{F}_t^{\text{min-plus}} := \left\{ \mathcal{V}_F \;\middle|\; F \subset \mathbf{F}_t^{\text{min-plus}} \right\} \ .$$

As done in Section 3.5, one could also have considered as basic functions the quadratic functions bounded in the Loewner sense between 0 and $\alpha_t \mathrm{I}$, where $\alpha_t > 0$, $t \in [\![0, T]\!]$, are real numbers such that, if ϕ is a quadratic form bounded between 0 and $\alpha_{t+1}\mathrm{I}$, then $\mathcal{B}_t^v(\phi)$ is bounded between 0 and $\alpha_t \mathrm{I}$.

Moreover, using the aforementioned properties, one will be able to compute $\mathcal{B}_t^v(\mathcal{V}_F)$ for a given switching control v and $\mathcal{B}_t(\mathcal{V}_F)$, for any finite set F of convex quadratic functions. Therefore, given a time $t \in [\![0, T-1]\!]$, we define the selection function $\phi_t^{\text{min-plus}}$ as follows. For any given $F \subset \mathbf{F}_{t+1}^{\text{min-plus}}$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$,

$$\begin{aligned}
&\phi_t^{\text{min-plus}}(F, x, y) = \mathcal{B}_t^v(\phi) \\
&\text{for some} \quad (v, \phi) \in \operatorname*{arg\,min}_{(v,\phi) \in \mathbb{V} \times F} \mathcal{B}_t^v(\phi)(x, y) \, .
\end{aligned}$$

Moreover, at time $t = T$, for any $F \subset \mathbf{F}_T^{\text{min-plus}}$ and $(x, y) \in \mathbb{X} \times \mathbb{R}$, we set

$$\phi_t^{\text{min-plus}}(F, x, y) = \widetilde{\psi}(x, y) = \psi(x).$$

Motivated by the 2-homogeneity of the value functions, the random draws of TDP for the basic functions $\mathbf{F}_t^{\text{min-plus}}$, $1 \leq t \leq T$ and the selection functions $\phi_t^{\text{min-plus}}$ will be made uniformly on the unit Euclidean sphere, which satisfies Definition 7. Indeed, by 2-homogeneity, it is enough to know the value functions of (3.50) on the sphere to know them on the whole state space.

3.6.5 Upper and lower approximations of the value functions

For a large number of discretization points N (defined in §3.6.2), one would expect that the value functions $V_{t,N}$ of (3.49) approximate the value functions V_t of (3.48). Indeed, one can show that for every time step $t \in [\![0, T]\!]$, the approximation error is bounded by $C_t T / N^2$ in $\mathbb{X}$, for some constant $C_t > 0$. Thus, for large N, we have $V_{t,N} \approx V_t$ and by Equation (3.51), for every $N \geq 2$, we have

$$\widetilde{V}_{t,N}(\cdot, 1) = V_{t,N} \geq V_t.$$

In the following Proposition we approximate $\widetilde{V}_{t,N}$ from above by a min-plus algorithm and V_t from below by SDDP and using the convergence result of Theorem 15 (admitting that the result still holds for SDDP in this framework), we obtain the following one.

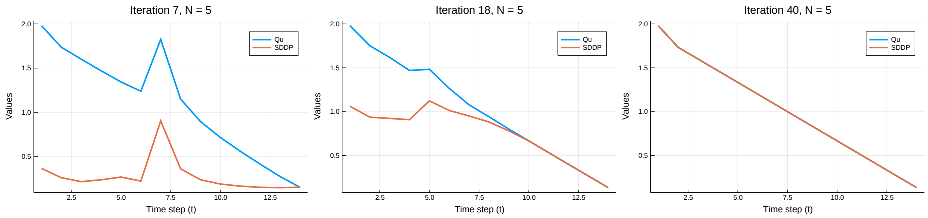

Figure 3.5: First example for $\beta = 1$, $\gamma = 5$ with $N = 5$ after 7 iterations (left), 18 iterations (middle) and 40 iterations (right).

Theorem 33. *For every $t \in [\![0, T]\!]$, denote by $\left(\overline{V}_t^k\right)_{k\in\mathbb{N}}$ (resp. $(\underline{V}_t^k)_{k\in\mathbb{N}}$) the sequence of functions generated by TDP with the selection function $\phi_t^{\text{min-plus}}$ (resp. ϕ_t^{SDDP}) and the draws made uniformly over the Euclidean sphere of $\mathbb{X}\times\mathbb{R}$ (resp. made as described in Proposition 22).*

Then the sequence $\left(\overline{V}_t^k\right)_{k\in\mathbb{N}}$ (resp. $(\underline{V}_t^k)_{k\in\mathbb{N}}$) is non-increasing (resp. non-decreasing), bounded from below (resp. above) by $\widetilde{V}_{t,N}$ (resp. V_t) and converges uniformly to $\widetilde{V}_{t,N}$ (resp. V_t) on any compact subset of $\mathbb{X}\times\mathbb{R}$ (resp. K_t^* defined in Proposition 22).

3.6.6 Numerical experiments

The following data was used as a specific case of (3.48). For every time $t \in [\![0, T-1]\!]$,

$$A_t = (1 - 0.1)\,\mathrm{Id} \qquad B_t = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \qquad b_t = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$C_t = 0.1\,\mathrm{Id} \qquad D_t = 0.1\,\mathrm{Id} \qquad d_t = 0.1.$$

The time horizon is $T = 15$, the states are in $\mathbb{X} = \mathbb{R}^n$ with $n = 25$, the unconstrained continuous controls are in $\mathbb{U} = \mathbb{R}^m$ with $m = 3$, the constrained continuous control is in $[\beta, \gamma]$, with $[\beta, \gamma] = [1, 5]$ in the first example and $[\beta, \gamma] = [-3, 5]$ in the second one. Moreover, we start from the initial point $x_0 = 0.2\,(1, \ldots, 1)^T$ when TDP is applied with the selection function ϕ_t^{SDDP} and the number of discretization points N is varying from 5 to 200, for TDP with the selection function $\phi_t^{\text{min-plus}}$. In Figures 3.5 and 3.6, we give graphs representing the values $\underline{V}_t^k(x_t^k)$ and $\overline{V}_t^k(x_t^k, 1)$ at each time step $t \in [\![0, T-1]\!]$ where the sequence of states $(x_t^k)_{k\in\mathbb{N}}$ is the optimal trajectory for the current lower approximations $(\underline{V}_t^k)_{k\in\mathbb{N}}$ defined in (22). From Theorem 33, we know that for every $t \in [\![0, T-1]\!]$ the gap $\overline{V}_t^k(x_t^k, 1) - \underline{V}_t^k(x_t^k)$ should be close to 0 as k increases assuming that N is large enough to have $V_t \approx V_{t,N}$.

On those two examples, we exhibit two convergence behaviors. On the first example, the constrained control has to be greater than 1, thus avoiding 0 which would have been (or almost) the optimal control if there were no constraint. The optimal constrained control is the projection on $\mathbb{U}\times[\beta, \gamma]$ of the optimal unconstrained control, thus the switching control is most of the time equal to the lower bound $\beta = 1$.

From this observation we deduce two properties. First, the upper approximation given by Qu algorithm is good, even for a small N, as the optimal switch is (most of the time) equal to β. Second, this implies that at iteration k, the set F_t^k is of small cardinality.

Moreover, in this example the number of switches is $N = 5$ thus few computations of $\mathcal{B}_t^v(\phi)(x)$ need to be done in order to compute $\mathcal{B}_t(\phi)(x)$ for some $x \in \mathbb{X}$ and $\phi \in F_t^k$. Thus,

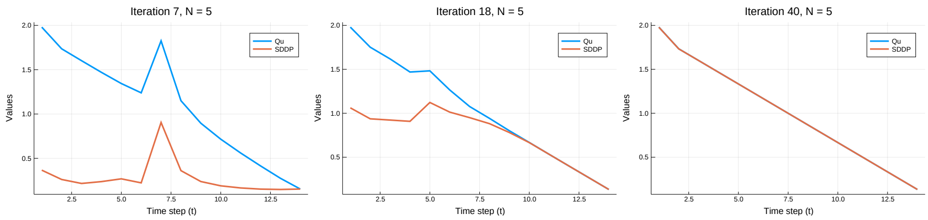

Figure 3.6: Second example for $\beta = -3$, $\gamma = 5$ with varying $N = 5$ (left), $N = 50$ (middle) and $N = 200$ (right) after 20 iterations.

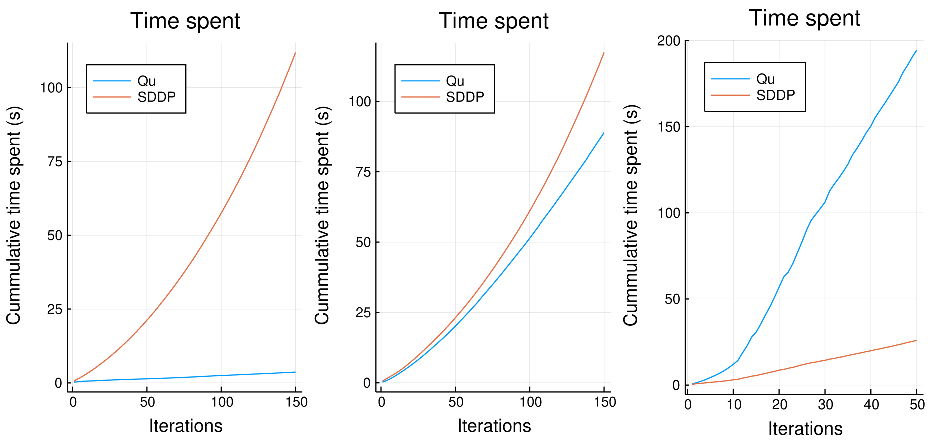

Figure 3.7: Time spent for the first example (left) and the second example when $N = 50$ (middle) and $N = 200$ (right).

as shown on the left of Figure 3.7, the computation time of an iteration of Qu's min-plus algorithm is small compared to SDDP which does not exploit this property.

On the second example, the constrained control is in an interval containing 0. The switching control often changes and this means more computations. A compromise between computational time and precision can be achieved (see Figure 3.7) in order to make the computational time of Qu algorithm similar to the one of SDDP algorithm.

Conclusion

In this chapter we have devised an algorithm, Tropical Dynamic Programming, that encompasses both a discrete time version of Qu's min-plus algorithm and the SDDP algorithm in the deterministic case. We have shown in the last section that Tropical Dynamic Programming can be applied to two natural frameworks: one for min-plus and one for SDDP. In the case where both framework intersects, one could apply Tropical Dynamic Programming with the selection functions $\phi_t^{\text{min-plus}}$ and get non-increasing upper approximations of the value function. Simultaneously, by applying Tropical Dynamic Programming with the selection function ϕ_t^{SDDP}, one would get non-decreasing lower approximations of the value function. Moreover, we have shown that the upper approximations are, almost surely, asymptotically equal to the value function on the whole space of states $\mathbb{X}$ and that the lower approximations are, almost surely, asymptotically equal to the value function on a set of interest.

Thus, in those particular cases we get converging bounds for $V_0(x_0)$, which is the value of the multistage optimization Problem 3.1, along with asymptotically exact minimizing policies. In those cases, we have shown a possible way to address the issue of computing efficient upper bounds when running the SDDP algorithm by running in parallel another algorithm (namely TDP with min-plus selection functions).

In Section 3.6 we studied a way to simultaneously build lower and upper approximations of the value functions using the results of the previous sections. However the discretization and homogenization scheme that was described is rapidly limited by the dimension of the control space, due to the need to discretize the constrained controls. We will provide in a future work, a systematic scheme to use simultaneously SDDP and a min-plus methods which is more efficient numerically and does not rely on discretization of the control space. Moreover we will extend the current framework to multistage stochastic optimization problems with finite white noises.

3.7 Algebraic Riccati Equation

This section gives complementary results for Section 3.5. We use the same framework and notations introduced in Section 3.5.

Proposition 34. *Fix a discrete control $v \in \mathbb{V}$ and a time step $t \in [\![0, T-1]\!]$.*

–(a) The operator $\mathcal{B}_t^v : \mathbb{S}_n \to \mathbb{S}_n^+$ restricted to the pure quadratic forms (identified with $\mathbb{S}_n$ the space of the symmetric semidefinite positive matrices) is given by the discrete time algebraic Riccati equation

$$\mathcal{B}_t^v(M) = C_t^v + (A_t^v)^T M A_t^v - (A_t^v)^T M B_t^v \big(D_t^v + B_t^v M (B_t^v)^T\big)^{-1} (B_t^v)^T M A_t^v \ . \tag{3.52}$$

–*(b) Moreover, when* $M \in \mathbb{S}_n^+$ *Equation* (3.52) *can be rewritten as*

$$\mathcal{B}_t^v(M) = (A_t^v)^T M \Big(I + B_t^v (D_t^v)^{-1} (B_t^v)^T M\Big)^{-1} A_t^v + C_t^v. \tag{3.53}$$

Proof.
• We prove Equation (3.52). Note that if M is symmetric, then $\mathcal{B}_t^v(M)$ is also symmetric. Now, let $t \in \{T-1, T-2, \dots, 0\}$ and $M \in \mathbb{S}_n$ be fixed. Let $x \in \mathbb{X}$, we have that

$$\begin{aligned}
\mathcal{B}_t^v(M)(x) &= \inf_{u\in\mathbb{U}} x^T C_t^v x + u^T D_t^v u + \|f_t^v(x,u)\|^2 \frac{f_t^v(x,u)^T}{\|f_t^v(x,u)\|} M \frac{f_t^v(x,u)}{\|f_t^v(x,u)\|} \\
&= \inf_{u\in\mathbb{U}} x^T C_t^v x + u^T D_t^v u + f_t^v(x,u)^T M f_t^v(x,u) \\
&= x^T C_t^v x + \inf_{u\in\mathbb{U}} u^T D_t^v u + f_t^v(x,u)^T M f_t^v(x,u).
\end{aligned} \tag{3.54}$$

As $u \mapsto f_t^v(x,u)$ is linear, $D_t^v \succ 0$ and $M \succeq 0$, we have that

$$g : u \in \mathbb{U} \mapsto u^T D_t^v u + f_t^v(x,u)^T M f_t^v(x,u) \in \mathbb{R}$$

is strictly convex, hence is minimal when $\nabla g(u) = 0$ *i.e.* for $u \in \mathbb{U}$ such that:

$$\big(D_t^v + (B_t^v)^T M B_t^v\big)u + (B_t^v)^T M (A_t^v) x = 0\,. \tag{3.55}$$

Now we will show that $D_t^v + (B_t^v)^T M B_t^v$ is invertible. As $M \in \mathbb{S}_n$ and $D_t^v \in \mathbb{S}_n^+$, for every $u \in \mathbb{U}$, we have:

$$u^T\big(D_t^v + (B_t^v)^T M B_t^v\big)u = \underbrace{u^T D_t^v u}_{>0} + \underbrace{(B_t^v u)^T M (B_t^v u)}_{\geq 0} > 0\,.$$

We have shown that the symmetric matrix $D_t^v + (B_t^v)^T M B_t^v$ is definite positive and thus invertible. We conclude that Equation (3.55) is equivalent to:

$$u = -\big(D_t^v + (B_t^v)^T M B_t^v\big)^{-1} (B_t^v)^T M (A_t^v) x\,. \tag{3.56}$$

Finally, replacing Equation (3.56) in Equation (3.54) we get after simplifications that

$$\begin{aligned}
\mathcal{B}_t^v(M)(x) = x^T\Big(&C_t^v + (A_t^v)^T M A_t^v \\
&- (A_t^v)^T M B_t^v \big(D_t^v + (B_t^v)^T M B_t^v\big)^{-1} (B_t^v)^T M A_t^v\Big)x\,,
\end{aligned}$$

which gives Equation (3.52).
• Equation (3.53) follows from [LR95, Proposition 12.1.1 page 271]. □

3.8 Smallest and greatest eigenvalues

Here we recall some formulas on the lowest and greatest eigenvalues of a matrix. Denote the smallest eigenvalue of a symmetric real matrix M by $\lambda_{\min}(M)$ (every eigenvalue of M is real) and by $\lambda_{\max}(M)$ its greatest eigenvalue.

Proposition 35. *Let $n > 0$ be given. We have the following matrix inequalities.*

$$\forall (A,B) \in \mathbb{S}_n^2 \ , \ \ \lambda_{\max}(A+B) \leq \lambda_{\max}(A) + \lambda_{\max}(B) \ . \tag{3.57a}$$

$$\forall (A,B) \in \mathbb{M}_n \times \mathbb{S}_n \ , \ \ \lambda_{\max}(A^T BA) \leq \lambda_{\max}(A^T A)\lambda_{\max}(B) \ . \tag{3.57b}$$

Proof. For any matrix $M \in \mathbb{M}_n$, the spectral norm of M, $\|M\|_{\text{sp}}$, (See [Cia89, Theorem 1.4.2]) is the subordinate matrix norm of the Euclidean norm on $\mathbb{R}^n$. When the matrix $M \in \mathbb{S}_n$ is real symmetric, we have that $\|M\|_{\text{sp}} = \lambda_{\max}(M)$ and for any real matrix $M \in \mathbb{M}_n$, we have that $\lambda_{\max}(M^T M) = \lambda_{\max}(MM^T) = \|M\|^2$.

• Fix $A, B \in \mathbb{S}_n$, we prove Equation (3.57a). As $A + B \in \mathbb{S}_n$ and using the fact that a subordinate matrix norm is a norm we have that $\lambda_{\max}(A+B) = \|A+B\|_{\text{sp}} \leq \|A\|_{\text{sp}} + \|B\|_{\text{sp}} = \lambda_{\max}(A) + \lambda_{\max}(B)$.

• Fix $(A,B) \in \mathbb{M}_n \times \mathbb{S}_n$. We prove Equation (3.57b) as follows

$$\begin{aligned}
\lambda_{\max}(A^T BA) &= \|A^T BA\| && (\text{as } A^T BA \in \mathbb{S}_n)\\
&\leq \|A^T\|_{\text{sp}} \|B\|_{\text{sp}} \|A\|_{\text{sp}} && (\|\cdot\|_{\text{sp}} \text{ is submultiplicative as a matrix norm})\\
&= \|A\|_{\text{sp}}^2 \|B\|_{\text{sp}} = \lambda_{\max}(A^T A)\lambda_{\max}(B) \ .
\end{aligned}$$

This ends the proof. □

3.9 Homogenization

We explain why, by adding another dimension to the state variable, there is no loss of generality induced by studying pure quadratic forms in Problem 3.37 instead of positive polynomial of degree 2, nor is there one for studying linear dynamics instead of affine dynamics.

First, we define the operator $\mathcal{H}_2$ that maps a function ϕ defined on a finite dimensional vector space $\mathbb{E}$ to a 2-homogeneous function $\mathcal{H}_2(\phi)$ defined on the extended domain $\mathbb{E} \times \mathbb{R}$ as follows

$$\begin{aligned}
\mathcal{H}_2 : \ \ \overline{\mathbb{R}}^{\mathbb{E}} \ &\longrightarrow \ \overline{\mathbb{R}}^{\mathbb{E}\times\mathbb{R}}\\
\phi \ &\longmapsto \ \mathcal{H}_2(\phi) : (z,y) \mapsto y^2 \phi(\tfrac{z}{y}) \text{ if } y \neq 0, \ 0 \text{ otherwise.}
\end{aligned} \tag{3.58}$$

Thus, if ϕ is a positive polynomial of degree 2, then $\mathcal{H}_2(\phi)$ is a 2-homogeneous convex quadratic form (with possibly a mixed term in x and u). In a similar way, we define the operator $\mathcal{H}_1$ that maps any function ϕ defined on a domain $\mathbb{E}$ and taking values in $\mathbb{E}$ to a 1-homogeneous function $\mathcal{H}_1(\phi)$ as follows

$$\begin{aligned}
\mathcal{H}_1 : \ \ \mathbb{E}^{\mathbb{E}} \ &\longrightarrow \ (\mathbb{E}\times\mathbb{R})^{\mathbb{E}\times\mathbb{R}}\\
\phi \ &\longmapsto \ \mathcal{H}_1(\phi) : (z,y) \mapsto (y\phi(\tfrac{z}{y}), y) \text{ if } y \neq 0, \ 0 \text{ otherwise.}
\end{aligned} \tag{3.59}$$

Now consider $(\mathcal{B}_t)_{t\in[\![0,T-1]\!]}$ the Bellman operators associated to Problem 3.37

$$\begin{aligned}
\mathcal{B}_t : \ \ \overline{\mathbb{R}}^{\mathbb{X}} \ &\longrightarrow \ \overline{\mathbb{R}}^{\mathbb{X}}\\
\phi \ &\longmapsto \ \mathcal{B}_t(\phi) : x \mapsto \min_{\substack{u\in\mathbb{U}\\ v\in\mathbb{V}}} c_t^v(x,u) + \phi(f_t^v(x,u)) \ ,
\end{aligned} \tag{3.60}$$

We denote by $(\mathcal{B}_t^{\mathcal{H}})_{t\in[\![0,T-1]\!]}$, the family of Bellman operators obtained through homogenization (with $\mathbb{E} = \mathbb{X} \times \mathbb{U}$) as follows

$$\begin{aligned}
\mathcal{B}_t^{\mathcal{H}} : \ \ \overline{\mathbb{R}}^{\mathbb{X}\times\mathbb{R}} \ &\longrightarrow \ \overline{\mathbb{R}}^{\mathbb{X}\times\mathbb{R}}\\
\varphi \ &\longmapsto \ \mathcal{B}_t^{\mathcal{H}}(\varphi) : (x,y) \mapsto \min_{\substack{u\in\mathbb{U}\\ v\in\mathbb{V}}} \mathcal{H}_2(c_t^v)(x,u,y)\\
&\qquad\qquad + \varphi(\mathcal{H}_1(f_t^v)(x,u,y)).
\end{aligned} \tag{3.61}$$

The next proposition relates the solution of Problem 3.37 with non-homogeneous functions to the solution of the associated homogenized problem.

Proposition 36. *Let $(V_t)_{t\in[\![0,T]\!]}$ (resp. $(\widetilde{V}_t)_{t\in[\![0,T]\!]}$) denote the solutions of the Dynamic Programming Equation (3.3) system of equations associated with the operators $(\mathcal{B}_t)_{t\in[\![0,T-1]\!]}$ defined by Equation (3.60) (resp. $(\mathcal{B}_t^{\mathcal{H}})_{t\in[\![0,T-1]\!]}$ defined by Equation (3.61)) and final cost ψ (resp. $\mathcal{H}_2\big(\psi\big)$). Then, for every $x\in\mathbb{X}$ and $t\in[\![0,T]\!]$, we have that $V_t(x)=\widetilde{V}_t(x,1)$.*

Proof. First, it is easy to prove by backward recursion on time $t\in[\![0,T]\!]$, that the mappings $\widetilde{V}_t$ for every $t\in[\![0,T]\!]$, are 2-homogeneous. Second, we will show by backward recursion on time that, for every $t\in[\![0,T]\!]$,

$$\widetilde{V}_t=\mathcal{H}_2\big(V_t\big). \tag{3.62}$$

Then, the result will follow by evaluating Equation (3.62) at $y=1$. At the final time $t=T$, we have that

$$\widetilde{V}_T:=\mathcal{H}_2\big(\psi\big)=\mathcal{H}_2\big(V_T\big).$$

Now, assume that for some $t\in[\![0,T-1]\!]$, we have that $\widetilde{V}_{t+1}=\mathcal{H}_2\big(V_{t+1}\big)$, for $(x,y)\in\mathbb{X}\times\mathbb{R}$ we successively obtain that

$$\begin{aligned}
\widetilde{V}_t(x,y)&=\mathcal{B}_t^{\mathcal{H}}(\widetilde{V}_{t+1})(x,y)\\
&=\min_{\substack{u\in\mathbb{U}\\ v\in\mathbb{V}}}\mathcal{H}_2\big(c_t^v\big)(x,u,y)+\widetilde{V}_{t+1}\Big(\mathcal{H}_1\big(f_t^v\big)(x,u,y)\Big) && \text{(2-homogeneity of } \widetilde{V}_{t+1})\\
&=\min_{\substack{u\in\mathbb{U}\\ v\in\mathbb{V}}}\mathcal{H}_2\big(c_t^v\big)(x,u,y)+\mathcal{H}_2\big(V_{t+1}\big)\Big(yf_t^v(\frac{x}{y},\frac{u}{y}),y\Big) && \text{(Induction hyp. and def.)}\\
&=\min_{\substack{u\in\mathbb{U}\\ v\in\mathbb{V}}}y^2c_t^v(\frac{x}{y},\frac{u}{y})+y^2V_{t+1}(f_t^v(\frac{x}{y},\frac{u}{y})) && \text{(by Equation (3.58))}\\
&=y^2\min_{\substack{u'\in\mathbb{U}\\ v\in\mathbb{V}}}c_t^v(\frac{x}{y},u')+V_{t+1}(f_t^v(\frac{x}{y},u')) && (u'=u/y)\\
&=y^2\mathcal{B}_t(V_{t+1})(\frac{x}{y})\\
&=\mathcal{H}_2\big(V_t\big)(x,y)\ . && \text{(by Equation (3.58))}
\end{aligned}$$

This ends the proof. □

Lastly, we briefly explain how to get rid of the possible mixed terms in both u and x in the cost functions after homogenization. That is, there is no loss of generality to consider the case of cost functions which are positive polynomials of degree 2 and affine cost than to consider the case studied in §3.5.1, *i.e.* pure quadratic costs and linear functions. From Proposition 36, we have seen that one can consider the case where the cost functions are 2-homogeneous with linear dynamics. Assume (for the sake of simplicity, we omit the discrete control v here) that the cost function c_t is of the form

$$c_t(x,u):=x^TP_1x+x^TP_2u+u^TP_3u,$$

where P_1, P_2 and P_3 are symmetric semidefinite positive matrices of coherent dimensions, with P_3 being definite positive. Moreover, fix a 2-homogeneous convex quadratic form ψ and assume the dynamic f_t to be linear of the form

$$f_t(x,u):=Ax+Bu.$$

Setting $Q_1 := P_1 - \frac{1}{4}P_2P_3^{-1}P_2^T$, $Q_2 := P_3$, $L := \frac{1}{2}P_3^{-1}P_2^T$, one has that the cost function $(x,u) \mapsto c_t'(x,u) := x^TQ_1x + u^TQ_2u$ is a quadratic function without mixing term and $(x,u) \mapsto f_t(x,u) := (A+L)x + Bu$ is linear. Furthermore, by straightforward computations, one can check that c_t and f_t satisfy:

$$c_t(x, u + Lx) = c_t'(x,u) \quad \text{and} \quad f_t(x, u+Lx) = f_t'(x,u). \tag{3.63}$$

Note that as $Q_2 = P_3$, the matrix Q_2 is symmetric definite positive and as c_t is positive and by Equation (3.63) for every $x \in \mathbb{X}$ and $u \in \mathbb{U}$

$$x^TQ_1x = c_t'(x,0) = c_t(x, 0+Lx) \geq 0,$$

then Q_1 is symmetric semidefinite positive. Thus the quadratic function c_t' is convex and a pure quadratic form in the sense of Definition 24.

Consider the Bellman operator associated with the costs c_t' and dynamics f_t':

$$\begin{array}{rccl} \mathcal{B}_t' : & \overline{\mathbb{R}}^{\mathbb{X}} & \longrightarrow & \overline{\mathbb{R}}^{\mathbb{X}} \\ & \phi & \longmapsto & \mathcal{B}_t'(\phi) : x \mapsto \min_{u \in \mathbb{U}} c_t'(x,u) + \phi(f_t'(x,u)) \,, \end{array} \tag{3.64}$$

Thus, for any function $\phi \in \overline{\mathbb{R}}^{\mathbb{X}}$ and every $x \in \mathbb{X}$, recall that $\mathbb{U} = \mathbb{R}^m$ is unconstrained, so we have that

$$\begin{aligned} \mathcal{B}_t(\phi)(x) &= \min_{u \in \mathbb{U}} c_t(x,u) + \phi(f_t(x,u)) \\ &= \min_{u' = u + Lx \in \mathbb{U}} c_t(x, u+Lx) + \phi(f_t(x, u+Lx)) \\ &= \min_{u' \in \mathbb{U}} c_t'(x,u') + \phi(f_t(x,u')) \\ \mathcal{B}_t(\phi)(x) &= \mathcal{B}_t'(\phi)(x). \end{aligned} \tag{3.65}$$

From Equation (3.65), one can deduce by backward recursion (as done in Proposition 36) on the time step $t \in [\![0, T]\!]$, that the value functions V_t (resp. V_t') of the Dynamic Programming problem with Bellman operators $\mathcal{B}_t$ (resp. $\mathcal{B}_t'$) and final cost function ψ (resp. ψ as well) satisfy $V_t = V_t'$.

Chapter 4

Tropical Dynamic Programming: toward the stochastic case

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4.1 Introduction

In this chapter we study multistage stochastic optimal control problems in the hazard-decision framework (hazard comes first, decision second). Starting from a given state x_0, a decision maker observes the outcome w_1 of a random variable $\mathbf{W_1}$, then decides on a control u_0 which induces a *known* cost $c_0^{w_1}(x_0, u_0)$ and the system evolves to a future state x_1 from a *known* dynamic: $x_1 = f_0^{w_1}(x_0, u_0)$. Having observed a new random outcome, the decision maker makes a new decision based on this observation which induces a known cost, then the system evolves to a known future state, and so on until T decisions have been made. At the last step, there are constraints on the final state x_T which are modeled by a final cost function ψ. The decision maker aims to minimize the average cost of her decisions.

Multistage Stochastic optimization Problems (MSP) can be formally described by the following optimization problem

$$
\begin{aligned}
&\min_{(\mathbf{X},\mathbf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t^{\mathbf{W_{t+1}}}(\mathbf{X_t}, \mathbf{U_t}) + \psi(\mathbf{X_T})\right], \\
&\text{s.t. } \mathbf{X_0} = x_0 \text{ given}, \forall t \in [\![0, T-1]\!], \\
&\mathbf{X_{t+1}} = f_t^{\mathbf{W_{t+1}}}(\mathbf{X_t}, \mathbf{U_t}), \\
&\sigma(\mathbf{U_t}) \subset \sigma(\mathbf{X_0}, \mathbf{W_1}, \dots, \mathbf{W_{t+1}}), \text{ (non-anticipativity)}
\end{aligned}
\tag{4.1}
$$

where $(\mathbf{W_t})_{t\in[\![1,T]\!]}$ is a given sequence of independent random variables each with values in some measurable set $(\mathbb{W}_t, \mathcal{W}_t)$. We refer to the random variable $\mathbf{W_{t+1}}$ as a *noise* and throughout the remainder of the chapter we assume the following on the sequence of noises.

Assumption 5. *Each random variable $\mathbf{W_t}$ in Problem (4.1) has finite support and the sequence of random variable $(\mathbf{W_t})_{t\in[\![1,T]\!]}$ is independent.*

One approach to solving MSP problems is by dynamic programming, see for example [Ber16, CCCDL15, PP14, SDR09]. For some integers $n, m \in \mathbb{N}$, denote by $\mathbb{X} = \mathbb{R}^n$ the *state space* and $\mathbb{U} = \mathbb{R}^m$ the *control space*. Both $\mathbb{X}$ and $\mathbb{U}$ are endowed with their Euclidean structure and Borelian structure. We define the pointwise Bellman operators $\mathcal{B}_t^w$ and the average Bellman operators $\mathfrak{B}_t$ for every $t \in [\![0, T-1]\!]$. For each possible realization $w \in \mathbb{W}_{t+1}$ of the noise $\mathbf{W_{t+1}}$, for every function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ taking extended real values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, the function $\mathcal{B}_t^w(\phi)(\cdot) : \mathbb{X} \to \overline{\mathbb{R}}$ is defined by

$$
\forall x \in \mathbb{X}, \ , \ \mathcal{B}_t^w(\phi)(x) = \min_{u\in\mathbb{U}} \Big(c_t^w(x, u) + \phi\big(f_t^w(x, u)\big)\Big) .
$$

Now, the average Bellman operator $\mathfrak{B}_t$ is the mean of all the pointwise Bellman operators with respect to the probability law of $\mathbf{W_{t+1}}$. That is, for every $\phi : \mathbb{X} \to \overline{\mathbb{R}}$, we have that

$$
\forall x \in \mathbb{X} \ , \ \mathfrak{B}_t(\phi)(x) = \mathbb{E}\big[\mathcal{B}_t^{\mathbf{W_{t+1}}}(\phi)(x)\big] = \mathbb{E}\Big[\min_{u\in\mathbb{U}} \Big(c_t^{\mathbf{W_{t+1}}}(x, u) + \phi\big(f_t^{\mathbf{W_{t+1}}}(x, u)\big)\Big)\Big] .
$$

The average Bellman operator can be seen as a one stage operator which computes the value of applying the best (average) control at a given state x. Note that in the hazard-decision framework assumed here, the control is taken after observing the noise. Now, the Dynamic Programming approach states that in order to solve MSP Problems (4.1), it suffices to solve the following system of *Bellman equations* (4.2),

$$
V_T = \psi \quad \text{and} \quad \forall t \in [\![0, T-1]\!], V_t = \mathfrak{B}_t(V_{t+1}) . \tag{4.2}
$$

Solving the Bellman equations means computing recursively backward in time the *(Bellman) value functions* V_t. Finally, the value $V_0(x_0)$ is the solution of the multistage Problem 4.1.

Grid-based approach to compute the value functions suffers from the so-called curse of dimensionality. Assuming that the value functions $\{V_t\}_{t\in[\![0,T]\!]}$ are convex, one approach to bypass this difficulty is proposed by Pereira and Pinto [PP91] with the Stochastic Dual Dynamic Programming (SDDP) algorithm which computes piecewise affine approximations of each value function V_t. At a given iteration $k \in \mathbb{N}^*$ of SDDP, for every time step $t \in [\![0,T]\!]$, the value function V_t is approximated by $\underline{V}_t^k = \max_{\phi\in \underline{F}_k} \phi$ where $\underline{F}_k$ is a finite set of affine functions. Then, given a realization of the noise process $(W_t)_{t\in[\![1,T]\!]}$, the decision maker computes an optimal trajectory associated with the approximations $(\underline{V}_t^k)_{t\in[\![0,T]\!]}$ and adds a new function, ϕ_t^{k+1} (named cut) to the current collection $\underline{F}_t^k$ which defines $\underline{V}_t^k$, that is $\underline{F}_t^{k+1} = \underline{F}_t^k \cup \left\{\phi_t^{k+1}\right\}$. Although SDDP does not involve discretization of the state space, one of its computational bottleneck is the lack of efficient stopping criterion: SDDP easily builds lower approximations of the value function but upper approximations are usually computed through a costly Monte-Carlo scheme.

In order to build upper approximations of the value functions, Min-plus methods were studied (*e.g.* [McE07, Qu14]) for optimal control problems in continuous time. When the value functions $\{V_t\}_{t\in[\![0,T]\!]}$ are convex (or more generally, semiconcave), discrete time adaptations of Min-plus methods build for each $t \in [\![1,T]\!]$ approximations of convex value function V_t as finite infima of convex quadratic forms. That is, at given iteration $k \in \mathbb{N}$, we consider upper approximations defined as $\overline{V}_t^k = \min_{\phi\in\overline{F}_t^k} \phi$, where $\overline{F}_t^k$ is a finite set of convex quadratic forms. Then, a sequence of *trial points* $(x_t^k)_{t\in[\![0,T]\!]}$ is drawn (*e.g.* uniformly on the unit sphere as in [Qu14]) and for every $t \in [\![0,T-1]\!]$ a new function ϕ_t^{k+1} is added, $\overline{F}_t^{k+1} = \overline{F}_t^k \cup \left\{\phi_t^{k+1}\right\}$. The function ϕ_t^{k+1} should be compatible with the Bellman equation, in particular it should be *tight*, *i.e.* the Bellman equations should be satisfied at the trial point,

$$\mathfrak{B}_t\left(\phi_{t+1}^{k+1}\right)\left(x_t^k\right) = \phi_t^{k+1}(x_t^k).$$

In [ACT18], the authors present a common framework for a deterministic version of SDDP and a discrete time version of Min-plus algorithms. Moreover, the authors give sufficient conditions on the way the trial points have to be sampled in order to obtain asymptotic convergence of either upper or lower approximations of the value functions. Under these conditions, the main reason behind the convergence of these algorithm was shown to be that the Bellman equations (4.2) are asymptotically satisfied on all cluster points of possible trial points. In this chapter, we would like to extend the work of [ACT18] by introducing a new algorithm called Tropical Dynamic Programming (TDP).

In [BDZ18, PdF13], are studied approximation schemes where lower approximations are given as a suprema of affine functions and upper approximations are given as a polyhedral function. We aim in this chapter to extend, with TDP, the approach of [BDZ18, PdF13] by considering more generally that lower approximations are max-plus linear combinations of some *basic functions* and upper approximations are min-plus linear combinations of other *basic functions* where basic functions are defined later. TDP can be seen as a tropical variant of parametric approximations used in Adaptive Dynamic Programming (see [Ber19, Pow11]) where the value functions are approximated by linear combinations of basis functions. In this chapter, we will:

1. Extend the deterministic framework of [ACT18] to Lipschitz MSP defined in Equation (4.1) and introduce TDP, see Section 4.2.

2. Ensure that upper and lower approximations converge to the true value functions on a common set of points, see Section 4.3. The main result of Section 4.3 generalizes to any min-plus/max-plus approximation scheme the result of [BDZ18] which was stated for a variant of SDDP.

3. Explicitly give several numerically efficient ways to build upper and lower approximations of the value functions, as min-plus and max-plus linear combinations of some simple functions, see Section 4.4.

4.2 Tropical Dynamical Programming on Lipschitz MSP

4.2.1 Lipschitz MSP with independent finite noises

For every time step $t \in [\![1, T]\!]$, we denote by $\operatorname{supp}(\mathbf{W_t})$ the *support*[1] of the discrete random variable $\mathbf{W}_t$ and for a given subset $X \subset \mathbb{X}$, we denote by π_X the euclidean projector on X. State and control constraints for each time t are modeled in the cost functions which may possibly take infinite values outside of some given sets. Now, we introduce a sequence of sets $\{X_t\}_{t \in [\![0,T]\!]}$ which only depend on the problem data and make the following compactness assumption:

Assumption 6 (Compact state space)**.** *For every time $t \in [\![0, T]\!]$, we assume that the set X_t is a nonempty compact set in $\mathbb{X}$ where the sequence of sets $\{X_t\}_{t \in [\![0,T]\!]}$ is defined, for all $t \in [\![0, T-1]\!]$, by*

$$X_t := \bigcap_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} \pi_{\mathbb{X}}(\operatorname{dom} c_t^w) \ , \tag{4.3}$$

and for $t = T$ by $X_T = \operatorname{dom} \psi$.

For each noise $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, $t \in [\![0, T-1]\!]$, we also introduce the *constraint set-valued mapping* $\mathcal{U}_t^w : \mathbb{X} \rightrightarrows \mathbb{U}$ defined for every $x \in \mathbb{X}$ by

$$\mathcal{U}_t^w(x) := \left\{ u \in \mathbb{U} \,\middle|\, c_t^w(x, u) < +\infty \text{ and } f_t^w(x, u) \in X_{t+1} \right\} . \tag{4.4}$$

We will assume that the data of Problem (4.1) is Lipschitz in the sense defined below. Let us stress that we do not assume structure on the dynamics or costs like linearity or convexity, only that they are Lipschitz.

Assumption 7 (Lipschitz MSP)**.** *For every time $t \in [\![0, T-1]\!]$, we assume that for each $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, the dynamic f_t^w, the cost c_t^w are Lipschitz continuous on $\operatorname{dom} c_t^w$ and the set-valued mapping constraint $\mathcal{U}_t^w$ is Lipschitz continuous on X_t,* i.e. *for some constant $L_{\mathcal{U}_t^w} > 0$, for every $x_1, x_2 \in X_t$, we have*[2]

$$d_{\mathcal{H}}\big(\mathcal{U}_t^w(x_1), \mathcal{U}_t^w(x_2)\big) \leq L_{\mathcal{U}_t^w} \|x_1 - x_2\|. \tag{4.5}$$

[1] The support of the discrete random variable $\mathbf{W}_t$ is equal to the set $\{w \in \mathbb{W}_t \mid \mathbb{P}(\mathbf{W_t} = w) > 0\}$.

Computing a (sharp) Lipschitz constant for the set-valued mapping $\mathcal{U}_t^w : \mathbb{X} \rightrightarrows \mathbb{U}$ is difficult. However, when the graph of the set-valued mapping $\mathcal{U}_t^w$ is polyhedral, as in the linear-polyhedral framework studied in Section 4.4, one can compute a Lipschitz constant for $\mathcal{U}_t^w$. We make the following assumption in order to ensure that the domains of the value functions V_t are chosen by the decision maker. It can be seen as a recourse assumption.

Assumption 8 (Recourse assumption)**.** *Given $t \in [\![0, T-1]\!]$, for every noise realization $w \in \operatorname{supp}(\mathbf{W_t})$ the set-valued mapping $\mathcal{U}_t^w : \mathbb{X} \rightrightarrows \mathbb{U}$ defined in (4.4) is nonempty compact valued.*

A priori, it might be difficult to compute the domain of each value function V_t. However, under the recourse Assumption 8, we have that $\mathrm{dom} V_t := X_t$ and thus the domain of each value function is known to the decision maker.

Lemma 37 (Known domains of V_t)**.** *Under Assumptions 5 and 8, for every $t \in [\![0, T]\!]$, the domain of V_t is equal to X_t.*

Proof. We make the proof by backward induction on time. At time $t = T$, we have $V_T = \psi$ and thus $\mathrm{dom} V_T = \mathrm{dom}\psi = X_T$. Now, for a given $t \in [\![0, T-1]\!]$, we assume that $\mathrm{dom} V_{t+1} = X_{t+1}$ and we prove that $\mathrm{dom} V_t = X_t$.

First, fix $x \in X_t$. Then, for every $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, using Assumption 8, $\mathcal{U}_t^w(x)$ is nonempty and thus $V_t(x) < +\infty$. Moreover, by Assumptions 7 and Assumptions 8 the optimization problem

$$\min_{u \in \mathbb{U}} \Big(c_t^w(x,u) + V_{t+1}\big(f_t^w(x,u)\big) \Big) = \min_{u \in \mathcal{U}_t^w(x)} \Big(c_t^w(x,u) + V_{t+1}\big(f_t^w(x,u)\big) \Big),$$

consists in the minimization of a continuous function in u over a nonempty compact set. Denote by $u^w \in \mathcal{U}_t^w(x)$ a minimizer of this optimization problem. We have, denoting by $\{p_w\}_{w \in \operatorname{supp}(\mathbf{W_{t+1}})}$ the discrete probability law of the random variable $\mathbf{W_{t+1}}$, that

$$\begin{aligned} V_t(x) &= \mathfrak{B}_t\left(V_{t+1}\right)(x) \\ &= \mathbb{E}\big[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(V_{t+1}\right)(x)\big] \\ &= \sum_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} p_w \inf_{u \in \mathbb{U}} \Big(c_t^w(x,u) + V_{t+1}\big(f_t^w(x,u)\big) \Big) \\ &= \sum_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} p_w \Big(c_t^w(x,u^w) + V_{t+1}\big(f_t^w(x,u^w)\big) \Big) . \end{aligned}$$

As every term in the right hand side of the previous equation is finite, we have $V_t(x) < +\infty$ and thus $x \in \mathrm{dom} V_t$.

Second, fix $x \notin X_t$. Then, there exists an element $w \in \operatorname{supp}(\mathbf{W_{t+1}})$ such that $c_t^w(x,u) = +\infty$ for every control $u \in \mathbb{U}$. We therefore have that $V_t(x) = +\infty$ and $x \notin \mathrm{dom} V_t$.

We conclude that $\mathrm{dom} V_t = X_t$ which ends the proof. □

[2] The Hausdorff distance $d_{\mathcal{H}}$ between two nonempty compact sets X_1, X_2 in $\mathbb{X}$ is defined by

$$d_{\mathcal{H}}(X_1, X_2) = \max\big(\max_{x_1 \in X_1} d(x_1, X_2), \max_{x_2 \in X_2} d(X_1, x_2)\big) = \max\big(\max_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2), \max_{x_2 \in X_2} \min_{x_1 \in X_1} d(x_1, x_2)\big).$$

In Section 4.4, it will be crucial for numerical efficiency to have a good estimation of the Lipschitz constant of the function $\mathfrak{B}_t\left(V_{t+1}^k\right)$.

We now prove that under Assumptions 7 and Assumptions 8, the operators $\mathfrak{B}_t$ preserve Lipschitz regularity. Given a L_{t+1}-Lipschitz function ϕ and $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, in order to compute a Lipschitz constant of the function $\mathcal{B}_t^w(\phi)(\cdot)$ we exploit the fact that the set-valued constraint mapping $\mathcal{U}_t^w$ and the data of Problem 4.1 are Lipschitz in the sense of Assumptions 7. This was mostly already done in [ACT18], but for the sake of completeness, we will slightly adapt its statement and proof.

Proposition 38 ($\mathfrak{B}_t$ is Lipschitz regular)**.** *Let $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ be given. Under Assumptions 5 to 8, if for some $L_{t+1} > 0$, ϕ is L_{t+1}-Lipschitz on X_{t+1}, then the function $\mathfrak{B}_t(\phi)$ is L_t-Lipschitz on X_t for some constant $L_t > 0$ which only depends on the data of Problem 4.1 and on L_{t+1}.*

Proof. Let $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ be a L_{t+1}-Lipschitz function on X_{t+1}. We will show that for each $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, the mapping $\mathcal{B}_t^w(\phi)(\cdot)$ is L_w-Lipschitz for some constant L_w which only depends on the data of problem (4.1). Fix $w \in \operatorname{supp}(\mathbf{W_{t+1}})$ and $x_1, x_2 \in X_t$. Denote by u_2^* an optimal control at x_2 and w, that is $u_2^* \in \arg\min_{u \in \mathcal{U}_t^w(x_2)} \Big(c_t^w(x_2,u) + \phi\big(f_t^w(x_2,u)\big)\Big)$, or equivalently, u_2^* satisfies

$$c_t^w(x_2,u_2^*) + \phi\big(f_t^w(x_2,u_2^*)\big) = \mathcal{B}_t^w(\phi)(x_2)\,. \tag{4.6}$$

Then, for every $u_1 \in \mathcal{U}_t^w(x_1)$ we successively have

$$\begin{aligned}
\mathcal{B}_t^w(\phi)(x_1) &\le c_t^w(x_1,u_1) + \phi\big(f_t^w(x_1,u_1)\big) && \text{(as } u_1 \in \mathcal{U}_t^w(x_1) \text{ is admissible)}\\
&\le \mathcal{B}_t^w(\phi)(x_2) + c_t^w(x_1,u_1) + \phi\big(f_t^w(x_1,u_1)\big) - \mathcal{B}_t^w(\phi)(x_2)\\
&= \mathcal{B}_t^w(\phi)(x_2) + \big(c_t^w(x_1,u_1) - c_t^w(x_2,u_2^*)\big) + \Big(\phi\big(f_t^w(x_1,u_1)\big) - \phi\big(f_t^w(x_2,u_2^*)\big)\Big)\\
& && \text{(using (4.6))}\\
&\le \mathcal{B}_t^w(\phi)(x_2) + L\big(\|x_1 - x_2\| + \|u_1 - u_2^*\|\big), && \text{(by Assumption 7)}
\end{aligned}$$

where $L = \max(L_{c_t^w}, L_{t+1}L_{f_t^w})$. Now, as the set-valued mapping $\mathcal{U}_t^w$ is $L_{\mathcal{U}_t^w}$-Lipschitz, there exists $\tilde{u}_1 \in \mathcal{U}_t^w(x_1)$ such that

$$\|\tilde{u}_1 - u_2^*\| \le L_{\mathcal{U}_t^w}\|x_1 - x_2\|.$$

Hence, setting $L_w := \max(L_{c_t^w}, L_{t+1}L_{f_t^w})(1 + L_{\mathcal{U}_t^w})$, we obtain

$$\mathcal{B}_t^w(\phi)(x_1) - \mathcal{B}_t^w(\phi)(x_2) \le L_t\|x_1 - x_2\|.$$

Reverting the role of x_1 and x_2 we get the converse inequality. Hence, we have shown that, for every $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, the mapping $\mathcal{B}_t^w(\phi)$ is L_w-Lipschitz. Thus, setting $L_t = (\sum_w p_w L_w)$, we have

$$\begin{aligned}
\big|\mathfrak{B}_t(\phi)(x_1) - \mathfrak{B}_t(\phi)(x_2)\big| &\le \sum_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} p_w\big|\mathcal{B}_t^w(\phi)(x_1) - \mathcal{B}_t^w(\phi)(x_2)\big|\\
&\le \Big(\sum_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} p_w L_w\Big)\|x_1 - x_2\|\,,
\end{aligned}$$

as $\mathcal{B}_t^w(\phi)$ is L_w-Lipschitz. We obtain that the mapping $\mathfrak{B}_t(\phi)$ is L_t-Lipschitz continuous on $\operatorname{dom}V_t$ and this concludes the proof. $\square$

The explicit constant L_t computed in the proof of Proposition 38 does not exploit any possible structure of the data, *e.g.* linearity. In the presence of such structure or possible decomposition, it is possible to greatly reduce the value of the constant L_t. However, in the sequel, we only care for the regularity result given in Proposition 38 and computing sharper bounds under some specific structure is left for future works.

Using the fact that the final cost function $\psi = V_T$ is Lipschitz on X_T, by successive applications of Proposition 38, one gets the following corollary.

Corollary 39 (The value functions of a Lipschitz MSP are Lipschitz continuous)**.** *For every time step $t \in [\![0, T]\!]$, the value function V_t is L_{V_t}-Lipschitz continuous on X_t where $L_{V_t} > 0$ is a constant which only depends on the data of Problem 4.1.*

4.2.2 Tight and valid selection functions

We formally define now what we call *basic functions*. In the sequel, the notation in bold $\mathbf{F}_t$ will stand for a set of basic functions and F_t will stand for a subset of $\mathbf{F}_t$.

Definition 40 (Basic functions)**.** *Given $t \in [\![0, T]\!]$, a* basic function *$\phi : \mathbb{X} \to \overline{\mathbb{R}}$ is a L_{V_t}-Lipschitz continuous function on X_t, where the constant $L_{V_t} > 0$ is defined in Corollary 39.*

In order to ensure the convergence of the scheme detailed in the introduction, at each iteration of TDP algorithm a basic functions which is be *tight* and *valid* in the sense below is added to the current sets of basic functions. The idea behind these assumptions is to ensure that the Bellman equations (4.2) will gradually be satisfied: it is too numerically hard to find functions satisfying the Bellman equations (4.2), however tightness and validity can be checked efficiently and this will be enough to ensure asymptotic convergence of our TDP algorithm.

There is a dissymmetry for the validity assumption which depends on whether the decision maker wants to build upper or lower approximations of the value functions. In §4.2.4, we will assume that the decision maker has, at hist disposal, two sequences of selection functions $(\overline{S}_t)_{t\in[\![0,T]\!]}$ and $(\underline{S}_t)_{t\in[\![0,T]\!]}$. The former to select basic functions for the upper approximations and the latter for the lower approximations of V_t. We write S_t when designing either $\overline{S}_t$ or $\underline{S}_t$ and denote by $\overline{\mathcal{V}}_{\overline{F}_t}$ (resp. $\underline{\mathcal{V}}_{\underline{F}_t}$) the pointwise infimum (resp. pointwise supremum) of basic functions in $\underline{F}_t$ (resp. in $\overline{F}_t$) when approximating from above (resp. below) a maping V_t. The Figure 4.1 illustrates the formal definition of selection functions given below. Given a set Z, we denote by $\mathcal{P}(Z)$ its *power set*, *i.e.* the set of all subsets included in Z.

Definition 41 (Selection functions)**.** *Let a time step $t \in [\![0, T-1]\!]$ be fixed. A* selection function *or simply* selection function *is a mapping S_t from $\mathcal{P}(\mathbf{F}_{t+1}) \times X_t$ to $\mathbf{F}_t$ satisfying the following properties*

- ***Tightness**: for every set of basic functions $F_{t+1} \subset \mathbf{F}_{t+1}$ and $x \in X_t$, the mappings $S_t\left(F_{t+1}, x\right)$ and $\mathcal{B}_t^w\left(V_{F_{t+1}}\right)(\cdot)$ coincide at point x, that is*

$$S_t\left(F_{t+1}, x\right)(x) = \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(x)\,.$$

- ***Validity**: for every set of basic functions $F_{t+1} \subset \mathbf{F}_{t+1}$ and for every $x \in X_t$ we have*

$$\overline{S}_t\left(F_{t+1}, x\right) \geq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(\cdot)\,, \qquad \text{(when building upper approximations)}$$
$$\underline{S}_t\left(F_{t+1}, x\right) \leq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)(\cdot)\,. \qquad \text{(when building lower approximations)}$$

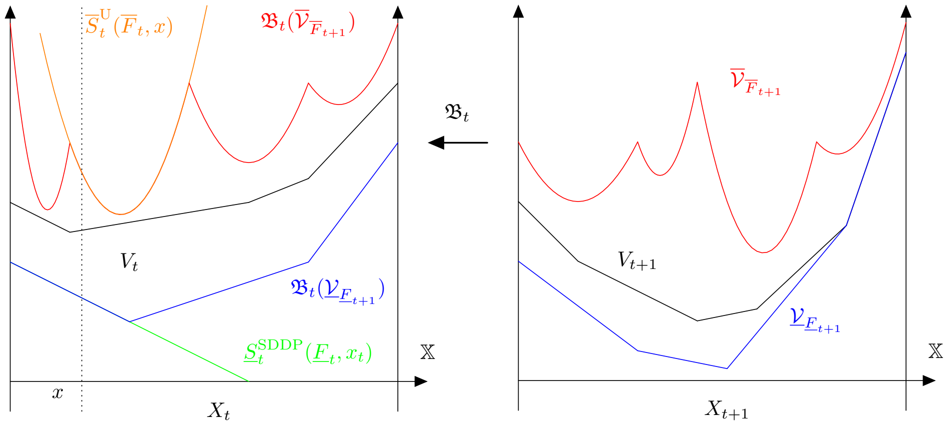

Figure 4.1: Given a time step $t \in [\![0, T-1]\!]$, we illustrate the notions of tightness and validity of selection functions. A selection function takes as input a trial point x in the domain X_t of V_t and a set of basic functions $F_t \subset \mathbf{F}_{t+1}$ building the approximations at the future time step $t+1$ (right: pointwise suprema or infima of the basic functions). Then, the Bellman operator $\mathfrak{B}_t$ translates one step backward in time the right picture to the picture on the left.
Tightness of the selection function enforces that the output is a function equal to the Bellman image of the future approximation of V_{t+1} at x; it is a local property.
Validity enforces that the output of the selection function remains below, or above, the Bellman image the approximation of V_{t+1} everywhere on the domain of V_t; it is a global property.
More details on these examples of selection functions in Section 4.4.

For $t = T$, we also say that $S_T : X_T \to \mathbf{F}_T$ is a selection function *if the mapping S_T is* tight *and* valid *with a modified definition of* tight *and* valid *defined now. The mapping S_T is said to be* valid *if, for every $x \in X_T$, the function $S_T(x)$ remains above (resp. below) the value function at time T when building upper approximations (resp. lower approximations). The mapping S_T is said to be* tight *if it coincides with the value function at point x, that is, for every $x \in X_T$ we have*

$$S_T(x)(x) = V_T(x).$$

Remark 42. *Note that the validity and tightness assumptions at time $t = T$ are stronger than at times $t < T$ as the final cost function is a known data, we are allowed to enforce conditions directly on the value function V_T and not just the on the image of the current approximations at time $t + 1$ as it is the case when $t < T$.*

4.2.3 The problem-child trajectory

From the previous section, given a set of basic functions and a point in $\mathbb{X}$, a selection function is used to computes a new basic function. We explain in this section the algorithm used to select the points which are used for searching new basic functions.

In this section, we present how to build a trajectory of states, without discretization of the whole state space. Selection functions for both upper and lower approximations of V_t will be evaluated along it. This trajectory of states, coined *problem-child* trajectory, was introduced by Baucke, Downward and Zackeri in 2018 (see [BDZ18]) for a variant of SDDP first studied by Philpott, de Matos and Finardi in 2013 (see [PdF13]).

We present in Algorithm 2 a generalized problem-child trajectory, it is the sequence of states on which we evaluate selection functions.

Algorithm 2 Problem-child trajectory

Input: Two sequences of functions from $\mathbb{X}$ to $\overline{\mathbb{R}}$, $\overline{\phi}_0, \dots, \overline{\phi}_T$ and $\underline{\phi}_0, \dots, \underline{\phi}_T$ with respective domains equal to $\mathrm{dom} V_t$.

Output: A sequence of states $(x_0^*, \dots, x_T^*)$.

Set $x_0^* := x_0$.

for $t \in [\![0, T-1]\!]$ **do**

for $w \in \mathrm{supp}\,(\mathbf{W_{t+1}})$ **do**

Compute an optimal control u_t^w for $\underline{\phi}_{t+1}$ at x_t^* for the given w

$$u_t^w \in \operatorname*{arg\,min}_{u \in \mathbb{U}} \Big(c_t^w(x_t^*, u) + \underline{\phi}_{t+1}\big(f_t^w(x_t^*, u)\big)\Big). \tag{4.7}$$

end for

Compute "the worst" noise $w^* \in \mathrm{supp}\,(\mathbf{W_{t+1}})$. *i.e.* the one which maximizes the "future" gap

$$w^* \in \operatorname*{arg\,max}_{w \in \mathrm{supp}(\mathbf{W_{t+1}})} \big(\overline{\phi}_{t+1} - \underline{\phi}_{t+1}\big)\big(f_t^w(x_t^*, u_t^w)\big).$$

Compute the next state dynamics for noise w^* and associated optimal control $u_t^{w^*}$:

$$x_{t+1}^* = f_t^{w^*}(x_t^*, u_t^{w^*}) \ .$$

end for

One can interpret the problem child trajectory as the worst (for the noises) optimal trajectory (for the controls) of the lower approximations. It is worth mentioning that the problem-child trajectory is deterministic. The approximations of the value functions will be refined along the problem-child trajectory only, thus avoiding a discretization of the state space. The main computational drawback of such approach is the need to solve Problem (4.7) $|\mathrm{supp}\,(\mathbf{W_1})| + \ldots + |\mathrm{supp}\,(\mathbf{W_T})|$ times. Except on special instances like the linear-quadratic case, one cannot expect to find a closed form expression for solutions of Equation (4.7). However, we will see in Section 4.4 examples where Problem (4.7) can be solved by Linear Programming or Quadratic Programming. Simply put, if one can solve efficiently the deterministic problem (4.7) and if at each time step the set $\mathrm{supp}\,(\mathbf{W_t})$ remains of small cardinality, then using the problem-child trajectory and the Tropical Dynamical Algorithm presented below in Section 4.2.4, one can solve MSP problems with finite independent noises efficiently. This might be an interesting framework in practice if at each step the decision maker has a few different forecasts on which her inputs are significantly different.

4.2.4 Tropical Dynamic Programming

Algorithm 3 Tropical Dynamic Programming (TDP)

Input: For every $t \in [\![0, T]\!]$, two compatible selection functions $\overline{S}_t$ and $\underline{S}_t$. A sequence of independent random variables $(\mathbf{W_t})_{t \in [\![0,T-1]\!]}$, each with finite support.

Output: For every $t \in [\![0, T]\!]$, two sequence of sets $(\overline{F}_t^k)_{k \in \mathbb{N}}$, $(\underline{F}_t^k)_{k \in \mathbb{N}}$ and the associated functions $\overline{V}_t^k = \inf_{\phi \in \overline{F}_t^k} \phi$ and $\underline{V}_t^k = \sup_{\phi \in \underline{F}_t^k} \phi$.

Define for every $t \in [\![0, T]\!]$, $\overline{F}_t^0 := \emptyset$ and $\underline{F}_t^0 := \emptyset$.

for $k \geq 0$ **do**

Forward phase

Compute the problem-child trajectory $(x_t^k)_{t \in [\![0,T]\!]}$ for the sequences $(\overline{\mathcal{V}}_{\overline{F}_t^k})_{t \in [\![0,T]\!]}$ and $(\underline{\mathcal{V}}_{\underline{F}_t^k})_{t \in [\![0,T]\!]}$ using Algorithm 2.

Backward phase

At $t = T$, compute new basic functions $\overline{\phi}_T := \overline{S}_T\left(x_T^k\right)$ and $\underline{\phi}_T := \underline{S}_T(x_T^k)$.

Add them to current collections, $\overline{F}_T^{k+1} := \overline{F}_T^k \cup \left\{\overline{\phi}_T\right\}$ and $\underline{F}_T^{k+1} := \underline{F}_T^k \cup \left\{\underline{\phi}_T\right\}$.

for t from $T-1$ to 0 **do**

Compute new basic functions: $\overline{\phi}_t := \overline{S}_t\left(\overline{F}_{t+1}^{k+1}, x_t^k\right)$ and $\underline{\phi}_t := \underline{S}_t\left(\underline{F}_{t+1}^{k+1}, x_t^k\right)$.

Add them to the current collections: $\overline{F}_t^{k+1} := \overline{F}_t^k \cup \left\{\overline{\phi}_t\right\}$ and $\underline{F}_t^{k+1} := \underline{F}_t^k \cup \left\{\underline{\phi}_t\right\}$.

end for

end for

4.3 Asymptotic convergence of TDP along the problem-child trajectory

In this section, we will assume that Assumptions (5) to (8) are satisfied. We recall that, under Assumption 8, the sequence of sets $\{X_t\}_{t \in [\![0,T]\!]}$ defined in Equation (4.3) is known and for all $t \in [\![0, T]\!]$ the domain of V_t is equal to X_t. We denote by $(x_t^k)_{k \in \mathbb{N}}$ the sequence of trial points generated by TDP algorithm at time t for every $t \in [\![0, T]\!]$, and by $(u_t^k)_{k \in \mathbb{N}}$ and $(w_t^k)_{k \in \mathbb{N}}$ the optimal control and worst noises sequences associated for each time t with x_t^k in the problem-child trajectory in Algorithm 2.

Now, observe that for every $t \in [\![0, T]\!]$, the approximations of V_t generated by TDP, $(\overline{V}_t^k)_{k \in \mathbb{N}}$ and $(\underline{V}_t^k)_{k \in \mathbb{N}}$, are respectively non-increasing and non-decreasing. Moreover, for every index $k \in \mathbb{N}$ we have

$$\underline{V}_t^k \leq V_t \leq \overline{V}_t^k .$$

We refer to [ACT18, Lemma 7] for a proof. Observing that the basic functions are all L_{V_t}-Lipschitz continuous on X_t one can prove using Arzelà-Ascoli Theorem the following proposition.

Proposition 43 (Existence of an approximating limit)**.** *Let $t \in [\![0, T]\!]$ be fixed, the sequences of functions $\left(\underline{V}_t^k\right)_{k \in \mathbb{N}}$ and $\left(\overline{V}_t^k\right)_{k \in \mathbb{N}}$ generated by Algorithm 3 converge uniformly on X_t to two functions $\underline{V}_t^*$ and $\overline{V}_t^*$. Moreover, $\underline{V}_t^*$ and $\overline{V}_t^*$ are L_{V_t}-Lipschitz continuous on X_t and satisfy $\underline{V}_t^* \leq V_t \leq \overline{V}_t^*$.*

Proof. Omitted as it is slight rewriting of [ACT18, Proposition 9]. $\square$

If we extract a converging subsequence of trial points, then using compactness, extracting a subsubsequence if needed, one can find a subsequence of trial points, and associated controls that jointly converge.

Lemma 44. *Fix $t \in [\![0, T-1]\!]$ and denote by $(x_t^k)_{k\in\mathbb{N}}$ the sequence of trial points generated by Algorithm 3 and by $(u_t^k)_{k\in\mathbb{N}}$ the sequence of associated optimal controls. There exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and a state-control ordered pair $(x_t^*, u_t^*) \in X_t \times \mathbb{U}$ such that*

$$\begin{cases} x_t^{\sigma(k)} \underset{k\to+\infty}{\longrightarrow} x_t^*, \\ u_t^{\sigma(k)} \underset{k\to+\infty}{\longrightarrow} u_t^*. \end{cases} \tag{4.8}$$

Proof. Fix a time step $t \in [\![0, T-1]\!]$. First, by construction of the problem-child trajectories, the sequence $(x_t^k)_{k\in\mathbb{N}}$ remains in the subset X_t that is $x_t^k \in X_t$ for all $k \in \mathbb{N}$.

Second, we show that the sequence of controls $(u_t^k)_{k\in\mathbb{N}}$ is included in a compact subset of $\mathbb{U}$. Under Assumption 6, X_t is a nonempty compact subset of $\mathbb{X}$. For every $w \in \operatorname{supp}(\mathbf{W_{t+1}})$ the set-valued mapping $\mathcal{U}_t^w$ is Lipschitz continuous on X_t under Assumption 7, hence upper semicontinuous[3] on X_t. Moreover, under recourse Assumption 8, $\mathcal{U}_t^w$ is nonempty compact valued. Thus, by [AE84, Proposition 11 p.112], its image $\mathcal{U}_t^w(X_t)$ of the compact X_t is a nonempty compact subset of $\mathbb{U}$. Finally as the random variable $\mathbf{W_{t+1}}$ has a finite support under Assumption 5, the set $U_t := \cup_{w\in\operatorname{supp}(\mathbf{W_{t+1}})} \mathcal{U}_t^w(X_t)$ is a compact subset of $\mathbb{U}$. The sequence $(u_t^k)_{k\in\mathbb{N}}$ remains in U_t and therefore we conclude that it remains in a compact subset of $\mathbb{U}$.

Finally, as the sequence $(x_t^k, u_t^k)_{k\in\mathbb{N}}$ is included in the compact subset $X_t \times U_t$ of $\mathbb{X}\times\mathbb{U}$, one can extract a converging subsequence, hence the result. $\square$

Lastly, we will use the following elementary lemma, whose proof is omitted.

Lemma 45. *Let $(g^k)_{k\in\mathbb{N}}$ be a sequence of functions that converges uniformly on a compact K to a function g^*. If $(y^k)_{k\in\mathbb{N}}$ is a sequence of points in K that converges to $y^* \in K$ then one has*

$$g^k(y^k) \underset{k\to+\infty}{\longrightarrow} g^*(y^*).$$

We now state the main result of this chapter. For a fixed $t \in [\![0, T]\!]$, as the Bellman value function V_t is always sandwiched between the sequences of upper and lower approximations, if the gap between upper and lower approximations vanishes at a given state value x, then upper and lower approximations will both converge to $V_t(x)$. Note that, even though a MSP is a stochastic optimization problem, the convergence result below is not. Indeed, we have assumed (see Assumption 5) that the noises have finite supports, thus under careful selection of scenario as done by the Problem-child trajectory, we get a "sure" convergence.

Theorem 46 (Vanishing gap along problem-child trajectories)**.** *Denote by $(\overline{V}_t^k)_{k\in\mathbb{N}}$ and $(\underline{V}_t^k)_{k\in\mathbb{N}}$ the approximations generated by the Tropical Dynamic Programming algorithm. For every $k \in \mathbb{N}$ denote by $(x_t^k)_{0\le t\le T}$ the current Problem-child trajectory.*

[3] The compact valued set-valued mapping $\mathcal{U}_t^w : \mathbb{X} \rightrightarrows \mathbb{U}$ is *upper semicontinuous* on X_t if, for all $x_t \in X_t$, if an open set $U \subset \mathbb{U}$ contains $\mathcal{U}_t^w(x_t)$ then $\{x \in \mathbb{X} \mid \mathcal{U}_t^w x \subset U\}$ contains a neighborhood of x_t.

Then, under Assumptions 5 to 8, we have that

$$\overline{V}_t^k(x_t^k) - \underline{V}_t^k(x_t^k) \underset{k\to+\infty}{\longrightarrow} 0 \quad \textit{and} \quad \overline{V}_t^*(x_t^*) = \underline{V}_t^*(x_t^*) \ ,$$

for every accumulation point x_t^* of the sequence $(x_t^k)_{k\in\mathbb{N}}$.

Proof. We prove by backward recursion that, for every $t \in [\![0, T]\!]$, for every accumulation point x_t^* of the sequence $(x_t^k)_{k\in\mathbb{N}}$, we have

$$\overline{V}_t^*(x_t^*) = \underline{V}_t^*(x_t^*). \tag{4.9}$$

By a direct consequence of the tightness of the selection functions one has that for every $k \in \mathbb{N}$, $\overline{V}_T^k(x_T^k) = V_T(x_T^k) = \underline{V}_T^k(x_T^k)$. Thus, the equality (4.9) holds for $t = T$ by Lemma 45.

Now assume that for some $t \in [\![0, T-1]\!]$, for every accumulation point x_{t+1}^* of $(x_{t+1}^k)_{k\in\mathbb{N}}$ we have

$$\overline{V}_{t+1}^*(x_{t+1}^*) = \underline{V}_{t+1}^*(x_{t+1}^*). \tag{4.10}$$

On the one hand, for every index $k \in \mathbb{N}$ one has

$$\begin{aligned}
\underline{V}_t^{k+1}(x_t^k) &= \mathfrak{B}_t\left(\underline{V}_{t+1}^{k+1}\right)\left(x_t^k\right), && \text{(Tightness)}\\
&\geq \mathfrak{B}_t\left(\underline{V}_{t+1}^{k}\right)\left(x_t^k\right), && \text{(Monotonicity)}\\
&= \mathbb{E}\Big[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(\underline{V}_{t+1}^{k}\right)\left(x_t^k\right)\Big] && \text{(by definition of } \mathfrak{B}_t)\\
&= \mathbb{E}\Big[c_t^{\mathbf{W_{t+1}}}(x_t^k, u_t^{\mathbf{W_{t+1}}}) + \underline{V}_{t+1}^k\big(f_t^{\mathbf{W_{t+1}}}(x_t^k, u_t^{\mathbf{W_{t+1}}})\big)\Big] && \text{(by Equation (4.7))}\\
&= \sum_{w\in\mathrm{supp}(\mathbf{W_{t+1}})} \mathbb{P}\big[\mathbf{W_{t+1}} = w\big]\Big(c_t^w(x_t^k, u_t^w) + \underline{V}_{t+1}^k\big(f_t^w(x_t^k, u_t^w)\big)\Big) \ .
\end{aligned}$$

On the other hand, for every index $k \in \mathbb{N}$ one has

$$\begin{aligned}
\overline{V}_t^{k+1}(x_t^k) &= \mathfrak{B}_t\left(\overline{V}_{t+1}^{k+1}\right)\left(x_t^k\right), && \text{(Tightness)}\\
&= \mathbb{E}\Big[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(\overline{V}_{t+1}^{k+1}\right)\left(x_t^k\right)\Big]\\
&\leq \mathbb{E}\Big[c_t^{\mathbf{W_{t+1}}}(x_t^k, u_t^{\mathbf{W_{t+1}}}) + \overline{V}_{t+1}^{k+1}\big(f_t^{\mathbf{W_{t+1}}}(x_t^k, u_t^{\mathbf{W_{t+1}}})\big)\Big] && \text{(Def. of pointwise } \mathcal{B}_t^w)\\
&\leq \mathbb{E}\Big[c_t^{\mathbf{W_{t+1}}}(x_t^k, u_t^{\mathbf{W_{t+1}}}) + \overline{V}_{t+1}^{k}\big(f_t^{\mathbf{W_{t+1}}}(x_t^k, u_t^{\mathbf{W_{t+1}}})\big)\Big] && \text{(Monotonicity)}\\
&= \sum_{w\in\mathrm{supp}(\mathbf{W_{t+1}})} \mathbb{P}[\mathbf{W_{t+1}} = w]\Big(c_t^w(x_t^k, u_t^w) + \overline{V}_{t+1}^k\big(f_t^w(x_t^k, u_t^w)\big)\Big) \ .
\end{aligned}$$

By definition of the problem-child trajectory, recall that $u_t^k := u_t^{w_t^k}$, thus we have $x_{t+1}^k := f_t^{w_t^k}(x_t^k, u_t^k)$ and for every $k \in \mathbb{N}$

$$\begin{aligned}
0 \leq \overline{V}_t^{k+1}(x_t^k) - \underline{V}_t^{k+1}(x_t^k) &\leq \sum_{w\in\mathrm{supp}(\mathbf{W_{t+1}})} \mathbb{P}\big[\mathbf{W_{t+1}} = w\big]\Big(\big(\overline{V}_{t+1}^k - \underline{V}_{t+1}^k\big)\big(f_t^w(x_t^k, u_t^w)\big)\Big)\\
&\leq \overline{V}_{t+1}^k(x_{t+1}^k) - \underline{V}_{t+1}^k(x_{t+1}^k) \ .
\end{aligned}$$

Thus, we get that for every function $\sigma : \mathbb{N} \to \mathbb{N}$

$$0 \leq \overline{V}_t^{\sigma(k)+1}(x_t^{\sigma(k)}) - \underline{V}_t^{\sigma(k)+1}(x_t^{\sigma(k)}) \leq \overline{V}_{t+1}^{\sigma(k)}(x_{t+1}^{\sigma(k)}) - \underline{V}_{t+1}^{\sigma(k)}(x_{t+1}^{\sigma(k)}) \ . \tag{4.11}$$

By Lemma 44 and continuity of the dynamics, there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that the sequence of future states $x_{t+1}^{\sigma(k)} = f_t^{w_{t+1}^{\sigma(k)}}(x_t^{\sigma(k)}, u_t^{\sigma(k)})$, $k \in \mathbb{N}$, converges to some future state $x_{t+1}^* \in X_{t+1}$. Thus, by Lemma 45 applied to the $2L_{V_{t+1}}$-Lipschitz functions $g^k := \overline{V}_{t+1}^{\sigma(k)} - \underline{V}_{t+1}^{\sigma(k)}$, $k \in \mathbb{N}$ and the sequence $y^k := x_{t+1}^{\sigma(k)}$, $k \in \mathbb{N}$ we have that

$$\overline{V}_{t+1}^{\sigma(k)}(x_{t+1}^{\sigma(k)}) - \underline{V}_{t+1}^{\sigma(k)}(x_{t+1}^{\sigma(k)}) \underset{k \to +\infty}{\longrightarrow} \overline{V}_{t+1}^*(x_{t+1}^*) - \underline{V}_{t+1}^*(x_{t+1}^*) \ .$$

Likewise, by Lemma 45 applied to the $2L_{V_t}$-Lipschitz functions $g^k := \overline{V}_t^{\sigma(k)+1} - \underline{V}_t^{\sigma(k)+1}$, $k \in \mathbb{N}$ and the sequence $y^k := x_t^{\sigma(k)}$, $k \in \mathbb{N}$ we have that

$$\overline{V}_t^{\sigma(k)+1}(x_t^{\sigma(k)}) - \underline{V}_t^{\sigma(k)+1}(x_t^{\sigma(k)}) \underset{k \to +\infty}{\longrightarrow} \overline{V}_t^*(x_t^*) - \underline{V}_t^*(x_t^*) \ .$$

Thus, taking the limit in k in Equation (4.11), we have that

$$0 \leq \overline{V}_t^*(x_t^*) - \underline{V}_t^*(x_t^*) \leq \overline{V}_{t+1}^*(x_{t+1}^*) - \underline{V}_{t+1}^*(x_{t+1}^*).$$

By induction hypothesis (4.10) we have that $\overline{V}_{t+1}^*(x_{t+1}^*) - \underline{V}_{t+1}^*(x_{t+1}^*) = 0$. Thus, we have shown that

$$\overline{V}_t^*(x_t^*) = \underline{V}_t^*(x_t^*) \ .$$

This concludes the proof. □

4.4 Illustrations in the linear-polyhedral framework

In this section, we first present a class of Lipschitz MSP that we call *linear-polyhedral* MSP where dynamics are linear and costs are polyhedral, *i.e.* functions with convex polyhedral epigraph. Second, we give three selection functions, one which generates polyhedral lower approximations (see §4.4.2) and two which generates upper approximations, one as infima of U-shaped functions (see §4.4.3) and one as infima of V-shaped functions (see §4.4.4).

In Table 4.1 we illustrate the flexibility made available by TDP to the decision maker to approximate value functions. Implementations were done in the programming language Julia 1.4.2 using the optimization interface JuMP 0.21.3, [DHL17]. The code is available online (`https://github.com/BenoitTran/TDP`) as a collection of Julia Notebooks.

Selection mapping	Tight	Valid	Averaged	Computational difficulty
SDDP	✓	✓	✓	$\text{Card}(\mathbf{W_{t+1}})$ LPs
U	✓	✗	✓	$\text{Card}(\mathbf{W_{t+1}}) \cdot \text{Card}(F)$ QPs
V	✓	✓	✗	one LP

Table 4.1: Summary of the three selection functions presented in Section 4.4.

4.4.1 Linear-polyhedral MSP

We want to solve MSPs where the dynamics are linear and the costs are polyhedral. That is, we want to solve optimization problems of the form (4.1) where for each time step $t \in [\![0, T-1]\!]$ the state dynamics is linear, $f_t^w(x,u) = A_t^w x + B_t^w u$ for some matrices A_t^w and B_t^w of suitable dimensions and the cost is polyhedral:

$$c_t^w(x,u) = \max_{i \in I_t} \left\langle c_t^{i,w},\, (x;u) \right\rangle + d_t^{i,w} + \delta_{P_t^w}(x,u) \;, \tag{4.12}$$

where I_t is a finite set, $c^{i,w} \in \mathbb{X}\times\mathbb{U}$, $d_t^{i,w}$ is a scalar and P_t^w is a convex polyhedron. The final cost function ψ is of the form $\psi(x) = \max_{i \in I_T} \langle c_T^i,\, x\rangle + d_T^i + \delta_{X_T}$ where X_T is a nonempty convex polytope. We assume that Assumption 5, 6 and 8 are satisfied.

Proposition 47 (Linear-polyhedral MSP are Lipschitz MSP)**.** *Linear-polyhedral MSP are Lipschitz MSP in the sense of Assumption 7.*

Proof. By construction, the costs c_t^w and the dynamics f_t^w are Lipschitz continuous with explicit constants. We show that, for every $t \in [\![0, T-1]\!]$ and every $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, the constraint set-valued mapping $\mathcal{U}_t^w$ is Lipschitz continuous. From [RW09, Example 9.35], it is enough to show that the graph of $\mathcal{U}_t^w$ is a convex polyhedron. By assumption $\operatorname{dom} c_t^w$ is a convex polyhedron and by recourse assumption, $\operatorname{Graph} \mathcal{U}_t^w$ is nonempty. As a nonempty intersection of convex polyhedrons is a convex polyhedron, we only have to show that $\{(x,u) \in \mathbb{X}\times\mathbb{U} \,|\, f_t^w(x,u) \in X_{t+1}\}$ is a convex polyhedron as well.

Using Equation (4.3) we have that X_{t+1} is given by $X_{t+1} = \cap_{w \in \operatorname{supp}(\mathbf{W_{t+2}})} \pi_{\mathbb{X}}\big(\operatorname{dom} c_{t+1}^w\big)$, which is the nonempty intersection of convex polyhedrons. Thus, X_{t+1} is a convex polyhedron which implies that there exists a matrix Q_{t+1} and a vector b_{t+1} such that $X_{t+1} = \big\{x \in \mathbb{X} \,\big|\, Q_{t+1}x \leq b_{t+1}\big\}$. Therefore, we obtain that the two following sets coincide

$$\big\{(x,u) \in \mathbb{X}\times\mathbb{U} \,\big|\, f_t^w(x,u) \in X_{t+1}\big\} = \big\{(x,u) \in \mathbb{X}\times\mathbb{U} \,\big|\, Q_{t+1}A_t^w x + Q_{t+1}B_t^w u \leq b_{t+1}\big\} \;.$$

The latter being convex polyhedral we obtain that the former is convex polyhedral. This ends the proof. □

Now, observe that as linear-polyhedral MSP are Lipschitz MSP, by Corollary 39, the value function V_t is L_{V_t}-Lipschitz continuous on X_t for all $t \in [\![0, T]\!]$. Moreover, under the recourse assumption 8 we can show that the Bellman operators $\mathfrak{B}_{t t \in [\![0, T-1]\!]}$ preserve polyhedrality in the sense defined below.

Lemma 48 ($\mathfrak{B}_t$ preserves polyhedrality)**.** *For every $t \in [\![0, T-1]\!]$, if $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ is a* polyhedral function*, i.e. its epigraph is a convex polyhedron, then $\mathfrak{B}_t(\phi)$ is a polyhedral function as well.*

Proof. For every $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, we have shown in the proof of Proposition 47 that the graph of $\mathcal{U}_t^w$ is a convex polyhedron. Thus, $(x,u) \mapsto c_t^w(x,u) + \phi\big(f_t^w(x,u)\big) + \delta_{\operatorname{Graph} \mathcal{U}_t^w}(x,u)$ is convex polyhedral and by [BL06, Proposition 5.1.8.e], $\mathcal{B}_t^w(\phi)$ is polyhedral as well. Finally, under Assumption 5, we deduce that $\mathfrak{B}_t(\phi) := \sum_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} \mathcal{B}_t^w(\phi)$ is polyhedral as a finite sum of polyhedral functions. This ends the proof. □

4.4.2 SDDP lower approximations

Stochastic Dual Dynamic Programming is a popular algorithm which was introduced by Perreira and Pinto in 1991 (see [PP91]) and studied extensively since then, *e.g.* [ACd19, BDZ18, BFFd20, Gui14, PG08, PdF13, Sha11, ZAS18].

Lemma 48 is the main intuitive justification of using SDDP in linear-polyhedral MSPs: if the final cost function is polyhedral, as the operators $\{\mathfrak{B}_t\}_{t\in[\![0,T-1]\!]}$ preserve polyhedrality, by backward induction on time, we obtain that the value function V_t is polyhedral for every $t \in [\![0,T]\!]$. Hence, the decision maker might be tempted to construct polyhedral approximations of V_t as well.

We now present a way to generate polyhedral lower approximations of value functions, as done in the literature of SDDP, by defining a proper selection mapping. When the value functions are convex, SDDP builds lower approximations as suprema of affine cuts. We put SDDP in TDP's framework by constructing a lower selection function.

First, for every time step $t \in [\![0,T]\!]$, define the set of basic functions,

$$\underline{\mathbf{F}}_t^{\mathrm{SDDP}} := \left\{ \langle a, \cdot \rangle + b + \delta_{X_t} \,\middle|\, (a,b) \in \mathbb{X}\times\mathbb{R} \text{ s.t. } \|a\| \leq L_{V_t} \right\} .$$

At time $t = T$, given a trial point $x \in X_T$, we define $\underline{S}_T^{\mathrm{SDDP}}(x) = \langle a_x, \cdot - x \rangle + b_x$, where a_x is a subgradient of the convex polyhedral function ψ at x and $b_x = \psi(x)$. Tightness and validity of $\underline{S}_T^{\mathrm{SDDP}}$ follow from the given expression. Now, for $t \in [\![0, T-1]\!]$, we compute a tight and valid cut for $\mathcal{B}_t^w$ for each possible value of the noise w, then average it to get a tight and valid cut for $\mathfrak{B}_t$. The details are given in Algorithm 4.

Algorithm 4 SDDP Selection function $\underline{S}_t^{\mathrm{SDDP}}$ for $t < T$

Input: A set of basic functions $\underline{F}_{t+1} \subset \underline{\mathbf{F}}_{t+1}^{\mathrm{SDDP}}$ and a trial point $x_t \in X_t$.
Output: A tight and valid basic function $\underline{\phi}_t \in \underline{\mathbf{F}}_t^{\mathrm{SDDP}}$.
 for $w \in \mathrm{supp}\,(\mathbf{W_{t+1}})$ **do**
  Solve by linear programming $b^w := \mathcal{B}_t^w\left(\underline{\mathcal{V}}_{\underline{F}_{t+1}}\right)(x)$ and compute a subgradient a^w of $\mathcal{B}_t^w\left(\underline{\mathcal{V}}_{\underline{F}_{t+1}}\right)$ at x.
 end for
 Set $\underline{\phi} := \langle a, \cdot \rangle + b + \delta_{X_t}$ where $a := \sum_{w\in\mathrm{supp}(\mathbf{W_{t+1}})} p_w a^w$ and $b = \sum_{w\in\mathrm{supp}(\mathbf{W_{t+1}})} p_w b^w$.

We say that S_t^w is a selection function for $\mathcal{B}_t^w$, for a given noise value $w \in \mathrm{supp}\,(\mathbf{W_{t+1}})$ if Definition 41 is satisfied when replacing $\mathfrak{B}_t$ by $\mathcal{B}_t^w$. We now prove that $\underline{S}_t^{\mathrm{SDDP}}$ is a selection function, *i.e.* it is tight and valid in the sense of Definition 41. It follows from the general fact that by averaging functions which are tight and valid for the pointwise Bellman operators $\mathcal{B}_t^w$, $w \in \mathrm{supp}\,(\mathbf{W_{t+1}})$, then one get a tight and valid function for the average Bellman operator $\mathfrak{B}_t$. Note that the average of affine functions is still an affine function, the set of basic functions $\underline{\mathbf{F}}_t^{\mathrm{SDDP}}$ is stable by averaging.

Lemma 49. *Let a time step $t \in [\![0, T-1]\!]$ be fixed and let be given for every noise value $w \in \mathrm{supp}\,(\mathbf{W_{t+1}})$ a selection function S_t^w for $\mathcal{B}_t^w$. Then, the mapping S_t defined by $S_t = \mathbb{E}[S_t^{\mathbf{W_{t+1}}}]$ is a selection mapping for $\mathfrak{B}_t$.*

Proof. Fix $t \in [\![0, T-1]\!]$. Given a trial point $x \in X_t$ and a set of basic functions F, the pointwise tightness (resp. validity) equality (resp. inequality) is satisfied for every realization

w of the noise $\mathbf{W_{t+1}}$, that is

$$S_t^w(F,x)(x) = \mathcal{B}_t^w(\mathcal{V}_F)(x), \qquad \text{(Pointwise tightness)}$$
$$S_t^w(F,x) \geq \mathcal{B}_t^w\left(\overline{\mathcal{V}}_F\right), \qquad \text{(Pointwise validity when building upper approximations)}$$
$$S_t^w(F,x) \leq \mathcal{B}_t^w\left(\underline{\mathcal{V}}_F\right). \qquad \text{(Pointwise validity when building lower approximations)}$$

Recall that $\mathfrak{B}_t(\mathcal{V}_F)(x) = \mathbb{E}[\mathcal{B}_t^{\mathbf{W_{t+1}}}(\mathcal{V}_F)(x)]$, thus taking the expectation in the above equality and inequalities, one gets the lemma. □

Proposition 50 (SDDP Selection function)**.** *For every $t \in [\![0,T]\!]$, the mapping $\underline{S}_t^{\mathrm{SDDP}}$ is a selection function in the sense of Definition 41.*

Proof. For $t=T$, for every $x_T \in X_T$, by construction we have

$$\underline{S}_T^{\mathrm{SDDP}}(x_T) = \psi(x_T) = V_T(x_T).$$

Thus, $\underline{S}_T^{\mathrm{SDDP}}$ is tight and it is valid as $\underline{S}_T^{\mathrm{SDDP}}(x_T) = \langle a, \cdot - x_T\rangle + \psi(x_T)$ is an affine minorant of the convex function ψ which is exact at x_T. Now, fix $t \in [\![0, T-1]\!]$, a set of basic functions $\underline{F}_t \subset \underline{\mathbf{F}}_t^{\mathrm{SDDP}}$ and a trial point $x_t \in X_t$. By construction, $\underline{S}_t^{\mathrm{SDDP}}$ is tight as we have

$$\underline{S}_t^{\mathrm{SDDP}}(\underline{F}_t, x_t)(x_t) = \langle a,\, x_t - x_t\rangle + \mathbb{E}\Big[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(\underline{\mathcal{V}}_{\underline{F}_t}\right)(x_t)\Big] = \mathfrak{B}_t\left(\underline{\mathcal{V}}_{\underline{F}_t}\right)(x_t).$$

Moreover, for every $w \in \mathrm{supp}(\mathbf{W_{t+1}})$, a^w (see Algorithm 4) is a subgradient of $\mathcal{B}_t^w\left(\underline{\mathcal{V}}_{\underline{F}_t}\right)$ at x_t. Thus as a is equal to $\mathbb{E}[a^{\mathbf{W_{t+1}}}]$ it is a subgradient of $\mathfrak{B}_t\left(\underline{\mathcal{V}}_{\underline{F}_t}\right)$ at x_t. Hence, the mapping $\underline{S}_t^{\mathrm{SDDP}}$ is valid. □

4.4.3 U-upper approximations

We have seen in Lemma 49, that in order to construct a selection function for $\mathfrak{B}_t$, it suffices to construct a selection function for each pointwise Bellman operator $\mathcal{B}_t^w$. In order to do so, for upper approximations we exploit the min-additivity of the pointwise Bellman operators $\mathcal{B}_t^w$. That is, given a set of functions F, we use the following decomposition

$$\forall t \in [\![0, T-1]\!], \forall x \in \mathbb{X}, \forall w \in \mathrm{supp}(\mathbf{W_{t+1}}),\ \mathcal{B}_t^w\left(\inf_{\phi\in F}\phi\right)(x) = \inf_{\phi\in F}\mathcal{B}_t^w(\phi)(x).$$

This is a decomposition of the computation of $\mathcal{B}_t^w\left(\overline{\mathcal{V}}_F\right)$ which is possible for upper approximations but not for lower approximations as for minimization problems, the Bellman operators (average or pointwise) are min-plus linear but generally not max-plus linear.

However, in linear-polyhedral MSP, the value functions are polyhedral. Approximating from above value function V_t by infima of convex quadratics is not suited: in particular, one cannot ensure validity of a quadratic at a kink of the polyhedral function V_t. Still, we present a selection function which is tight but not valid. In the numerical experiment of Figure 4.2, we illustrate that the selection function defined below might not be valid, but the error is still reasonable. Yet, this will motivate the use of other basic functions more suited to the linear-polyhedral framework, as done in §4.4.4.

We consider basic functions that are U-shaped, *i.e.* of the form $\frac{c}{2}\|x-a\|^2 + b$ for some constant $c > 0$, vector a and scalar b. We call such function a *c-function.* We now fix a sequence of constants $(c_t)_{t\in[\![0,T]\!]}$ such that $c_t > L_{V_t}$. For every time $t \in [\![0,T]\!]$, we define the set of basic functions

$$\overline{\mathbf{F}}_t^{\mathrm{U}} = \left\{ \frac{c_t}{2}\|x-a\|^2 + b + \delta_{X_t} \,\middle|\, (a,b) \in \mathbb{X}\times\mathbb{R} \right\} .$$

At time $t = T$, we select the c_T-quadratic mapping which is equal to ψ at point $x \in X_T$ and has same (sub)gradient at x, *i.e.* $\overline{S}_T^{\mathrm{U}}(x) = \frac{c_T}{2}\|\cdot - a\|^2 + b$ where $a = x - \frac{1}{c}\lambda$ and $b = \psi(x) - \frac{1}{2c}\|\lambda\|^2$ with λ being a subgradient of ψ at x.

The mapping $\overline{S}_t^{\mathrm{U}}$ defined in Algorithm 5 is tight but not necessarily valid, see an illustration in Figure 4.2. As with SDDP, in order to build a tight selection function at $t < T$ for $\mathfrak{B}_t$ we first compute a tight selection function for each $\mathcal{B}_t^w$, $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, which can be done numerically by quadratic programming.

Algorithm 5 U Selection function $\overline{S}_t^{\mathrm{U}}$ for $t < T$

Input: A set of L_{V_t}-Lipschitz continuous U-shaped functions $\overline{F}_{t+1} \subset \overline{\mathbf{F}}_{t+1}^{\mathrm{U}}$ and a trial point $x_t \in X_t$.

Output: A tight basic function $\overline{\phi}_t \in \overline{\mathbf{F}}_t^{\mathrm{U}}$.

for $w \in \operatorname{supp}(\mathbf{W_{t+1}})$ **do**

Solve by quadratic programming $v^w := \mathcal{B}_t^w\left(\overline{\mathcal{V}}_{\overline{F}_{t+1}}\right)(x) = \inf_{\overline{\phi}\in\overline{F}_{t+1}} \mathcal{B}_t^w\left(\overline{\phi}\right)(x)$ and compute $a^w = x - \frac{1}{c}\lambda$ and $b^w = v^w - \frac{1}{2c}\|\lambda\|^2$ with λ being a subgradient of $\mathcal{B}_t^w\left(\overline{\mathcal{V}}_{\overline{F}_{t+1}}\right)$ at x.

end for

Set $\overline{\phi} := \frac{c_t}{2}\|\cdot - a\|^2 + b + \delta_{X_t}$ where $a := \mathbb{E}[a^{\mathbf{W_{t+1}}}]$ and $b = \mathbb{E}[\frac{c_t}{2}\|\cdot - a\|^2 + b^{\mathbf{W_{t+1}}}]$.

4.4.4 V-upper approximations

We have seen in §4.4.3 that U-shaped basic functions may not be suited to approximate polyhedral functions. In [PdF13], upper approximations which were polyhedral as well were introduced. In this section we propose upper approximations of V_t as infima of V-shaped functions. Even though when V_t is polyhedral the approach of [PdF13] seems the most natural, their approximations cannot be easily expressed as a pointwise infima of basic functions.

In future works we will add a max-plus/min-plus projection step to TDP in order to broaden the possibilities of converging approximations available to the decision maker. In particular, polyhedral upper approximations as in [PdF13] will be covered.

In this section, by introducing a new tight and valid selection function, we would like to emphasize on the flexibility already available to the decision maker by adopting the framework of TDP.

We consider V-shaped functions, *i.e.* functions of the form $L\|x-a\|_1 + b$ with $a \in \mathbb{X} = \mathbb{R}^n$ and $b \in \mathbb{R}$ and a constant $L > 0$. We define for every time step $t \in [\![0,T]\!]$, the set of basic functions

$$\overline{\mathbf{F}}_t^{\mathrm{V}} := \left\{ \frac{L_{V_t}}{\sqrt{n}}\|\cdot - a\|_1 + b \,\middle|\, (a,b) \in \mathbb{X}\times\mathbb{R} \right\} .$$

At time $t = T$, we compute a V-shaped function at $\psi(x)$, *i.e.* given a trial point $x \in X_T$, using the expression $\overline{S}_T^{\mathrm{V}}(x) = \frac{L_{V_T}}{\sqrt{n}}\|\cdot - x\|_1 + \psi(x)$. For time $t \in [\![0,T-1]\!]$, the selection function

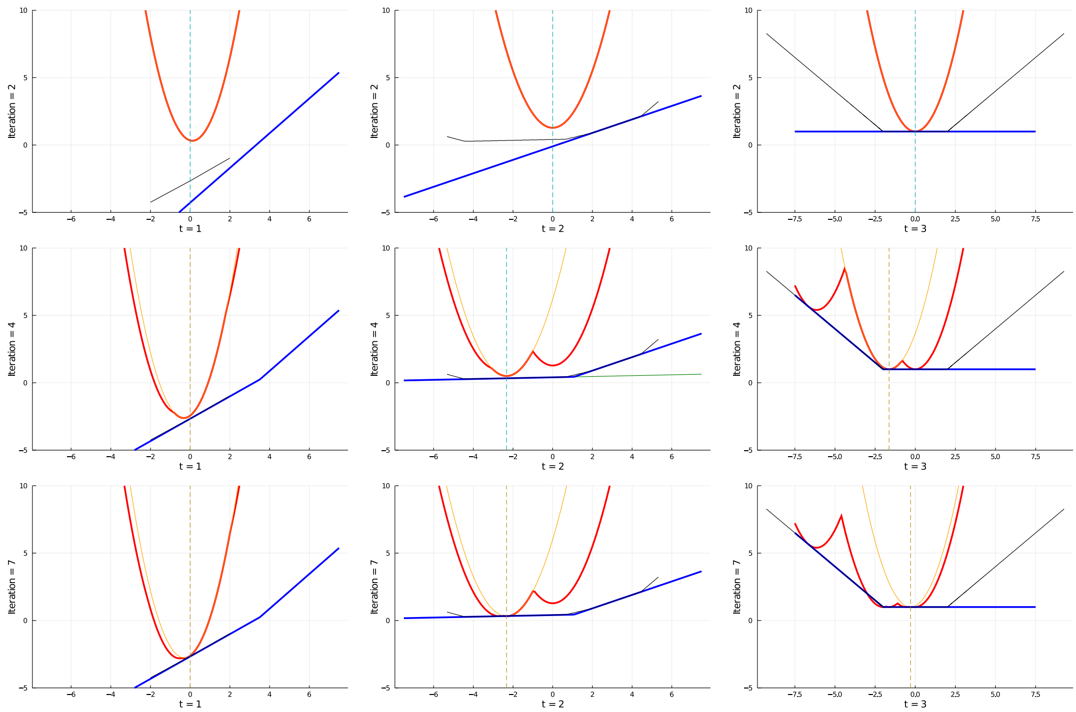

Figure 4.2: U-SDDP approximations of the value functions. In the bottom right, the U-shaped basic functions might not be valid when the trial point is associated with a kink of value function. Still, we observe that the gap between upper and lower approximations vanishes along the problem-child trajectory (in dashed lines).

is given in Algorithm 6. The main difference with the previous cases treated in §4.4.2 and in §4.4.3 is that V-shaped functions are not stable by averaging as the average of several V-shaped functions is a polyhedral function.

Algorithm 6 V Selection function $\overline{S}_t^{\mathrm{V}}$ for $t < T$

Input: A set of L_{V_t}-Lipschitz continuous V-shaped functions $\overline{F}_{t+1} \subset \overline{\mathbf{F}}_{t+1}^{\mathrm{V}}$ and a trial point $x_t \in X_t$.

Output: A tight and valid basic function $\overline{\phi}_t \in \overline{\mathbf{F}}_t^{\mathrm{V}}$.

Solve by linear programming $b := \mathfrak{B}_t \left(\overline{\mathcal{V}}_{\overline{F}_{t+1}}\right)(x_t)$.

Set $\overline{\phi}_t := \frac{L_{V_t}}{\sqrt{n}} \|\cdot - x_t\| + b$.

Proposition 51 (V Selection function)**.** *For every $t \in [\![0, T]\!]$, the mapping $\overline{S}_t^{\mathrm{V}}$ described in Algorithm 6 is a selection function in the sense of Definition 41.*

Proof. At time $t = T$, for every $x_T \in X_T$, we have $\overline{S}_T^{\mathrm{V}}(x_T) = \frac{L_{V_T}}{\sqrt{n}} \|\cdot - x_T\|_1 + \psi(x_T)$. Thus, $\overline{S}_T^{\mathrm{V}}(x_T)(x_T) = \psi(x_T)$ and $\overline{S}_T^{\mathrm{V}}$ is a tight mapping. As the polyhedral function $\psi(x) = \max_{i \in I_T} \langle c_T^i, x \rangle + d_T^i + \delta_{X_T}$ is L_{V_T}-Lipschitz continuous, by Cauchy-Schwarz inequality, for every $x \in X_T$ and $i \in I_T$, we have

$$\langle c_T^i, x - x_T \rangle \leq \|c_T^i\|_2 \|x - x_T\|_2 \leq L_{V_T} \frac{1}{\sqrt{n}} \|x - x_T\|_1 .$$

Adding $\langle c_T^i, x_T \rangle + d_T^i$ on both sides of the last inequality and taking the maximum over $i \in I_T$ we have that

$$\psi(x) = \max_{i \in I_T} \langle c_T^i, x \rangle + d_T^i \leq L_{V_T} \frac{1}{\sqrt{n}} \|x - x_T\|_1 + \psi(x_T) = \overline{S}_T^{\mathrm{V}}(x_T)(x),$$

which gives that $\overline{S}_T^{\mathrm{V}}$ is a valid mapping.

Now, fix $t < T$, we show that the mapping $\overline{S}_t^{\mathrm{V}}$ is tight and valid as well. By construction, for every set of basic functions $\overline{F}_{t+1} \subset \overline{\mathbf{F}}_{t+1}^{\mathrm{U}}$ and trial point $x_t \in X_t$, we have

$$\overline{S}_t^{\mathrm{V}}(\overline{F}_{t+1}, x_t)(x_t) = b = \mathfrak{B}_t \left(\overline{\mathcal{V}}_{\overline{F}_{t+1}}\right)(x_t) .$$

Hence, $\overline{S}_t^{\mathrm{V}}$ is a tight mapping.

We check that $\overline{S}_t^{\mathrm{V}}$ is a valid mapping. First, as each basic function $\phi \in \overline{F}_{t+1}$ is $L_{V_{t+1}}$-Lipschitz continuous on X_t, we show that $\overline{\mathcal{V}}_{\overline{F}_{t+1}}$ is $L_{V_{t+1}}$-Lipschitz continuous on X_t as well. Given $x_1, x_2 \in X_t$, we have

$$\begin{aligned}
|\overline{\mathcal{V}}_{\overline{F}_{t+1}}(x_1) - \overline{\mathcal{V}}_{\overline{F}_{t+1}}(x_2)| &= | \inf_{\phi \in \overline{F}_{t+1}} \phi(x_1) - \inf_{\phi \in \overline{F}_{t+1}} \phi(x_2)| \\
&\leq \sup_{\phi \in \overline{F}_{t+1}} |\phi(x_1) - \phi(x_2)| \\
&\leq L_{V_t} \|x_1 - x_2\| .
\end{aligned}$$

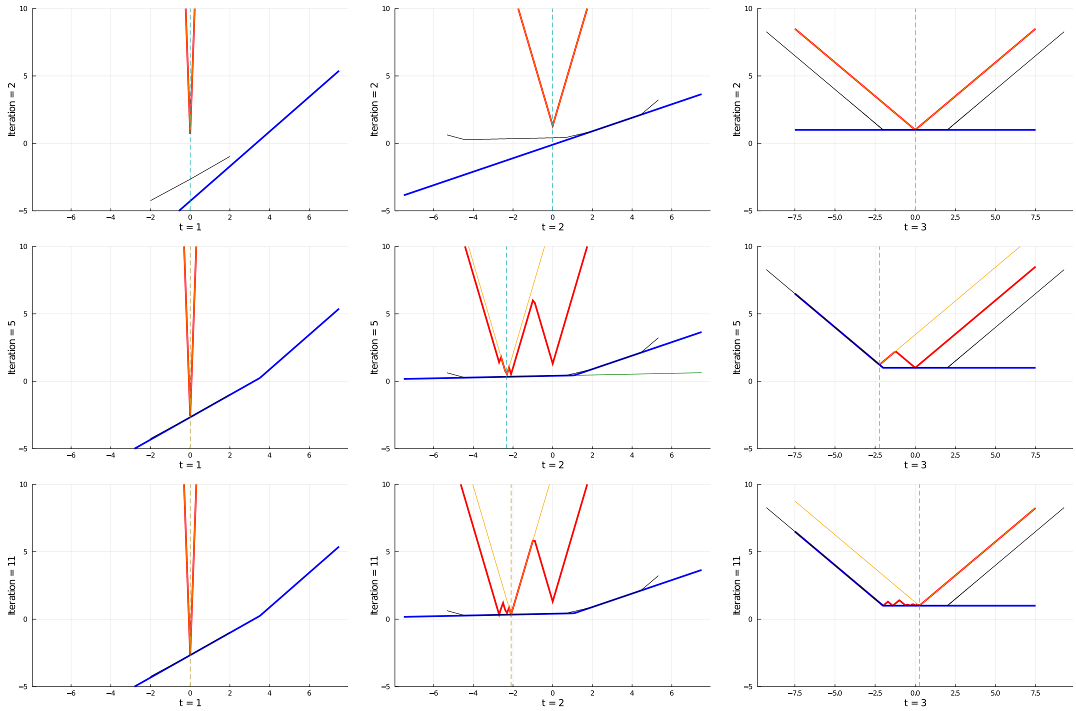

Figure 4.3: V-SDDP approximations of the value functions. As the selection function S_t^{V} does not average other basic functions to compute a new one (compare with S_t^{U} or S_t^{SDDP}), we lose the regularizing effect of averaging: the upper basic functions added are very sharp. We still observe that the gap between upper and lower approximations vanishes along the problem-child trajectory (in dashed lines).

As the Bellman operator $\mathfrak{B}_t$ is Lipschitz regular in the sense of Proposition 38, $\mathfrak{B}_t\left(\overline{\mathcal{V}}_{\overline{F}_{t+1}}\right)$ is L_{V_t}-Lipschitz continuous.

Second, by min-additivity of the Bellman operator $\mathfrak{B}_t$, we have that

$$\mathfrak{B}_t\left(\overline{\mathcal{V}}_{\overline{F}_{t+1}}\right)(x) = \mathfrak{B}_t\left(\inf_{\phi\in\overline{F}_{t+1}} \phi\right)(x) = \inf_{\phi\in\overline{F}_{t+1}} \mathfrak{B}_t\left(\phi\right)(x)\,.$$

Recall that by Lemma 48, the Bellman operator $\mathfrak{B}_t$ preserves polyhedrality. As $\phi \in \overline{F}_{t+1}$ is polyhedral, $\mathfrak{B}_t\left(\phi\right)$ is polyhedral as well and as in the case $t = T$, *mutatis mutandis* we have that $\overline{S}_t^{\mathrm{V}}$ is valid. $\square$

Conclusion

- TDP generates simultaneously *monotonic* approximations $(\underline{V}_t^k)_k$ and $(\overline{V}_t^k)_k$ of V_t.
- Each approximation is either a *min-plus* or *max-plus linear* combinations of basic functions.
- Each basic function should be *tight* and *valid*.

- The approximations are refined iteratively along the Problem-child trajectory *without discretizing the state space.*
- The *gap* between upper and lower approximation *vanishes along the Problem-child trajectory.*
- TDP generalizes a similar approach done in [PdF13] and proved by [BDZ18] for a variant of SDDP in convex MSPs.

Perpectives

- Consider an additional min-plus/max-plus projection step of suprema/infima of basic functions.
- Extensive numerical comparisons with existing methods, namely classical SDDP and the upper approximations obtained by Fenchel duality of [LCC+18].
- Extend the scope of TDP to encompass Partially Observed Markov Decision Processes.

A comparison Lemma: Let $\psi : \mathbb{X} \to \mathbb{R}$ be given and consider the set $U_t[\psi, x] : \mathbb{W}_{t+1} \rightrightarrows \mathbb{U}$ such that $U_t[\psi, x](w) = \arg\min_{u \in \mathbb{U}} \Big(c_t^w(x,u) + \psi\big(f_t^w(x,u)\big)\Big)$. Now, given ϕ and ϕ' such that $\phi \leq \phi'$ we obtain that forall $x \in \mathbb{X}$ and

$$\begin{aligned}
\mathfrak{B}_t\left(\phi'\right)(x) - \mathfrak{B}_t\left(\phi\right)(x) &= \mathbb{E}\big[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(\phi'\right)(x)\big] - \mathbb{E}\big[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(\phi\right)(x)\big] \\
&= \mathbb{E}\Big[\min_{u \in \mathbb{U}} \Big(c_t^{\mathbf{W_{t+1}}}(x,u) + \phi'\big(f_t^{\mathbf{W_{t+1}}}(x,u)\big)\Big)\Big] - \mathbb{E}\Big[\min_{u \in \mathbb{U}} \Big(c_t^{\mathbf{W_{t+1}}}(x,u) + \phi\big(f_t^{\mathbf{W_{t+1}}}(x,u)\big)\Big)\Big] \\
&\leq \mathbb{E}\Big[c_t^{\mathbf{W_{t+1}}}(x, u_t^{\mathbf{W_{t+1}}}) + \phi'\big(f_t^{\mathbf{W_{t+1}}}(x, u_t^{\mathbf{W_{t+1}}})\big)\Big] - \mathbb{E}\Big[c_t^{\mathbf{W_{t+1}}}(x, u_t^{\mathbf{W_{t+1}}}) + \phi\big(f_t^{\mathbf{W_{t+1}}}(x, u_t^{\mathbf{W_{t+1}}})\big)\Big] \quad \forall u
\end{aligned}$$

as we have for all $u_t^w \in U_t[\phi, x](w)$ that $\min_{u \in \mathbb{U}} \Big(c_t^w(x,u) + \phi\big(f_t^w(x,u)\big)\Big) = c_t^w(x, u_t^w) + \phi\big(f_t^w(x, u_t^w)\big)$ and $\min_{u \in \mathbb{U}} \Big(c_t^w(x,u) + \phi'\big(f_t^w(x,u)\big)\Big) \leq c_t^w(x, u_t^w) + \phi'\big(f_t^w(x, u_t^w)\big)$

$$\begin{aligned}
&\leq \mathbb{E}\Big[\phi'\big(f_t^{\mathbf{W_{t+1}}}(x, u_t^{\mathbf{W_{t+1}}})\big) - \phi\big(f_t^{\mathbf{W_{t+1}}}(x, u_t^{\mathbf{W_{t+1}}})\big)\Big] \\
&\leq \phi'\big(f_t^{w^*}(x, u_t^{w^*})\big) - \phi\big(f_t^{w^*}(x, u_t^{w^*})\big)
\end{aligned}$$

where $w^* \in \arg\max_w \phi'\big(f_t^w(x, u_t^w)\big) - \phi\big(f_t^w(x, u_t^w)\big)$.

Part III

Entropic regularization of the Nested Distance

Chapter 5

Entropic regularization of the Nested Distance

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5.1 Introduction: from the Wasserstein distance to the Nested Distance

In Multistage Stochastic Programming (MSP), Georg Pflug introduced in 2009 [Pfl09] the *Nested Distance*, which is a refinement of the Wasserstein distance to account for proximity in the filtrations between two discrete time stochastic processes. Following usual denomination in the Stochastic Programming community (see [HR09, PP14, SDR09]), we denote by *scenario tree* a discrete time stochastic process which is also discrete and finite in space.

There are many different distances between scenario trees however few are suited for MSP purposes: one would like to guarantee continuity of the value function of a MSP with respect to scenario trees, *i.e.* if two scenario trees are arbitrarily close to each other, then the value of the associated MSP (with the same structure except for the scenario trees) can be made arbitrarily close as well.

One possible distance between scenario tree is the so-called Wasserstein distance. Intuitively, the Wasserstein distance between two probabilities p and q (for scenario tree $(X_t)_{t\in[\![1,T]\!]}$, consider the probability law of the tuple $(X_1,\dots,X_T)$) corresponds to the optimal cost of splitting and transporting the mass from one to the other. We write $\mathbf{1}_k$, $k\in\mathbb{N}$, for the vector of $\mathbb{R}^k$ made of ones.

Definition 52 (Discrete optimal transport and Wasserstein distances)**.** *Let n, m be two integers and $\mathbb{X}=\{x_1,x_2,\dots,x_n\}$ and $\mathbb{Y}=\{y_1,\dots,y_m\}$ be two finite sets included in $\mathbb{R}^t$ for a real $t\geq 1$. Denote by $c=(c_{ij})_{i,j}$ a $n\times m$ positive matrix called* cost matrix*. The* optimal transport cost *between two probability measures p and q on respectively $\mathbb{X}$ and $\mathbb{Y}$, is the value of the following optimization problem*

$$\mathrm{OT}(p,q;c)=\min_{\pi\in\mathbb{R}_+^{n\times m}}\sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} c_{ij}\pi_{ij}\ \textit{s.t.}\ \pi\mathbf{1}_m=p\ \textit{and}\ \pi^T\mathbf{1}_n=q. \tag{5.1}$$

Moreover, defining the cost function *by $c(x_i,y_j)=c_{ij}$ for every indexes i,j, Problem (5.1) can be written using probabilistic vocabulary as*

$$\mathrm{OT}(p,q;c)=\min_{\substack{(X,Y)\ s.t.\\ X\sim p\\ Y\sim q}}\mathbb{E}_{(X,Y)}[c(X,Y)], \tag{5.2}$$

where the notation $X\sim p$ (resp. $Y\sim q$) means that the probability law of the random variable $X\in\mathbb{X}$ (resp. $Y\in\mathbb{Y}$) is equal to p (resp. q) and the notation $\mathbb{E}_{(X,Y)}$ is the expectation under the probability law of the couple of random variables (X,Y).

Lastly, when for some real $r\geq 1$, the cost function c is equal to d^r with d a metric on $\mathbb{X}\times\mathbb{Y}$, then $\mathrm{OT}(p,q;d^r)^{1/r}$ is the r-th Wasserstein distance *between p and q, denoted $\mathrm{W_r}(p,q)$.*

We refer to the textbooks [PC19, Vil09] for a presentation and references on optimal transport.

In two stage multistage optimization problems, under some regularity assumptions, the value function of a bilevel MSP is Lipschitz continuous with respect to the Wasserstein distances, see [PP14, Chapter 6]. However the value function of MSP with more than two stages is not continuous with respect to the Wasserstein distances, as seen in Example 4, where we show that for a three stages MSP, two scenario trees can be arbitrarily close to each other in the 1-Wasserstein metric but the gap in value of the associated MSP is arbitrarily large.

Example 4 (The Wasserstein distance is not suited for MSP)**.** *In this example we illustrate that the* 1*-Wasserstein is not a relevant metric to evaluate distance between scenario trees involved in a MSP: an arbitrary small Wasserstein distance between two scenario trees may yield an arbitrary large gap in values of the same MSP.*

Given a scenario tree Z (see Definition 53 for a formal definition) with natural filtration $(\mathcal{F}_t)_{t\in[\![0,2]\!]}$ [1]*, we want to buy a single object at the minimal average cost*

$$v(Z) = \min_{\mathbf{u}} \left\{ \mathbb{E}\left[\sum_{t=0}^{2} Z_t \mathbf{u_t}\right] \mid \begin{array}{c} \mathbf{u_t} \in \{0,1\}, \\ \mathbf{u_t} \textit{ is } \mathcal{F}_t \textit{-measurable,} \\ \sum_{t=0}^{T} \mathbf{u_t} = 1, \end{array} \right\}.$$

Fix $A \gg \epsilon > 0$, in Figure 5.1 are two scenario tree modeling the price of an object during 3 *time steps. Their natural filtrations are different. Intuitively, on the left scenario tree, the decision maker observes an ϵ variation of the price at $t = 1$ and knows that it will yield an explosion (upward or downward) of the price at $t = 2$. Whereas on the right scenario tree, the decision maker does not recognize such information at time $t = 1$. Example inspired from [HRS06].*

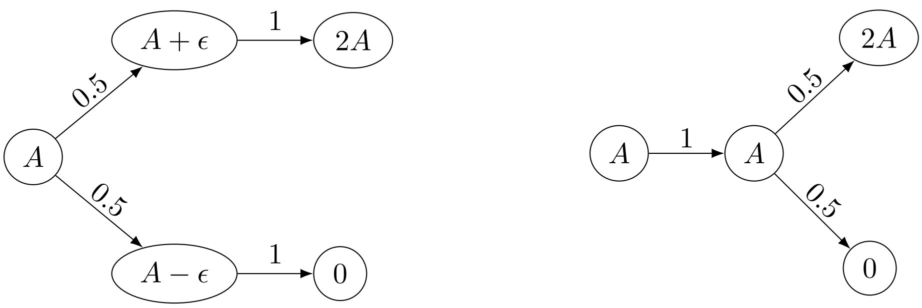

Figure 5.1: Left: scenario tree $X = (X_0, X_1, X_2)$. Right: scenario tree $Y = (Y_0, Y_1, Y_2)$.

On the one hand we have proximity in the 1*-Wasserstein metric* W *as*

$$\mathrm{W}(X, Y) = 2\epsilon.$$

On the other hand, the optimal values are $v(X) = \frac{A+\epsilon}{2}$ and $v(Y) = A$. Thus, we have an arbitrarily large gap in values

$$|v(X) - v(Y)| = \frac{A-\epsilon}{2} \underset{A\to+\infty}{\to} +\infty.$$

In 2012, Pflug and Pichler proved in [PP12] that the Nested Distance previously introduced by Pflug [Pfl09], is the correct adaptation of the Wasserstein distance for multistage stochastic programming: under regularity assumptions, the value function of MSPs is Lipschitz continuous with respect to the Nested Distance between scenario trees. Since then, it has been used as a a pruning tool to obtain reduced trees with a certain guaranty of the quality of the approximation. The Nested Distance both quantifies the quality of an approximating tree and the associated optimal transport plan also allows for reduction of scenario trees, see for example [KP15, HVKM20].

[1] For every $t \in [\![0,2]\!]$, $\mathcal{F}_t = \sigma(Z_0, \ldots, Z_t)$.

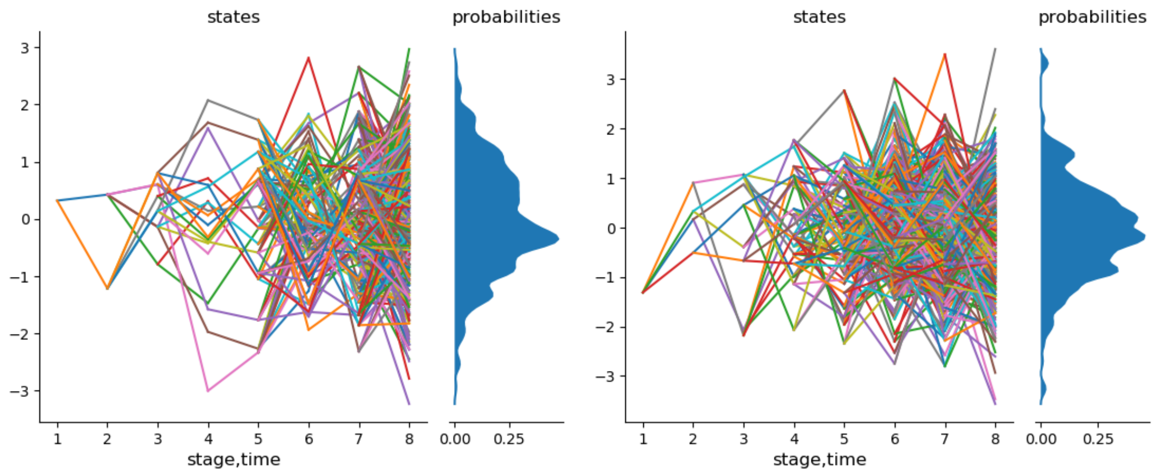

Figure 5.2: Two scenario trees X and Y with a continuous probability approximation of the histogram the leaves. Their Nested Distance is $\mathrm{ND}_2(X, Y) = 1.009$ and its entropic regularization is $\mathrm{END}_2(X, Y) = 1.011$, see Section 5.3. The trees were generated using the ScenTrees.jl package [KPP20].

Without additional structure (like independence) of the scenario tree, the Nested Distance is usually computed via a backward recursive algorithm (introduced in [PP12], see also [PS19, Definition 15]) which amounts to solve an exponential number (in T) number of optimal transportation problems. It decomposes over time the computation of the Nested Distance as the dynamic computation of a finite number of optimal transport problems between conditional probabilities with costs updated backward.

Optimal transport between discrete probabilities of size n can be solved by the Hungarian algorithm with complexity $O(n^3)$ (see [EK72]) or with the auction algorithm with complexity roughly $O(n^3 \log n)$, see [BC89].

By adding an entropic term to the primal of the optimal transport problem associated with the computation of a Wasserstein cost, an alternating projection scheme yield Sinkhorn's algorithm, introduced in Optimal Transport in [Cut13] to compute Wasserstein distances. By carefully selecting the entropic regularization term, Sinkhorn's algorithm computes an ϵ-overestimation of the Wasserstein distance in $O(n^2 \log(n)\epsilon^{-3})$ operations.

Relaxing each optimal transport problem involved in the recursive computation of the Nested Distance, we end up with an entropic regularization of the Nested Distance.

The remainder of the chapter is organized as follows

- In Section 5.2, we first formally define the Nested Distance as the value of a dynamic system of optimal transport problems between conditional probabilites and varying costs. Then, we present an entropic regularization of the discrete optimal transport Problem (5.1) and how this relaxed OT problem can be solved efficiently by Sinkhorn's algorithm. Lastly, we define a natural entropic regularization of the Nested Distance by relaxing each OT problem involved in its dynamic formulation.

- In Section 5.3, we end this chapter with a numerical experiment showing both the speedup of our approach to compute Nested Distances and also its relative preciseness.

5.2 The Nested Distance and its entropic regularization

5.2.1 Dynamic computation of the Nested Distance

Throughout the remainder of the chapter, we fix an integer $T > 1$ and we consider that the set of reals $\mathbb{R}$ is endowed with its usual distance and borelian structure. Moreover, for every $t \in [\![1, T]\!]$, $\mathbb{R}^t = \mathbb{R} \times \cdots \times \mathbb{R}$ is seen as a filtered space endowed with its cylinder σ-algebra.

Definition 53 (Scenario tree)**.** *Let $(X_t)_{t\in[\![1,T]\!]}$ be a discrete time stochastic process defined on some probability space. The stochastic process $(X_t)_{t\in[\![1,T]\!]}$ is a* scenario tree *if it is also finite and discrete in space,* i.e. *for every time indexes $1 \leq s \leq t \leq T$, the support $\mathbb{X}_{s:t}$ of $X_{s:t} = (X_s, \dots, X_t)$ defined by*

$$\mathbb{X}_{s:t} := \left\{ x_{s:t} = (x_s, \dots, x_t) \in \mathbb{R}^{t-s} \mid \mathbb{P}(X_s = x_s, \dots, X_t = x_t) > 0 \right\}$$

is non-empty, finite and $\sum_{x_{s:t}\in\mathbb{X}_{s:t}} \mathbb{P}(X_s = x_s, \dots, X_t = x_t) = 1$.

Following [PS19], we define the Nested Distance between scenario trees as the value of a recursive computation of optimal transportation between conditional probabilities with updated costs. Given two scenario trees $X = (X_t)_{t\in[\![1,T]\!]}$ and $Y = (Y_t)_{t\in[\![1,T]\!]}$, for every $s, t \in [\![1, T]\!]$, we define the tuple of random variable variables $X_{s:t} = (X_s, \dots X_t)$ and $Y_{s:t} = (Y_s, \dots, Y_t)$. Denote by $x_{s:t}$ and $y_{s:t}$ any element of their support $\mathbb{X}_{s:t}$ and $\mathbb{Y}_{s:t}$ (see Definition 53). Lastly, for every $t \in [\![1, T]\!]$, denote by P_t and $\tilde{P}_t$ the probability laws of $X_{1:t} = (X_1, \dots, X_t)$ and $Y_{1:t} = (Y_1, \dots, Y_t)$, respectively.

Definition 54 (Nested Distance between scenario trees)**.** *Let X and Y be two scenario trees. Given $r \geq 1$, and the metric $d(x, y) = \|x-y\|_r$ on $\mathbb{R}^T$, for every $t \in [\![1, T]\!]$, compute recursively backward in time functions $c_t : \mathbb{X}_{1:T} \times \mathbb{Y}_{1:T} \to \overline{\mathbb{R}}$ by*

$$\begin{cases} c_T(x_{1:T}, y_{1:T}) = d(x_{1:T}, y_{1:T}), \;\; \forall (x_{1:T}, y_{1:T}) \in \mathbb{X}_{1:T} \times \mathbb{Y}_{1:T}, \\ c_t(x_{1:T}, y_{1:T}) = \mathrm{OT}(P_{t+1}(\cdot \mid X_{1:t} = x_{1:t}), \tilde{P}_{t+1}(\cdot \mid Y_{1:t} = y_{1:t}); c^r_{t+1})^{1/r}, \\ \forall t \in [\![1, T-1]\!], \;\; \forall (x_{1:T}, y_{1:T}) \in \mathbb{X}_{1:T} \times \mathbb{Y}_{1:T}. \end{cases} \tag{5.3}$$

*Set $\mathrm{ND}_r(X, Y) := \mathrm{OT}(P_T, \tilde{P}_T, c^r_1)^{1/r}$, it is the r-*Nested Distance *between the scenario trees X and Y.*

Although for every $t \in [\![1, T]\!]$ the domain of c_t is $\mathbb{X}_{1:T} \times \mathbb{Y}_{1:T}$, only the process up to t matters *i.e.* for every $x_{1:T}, x'_{1:T} \in \mathbb{X}_{1:T}$ and $y_{1:T}, y'_{1:T} \in \mathbb{Y}_{1:T}$ such that $x_{1:t} = x'_{1:t}$ and $y_{1:t} = y'_{1:t}$ we have $c_t(x_{1:T}, y_{1:T}) = c_t(x'_{1:T}, y'_{1:T})$. It follows from [PS19, Proposition 20] and [PP12, Theorem 19] that the Nested Distances ND_r introduced in Definition 54, are distances on the space of scenario trees.

Remark 55. *Solving Problem (5.3) amounts to solving an exponential (in T) number of Linear optimization Problems where the dimension of the variable to optimize is bounded by $n \cdot m$ where n (resp. m) is $\max_{t\in[\![1:T-1]\!]}|\mathbb{X}_{t:t+1}|$ (resp. $\max_{t\in[\![1:T-1]\!]}|(\mathbb{Y}_{t:t+1})|$).*

5.2.2 Entropic regularization of optimal transport problems

We will relax the OT Problem 5.1 by adding an entropy term to the objective function. The *Shannon entropy* or simply *entropy* of a random variable Z with values in a finite subset $\mathbb{Z}$

of cardinal $k \in \mathbb{N}$ in $\mathbb{R}^t$, $t \geq 1$ and probability vector $(p_1; \dots; p_k) \in (\mathbb{R}_+^*)^k$ is defined as

$$H(Z) = \mathbb{E}[-\log Z] = -\sum_{i=1}^{k} p_i \log(z_i).$$

By adding an entropy regularization term to the objective of an optimal transport Problem 5.2 (using the probabilistic notations), the linear objective function of a discrete OT problem becomes strongly convex, hence damping the combinatorial aspects of OT.

Definition 56 (Regularized Optimal Transport)**.** *With the notations of Definition 52, for every real $\gamma > 0$ we define the following* regularized optimal transport plan *between probabilities $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$ with cost matrix $c \in \mathbb{R}^{n\times m}$*

$$\pi_\gamma(p, q; c) = \operatorname*{arg\,min}_{\substack{(X,Y) s.t.\\ X\sim p\\ Y\sim q}} \mathbb{E}[c(X, Y) - \gamma H(\mathcal{L}(X, Y))], \tag{5.4}$$

where $\mathcal{L}(X, Y)$ is the probability law of the couple (X, Y) of random variables. Then, the associated value is the regularized optimal transport OT_γ *between p and q*

$$\mathrm{OT}_\gamma(p, q; c) = \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} c_{ij}(\pi_\gamma)_{ij}. \tag{5.5}$$

Remark 57. *Note that as the regularized optimal transport plan pi_γ also satisfies the constraints of the (unregeralized) optimal transport problem, we have for every $\gamma > 0$ that $\mathrm{OT}(p, q; c) \leq \mathrm{OT}_\gamma(p, q; c)$. Moreover when γ tends to 0, one recover the optimal transport value,* i.e.

$$\mathrm{OT}_\gamma(p, q; c) \underset{\gamma\to 0}{\longrightarrow} \mathrm{OT}(p, q; c).$$

Given an integer $t \geq 1$, let p and q be two probabilities on $\mathbb{R}^t$ with respective finite support of size $n \in \mathbb{N}$ and $m \in \mathbb{N}$. We say that a $n \times m$ matrix π is a *transport plan between p and q* if it is admissible in Problem (5.1), *i.e.* π satisfies the *mass conservation constraints*:

$$\pi \mathbf{1}_m = p \text{ and } \pi^T \mathbf{1}_n = q. \tag{5.6}$$

The set of transport plans between p and q is denoted by $\mathcal{P}(p, q)$.

We now present Sinkhorn's algorithm and prove its convergence. This algorithm was (re)discovered by Cuturi in [Cut13] who used it to solve the regularized optimization Problem (5.4). Proofs of the different following statements can be found in [Cut13] and [PC19, Chapter 4], see also Appendix 5.4 for a sketch of proof with detailed references.

Theorem 58 (Sinkhorn's algorithm and its convergence)**.** *Fix $\gamma > 0$, an integer $t \geq 1$ and let p and q be two probabilities on $\mathbb{R}^t$ with respective finite support of size $n \in \mathbb{N}$ and $m \in \mathbb{N}$. The following assertions are true:*

- Existence and uniqueness of an optimal transport plan π^*. *There exists a unique transport plan π^* which minimizes the regularized optimal transport Problem (5.4) with cost matrix $c = (c_{ij})_{i,j} \in \mathbb{R}^{n\times m}$.*

- The optimal transport plan is a rescaling of the Gibbs kernel G. *There exists two positive vectors* $u^* \in (\mathbb{R}_+^*)^n$, $v^* \in (\mathbb{R}_+^*)^m$ *such that*

$$\pi^* = \operatorname{diag}(u^*) G \operatorname{diag}(v^*),$$

where G *is the Gibbs kernel defined by* $G_{ij} = \exp(-\frac{c_{ij}}{\gamma})$.

- Alternatively rescaling the lines and columns (Sinkhorn's algorithm) of G converges to π^*. *Alternatively rescaling the lines and columns of* G *in order to satisfy the mass conservation constraints of Equation* (5.6) *converges to the optimal transport plan* π^*. *More precisely, iterates* $(u_k, v_k) \in (\mathbb{R}_+^*)^n \times (\mathbb{R}_+^*)^m$, $k \in \mathbb{N}$, *defined by* $u_0 = \mathbf{1}_n$, $v_0 = \mathbf{1}_m$, *and*

$$\begin{cases} u_{k+1} = \mathbf{1}_n \ ./(G v_k) \quad \textit{(where ./ is the entrywise division)} \\ v_{k+1} = \mathbf{1}_m \ ./(G u_{k+1}), \end{cases} \tag{5.7}$$

converge to the optimal scaling vectors u^* *and* v^*.

- Sinkhorn's algorithm converges linearly to π^*. *The Gibbs kernel is a positive matrix and thus is contractant w.r.t. the Hilbert projective metric on the cone of positive vectors. As a consequence, the convergence of Sinkhorn's iterates* $(u_k, v_k)_{k \in \mathbb{N}}$ *defined in Equation* (5.7) *converge linearly to the optimal scaling vectors* (u^*, v^*).

- Overall complexity. *For every* $\epsilon > 0$, *considering for simplicity* $n = m$, *setting* $\gamma = \frac{4\log(n)}{\epsilon}$, *Sinkhorn's algorithm computes* $\pi^* \in \mathcal{P}(p,q)$ *in* $O(n^2 \log(n) \epsilon^{-3})$ *operations which satisfies*

$$\sum_{ij} \pi_{ij}^* c_{ij} \leq \mathrm{OT}(p, q; c) + \epsilon.$$

Remark 59. *The Sinkhorn updates can be interpreted as alternatively projecting the* $n \times m$ *matrix* $\pi_k = \operatorname{diag}(u_k) G \operatorname{diag}(v_k)$ *to the affine sets* $A_1 := \{\pi \in \mathbb{R}_+^{n \times m} \mid \pi \mathbf{1}_m = p\}$ *and* $A_2 := \{\pi \in \mathbb{R}_+^{n \times m} \mid \pi^T \mathbf{1}_n = q\}$. *The projections are to be understood in Bregman's sense:* $\pi_{2k} = \arg\min_{\pi \in A_1} \mathrm{KL}(\pi \| \pi_{2k-1})$ *and* $\pi_{2k+1} = \arg\min_{\pi \in A_2} \mathrm{KL}(\pi \| \pi_{2k})$ *where* KL *is a renormalization of the usual Kullback-Leibler divergence. We refer to [BCC+15, PC19] for details on this interpretation.*

We comment on the optimal transport plan associated with the relaxed OT Problem (5.4). When the regularization parameter γ is large, then the optimal transport plan is very diffuse: in the left part of Figure 5.3, this means that the mass of each red dot is spread along many different blue dots. The closer the regularization parameter γ gets to 0, then the combinatorial aspect of discrete OT appears gradually: each red dot is spread along few different blue dots. This is expected, as if there were the same number of blue and red dots in Figure 5.3, then the OT problem is an assignment problem. Informally, the entropic regularization of Equation (5.4) dampens the combinatorial aspects of the optimal transport problem of Equation (5.1).

5.2.3 Entropic regularization of the Nested Distance

We have seen in §5.2.2 how to compute efficient upper bound OT_γ of the discrete optimal transport problem OT. Hence, by replacing optimal transport problems by their relaxed

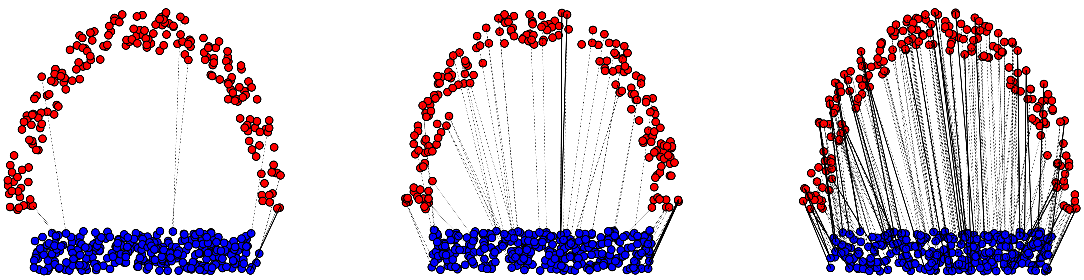

Figure 5.3: Left: $\gamma = 0.008$. Middle: $\gamma = 0.005$. Right: $\gamma = 0.003$. Effect of the regularization parameter on the optimal transport plan of the relaxed OT Problem (5.4), between the red cloud and the blue cloud. In all cases, a continuous edge (resp. dashed line) exists if more than 30 percent (resp. 20 percent) of the red dot mass is moved to the associated blue one.

counterpart in the dynamic computation of the Nested Distance in Equation (5.3), we have an entropic regularization of the Nested Distance noted END.

Note that a regularizing parameter $\gamma > 0$ must be chosen for each optimal transport problem in Equation (5.3). On the one hand, one would like to put γ as small as possible in order to have the best approximation of the unregularized OT problem. On the other hand, as seen in Theorem 58, the optimal transport plan of the regularized OT problem OT_γ is a rescaling of the Gibbs kernel $(G_{ij}) = \exp(-\frac{c_{ij}}{\gamma})$ and the Sinkhorn iterates involve this kernel as well. When γ is too close to 0, Sinkhorn's algorithm shows numerical instabilities. So we refrain from using a single regularizing parameter for every OT problem involved in Equation (5.3). We simply put one that seems big enough to avoid numerical issues, namely we set $\gamma = \frac{\max_{ij} c_{ij}}{100}$, which changes as the cost matrix is updated. Hence the regularizing parameters do not explicitly appear in the notation END of the Entropic regularization of the Nested Distance. For every time $t \in [\![1 : T-1]\!]$ and every *node* $x_{1:t} \in \mathbb{X}_{1:t}$, we define its set of *children* $x_{1:t}^+ := \{\tilde{x}_{1:t+1} \in \mathbb{X}_{1:t+1} \mid \tilde{x}_{1:t} = x_{1:t}\}$.

Definition 60 (Entropic regularization of Nested Distance between scenario trees)**.** *Let X and Y be two scenario trees. Given $r \geq 1$, and the metric $d(x,y) = \|x-y\|_r$ over $\mathbb{R}^T$, for every $t \in [\![1,T]\!]$, compute recursively backward in time functions $c_t : \mathbb{X}_{1:t} \times \mathbb{Y}_{1:t} \to \overline{\mathbb{R}}$ by*

$$\begin{cases} c_T(x_{1:T}, y_{1:T}) = d(x_{1:T}, y_{1:T}),\ \forall (x_{1:T}, y_{1:T}) \in \mathbb{X}_{1:T} \times \mathbb{Y}_{1:T}, \\ c_t(x_{1:t}, y_{1:t}) = \mathrm{OT}_\gamma(P_{t+1}(\cdot \mid X_{1:t} = x_{1:t}), \tilde{P}_{t+1}(\cdot \mid Y_{1:t} = y_{1:t}); c_{t+1}^r)^{1/r}, \\ \forall t \in [\![1, T-1]\!],\ \forall (x_{1:t}, y_{1:t}) \in \mathbb{X}_{1:t} \times \mathbb{Y}_{1:t},\ \gamma = \max\limits_{\substack{x_{1:t+1} \in x_{1:t}^+ \\ y_{1:t+1} \in y_{1:t}^+}} c_{t+1}^r(x_{1:t+1}, y_{1:t+1})/100. \end{cases} \tag{5.8}$$

Set $\mathrm{END}_r(X,Y) := \mathrm{OT}_\gamma(P_T, \tilde{P}_T, c_1^r)^{1/r}$, *with* $\gamma = \max_{\substack{x_{1:1} \in \mathbb{X}_{1:1} \\ y_{1:1} \in \mathbb{Y}_{1:1}}} c_{t+1}^r(x_{1:1}, y_{1:1})/100$*, it is the* Entropic regularization of the r-Nested Distance *between the scenario trees X and Y.*

Note that by Remark 59, for every $r \geq 1$ and scenario trees X and Y,

$$\mathrm{ND}_r(X,Y) \leq \mathrm{END}_r(X,Y).$$

Horizon T	Time ND_2 (s)	Time END_2 (s)	Speedup	Relative error (%)
2	$< 10^{-3}$	$< 10^{-4}$	6.20	3.30
4	0.035	0.0093	6.78	0.93
6	1.71	0.44	4.20	0.66
8	76.64	16.07	4.64	0.15
10	2328.34	550.93	4.03	0.048

Table 5.1: Average results after 10 runs. A Jupyter notebook in Julia 1.4.2 of this experiment is available at https://github.com/BenoitTran/END.

Hence, even though END_r is not a distance between scenario trees, it still quantifies proximity between scenario trees and maintains the main desirable feature of the Nested Distance: denoting by v the value of a MSP satisfying the regularity assumptions of [PP12, Theorem 11], there exists a constant $L > 0$ such that for every scenario trees X and Y we have

$$|v(X) - v(Y)| \leq L \cdot \text{ND}_r(X, Y) \leq L \cdot \text{END}_r(X, Y).$$

5.3 Numerical experiment

We compare an implementation of the Nested Distance and an implementation of its regularized counterpart.

First, we randomly generate a scenario tree of given depth T by a forward procedure. Starting from a root note, at each time step draw a uniformly random number of children between 1 and 5. Every node at time t has the given number of children whose values are random as well. The tree generation is done using the Julia package ScenTrees.jl, see [KPP20]. The discrete optimal transport problems are solved using Gurobi (simplex method) instead of an implementation of the Hungarian or auction algorithm.

We compute the Nested Distance and the Entropic regularization of the Nested Distance for pairs of tree generated as above. In Figure 5.1 we give the average of 10 pairs of comparisons for a given horizon T.

In Figure 5.1 the column "Relative error" represents the ratio $\frac{\text{END}_2 - \text{ND}_2}{\text{END}_2}$. The results of Figure 5.1 show that, even without carefully tuning the regularizing parameter γ involved in each intermediate optimal transport problem, the Entropic regularization of the Nested Distance gives values that are close to its unregularized counterpart. The speedup (ratio between the running time of ND_2 over END_2) is interesting but needs to be compared with an implementation of the Hungarian algorithm or the auction algorithm for optimal transport purposes.

5.4 Proof of Theorem 58

Sketch of proof of Theorem 58. Fix $\gamma > 0$, an integer $t \geq 1$ and let p and q be two probabilities on $\mathbb{R}^t$ with respective finite support of size $n \in \mathbb{N}$ and $m \in \mathbb{N}$. The objective of Problem (5.1) is strongly convex and the constraints are affine. As the marginal product $p \otimes q$ satisfies the mass conservation constraints of Equation 5.6, it is admissible for Problem (5.4). Thus, there exists a unique minimizer $\pi^* \in \mathbb{R}^{n \times m}$ to Problem (5.4).

We now compute the minimizer $\pi^* \in \mathbb{R}^{n\times m}$ of Problem (5.4). For every $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^m$, the Lagrangian of Problem (5.1) is

$$\mathcal{L}(\pi;\alpha,\beta) = \sum_{\substack{1\leq i\leq n\\1\leq j\leq m}} \pi_{ij}(c_{ij} + \gamma\ \log(\pi_{ij})) + \langle \alpha, \pi \mathbf{1}_m - p\rangle + \langle \beta, \pi^T \mathbf{1}_n - q\rangle.$$

The first order necessary condition of optimality

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,m]\!], \frac{\partial \mathcal{L}}{\partial \pi_{ij}}(\pi^*;\alpha^*,\beta^*) = 0,$$

is equivalent to for every $(i,j) \in [\![1,n]\!] \times [\![1,m]\!]$,

$$\pi_{ij} = \exp(-\frac{1}{2} - \frac{\alpha_i^*}{\gamma})\exp(-\frac{c_{ij}}{\gamma})\exp(-\frac{1}{2} - \frac{\beta_j^*}{\gamma}).$$

This implies that solutions of Problem (5.1) π^* are of the form

$$\pi^* = \mathrm{diag}(u^*)G\,\mathrm{diag}(v^*),$$

where the Gibbs kernel $G \in \mathbb{R}^{n\times m}$ is a positive matrix $G_{ij} = \exp(-\frac{c_{ij}}{\gamma})$ and the vectors $u^* = (\exp(-\frac{1}{2} - \frac{\alpha_i^*}{\gamma}))_{1\leq i\leq n} \in (\mathbb{R}_+^*)^n$, $v^* = (\exp(-\frac{1}{2} - \frac{\beta_j^*}{\gamma}))_{1\leq j\leq m} \in (\mathbb{R}_+^*)^m$ are positive. Moreover, π^* has to satisfy the mass constraints of Equation (5.6), *i.e.* its row sums and column sums are prescribed. Hence, by Sinkhorn's Theorem [Sin67], π^* is unique and the Sinkhorn's algorithm which starts from $u_0 = \mathbf{1}_n$ and $v_0 = \mathbf{1}_m$ and updates

$$\begin{cases} u_{k+1} = \mathbf{1}_n \ ./(Gv_k) \quad \text{(where ./ is the entrywise division)} \\ v_{k+1} = \mathbf{1}_m \ ./(Gu_{k+1}), \end{cases}$$

converges to the optimal scaling vectors u^* and v^*.

Now, from [Bir57], the positive matrix G maps the convex cone of positive vectors into itself. Exploiting this fact, the linear convergence of Sinkhorn's algorithm was established in [FL89].

The overall complexity of Sinkhorn's algorithm was proved in [ANR17]. □

Part IV

Interchange between integration and minimization

Chapter 6

Interchange between integration and minimization

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6.1 Introduction

The question of interchanging integration and minimization is an important issue in stochastic optimization (where integration corresponds to mathematical expectation). Loosely stated, given a measured space $(\Omega, \mathcal{F}, \mu)$ and a subset $X \subset \overline{\mathbb{R}}^{\Omega}$ of functions, we wonder when does the following equality hold

$$\inf_{x \in X} \int_{\Omega} x \, \mathrm{d}\mu = \int_{\Omega} \inf_{x \in X} x \, \mathrm{d}\mu \,. \tag{6.1}$$

Mathematical framework and conditions to get Equation (6.1) can be found in [BG01, EKT13, Gin09, RW09, SDR09]. We focus on [Gin09] and [RW09].

To begin with, in Equation (6.1) one needs to clarify in which sense the integral $\int$ is to be understood and in which sense the infima $\inf_{X \in X} x$ or $\inf_{x \in X} \int x \, \mathrm{d}\mu$ are defined. Then, when the subset X, over which minimization is performed, is a subset of $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and when the integral $\int$ is the usual Lebesgue integral, Giner obtained in [Gin09] a necessary and sufficient condition for (6.1) as follows. In this case, the space $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ is endowed with the usual μ-pointwise order, and the infimum is $\inf_{x \in X} x = \operatorname{ess\,inf}_{x \in X} x$, which is well-defined by [Nev70, Proposition II.4.1]. Given a subset $X \subset L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ of functions, Giner establishes that Equation (6.1) holds true if and only if, for every finite family $x_1, \ldots, x_n$ in X, we have

$$\inf_{x \in X} \int_{\Omega} (x - \inf_{1 \leq i \leq n} x_i) \, \mathrm{d}\mu \leq 0 \,.$$

However, checking the above condition is not an easy task, as it depends jointly on the integral $\int$ and on the subset X. Moreover, one may wonder if we can still have Equation (6.1) for more general subsets X which are integrable in a weaker sense than Lebesgue integrable?

When a subset of functions $X \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ is the image of a set U by a mapping $f : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, *i.e.* $X = f(U)$, a celebrated theorem of Rockafellar and Wets ([RW09, Theorem 14.60]) gives a condition on the mapping f and a condition on the set U so that Equation (6.1) holds. In this case, we deal with minimization over subsets X of $L^0(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and interchange with the outer integral, a generalization of the Lebesgue integral to $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. We study the outer integral and its properties in Appendix 6.5.

The Chapter is organized as follows. Sect. 6.2 is devoted to a minimization interchange theorem on posets. More precisely, we provide an abstract interchange theorem of the form

$$\bigwedge_{x \in X} \Phi(x) = \Phi\big(\bigwedge_{x \in X} x\big) \,. \tag{6.2}$$

Once assumed conditions on the mapping $\Phi : \mathbb{X} \to \mathbb{Y}$ and structural properties of the sets $\mathbb{X}$, $\mathbb{Y}$, we provide a necessary and sufficient condition so that Equation (6.2) holds true. Our result is in the lineage of Giner's, as our necessary and sufficient condition involves both the mapping Φ and the set X.

Sect. 6.3 then tackles the original question of interchange between minimization and integration by specifying the results of Section 6.2.

We hope that either our abstract interchange theorem or its application to the extended Lebesgue integral provide insight as to how one may obtain the interchange between integration and minimization as in Equation (6.1), and as to how we can go beyond the integral case (risk measures in stochastic optimization).

6.2 Minimization interchange theorem on posets

In §6.2.1 we present our main result, namely Theorem 63, which provides an abstract interchange result in the form of Equation (6.2) for a mapping $\Phi : \mathbb{X} \to \mathbb{Y}$ (generalization of the integral) between specific posets. We give necessary and sufficient condition on a subset $X \subset \mathbb{X}$, called Φ-inf-directed such that the abstract interchange holds. Then, in §6.2.2 we a sufficient condition, for the subset $X \subset \mathbb{X}$ to be Φ-inf-directed, that is more practical to check.

6.2.1 Main result

Before stating Theorem 63, we provide background on posets and lattices, as well as two new definitions.

We say that $(\mathbb{X}, \preceq)$ is a *poset* when $\mathbb{X}$ is a set and $\preceq$ is a partial order on $\mathbb{X}$, that is, a reflexive, antisymmetric and transitive binary relation.

Consider a poset $(\mathbb{X}, \preceq)$ and a subset $X \subset \mathbb{X}$. Any $x' \in \mathbb{X}$ such that, for all $x \in X$, we have that $x \preceq x'$ is called an *upper bound* of the set X. If an upper bound $x' \in \mathbb{X}$ of the set X is such that $x' \preceq x''$, for any other upper bound $x'' \in \mathbb{X}$ of the set X, then x' is unique and is called the *least upper bound* of the set X. It is denoted by $\sup X$ or, more explicitly, by $\bigvee_{x\in X} x$. In the same way, we define the *greatest lower bound* $\inf X$ or $\bigwedge_{x\in X} x$. We say that a poset $(\mathbb{X}, \preceq)$ is a *sup-semilattice* (resp. *inf-semilattice*), or *upper semilattice* (resp. *lower semilattice*), if every nonempty finite subset of $\mathbb{X}$ has a least upper bound (resp. greatest lower bound). A *lattice* is both a sup-semilattice and an inf-semilattice.

We say that a poset $(\mathbb{X}, \preceq)$ is a *complete sup-semilattice* (resp. *complete inf-semilattice*), or *complete upper semilattice* (resp. *complete lower semilattice*), if every nonempty subset of $\mathbb{X}$ has a least upper bound (resp. greatest lower bound). A *complete lattice* is both a complete sup-semilattice and a complete inf-semilattice.

We say that a subset $X \subset \mathbb{X}$ has the *countable sup property* if $\bigvee_{x\in X} x$ exists in $\mathbb{X}$ and if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\bigvee_{n\in\mathbb{N}} x_n = \bigvee_{x\in X} x$. In the same way, we define the *countable inf property*.

Now, we introduce two new notions. The first one is that of *sequentially-inf continuity*. The name is suggested by the fact that, when a mapping Φ is order preserving and both *sequentially-inf continuous* and *sequentially-sup continuous* as defined here, then it is sequentially order continuous (denoted as "continuité monotone séquentielle" in [Nev70, p. 37]).

Definition 61. *Let $(\mathbb{X}, \preceq_{\mathbb{X}})$ and $(\mathbb{Y}, \preceq_{\mathbb{Y}})$ be two inf-semilattice (resp. sup-semilattice) and $\Phi : \mathbb{X} \to \mathbb{Y}$ be a mapping. We say that the mapping Φ is* sequentially-inf continuous *(resp.* sequentially-sup continuous*) when, for every nonincreasing (resp. nondecreasing) sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, if $\bigwedge_{n\in\mathbb{N}} x_n$ exists in $\mathbb{X}$ and $\bigwedge_{n\in\mathbb{N}} \Phi(x_n)$ exists in $\mathbb{Y}$ (resp. if $\bigvee_{n\in\mathbb{N}} x_n$ exists in $\mathbb{X}$ and $\bigvee_{n\in\mathbb{N}} \Phi(x_n)$ exists in $\mathbb{Y}$), then we have*

$$\bigwedge_{n\in\mathbb{N}} \Phi(x_n) \preceq_{\mathbb{Y}} \Phi(\bigwedge_{n\in\mathbb{N}} x_n) \quad \Big(\textit{resp.} \quad \Phi(\bigvee_{n\in\mathbb{N}} x_n) \preceq_{\mathbb{Y}} \bigvee_{n\in\mathbb{N}} \Phi(x_n)\Big) . \tag{6.3}$$

The second notion is that of Φ*-inf-directed*.

Definition 62. *Let $(\mathbb{X}, \preceq_{\mathbb{X}})$ be a poset and $(\mathbb{Y}, \preceq_{\mathbb{Y}})$ be a complete inf-semilattice (resp. sup-semilattice) and $\Phi : \mathbb{X} \to \mathbb{Y}$ be a mapping. Let $X \subset \mathbb{X}$ be a subset of $\mathbb{X}$. We say that the* *subset* X is Φ-inf-directed *(resp.* X is Φ-sup-directed*) if, for every finite subset* $\tilde{X} \subset X$*, we have that*

$$\bigwedge_{x \in X} \Phi(x) \preceq_{\mathbb{Y}} \Phi(\bigwedge_{x \in \tilde{X}} x) \quad \Big(\textit{resp.} \quad \Phi(\bigvee_{x \in \tilde{X}} x) \preceq_{\mathbb{Y}} \bigvee_{x \in X} \Phi(x)\Big) \; . \tag{6.4}$$

With these two definitions, we can now state our main theorem.

Theorem 63 (Minimization Interchange Theorem)**.** *Let* $(\mathbb{X}, \preceq_{\mathbb{X}})$ *be a poset and* $(\mathbb{Y}, \preceq_{\mathbb{Y}})$ *be a complete inf-semilattice. Let* $X \subset \mathbb{X}$ *be a subset of* $\mathbb{X}$*,* $\widetilde{\mathbb{X}} \subset \mathbb{X}$ *be an inf-semilattice such that* $X \subset \widetilde{\mathbb{X}} \subset \mathbb{X}$*, and* $\Phi : \mathbb{X} \to \mathbb{Y}$ *be a mapping such that*

1. *the mapping* Φ *is order preserving,* i.e. *for every* $x, x' \in \mathbb{X}$*,*

$$x \preceq_{\mathbb{X}} x' \Rightarrow \Phi(x) \preceq_{\mathbb{Y}} \Phi(x') \; , \tag{6.5}$$

2. *the mapping* Φ *is* sequentially-inf continuous*, when restricted to the inf-semilattice* $\widetilde{\mathbb{X}}$ *(see Definition 61),*

3. *the subset* X *has the countable inf property,* i.e. $\wedge_{x \in X} x$ *exists in* $\mathbb{X}$ *and there exists a sequence* $(x_n)_{n \in \mathbb{N}}$ *in* X *such that*

$$\bigwedge_{n \in \mathbb{N}} x_n = \bigwedge_{x \in X} x \; . \tag{6.6}$$

Then, we have the interchange formula

$$\bigwedge_{x \in X} \Phi(x) = \Phi\big(\bigwedge_{x \in X} x\big) \tag{6.7}$$

if and only if the subset X *is* Φ*-inf-directed (as in Definition 62).*

Proof. Let $\Phi : \mathbb{X} \to \mathbb{Y}$ and $X \subset \mathbb{X}$ be given satisfying Assumptions 1-2-3.

• We assume that the subset X is Φ-inf-directed and we prove the interchange formula (6.7) by means of two inequalities.

First, using the fact that Φ is order preserving, we have that

$$\Phi\big(\bigwedge_{x \in X} x\big) \preceq_{\mathbb{Y}} \Phi(x') \; , \;\; \forall x' \in X \; . \tag{6.8}$$

Thus, using that $(\mathbb{Y}, \preceq_{\mathbb{Y}})$ is a complete inf-semilattice, we obtain that

$$\Phi\big(\bigwedge_{x \in X} x\big) \preceq_{\mathbb{Y}} \bigwedge_{x \in X} \Phi(x) \; .$$

Second, we prove the reverse inequality. Using Assumption 3, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$, whose terms are in X, and such that $\wedge_{n \in \mathbb{N}} x_n = \wedge_{x \in X} x$. Now, we define a new sequence $\{x'_n\}_{n \in \mathbb{N}}$ by $x'_n = \wedge_{k \leq n} x_k$, for all $n \in \mathbb{N}$. So defined, x'_n does not necessarily belong to the subset X, but belongs to the lower semilattice $\widetilde{\mathbb{X}}$ which contains X. Then, the sequence $(x'_n)_{n \in \mathbb{N}}$ is nonincreasing and satisfies the equalities

$$\bigwedge_{n \in \mathbb{N}} x'_n = \bigwedge_{n \in \mathbb{N}} x_n = \bigwedge_{x \in X} x \; . \tag{6.9}$$

Then, we get

$$\bigwedge_{x \in X} \Phi(x) \preceq_{\mathbb{Y}} \Phi(\bigwedge_{k \leq n} x_k)$$

as the subset X is Φ-inf-directed, by assumption, and $\{x_k \,|\, k \leq n\} \subset X$ is finite

$$\preceq_{\mathbb{Y}} \Phi(x'_n)$$

by definition of x'_n, so that we deduce

$$\bigwedge_{x \in X} \Phi(x) \preceq_{\mathbb{Y}} \bigwedge_{n \in \mathbb{N}} \Phi(x'_n) \qquad \text{(as } (\mathbb{Y}, \preceq_{\mathbb{Y}}) \text{ is a complete inf-semilattice by assumption)}$$

$$\preceq_{\mathbb{Y}} \Phi(\bigwedge_{n \in \mathbb{N}} x'_n)$$

by (6.3) as the mapping Φ is sequentially-inf continuous on the inf-semilattice $\widetilde{\mathbb{X}}$, and as $x'_n \in \widetilde{\mathbb{X}}$

$$= \Phi\big(\bigwedge_{x \in X} x\big) \, . \qquad \text{(using Equation (6.9))}$$

• Conversely, we assume that the interchange formula (6.7) holds for the subset $X \subset \mathbb{X}$, and we show that X is Φ-inf-directed.

For this purpose, we consider a finite subset $\tilde{X} \subset X$, and we get

$$\bigwedge_{x \in X} \Phi(x) = \Phi(\bigwedge_{x \in X} x) \qquad \text{(by the interchange formula)}$$

$$\preceq_{\mathbb{Y}} \Phi\big(\bigwedge_{x \in \tilde{X}} x\big)$$

since the mapping Φ is order preserving and $\bigwedge_{x \in X} x \preceq_{\mathbb{X}} \bigwedge_{x \in \tilde{X}} x$.

This concludes the proof. □

6.2.2 A sufficient condition for Φ-directed sets

Given an order preserving and sequentially-inf-continuous mapping $\Phi : \mathbb{X} \to \mathbb{Y}$, where the posets $\mathbb{X}$, $\mathbb{Y}$ have sufficient structure, the Minimization Interchange Theorem 63 shows that a subset $X \in \mathbb{X}$ is Φ-inf-directed if, and only if, we have the abstract interchange formula $\bigwedge_{x \in X} \Phi(x) = \Phi\big(\bigwedge_{x \in X} x\big)$. However, as made apparent in its name, checking if X is Φ-inf-directed is a condition that involves both X and its image by Φ. We give a simple sufficient condition on the subset X only which ensures that X is Φ-inf-directed for every order preserving mapping Φ.

Let $(\mathbb{X}, \preceq)$ be a poset. A *sup-directed* set $X \subset \mathbb{X}$ is a nonempty set with the property that, for every x, $x' \in X$, there exists $x'' \in X$ such that $x \preceq x''$ and $x' \preceq x''$. An *inf-directed* set $X \subset \mathbb{X}$ is a nonempty set with the property that, for every x, $x' \in X$, there exists $x'' \in X$ such that $x'' \preceq x$ and $x'' \preceq x'$.

We now prove in Lemma 64 that any inf-directed (resp. sup-directed) subset $X \subset \mathbb{X}$ is Φ-inf-directed (resp. Φ-sup-directed). Informally, X being inf-directed is a sufficient condition to ensure that a subset X is rich enough from below, namely Φ-inf-directed for any order preserving mapping Φ.

Lemma 64 (Inf-directed implies Φ-inf-directed)**.** *Let* $(\mathbb{X}, \preceq_{\mathbb{X}})$ *be a poset,* $X \subset \mathbb{X}$ *be a subset, and* $(\mathbb{Y}, \preceq_{\mathbb{Y}})$ *be a complete inf-semilattice. If the subset* X *is inf-directed then* X *is* Φ*-inf-directed for any order preserving mapping* $\Phi : (\mathbb{X}, \preceq_{\mathbb{X}}) \to (\mathbb{Y}, \preceq_{\mathbb{Y}})$*.*

Proof. Suppose that $X \subset \mathbb{X}$ is an inf-directed subset of $(\mathbb{X}, \preceq)$, and let $\Phi : (\mathbb{X}, \preceq_{\mathbb{X}}) \to (\mathbb{Y}, \preceq_{\mathbb{Y}})$ be an order preserving mapping. We prove that the subset X is Φ-inf-directed.

For this purpose, we consider a finite subset $\tilde{X} \subset X$. Then, by repeated application of the inf-directed property, we get that there exists $\tilde{x} \in \tilde{X}$ such that $\tilde{x} \preceq_{\mathbb{X}} \wedge_{x \in \tilde{X}} x$. We therefore obtain that

$$\begin{aligned} \bigwedge_{x \in X} \Phi(x) &\preceq_{\mathbb{Y}} \Phi(\tilde{x}) && (\text{as } \tilde{x} \in X) \\ &\preceq_{\mathbb{Y}} \Phi\big(\wedge_{x \in \tilde{X}} x\big) \;, && (\text{as } \Phi \text{ is order preserving and } \tilde{x} \preceq_{\mathbb{X}} \wedge_{x \in \tilde{X}} x) \end{aligned}$$

which ensures that X is Φ-inf-directed and concludes the proof. □

The converse is false, *i.e.* Φ-inf-directed subsets are not necessarily inf-directed subsets as detailed now in Example 5.

Example 5 (The converse of Lemma 64 is false)**.** *Consider* $\Omega = \mathbb{R}$ *equipped with its Borel* σ*-algebra* $\mathcal{B}(\mathbb{R})$ *and Lebesgue measure* λ*. Define the poset* $\mathbb{X} = L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ *with the* μ*-pointwise order and the mapping* $\Phi : \mathbb{X} \to \overline{\mathbb{R}}$ *being the Lebesgue integral (See Appendix 6.5 for details). We claim that the subset* $X \subset \mathbb{X}$ *defined by* $X = \big(-n\mathbf{1}_{(n,n+1)},\, n \in \mathbb{N}\big) \subset \mathbb{R}^{\mathbb{R}}$*, is* Φ*-inf-directed but not inf-directed.*

First we calculate $\wedge_{x \in X} \Phi(x) = \wedge_{x \in X} \int_{\mathbb{R}} x(y)\, \lambda(\mathrm{d}y) = \wedge_{n \in \mathbb{N}} (-n) = -\infty$*. Second, for every finite subset of functions* $\tilde{X} = \{x_{n_1}, \dots x_{n_k}\} \subset X$*, we have* $\Phi(\wedge_{x \in \tilde{X}} x) \geq -k \max_{1 \leq i \leq k} n_i$*. Thus, we get that* $\wedge_{x \in X} \Phi(x) \leq \Phi(\wedge_{x \in \tilde{X}} x)$*, hence the subset* X *is* Φ*-inf-directed.*

But X *is not an inf-directed subset of* $(\mathbb{X}, \preceq)$*. Indeed, let for all* $k \in \mathbb{N}$*, the function* ψ_k *be defined by* $\psi_k = -k\mathbf{1}_{(k,k+1)}$ *and let* n *and* n' *in* $\mathbb{N}$ *be fixed such that* $n \neq n'$*. Assume that there exists* $n'' \in \mathbb{N}$ *such that* $\psi_{n''} \leq \psi_n \wedge \psi_{n'}$*. Then, we should have using the definition of the functions* $\{\psi_k\}_{k \in \mathbb{N}}$*, that the support of* $\psi_{n''}$ *should contain the set* $(n, n+1) \cup (n', n'+1)$*. However no function of* X *has for support the union of two unit intervals.*

In this case observe that the interchange between integration and minimization holds true. Indeed, on the one hand we have shown above that $\wedge_{x \in X} \Phi(x) = -\infty$ *and on the other hand, we have that*

$$\Phi(\wedge_{x \in X} x) \leq \int_0^{+\infty} (1 - y) \lambda(\mathrm{d}y) = -\infty,$$

hence $\wedge_{x \in X} \Phi(x) = -\infty = \Phi(\wedge_{x \in X} x)$*.*

6.3 Interchange between minimization and integration

Throughout this section, we consider a measure space $(\Omega, \mathcal{F}, \mu)$ and we refer the reader to Appendix 6.5 for material regarding extended Lebesgue integrals. We apply the abstract results of Section 6.2 to the case of subsets of $\tilde{\mathbb{X}} = L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, the set of measurable functions with Lebesgue integrable positive part, itself a subset of $\mathbb{X} = L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, the set of measurable functions, for which we consider the interchange with the mapping $\Phi : \mathbb{X} \to \overline{\mathbb{R}}$, which is an extension of the Lebesgue integral to $\mathbb{X}$ called the outer integral.

In §6.3.1, we state a result on the interchange between (an extension of) the Lebesgue integral and minimization for subsets of functions in $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ by specifying the Minimization Interchange Theorem of §6.2.1 to the aforementioned case. Then, in §6.3.2, we give sufficient conditions to ensure that a subset $X \subset \mathbb{X}$ is Φ-inf-directed. Proofs are to be found in §6.3.3. In §6.3.4, we use these sufficient conditions to recover both theorems of Giner and Rockafellar-Wets as applications of the Interchange theorem stated in §6.3.1. Lastly, in §6.3.5, we give another specialisation of the Minimization Interchange Theorem 63 to the case of the poset $\mathbb{X} = \{x : \Omega \to \overline{\mathbb{R}} \mid x \geq 0\}$ with the pointwise order and the Choquet integral $\Phi = \int^{\mathcal{C}}$.

6.3.1 Main result with integrals

We state the main result about the interchange between the outer integral $\int_{\Omega} : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ and minimization for subsets of measurable functions whose positive part has a finite integral. It is the specification of the Minimization Interchange Theorem 63 to the case of $\Phi : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ for subsets of $\tilde{X} = L^0_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ with the mapping Φ being the outer integral. Note that, from Proposition 80, outer integral and extended Lebesgue integrals coincide on either $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ or $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. Thus, Theorem 65 is stated with the Lebesgue integral.

Theorem 65. *Let X be a subset of $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. The following equality*

$$\inf_{x \in X} \int_{\Omega} x \, \mathrm{d}\mu = \int_{\Omega} \operatorname*{ess\,inf}_{x \in X} x \, \mathrm{d}\mu \tag{6.10}$$

is valid if an only if X is integrably inf-directed, i.e. *for every finite family $x_1, \ldots, x_n$ in X we have*

$$\inf_{x \in X} \int_{\Omega} x \, \mathrm{d}\mu \leq \int_{\Omega} \inf_{1 \leq i \leq n} x_i \, \mathrm{d}\mu \,. \tag{6.11}$$

Proof. As being *integrably inf-directed* defined here coincide with being Φ-inf-directed (see Definition 62) when $\Phi = \int_{\Omega}$ is the outer integral, we will show that the assumptions of Theorem 63 are fulfilled to obtain Theorem 65 as a specialization of Theorem 63. We prove in §6.3.3 that the assumptions of Theorem 63 are satisfied. Indeed, by Proposition 71, the structural assumptions on the domain of $\Phi : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ are satisfied:

- The set $\mathbb{X} = L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ with the μ-pointwise order is a poset;
- The subset $\tilde{\mathbb{X}} = L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ of $\mathbb{X}$, is a complete inf-semilattice;
- Every subset $X \subset \tilde{\mathbb{X}} \subset \mathbb{X}$ has the countable inf property.

Moreover, as $\mathbb{Y} = \overline{\mathbb{R}}$ with the usual order is a complete inf-semilattice, the structural assumptions on the codomain of Φ are satisfied as well. Lastly, by Proposition 72, the outer integral $\int_{\Omega} : L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ is

- order preserving,
- sequentially-inf continuous when restricted to the inf-semilattice $\tilde{\mathbb{X}} = L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$.

This ends the proof. $\square$

Note that as semi-integrable functions are linked by the relation (see Lemma 77)

$$x \in L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \Leftrightarrow -x \in L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}),$$

one can deduce a symmetric result about the interchange between outer integral and maximization.

Theorem 66. *Let X be a subset of $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. The following equality*

$$\sup_{x \in X} \int_{\Omega} x \, \mathrm{d}\mu = \int_{\Omega} \operatorname*{ess\,sup}_{x \in X} x \, \mathrm{d}\mu \tag{6.12}$$

is valid if an only if X is integrably sup-directed, i.e. *for any finite family $x_1, \ldots, x_n$ in X we have*

$$\int_{\Omega} \sup_{1 \le i \le n} x_i \, \mathrm{d}\mu \le \sup_{x \in X} \int_{\Omega} x \, \mathrm{d}\mu \ . \tag{6.13}$$

6.3.2 Corollaries

Here, we propose corollaries of Theorem 65 by providing conditions to obtain integrably inf-directed subsets of $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ using the notion of *decomposable subsets*. The notion of decomposable subsets is widely used in L^p spaces and we refer to [Gin09] for a survey on various related definitions.

Decomposable subsets

Here, in Definition 67, we consider the decomposable subset definition of [HU77].

Definition 67 (Decomposable subsets [HU77])**.** *A subset $X \subset L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ is* decomposable *if we have*

$$\forall x, x' \ \in U \ , \ \ \forall A \in \mathcal{F} \ , \ \ x\mathbf{1}_A + x'\mathbf{1}_{A^c} \in X \ . \tag{6.14}$$

We now prove that decomposable subsets are Φ-inf-directed, for any order preserving mapping Φ.

Lemma 68 (Decomposable subsets are Φ-inf-directed)**.** *Any decomposable subset $X \subset L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ is Φ-inf-directed for any order preserving mapping Φ from $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, equipped with the μ-pointwise order, to the extended reals $(\overline{\mathbb{R}}, \le)$.*

Proof. Let x and x' in X be given and consider the measurable set $A = \{x \le x'\}$. We have that $x \wedge x' = x\mathbf{1}_A \dotplus x'\mathbf{1}_{A^c} \in X$ since X is decomposable. We obtain that X is an inf-semilattice and thus also an inf-directed set. Thus, using Lemma 64, the subset X is Φ-inf-directed for any order preserving mapping Φ, from $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ equipped with the μ-pointwise order, to the extended reals $(\overline{\mathbb{R}}, \le)$. □

Rockafellar-Wets decomposable subsets

We have just seen that decomposability implies inf-directed. Rockafellar and Wets introduced a weaker notion of decomposability that Giner relates to Φ-inf-directed where Φ is the outer integral. We recall a few definitions.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with μ being a σ-finite measure. The upper set of U in $L^1(\Omega, \mathfrak{F}, \mu; \mathbb{R})$, denoted by $\underset{(1)}{\uparrow} U$, is defined by

$$\underset{(1)}{\uparrow} U = \left\{ v \in L^1(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}}) \mid \exists u \in U, v \geq u \right\} .$$

For $d \in \mathbb{N}^*$ and for any $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$-measurable function $g : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$, one can define a mapping $\tilde{g}$ on $L^0(\Omega, \mathfrak{F}, \mu; \mathbb{R}^d)$ to $L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$ by

$$\tilde{g} : u \in L^0(\Omega, \mathfrak{F}, \mu; \mathbb{R}^d) \mapsto g\big(\cdot, u(\cdot)\big) \in L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}}).$$

Definition 69 (Rockafellar-Wets decomposable)**.** *A subset $U \subset L^0(\Omega, \mathfrak{F}, \mu; \mathbb{R}^d)$ is* Rockafellar-Wets decomposable *(w.r.t. the σ-finite measure μ) if*

$$y\mathbf{1}_A + u\mathbf{1}_{A^c} \in U \ , \ \ \forall y \in L^\infty(A, \mathfrak{F}, \mu; \mathbb{R}^d) \ , \ \ \forall A \in \mathfrak{F} \ , \ \ \mu(A) < +\infty \ , \ \ \forall u \in U \ . \tag{6.15}$$

Proposition 70 (Proposition 5.4 - [Gin09])**.** *For any measurable function $g : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$, if a subset $U \subset L^0(\Omega, \mathfrak{F}, \mu; \mathbb{R}^d)$ is Rockafellar-Wets decomposable then $X = \underset{(1)}{\uparrow} \tilde{f}(U)$ is an integrably inf-directed subset of $L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$.*

6.3.3 Proofs of Theorem 65 and Theorem 66

We check in Proposition 71 (structural properties of the spaces of measurable and semi-integrable functions) and Proposition 72 (properties of the outer integral) that the assumptions of the Minimization Interchange Theorem 63 are satisfied.

Proposition 71 (Structural properties of the spaces of measurable and semi-integrable functions)**.**

- *The set $L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$, equipped with the μ-pointwise order, is a complete lattice which has both the countable sup and countable inf properties.*
- *The subset $L^1_\ominus(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$ (resp. $L^1_\oplus(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$) is a complete sup-semilattice (resp. inf-semilattice) which has the countable sup property (resp. countable inf property).*

Proof.
• We consider the set $L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$. First, the fact that it is a complete lattice is a consequence of the existence of the essential supremum and essential infimum for any family (countable or not) of class of random variables as proved in [Nev70, Proposition II.4.1] (the proof is for probability measures but it extends easily to σ-finite measures). We rephrase here the existence result of [Nev70, Proposition II.4.1] for the essential infimum case. For any class family (countable or not) $\{x_i\}_{i\in I}$ in $L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$, there exists a unique class $\operatorname{ess\,inf}_{i\in I} x_i \in L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$ which is a *greatest lower bound* of the family $\{x_i\}_{i\in I}$. That is, for any function $\underline{x} \in L^0(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$, we have

$$\forall i \in I, \underline{x} \leq x_i \Leftrightarrow \underline{x} \leq \operatorname*{ess\,inf}_{i\in I} x_i \ .$$

The fact that there exists a countable subfamily $\{x_{i_n}\}_{n\in\mathbb{N}}$ such that

$$\operatorname*{ess\,inf}_{i\in I} x_i = \inf_{n\in\mathbb{N}} x_{i_n}$$

is not stated explicitly in [Nev70, Proposition II.4.1], but it is stated in the proof as an intermediate result to obtain the essential infimum. It is immediate that the countable subfamily can be chosen as a nonincreasing sequence. The case of the essential supremum is treated in the same way.

• We consider the set $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ and consider a class family (countable or not) $\{u_i\}_{i\in I}$ in $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. As $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ is a subset of $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ we obtain (using the first part of the proof) the existence of $\operatorname{ess\,sup}_{i\in I} u_i \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ and the existence of a nondecreasing countable subfamily $\{u_{i_n}\}_{n\in\mathbb{N}}$ such that

$$\operatorname*{ess\,sup}_{i\in I} u_i = \sup_{n\in\mathbb{N}} u_{i_n} \; .$$

Using the monotone convergence theorem for $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, postponed in Proposition 78, we obtain that $\operatorname{ess\,sup}_{i\in I} u_i \in L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, as the supremum of a sequence in $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. As a consequence, the subset $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ is a complete sup-semilattice which has the countable sup property.

The case of $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ can be treated in a similar way. and this ends the proof. □

Proposition 72 (Properties of the outer integral)**.**

- *The outer integral* (6.31a) *is an order preserving mapping between the posets* $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ *and* $\overline{\mathbb{R}}$*.*
- *The outer integral* (6.31a) *is sequentially-sup continuous on the sup-semilattice* $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ *and sequentially-inf continuous on the inf-semilattice* $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$*.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be an nondecreasing sequence of functions in $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. We put $f = \vee_{n\in\mathbb{N}} f_n$, which belongs to the complete lattice $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. By Proposition 78, we get that $f \in L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ and that $\vee_{n\in\mathbb{N}} \int f_n d\mu = \int f d\mu$ by (6.29). Thus, the outer integral (6.31a) is sequentially-sup continuous on the sup-semilattice $L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$.

By using the property that $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) = -L^1_{\ominus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ and (77), we prove that the outer integral (6.31a) is sequentially-inf continuous on the inf-semilattice $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$. □

6.3.4 Comparison with the literature

Now, we combine the general interchange result of §6.3.1 with the conditions of §6.3.2 in order to recover the interchange theorems of Giner and Rockafellar-Wets.

Comparison with Giner [Gin09]

Theorem 73 (Theorem 4.2. – [Gin09])**.** *Let X be a subset of $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$, the following equality*

$$\inf_{x\in X} \int_\Omega x \,\mathrm{d}\mu = \int_\Omega \operatorname*{ess\,inf}_{x\in X} x \,\mathrm{d}\mu \; , \tag{6.16}$$

is valid if an only if X is integrably inf-directed*,* i.e. *for any finite family $x_1, \ldots, x_n$ in X we have*

$$\inf_{x\in X} \int_\Omega (x - \inf_{1\le i\le n} x_i) \,\mathrm{d}\mu \le 0 \; . \tag{6.17}$$

As $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}) \subset L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, the interchange formula in Theorem 65 is a slight generalization to $L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ of Giner's Theorem 73 stated for subsets of $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. This is no surprise, as we are indebted to Giner since Theorem 65 was greatly inspired by Giner's result.

Comparison with Rockafellar and Wets [RW09]

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with μ being a σ-finite measure. As we work with subsets of measurable functions, the integral used in this section is the outer integral as defined in Appendix in Definition 79.

Theorem 74 (Theorem 14.60 [RW09])**.** *Let U be a subset of $L^0(\Omega, \mathcal{F}, \mu; \mathbb{R}^d)$ that is Rockafellar-Wets decomposable. Let $g : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ be a normal integrand. If there exists $\bar{u} \in U$ such that $g\big(\cdot, \bar{u}(\cdot)\big) \in L^1_{\oplus}(\Omega, \mathcal{F}, \mu; \mathbb{R}^d)$, one has that*

$$\inf_{u \in U} \int^*_\Omega g\big(\omega, u(\omega)\big) \, \mathrm{d}\mu(\omega) = \int^*_\Omega \big(\inf_{u \in \mathbb{R}^n} g(\omega, u)\big) \, \mathrm{d}\mu(\omega) \, . \tag{6.18}$$

Moreover, as long as this common value is not $-\infty$, one has for any $\underline{u} \in U$ that

$$\underline{u} \in \arg\min_{u \in U} \int^*_\Omega g\big(\omega, u(\omega)\big) \, \mathrm{d}\mu(\omega) \iff \underline{u} \in \arg\min_{u \in \mathbb{R}^n} g(\omega, u) \ \ \mu\text{-a.s.} \ \ .$$

We prove that the Rockafellar-Wets interchange theorem can be deduced from Theorem 65 combined with [Gin09, Theorem 3.1].

Proof. (Equation (6.18) as a consequence of [Gin09, Theorem 3.1] and the Minimization Interchange Theorem 63)

• We introduce the set $X = \big\{\omega \mapsto g\big(\omega, u(\omega)\big) \,\big|\, u \in U\big\}$. Using the fact that g is a normal integrand and that U is a subset of $L^0(\Omega, \mathcal{F}, \mu; \mathbb{R}^d)$, we obtain that X is a subset of $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ [RW09, Theorem 14.37] and we can write

$$\inf_{u \in U} \int^*_\Omega g\big(\omega, u(\omega)\big) \, \mathrm{d}\mu(\omega) = \bigwedge_{x \in X} \int^*_\Omega x(\omega) \, \mathrm{d}\mu(\omega) \, . \tag{6.19}$$

Now, using the definition (6.31a) of the outer integral we have that

$$\bigwedge_{x \in X} \int^*_\Omega x(\omega) \, \mathrm{d}\mu(\omega) = \bigwedge_{x \in X} \inf_{\substack{x' \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}) \\ x' \geq x}} \int_\Omega x'(\omega) \, \mathrm{d}\mu(\omega) \, . \tag{6.20}$$

We therefore introduce the upper set of X in $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ denoted by $\underset{(1)}{\uparrow} X$ and defined by

$$\underset{(1)}{\uparrow} X = \big\{x' \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}) \,\big|\, \exists x \in X \text{ s.t. } x \leq x' \ \ \mu\text{-a.s. } \big\} \, . \tag{6.21}$$

Combining Equation (6.19)-(6.21), we therefore get that

$$\inf_{u \in U} \int^*_\Omega g\big(\omega, u(\omega)\big) \, \mathrm{d}\mu(\omega) = \bigwedge_{x' \in \underset{(1)}{\uparrow} X} \int_\Omega x'(\omega) \, \mathrm{d}\mu(\omega) \, . \tag{6.22}$$

In addition, using the fact that there exists $\bar{u} \in U$ such that $g\big(\cdot, \bar{u}(\cdot)\big) \in L^1_{\oplus}(\Omega, \mathfrak{F}, \mu; \overline{\mathbb{R}})$ we have that $\overline{x} = g\big(\cdot, \bar{u}(\cdot)\big)$ belongs to X. Therefore, the set $\underset{(1)}{\uparrow} X$ is not empty as $\overline{x}_+ \in \underset{(1)}{\uparrow} X$.

- By [Gin09, Propositon 5.4], as g is a normal integrand and thus measurable, the set $\underset{(1)}{\uparrow} U$ is integrably inf-directed.
- The last step to obtain (6.18) is to prove that

$$\inf_{u \in \mathbb{R}^n} g(\omega, u) = \bigwedge_{x' \in \underset{(1)}{\uparrow} X} x'(\omega) ,$$

which is obtained using [Gin09, Theorem 3.1]. □

6.3.5 Interchange between minimization and Choquet's integral

Fix $(\Omega, \mathcal{F})$ a measurable space. We specialize the Minimization Interchange Theorem 63 to the poset of nonnegative measurable functions

$$\mathbb{X} = \big\{x : \Omega \to \overline{\mathbb{R}} \mid x \geq 0 \text{ and measurable}\big\}$$

with the pointwise order and the Choquet integral $\Phi = \int^{\mathcal{C}}$ that we define below. We suggest [Kaw18] and the references therein for properties of the Choquet integral.

A *capacity* $c : \mathcal{F} \to \overline{\mathbb{R}}$ is a function which is order preserving $(\forall F_1, F_2 \in \mathcal{F}, F_1 \subset F_2 \Rightarrow c(F_1) \leq c(F_2))$ and such that $c(\emptyset) = 0$. Given a capacity c, the Choquet integral of a nonnegative measurable function $x \in \mathbb{X}$ is defined by

$$\int_{\Omega}^{\mathcal{C}} x(\omega) \,\mathrm{d}c(\omega) = \int_{\mathbb{R}_+} c\big(x > t\big) \,\mathrm{d}t ,$$

where the integral on the right-hand side is the Lebesgue integral of an nonincreasing function. We say that a capacity c is *continuous from above* if for any nondecreasing sequence of sets $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ such that $F = \cap_{n \in \mathbb{N}} F_n \in \mathcal{F}$ then we have $c(F_n) \underset{n \to +\infty}{\longrightarrow} F$. Lastly, a subset of functions $X \subset \mathbb{X}$ is *Choquet integrably inf-directed* if it is integrably inf-directed with the Choquet integral.

Proposition 75. *Let c be a continuous from above capacity and $X = \{x_i\}_{i \in I} \subset \mathbb{X}$ a family of nonnegative measurable functions with the countable-inf-property. We have*

$$\bigwedge_{i \in I} \int_{\Omega}^{\mathcal{C}} x_i \,\mathrm{d}c = \int_{\Omega}^{\mathcal{C}} \bigwedge_{i \in I} x_i \,\mathrm{d}c$$

if, and only if, X is Choquet integrably inf-directed.

Proof. We check that the assumptions of Theorem 63 are satisfied. The set of nonnegative measurable functions $\mathbb{X} = \big\{x : \Omega \to \overline{\mathbb{R}} \mid x \geq 0 \text{ and measurable}\big\}$ endowed with the pointwise order is an inf-semilattice.

The Choquet integral is order preserving on $\mathbb{X}$ (see [Kaw18, Proposition 2.3]).

As the capacity c is countinuous from above, the monotone decreasing pointwise convergence theorem holds (see [Kaw18, Theorem 3.2.(2)]): for every nonincreasing sequence of functions $\{x_n\}_{n \in \mathbb{N}}$ converging pointwise to $x \in \mathbb{X}$, we have

$$\bigwedge_{n \in \mathbb{N}} \int_{\Omega}^{\mathcal{C}} x_n \,\mathrm{d}c = \lim_{n \in \mathbb{N}} \int_{\Omega}^{\mathcal{C}} x_n \,\mathrm{d}c = \int_{\Omega}^{\mathcal{C}} x \,\mathrm{d}c ,$$

so the Choquet integral is sequentially-inf-continuous on $\mathbb{X}$.

Hence by Theorem 63, given $X = \{x_i\}_{i \in I} \subset \mathbb{X}$ a family of nonnegative functions with the countable-inf-property, we have

$$\bigwedge_{i \in I} \int_{\Omega}^{\mathcal{C}} x_i \, \mathrm{d}c = \int_{\Omega}^{\mathcal{C}} \bigwedge_{i \in I} x_i \, \mathrm{d}c$$

if, and only if, X is Choquet integrably inf-directed. $\square$

6.4 Conclusion and perspectives

We were initially interested in minimization of functions and interchange with mappings Φ which are not the integral. As said in the introduction, the question of interchanging integration and minimization is an important issue in stochastic optimization (where integration corresponds to mathematical expectation). Now, when the mathematical expectation is replaced with a risk measure, the question of interchange is less examined [SDR09]. An important class of risk measures is made of suprema of integral expressions. This is why in Section 6.2.1 we started with an abstract result on interchange and optimization followed by an analysis of the integral case. There now remains to study when our abstract results apply to suprema of integrals.

6.5 Extended Lebesgue and outer integrals

The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is endowed with its Borel σ-algebra (see [BP12, Application 4.2] or [Nev70, Chap. II]), and with the following extended additions and multiplication. We still denote by $+$ the usual addition when extended to $\overline{\mathbb{R}}_+ = \mathbb{R} \cup \{+\infty\}$ by $+\infty$ being absorbant, and to $\overline{\mathbb{R}}_- = \mathbb{R} \cup \{-\infty\}$ by $-\infty$ being absorbant. Then, we denote by $\underset{\cdot}{+}$ the addition on $\overline{\mathbb{R}}$ for which $-\infty$ is absorbant, *i.e.* $(+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty$ and by $\overset{\cdot}{+}$ the addition for which $+\infty$ is absorbant, *i.e.* $(+\infty) \overset{\cdot}{+} (-\infty) = (-\infty) \overset{\cdot}{+} (+\infty) = +\infty$. We set $\lambda \times (\pm\infty) = \pm\infty$ for $\lambda \in]0, +\infty[$, $\lambda \times (\pm\infty) = \mp\infty$ for $\lambda \in]-\infty, 0[$, and $0 \times (\pm\infty) = 0$.

Throughout this section, we fix a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$. The classical Lebesgue integral w.r.t. the σ-finite measure μ is defined for functions with values in $\mathbb{R}$ (real-valued functions). As we are motivated by optimization, we need results for integrals of functions with values in $\overline{\mathbb{R}}$ (extended real-valued functions). For integration of measurable real-valued functions w.r.t. a σ-finite measure μ, we refer the reader to [AB06, Chapter 11]; for integration of measurable extended real-valued functions w.r.t. a probability measure μ, we refer the reader to [Nev70]; for integration of measurable extended real-valued functions w.r.t. a σ-finite measure μ, we refer the reader to [Hal50, Chapter V]; for outer integration of extended real-valued functions w.r.t. a σ-finite measure μ, we refer the reader to [BS96].

It happens that results about monotonicity, additivity, external multiplication and monotone convergence of the integral are either scattered in the literature, or sometimes not formulated. This is due to the fact that the extension of the Lebesgue integral to extended real-valued functions gives rise to different expressions, which renders the exposition less systematic and elegant than with the Lebesgue integral of integrable real-valued functions. Also, some results belong to folklore and its is hard to find trace of their proof, as they are considered obvious. However, for the purpose of optimizing integral expressions, we provide below a systematic exposition of functional spaces $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $L^1_{\oplus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and $L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, and how the Lebesgue integral can be extended.

6.5.1 Functional space $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and the Lebesgue integral

We endow the set $\overline{\mathbb{R}}^{\Omega}$ of functions $f:\Omega\to\overline{\mathbb{R}}$ with the *μ-pointwise order* $\leq$ as follows: for any $f,g\in\overline{\mathbb{R}}^{\Omega}$,

$$f\leq g \iff \exists A\in\mathcal{F}\ ,\ \ \mu(A)=0\ ,\ \ f(\omega)\leq g(\omega)\ ,\ \ \forall\omega\in\Omega\setminus A\ . \tag{6.23}$$

We denote by $\mathcal{L}^0(\Omega,\mathcal{F};\overline{\mathbb{R}})$ the set of measurable functions from Ω to $\overline{\mathbb{R}}$ and by $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ the quotient $\mathcal{L}^0(\Omega,\mathcal{F};\overline{\mathbb{R}})/\sim$ where for any $f,g\in\mathcal{L}^0(\Omega,\mathcal{F};\overline{\mathbb{R}})$, $f\sim g$ if, and only if, $f=g$ μ-almost everywhere. The μ-pointwise order (6.23) induces an order on the set $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ of equivalence classes, that we will also denote by $\leq$ and call the μ-pointwise order. Thus, the expression $f\geq 0$ makes sense for $f\in L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$. In the same way, we introduce the *μ-pointwise order* $<$ on the set $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ of equivalence classes. Thus, the expressions $-\infty<f$, $f<+\infty$ and $-\infty<f<+\infty$ make sense for $f\in L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$.

The set $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ is stable under the two additions $\underset{\cdot}{+}$ or $\dot{+}$, and under external multiplication. We say that a subset of $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ is a *convex cone*, if it is stable under the addition $+$ and under external multiplication by a scalar in $\mathbb{R}_+$.

We write $\int$ for the Lebesgue integral deduced from the σ-finite measure space $(\Omega,\mathcal{F},\mu)$. The Lebesgue integral $\int$ is defined on the convex cone

$$L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})=\left\{f\in L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})\,\middle|\,f\geq 0\right\}\ , \tag{6.24}$$

where it takes values in $\overline{\mathbb{R}}_+$, given by the formula (see [AB06, Footnote 3, p. 411] for real-valued functions)

$$\int f=\int f\,\mathrm{d}\mu=\sup\left\{\int_{\Omega}\varphi\,\mathrm{d}\mu\,\middle|\,0\leq\varphi\leq f\ ,\ \ \varphi\text{ simple and nonnegative}\right\}\ , \tag{6.25}$$

where simple nonnegative functions (or μ-step functions) are functions of the form $\varphi(\cdot)=\sum_{i\in I}\alpha_i\mathbf{1}_{A_i}(\cdot)$ with I finite and $\{A_i\}_{i\in I}$ a sequence of measurable sets such that $\mu(A_i)<+\infty$ for all $i\in I$ and the coefficients $\{\alpha_i\}_{i\in I}$ are positive and finite reals and the indicator function $\mathbf{1}_A$ of a subset of Ω is defined by $\mathbf{1}_A(x)=1$ if $x\in A$ and $\mathbf{1}_A(x)=0$ if $x\notin A$.

The (extended) Lebesgue integral on $L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ has the following properties

- monotone: $\forall f,g\in L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $f\leq g \implies \int f\leq\int g$,
- additive: $\forall f,g\in L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $\int(f+g)=\int f+\int g$,
- positively homogeneous: $\forall f\in L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $\forall\lambda\in\mathbb{R}_+$, $\int(\lambda f)=\lambda\int f$,
- monotone convergence: for any nondecreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, then $f=\sup_{n\in\mathbb{N}}f_n\in L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and $\lim_{n\to+\infty}\int f_n=\int f$.

6.5.2 Functional spaces $L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and the extended Lebesgue integral

For any function $f : \Omega \to \overline{\mathbb{R}}$, we define its *positive part* $f_+ = \sup(0, f)$ and its *negative part* $f_- = \sup(0, -f)$. Obviously, we have $f = f_+ + (-f_-)$ (where we use the addition $+$ as one of the terms is zero for any value taken by the argument of the function f). We define the set

$$\mathcal{L}^1_\oplus(\Omega,\mathcal{F};\overline{\mathbb{R}}) = \left\{ f \in \mathcal{L}^0(\Omega,\mathcal{F};\overline{\mathbb{R}}) \,\middle|\, \int_\Omega f_+ \,\mathrm{d}\mu < +\infty \right\} , \tag{6.26a}$$

and the quotient set $\mathcal{L}^1_\oplus(\Omega;\overline{\mathbb{R}}) \backslash \sim$ by

$$L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) = \left\{ f \in L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \,\middle|\, \int_\Omega f_+ \,\mathrm{d}\mu < +\infty \right\} , \tag{6.26b}$$

with the property that

$$f \in L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \implies f < +\infty \tag{6.26c}$$

because $\int_\Omega f_+ \,\mathrm{d}\mu < +\infty \implies f_+ < +\infty \implies f \leq f_+ < +\infty$. In the same way, we define

$$\mathcal{L}^1_\ominus(\Omega,\mathcal{F};\overline{\mathbb{R}}) = \left\{ f \in \mathcal{L}^0(\Omega,\mathcal{F};\overline{\mathbb{R}}) \,\middle|\, \int_\Omega f_- \,\mathrm{d}\mu < +\infty \right\} , \tag{6.27a}$$

$$L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) = \left\{ f \in L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \,\middle|\, \int_\Omega f_- \,\mathrm{d}\mu < +\infty \right\} , \tag{6.27b}$$

with the properties that $L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) = -L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and that $f \in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \implies -\infty < f$.

We say that a (class of) function(s) $f \in L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ is *semi-integrable* if it belongs to $L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \cup L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, that is, if either $\int_\Omega f_+ \,\mathrm{d}\mu < +\infty$ or $\int_\Omega f_- \,\mathrm{d}\mu < +\infty$. The Lebesgue integral is extended from the convex cone $L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, to semi-integrable functions by ([Nev70, Proposition II-3-2], [AB06, Chapter 11], [Hal50, Chapter V])

$$\int f = \int f_+ + \left(-\int f_- \right) , \quad \forall f \in L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \cup L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) . \tag{6.28}$$

The extended Lebesgue integral on semi-integrable functions has the following properties (listed in [Nev70, Proposition II-3-3])

- monotone: $\forall f, g \in L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \cup L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $f \leq g \implies \int f \leq \int g$,
- additive on $L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$: $\forall f, g \in L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $\int (f+g) = \int f + \int g$,
- additive on $L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$: $\forall f, g \in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $\int (f+g) = \int f + \int g$,
- positively and negatively homogeneous: $\forall f \in L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \cup L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, $\forall \lambda \in \mathbb{R}$, $\int (\lambda f) = \lambda \int f$,
- monotone convergence on $L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$: for any nonincreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, then $f_n \uparrow f$ and $f = \inf_{n\in\mathbb{N}} f_n \in L^1_\oplus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and $\lim_{n\to+\infty} \int f_n = \int f$,
- monotone convergence on $L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$: for any nondecreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, then $f = \sup_{n\in\mathbb{N}} f_n \in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and $f_n \downarrow f$ and $\lim_{n\to+\infty} \int f_n = \int f$.

We provide some of the proofs.

Lemma 76. *For any functions f and g in $L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, we have $f+g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and*

$$\int_\Omega (f+g)\,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu + \int_\Omega g\,\mathrm{d}\mu\ ,\ \ \forall f\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})\ ,\ \ g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})\ .$$

Proof. We consider $f,g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$. Notice that, as $f,g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, we have that $-\infty<f$ and $-\infty<g$, so that we will use the addition $+$.

• We show that $\int_\Omega (f+g)\,\mathrm{d}\mu<+\infty$. On the one hand, we have

$$(f+g)_- = \sup\big(0,-(f+g)\big) = \sup\big(0,(-f)+(-g)\big)\ .$$

On the other hand, we have $(-f)\le f_-$ and $(-g)\le g_-$, hence $(-f)+(-g)\le f_-+g_-$ and thus $(f+g)_-\le f_-+g_-$. By monotonicity and additivity of the Lebesgue integral on $L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, we deduce that

$$\int_\Omega (f+g)_-\,\mathrm{d}\mu \le \int_\Omega f_-\,\mathrm{d}\mu + \int_\Omega g_-\,\mathrm{d}\mu < +\infty\ ,$$

because $\int_\Omega f_-\,\mathrm{d}\mu<+\infty$ and $\int_\Omega g_-\,\mathrm{d}\mu<+\infty$ by assumption ($f,g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$). Hence, $f+g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\mathbb{R})$.

• We prove the additivity of the integral. Notice that, as $f,g,f+g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, we have that $-\infty<f$ and $-\infty<g$, and also that $0\le f_-<+\infty$, $0\le g_-<+\infty$, $0\le (f+g)_-<+\infty$, $0\le \int_\Omega f_-\,\mathrm{d}\mu<+\infty$, $0\le\int_\Omega g_-\,\mathrm{d}\mu<+\infty$, $0\le\int_\Omega (f+g)_-<+\infty$, so that we will use the addition $+$.

As, for any function h, we have that $h=h_++(-h_-)$, we immediately get that

$$(f+g)_+ + \big(-(f+g)_-\big) = f+g = f_+ + (-f_-) + g_+ + (-g_-)\ .$$

Now, if we add, to the left and right hand side of the above equality, the three nonnegative reals $(f+g)_-$, f_- and g_- (none of them being $+\infty$), we obtain the equality

$$(f+g)_+ + f_- + g_- = f_+ + g_+ + (f+g)_-\ .$$

As this is an equality between sums of nonnegative functions, we apply the Lebesgue integral on $L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, and get

$$\int_\Omega (f+g)_+\,\mathrm{d}\mu + \int_\Omega f_-\,\mathrm{d}\mu + \int_\Omega g_-\,\mathrm{d}\mu = \int_\Omega f_+\,\mathrm{d}\mu + \int_\Omega g_+\,\mathrm{d}\mu + \int_\Omega (f+g)_-\,\mathrm{d}\mu\ ,$$

by additivity of the Lebesgue integral on $L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$. Now, the quantities $\int_\Omega f_-\,\mathrm{d}\mu$, $\int_\Omega g_-\,\mathrm{d}\mu$ and $\int_\Omega (f+g)_-\,\mathrm{d}\mu$ are three nonnegative reals (none of them being $+\infty$) by assumption ($f,g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and property $f+g\in L^1_\ominus(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$). Thus, we get, by subtracting these three finite terms,

$$\begin{aligned} &\int_\Omega (f+g)_+\,\mathrm{d}\mu + \big(-\int_\Omega (f+g)_-\,\mathrm{d}\mu\big) \\ &= \int_\Omega f_+\,\mathrm{d}\mu + \big(-\int_\Omega f_-\,\mathrm{d}\mu\big) + \int_\Omega g_+\,\mathrm{d}\mu + \big(-\int_\Omega g_-\,\mathrm{d}\mu\big), \end{aligned}$$

hence, by (6.28),

$$\int_{\Omega}(f+g)\,\mathrm{d}\mu = \int_{\Omega} f\,\mathrm{d}\mu + \int_{\Omega} g\,\mathrm{d}\mu\ .$$

This ends the proof. □

Lemma 77. *We have*

$$\int_{\Omega}(-f)\,\mathrm{d}\mu = -\int_{\Omega} f\,\mathrm{d}\mu\ ,\quad \forall f \in L^1_{\oplus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}) \cup L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})\ .$$

Proof. This is an obvious consequence of (6.28), and of $(-f)_+ = f_-$ and $(-f)_- = f_+$. □

Proposition 78 (Extended monotone convergence theorem for $L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be an nondecreasing sequence of functions in $L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, converging to $f \in \overline{\mathbb{R}}^{\Omega}$, that is, $f_n \uparrow f$. Then, $f \in L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and we have*

$$\lim_{n\to+\infty}\int f_n d\mu = \int f d\mu\ . \tag{6.29}$$

Proof. Notice that, as $f_n \in L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, we have that $-\infty < f_n$ for all $n \in \mathbb{N}$, so that we will use the addition $+$.

As $f \geq f_1$, we have that $\sup(0,-f) = f_- \leq (f_1)_- = \sup(0,-f_1)$, hence $\int f_- d\mu \leq \int (f_1)_- d\mu < +\infty$, where the last strict inequality is by assumption ($f_1 \in L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$). We conclude that $f \in L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$.

As, by assumption, $\int (f_1)_- d\mu < +\infty$, we conclude that $(f_1)_- < +\infty$. We consider two cases.

We suppose that $\int (f_1)_+ d\mu = +\infty$. As $\sup(0,f) = f_+ \geq (f_1)_+ = \sup(0,f_1)$, we also have that $\int f_+ d\mu = +\infty$. As a consequence, we get that $\int f_+ d\mu = \int (f_1)_+ d\mu = +\infty$, hence $\int f d\mu = \int f_1 d\mu = +\infty$, by definition of the integral $\int$ on $L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$. By monotonicity of the integral $\int$, we conclude that $+\infty = \int f_1 d\mu \leq \lim_{n\to+\infty}\int f_n d\mu \leq \int f_1 d\mu = +\infty$, hence that (6.29) holds true.

We now suppose that $\int (f_1)_+ d\mu < +\infty$. We deduce that $(f_1)_+ < +\infty$. As we had $(f_1)_- < +\infty$, we deduce that $-\infty < f_1 < +\infty$. Thus, we can define $\varphi_n = f_n + (-f_1)$ and $\varphi = f + (-f_1)$, which are functions in $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ such that $\varphi = f + (-f_1) \geq \varphi_n = f_n + (-f_1) \geq 0$, because $f_n \geq f_1$. As f_1 takes values in $\mathbb{R}$, we have that $\sup_n(f_n + (-f_1)) = \sup_n f_n + (-f_1)$, hence we obtain that $\varphi_n \uparrow \varphi$. As $\varphi_n \geq 0$, by the monotone convergence theorem for $(L^0_+(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}}), \int)$, we get that

$$\sup_n \int \varphi_n d\mu = \lim_{n\to+\infty}\int \varphi_n d\mu = \int \varphi d\mu\ .$$

As, by assumption, $(f_1)_- d\mu < +\infty$ and $\int (f_1)_+ d\mu < +\infty$, we get that $f_1 \in L^1(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$ and that $-\infty < \int f_1 d\mu < +\infty$, hence obtaining

$$\sup_n \Big(\int \varphi_n d\mu + \int f_1 d\mu\Big) = \sup_n \int \varphi_n d\mu + \int f_1 d\mu = \int \varphi d\mu + \int f_1 d\mu\ .$$

As $\varphi_n \geq 0$ and belongs to $L^0(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$, we have that $\varphi_n \in L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$. In the same way, we obtain that $\varphi \in L^1_{\ominus}(\Omega,\mathcal{F},\mu;\overline{\mathbb{R}})$. By the $+$-additivity property of the integral $\int$ on

$L^1_\ominus(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, we deduce that the first and last terms of the above equality are given by the following expressions

$$\sup_n \int (\varphi_n + f_1) d\mu = \int (\varphi + f_1) d\mu \, .$$

We obtain (6.29) because $\varphi_n + f_1 = f_n + (-f_1) + f_1 = f_n$ since f_1 takes values in $\mathbb{R}$, and, in the same way, $\varphi + f_1 = f + (-f_1) + f_1 = f$.

□

The classical *vector space of integrable functions* is

$$L^1(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) = L^1_\oplus(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \cap L^1_\ominus(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \, , \tag{6.30}$$

with the property that $f \in L^1(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \implies -\infty < f < +\infty$, that is, $L^1(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) = L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

6.5.3 Outer integral on $L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$

We follow [BS96] for the following definitions.

Definition 79. *We define the* outer integral *of a function by*

$$\int_\Omega^* f \, \mathrm{d}\mu = \inf \Big\{ \int_\Omega \psi \, \mathrm{d}\mu \, \Big| \, \psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}) \text{ and } f \le \psi \Big\} \, , \ \ \forall f \in \overline{\mathbb{R}}^\Omega \, , \tag{6.31a}$$

and the inner integral *by*

$$\int_*^\Omega f \, \mathrm{d}\mu = \sup \Big\{ \int_\Omega \psi \, \mathrm{d}\mu \, \Big| \, \psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}) \text{ and } f \ge \psi \Big\} \, , \ \ \forall f \in \overline{\mathbb{R}}^\Omega \, , \tag{6.31b}$$

where $\int_\Omega \psi \, \mathrm{d}\mu$ is the classical Lebesgue integral for $\psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

It is straightforward that

$$\int_*^\Omega f \, \mathrm{d}\mu \le \int_\Omega^* f \, \mathrm{d}\mu \, , \ \ \forall f \in \overline{\mathbb{R}}^\Omega \, , \tag{6.32a}$$

$$-\int_\Omega^* f \, \mathrm{d}\mu \le \int_\Omega^* (-f) d\mu \, , \ \ \forall f \in \overline{\mathbb{R}}^\Omega \, , \tag{6.32b}$$

$$\int_*^\Omega f \, \mathrm{d}\mu = -\Big(\int_\Omega^* (-f) \, \mathrm{d}\mu \Big) \, , \ \ \forall f \in \overline{\mathbb{R}}^\Omega \, . \tag{6.32c}$$

These outer and inner integrals extend the classical Lebesgue integral to the uncovered case where both $\int_\Omega f_+ \, \mathrm{d}\mu$ and $\int_\Omega f_- \, \mathrm{d}\mu$ equal $+\infty$ as shown in the following Proposition.

Proposition 80. *We have that*

$$\int_\Omega^* f \, \mathrm{d}\mu = \int_\Omega f_+ \, \mathrm{d}\mu \dot{+} \Big(- \int_\Omega f_- \, \mathrm{d}\mu \Big) \, , \ \ \forall f \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \, , \tag{6.33a}$$

$$\int_*^\Omega f \, \mathrm{d}\mu = \int_\Omega f_+ \, \mathrm{d}\mu \underset{\cdot}{+} \Big(- \int_\Omega f_- \, \mathrm{d}\mu \Big) \, , \ \ \forall f \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \, . \tag{6.33b}$$

As a consequence, the outer integral of f coincides with the extended Lebesgue integral (6.28) *on $L^1_\oplus(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}}) \cup L^1_\ominus(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$, that is, when f is semi-integrable.*

Proof. We consider $f \in L^0(\Omega, \mathcal{F}, \mu; \overline{\mathbb{R}})$ and we examine four possible cases in order to prove Equation (6.33a) (then Equation (6.33b) is obtained from (6.32c)).

• Suppose that $\int_\Omega f_+ \,\mathrm{d}\mu < +\infty$ and $\int_\Omega f_- \,\mathrm{d}\mu < +\infty$ (that is, $f \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$). Then we have that $\mu[\{f = \pm\infty\}] = 0$, and thus there exists a representant $\tilde{f} \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ in the class, which is equal to f (μ-a.s.). Thus, we have that $\int_\Omega^* f \,\mathrm{d}\mu \leq \int_\Omega \tilde{f} \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu$ as we can use $\psi = \tilde{f}$ in the definition of the outer integral. Now, in order to prove the reverse inequality, we have to consider two cases, depending whether $\int_\Omega^* f \,\mathrm{d}\mu$ is finite or is equal to $-\infty$.

$\diamond$ In the case where $\int_\Omega^* f \,\mathrm{d}\mu$ is finite, we fix $\epsilon > 0$. Using Equation (6.31a), there exists $\psi_\epsilon \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that $f \leq \psi_\epsilon$ and $\int_\Omega \psi_\epsilon \,\mathrm{d}\mu \leq \int_\Omega^* f \,\mathrm{d}\mu + \epsilon$. Using the fact that $f \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and the monotonicity of the Lebesgue integral, we obtain

$$\int_\Omega f \,\mathrm{d}\mu \leq \int_\Omega \psi_\epsilon \,\mathrm{d}\mu \leq \int_\Omega^* f \,\mathrm{d}\mu + \epsilon ,$$

which finally gives $\int_\Omega f \,\mathrm{d}\mu \leq \int_\Omega^* f \,\mathrm{d}\mu$ and therefore the equality $\int_\Omega f \,\mathrm{d}\mu = \int_\Omega^* f \,\mathrm{d}\mu$. Equation (6.33a) follows using Equation (6.28) as we have

$$\int_\Omega^* f \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu = \int_\Omega f_+ \,\mathrm{d}\mu + (-\int_\Omega f_- \,\mathrm{d}\mu) = \int_\Omega f_+ \,\mathrm{d}\mu \dot{+} \big(-\int_\Omega f_- \,\mathrm{d}\mu\big) .$$

$\diamond$ In the case where $\int_\Omega^* f \,\mathrm{d}\mu = -\infty$, then using Equation (6.31a) there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that $f \leq \psi_n$ and $\int_\Omega \psi_n \,\mathrm{d}\mu \leq -n$ for all $n \in \mathbb{N}$. This implies that $\int_\Omega f \,\mathrm{d}\mu = -\infty$, which contradicts the fact that $f \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

• Suppose that $\int_\Omega f_+ \,\mathrm{d}\mu < +\infty$ and and $\int_\Omega f_- \,\mathrm{d}\mu = +\infty$. Using the fact that $f \leq f_+$ we have that $\int_\Omega^* f \,\mathrm{d}\mu \leq \int_\Omega f_+ \,\mathrm{d}\mu$ as we can use $\psi = f_+ \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ in the definition (6.31a) of the outer integral. Moreover, as $\int_\Omega f_- \,\mathrm{d}\mu = +\infty$, we can find a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of nonnegative functions such that $\psi_n \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$, $\psi_n \leq f_-$ and such that $\lim_{n\to\infty} \int_\Omega \psi_n \,\mathrm{d}\mu = +\infty$ for all $n \in \mathbb{N}$ (take $\psi_n = \mathbf{1}_{\Omega_n} \min(n, f_-)$, where $(\Omega_n)_{n\in\mathbb{N}}$ is a monotone sequence of $\mathcal{F}$-measurable subsets of Ω covering Ω such that $\mu(\Omega_n) < +\infty$ which exists by σ-finite property). Using the fact that $\int_\Omega f_+ \,\mathrm{d}\mu < +\infty$, we can find $\tilde{f}_+ \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that $f_+ = \tilde{f}_+$ μ-a.s. Thus, for all $n \in \mathbb{N}$, we have that $f \leq (\tilde{f}_+ - \psi_n)$ and $(\tilde{f}_+ - \psi_n) \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. We obtain by monotone convergence that

$$\int_\Omega^* f \,\mathrm{d}\mu \leq \int_\Omega \tilde{f}_+ - \psi_n \,\mathrm{d}\mu = \int_\Omega f_+ \,\mathrm{d}\mu - \int_\Omega \psi_n \,\mathrm{d}\mu \underset{n\to+\infty}{\to} -\infty .$$

We therefore obtain Equation (6.33a) since both members of the equality are equal to $-\infty$.

• Suppose that $\int_\Omega f_+ \,\mathrm{d}\mu = +\infty$ and and $\int_\Omega f_- \,\mathrm{d}\mu < +\infty$. Then we prove that

$$\{\psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}) \,|\, f \leq \psi \;\; \mu\text{-a.s. }\} = \emptyset .$$

Indeed, assuming the existence of $\psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that $f \leq \psi$, we would obtain that $f_+ \leq \psi + f_-$ which, using the fact that $\psi + f_- \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$, would imply that $f_+ \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$, contradicting the assumption that $\int_\Omega f_+ \,\mathrm{d}\mu = +\infty$.

• Suppose that $\int_\Omega f_+ \,\mathrm{d}\mu = +\infty$ and and $\int_\Omega f_- \,\mathrm{d}\mu = +\infty$. Using the definition of $\dot{+}$, we get that the right hand side of Equation (6.33a) is equal to $+\infty$. Now, we show that

Equation (6.33a) holds true by proving that the set of functions $\psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that $f \leq \psi$ is empty. We proceed by contradiction. Assuming the existence of $\psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that $f \leq \psi$, we would have

$$+\infty = \int_\Omega f_+ \, \mathrm{d}\mu = \int_\Omega f \mathbf{1}_{f \geq 0} \, \mathrm{d}\mu \leq \int_\Omega \psi \mathbf{1}_{f \geq 0} \, \mathrm{d}\mu \leq \int_\Omega \psi \, \mathrm{d}\mu \ ,$$

contradicting the assumption that $\psi \in L^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. Therefore, in Equation (6.31a) we obtain that $\int_\Omega^* f \, \mathrm{d}\mu = +\infty$ and thus equality is ensured in Equation (6.33a).

This ends the proof. $\square$

Miscellaneous results

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A Uniform sampling on the unit sphere

In the numerical implementation of the min-plus algorithm described in Chapter 3, one needs to sample uniformly on the unit euclidean sphere. It can be done by simply renormalizing a uniform sample of gaussian distributions, or more generally, by renormalizing any sample of random variables with a radial density w.r.t. the Lebesgue measure. Throughout this section we fix $n \in \mathbb{N}$ and consider the vector space $\mathbb{R}^n$ to be endowed with its euclidean and borelian structures.

Proposition 81. *Let X be a random variable from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^n$. Assume that X has a density f with respect to the Lebesgue measure λ on $\mathbb{R}^n$ and that f is radial, that is there exist $\tilde{f} : \mathbb{R} \to \mathbb{R}_+$ such that for every $x \in \mathbb{R}^n$ we have*

$$f(x) = \tilde{f}(\|x\|).$$

Now denote by π_S the projection on the unit sphere S. Then the random variable $Y := \pi_S \circ X$ from $(\Omega, \mathcal{F}, \mathbb{P})$ to the unit sphere is uniform in the sense that for every borelian A of the unit sphere, we have that

$$\mathbb{P}(Y \in A) = \frac{\sigma(A)}{\sigma(S)},$$

where σ is the pushforward measure of the Lebesgue measure λ by the restriction to the unit ball (without 0) of π_S.

Remark 82. *By [LG06, p. 85-86] the measure σ on $(S, \mathcal{B}(S))$ defined in Proposition 81 is finite and invariant by unitary transformations. Moreover it is unique in the sense that every finite measure on $(S, \mathcal{B}(S))$ invariant by unitary transformations is proportional to σ.*

Proof. Fix A a borelian of the unit sphere. We have that

$$\begin{aligned}
\mathbb{P}(Y \in A) &= \int_\Omega \mathbf{1}_A(Y(\omega))\mathbb{P}(\mathrm{d}\omega) \\
&= \int_{\mathbb{R}^n} \mathbf{1}_A(y)\mathbb{P}_Y(\mathrm{d}y) \\
&= \mathbb{P}_{\pi_S \circ X}(Y \in A) \\
&= \mathbb{P}_X(\pi_S^{-1}(A)) \\
&= \int_{\mathbb{R}^n} \mathbf{1}_{\pi_S^{-1}(A)}(x) f(x)\,\mathrm{d}x && \text{(X has density f)} \\
&= \int_0^{+\infty} \int_S \mathbf{1}_{\pi_S^{-1}(A)}(rz) f(rz) r^{n-1} \sigma(\mathrm{d}z)\,\mathrm{d}r && \text{([LG06, Theorem 7.2.1])} \\
&= \int_0^{+\infty} \tilde{f}(r) r^{n-1} \int_S \mathbf{1}_{\pi_S^{-1}(A)}(rz)\sigma(\mathrm{d}z)\,\mathrm{d}r && \text{(f radial)} \\
&= \int_0^{+\infty} \tilde{f}(r) r^{n-1} \underbrace{\int_S \mathbf{1}_A(z)\sigma(\mathrm{d}z)}_{=\sigma(A)}\,\mathrm{d}r \\
&= \frac{\sigma(A)}{\sigma(S)} \int_0^{+\infty} \int_S \tilde{f}(r) r^{n-1}\sigma(\mathrm{d}z)\,\mathrm{d}r \\
&= \frac{\sigma(A)}{\sigma(S)}. && \text{(f radial density)}
\end{aligned}$$

□

Corollary 83. *Let $n \geq 1$ and $X = (X_1, X_2, \dots, X_n)$ be an i.i.d. sampling of the standard normal distribution. Then $\frac{X}{\|X\|}$ is an uniform random variable on the unit sphere.*

Proof. By independance of $(X_1, X_2, \dots, X_n)$, the random variable X has a radial density with respect to the Lebesgue measure on $\mathbb{R}^n$. Thus we can apply Proposition 81. □

B Approximating by independent scenario trees

Given a (non-independant) scenario tree $X = (X_1, \dots, X_{T-1})$ one can build its marginal process $X' = (X'_1, \dots, X'_{T-1})$. The process X' is independent and one may wonder if X' minimizes $\mathrm{d}_{\mathrm{ND}}(X, \cdot)$ for all independent scenario processes ? We give a counter example here. Informally, the idea behind is that the Nested Distance penalizes processes with different different flow of information.

Example 6 (The marginal process does not minimizes the Nested Distance)**.** *When p is big enough, one can drop the case where the decision maker gets 2 after getting 1: this will yield a better approximation than the marginal process. In fact we have that*

- *When $p < \frac{1}{3}$, then* $\mathrm{ND}(W, W'') > \mathrm{ND}(W, W')$.
- *When $p = \frac{1}{3}$, then* $\mathrm{ND}(W, W'') = \mathrm{ND}(W, W'')$.
- *When $p > \frac{1}{3}$, then* $\mathrm{ND}(W, W'') < \mathrm{ND}(W, W')$, *for instance for $p = \frac{1}{2}$:*

$$\mathrm{ND}(W, W'') = 1 < \frac{3}{2} = \mathrm{ND}(W, W').$$

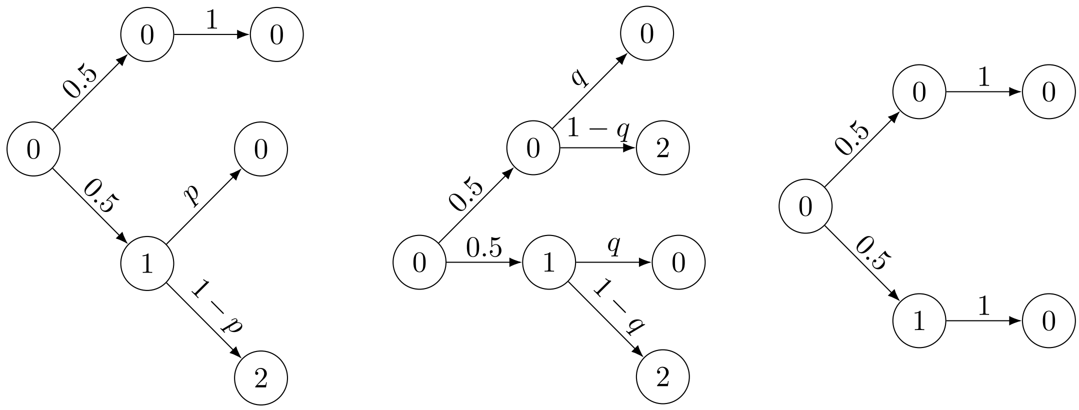

Figure 1: Left: initial scenario process X with $0 < p < 1$. Middle: marginal process of X', noted X' where $q = \frac{1}{2}(1+p)$. Right: independant process Y such that if $p > \frac{1}{3}$, then $\mathrm{ND}(X, Y) < \mathrm{ND}(X, X')$.

C Tropical Dynamic Programming for POMDP

In this section, we present an on-going work to apply TDP on Partially Observed Markov Decision Processes (POMDP).

C.1 Recalls on POMDP

Formally, a POMDP is described (in the finite settings) by a finite set of states $\mathbb{X} = \{x_1, ..., x_{|\mathbb{X}|}\}$, a finite set of actions $\mathbb{U} = \{u_1, ..., u_{|\mathbb{U}|}\}$, a finite set of observations $\mathbb{O} = \{o_1, ..., o_{|\mathbb{O}|}\}$, transition probabilities of the Markov chain

$$P_t^u(x_i, x_j) = \mathbb{P}\{X_{t+1} = x_j \,|\, X_t = x_i, U_t = u\} \;, \tag{34}$$

and conditional law of the observations

$$Q_{t+1}(o \,|\, x, u) = \mathbb{P}\{O_{t+1} = o \,|\, X_{t+1} = x, U_t = u\} \;, \tag{35}$$

a real-valued cost function $L_t(x, u)$ for any $t \in [\![0, T-1]\!]$, a final cost $K(x)$ and an initial probability law in the simplex of $\mathbb{R}^{|\mathbb{X}|}$ called the initial belief b_0. We assume here that the state space the control space and the observation space dimensions do not vary with time but for the sake of clarity we will use the notation $\mathbb{X}_t$ to designate the state space at time t even if it is equal to $\mathbb{X}$ and the same for control and observation states.

Under Markov assumptions, we can use at time t a probability distribution b_t, whose name is a reminder of belief, over current states as a sufficient statistic for the history of actions and observations up to time t. The space of beliefs is the simplex of $\mathbb{R}^{|\mathbb{X}|}$, denoted $\Delta_{|\mathbb{X}|}$. The belief dynamics, at time t, driven by action u_t and observation o_{t+1} is given by by the equation

$$b_{t+1} = \tau_t(b_t, u_t, o_{t+1}) \tag{36}$$

with $b_{t+1} \in \Delta_{|\mathbb{X}|}$ given by

$$b_{t+1}(x_{t+1}) = \beta_{t+1} Q_{t+1}(o_{t+1} \,|\, x_{t+1}, u_t) \Big(\sum_{x_t \in \mathbb{X}_t} b(x_t) P_t^{u_t}(x_t, x_{t+1}) \Big) \quad \forall x_{t+1} \in \mathbb{X}_{t+1} \;, \tag{37}$$

where β_{t+1} is a normalization constant to ensure that $b_{t+1} \in \Delta_{|\mathbb{X}|}$, that is

$$\beta_{t+1}^{-1} = \sum_{x_{t+1} \in \mathbb{X}_{t+1}} Q_{t+1}(o_{t+1} \,|\, x_{t+1}, u_t) \Big(\sum_{x_t \in \mathbb{X}_t} P_t^{u_t}(x_t, x_{t+1}) b(x_t) \Big) .$$

To simplify the notation we introduce the (sub-stochastic) matrix defined as follows

$$M_t^{u_t, o_{t+1}}(x_t, x_{t+1}) = Q_{t+1}(o_{t+1} \,|\, x_{t+1}, u_t) P_t^{u_t}(x_t, x_{t+1}) \quad \forall (x_t, x_{t+1}) \in \mathbb{X}_t \times \mathbb{X}_{t+1} ,$$

where we have $\sum_{o_{t+1}} \sum_{x_{t+1}} M_t^{u_t, o_{t+1}}(x_t, x_{t+1}) = 1$. Using matrix notations, where beliefs are represented by row vector and $\mathbf{1}$ is a column vector full of ones, we can rewrite the beliefs dynamics as

$$\tau_t(b_t, u_t, o_{t+1}) = \frac{b_t M_t^{u_t, o_{t+1}}}{b_t M_t^{u_t, o_{t+1}} \mathbf{1}} \in \Delta_{|\mathbb{X}|} .$$

In general the object of the optimization problem is to generate a policy that minimizes expected finite horizon cost for the controlled Markov chain $\{X_t^u\}_{t \in \mathbb{N}}$ with transition matrix P^u. That is consider the minimization problem

$$J(b_0) = \min_{U_1, \dots, U_{T-1}} \mathbb{E} \Big[\sum_{t=0}^{T-1} L_t(X_t, U_t) + K(X_T) \,\Big|\, b_0 \Big] . \tag{38}$$

It is classical to derive a Bellman equation for the beliefs given by the bellman operators for $t \in [\![0, T-1]\!]$

$$\mathcal{B}_t(V) = \inf_{u \in \mathbb{U}} \mathcal{B}_t^u(V) , \tag{39}$$

where for each $u \in \mathbb{U}$ and $t \in [\![0, T-1]\!]$, the Bellman operator $\mathcal{B}_t^u$ is defined by

$$\mathcal{B}_t^u(V)(b) = b L_t^u + \sum_{o \in \mathbb{O}_{t+1}} (b M_t^{u,o} \mathbf{1}) V \Big(\frac{b M_t^{u,o}}{b M_t^{u,o} \mathbf{1}} \Big) . \tag{40}$$

where L_t^u is the column vector $\big(L_t^u(x_t) \big)_{x_t \in \mathbb{X}_t}$. Note that the mapping $o_{t+1} \in \mathbb{O}_{t+1} \mapsto (b_t M_t^{u_t, o_{t+1}} \mathbf{1})$ is a probability distribution on $\mathbb{O}_{t+1}$ ($\sum_{o \in \mathbb{O}_{t+1}} b_t M_t^{u_t, o_{t+1}} \mathbf{1} = 1$).

The Bellman operator can be also written as

$$\mathcal{B}_t^u(V)(b) = b L_t^u + \sum_{b' \in \Delta_{|\mathbb{X}|}} \overline{P}_t^u(b, b') V(b') , \tag{41}$$

where, $\overline{P}^u$ is a controlled Markov chain transition matrix in the belief space. Indeed

$$\overline{P}_t^u(b, b') = \begin{cases} (b M_t^{u,o} \mathbf{1}) & \text{when } b' = \frac{b_t M_t^{u,o}}{b_t M_t^{u,o} \mathbf{1}} \text{ with } o \in \mathbb{O}_{t+1} , \\ 0 & \text{if not} , \end{cases} \tag{42}$$

which is a classical Bellman equation of a controlled Markov chain but with a state space in the belief space.

We conclude this section by the following lemma

Proposition 84. *The value functions $\{V_t\}_{t \in [\![0,T]\!]}$ solutions of the Bellman Equation*

$$\forall b \in \mathbb{R}_+^{|\mathbb{X}|} \quad V_T(b) = bK \quad \text{and} \quad \forall t \in [\![0, T-1]\!] \quad V_t(b) = \inf_{u \in \mathbb{U}} \mathcal{B}_t^u(V_{t+1})(b) , \tag{43}$$

where the operator $\mathcal{B}_t^u$ is given by Equation 40 are such that $V_0(b_0)$ is the optimal value of the minimization problem given by Equation 38.

C.2 The Bellman operator defined in Equation (39) propagate Lipschitz mappings

Proposition 85. *For $t \in [\![0, T-1]\!]$, assume that the mappings $L_t(u, \cdot)$ satisfy $\|L_t(u, \cdot)\|_\infty \leq \mathcal{L}$*[1] *for all $u \in \mathbb{U}$ and assume that a mapping K satisfy $\sup_{x \in \mathbb{X}} |K(x)| = \mathcal{K} < +\infty$. Then the solution of the Bellman Equation (43) are Lipschitz mappings.*

Proof.

• We consider the operator $\widetilde{\mathcal{B}}_t^u$ defined for mappings $\widetilde{V} : \mathbb{R}_+^{|\mathbb{X}|} \to \mathbb{R}$ by

$$\widetilde{\mathcal{B}}_t^u(\widetilde{V})(c) = cL_t^u + \sum_{o \in \mathbb{O}_{t+1}} \widetilde{V}\big(cM_t^{u,o}\big) \quad \forall c \in \mathbb{R}^{|\mathbb{X}|} , \tag{44}$$

where L_t^u stands for the column vector $(L_t(x, u))_{x \in \mathbb{X}_t}$ and we recall that beliefs are row vectors. We consider $\{\widetilde{V}_t\}_{t \in [\![0,T]\!]}$ solution of the Bellman Equation

$$\forall c \in \mathbb{R}_+^{|\mathbb{X}|} \quad \widetilde{V}_T(c) = cK \quad \text{and} \quad \forall t \in [\![0, T-1]\!] \quad \widetilde{V}_t(c) = \inf_{u \in \mathbb{U}} \widetilde{\mathcal{B}}_t^u(\widetilde{V}_{t+1})(c) . \tag{45}$$

First, we straightforwardly obtain by backward induction that the value functions $(\widetilde{V}_t)_{t \in [\![0,T]\!]}$ are homogeneous of degree 1. Second we prove that the operator $\widetilde{\mathcal{B}}_t^u$ preserves Lispchitz regularity. We proceed as follows. Consider c and c' in $\mathbb{R}_+^{|\mathbb{X}|}$ and suppose that $|\widetilde{V}(c) - \widetilde{V}(c')| \leq \mathcal{V}\|c' - c\|_1$. Then we have that

$$\begin{aligned}
\widetilde{\mathcal{B}}_t^u(\widetilde{V})(c') - \widetilde{\mathcal{B}}_t^u(\widetilde{V})(c) &= (c' - c)L_t^u + \sum_{o \in \mathbb{O}_{t+1}} \widetilde{V}\big(c'M_t^{u,o}\big) - V\big(cM_t^{u,o}\big) \\
&\leq \mathcal{L}\|c' - c\|_1 + \sum_{o \in \mathbb{O}_{t+1}} \mathcal{V}\|c'M_t^{u,o} - cM_t^{u,o}\|_1 \\
&\leq \mathcal{L}\|c' - c\|_1 + \mathcal{V} \sum_{\substack{o \in \mathbb{O}_{t+1} \\ x' \in \mathbb{X}}} \Big| \sum_{x \in \mathbb{X}} \big(c'(x) - c(x)\big) M_t^{u,o}(x, x') \Big| \\
&\leq \mathcal{L}\|c' - c\|_1 + \mathcal{V} \sum_{x \in \mathbb{X}} |c'(x) - c(x)| \sum_{\substack{o \in \mathbb{O}_{t+1} \\ x' \in \mathbb{X}}} M_t^{u,o}(x, x') \\
&\leq \mathcal{L}\|c' - c\|_1 + \mathcal{V} \sum_{x \in \mathbb{X}} |c'(x) - c(x)| \\
&\leq \Big(\mathcal{L} + \mathcal{V}\Big)\|c' - c\|_1 .
\end{aligned}$$

As a pointwise minimum of Lipschitz mappings having the same Lipschitz constant is Lipschitz, we obtain the same Lispchitz constant for the operators $\inf_{u \in \mathbb{U}} \widetilde{\mathcal{B}}_t^u$. Then, using the fact that $\overline{V}_T = K$ we obtain by backward induction that the Bellman value function $\widetilde{V}_t$ is $(\mathcal{L}(T - t) + \mathcal{K})$-Lipschitz for $t \in [\![0, T]\!]$ where $\mathcal{K} = \|K(\cdot)\|_\infty$.

• We prove now an intermediate result to link the solutions of the Bellman Equation (45) to the Bellman Equation (43). Suppose that $\widetilde{V}$ is 1-homogeneous and such that $\widetilde{V}(b) = V(b)$ for all $b \in \Delta_{|\mathbb{X}|}$. Then, We prove that $\widetilde{\mathcal{B}}_t^u(\widetilde{V})(b) = \mathcal{B}_t^u(V)(b)$ for all $b \in \Delta_{|\mathbb{X}|}$. For $b \in \Delta_{|\mathbb{X}|}$, we

[1] Since the state space if finite we identify mappings $\phi : \mathbb{X} \to \mathbb{R}$ with vectors in $\mathbb{R}^{|\mathbb{X}|}$

successively have that

$$
\begin{aligned}
\widetilde{\mathcal{B}}_t^u(\widetilde{V})(b) &= bL_t^u + \sum_{o\in\mathbb{O}_{t+1}} \widetilde{V}\big(bM_t^{u,o}\big) && (46)\\
&= bL_t^u + \sum_{o\in\mathbb{O}_{t+1}} (bM_t^{u,o}\mathbf{1})\widetilde{V}\big(\frac{bM_t^{u,o}}{bM_t^{u,o}\mathbf{1}}\big) && (\widetilde{V}\text{ is 1-homogeneous})\\
&= bL_t^u + \sum_{o\in\mathbb{O}_{t+1}} (bM_t^{u,o}\mathbf{1})V\big(\frac{bM_t^{u,o}}{bM_t^{u,o}\mathbf{1}}\big) && (\widetilde{V} = V\text{ on }\Delta_{|\mathbb{X}|})\\
&= \mathcal{B}_t^u(V)(b)\ . && (47)
\end{aligned}
$$

• Now we turn to solutions of Bellman Equation (43). Since $\widetilde{V}_T(c) = cK$ for all $c \in \mathbb{R}_+^{|\mathbb{X}|}$ and $V_T(b) = bK$ for all $b \in \Delta_{|\mathbb{X}|}$, the two mappings V_T and $\widetilde{V}_T$ coincide on the simplex of dimension $|\mathbb{X}|$. Then gathering the previous steps we obtain that V_t and $\widetilde{V}_t$ coincide also on the simplex of dimension $|\mathbb{X}|$ for all $t \in [\![0, T]\!]$. Finally, for all $t \in [\![0, T]\!]$ $\widetilde{V}_t$ being $(\mathcal{L}(T-t)+\mathcal{K})$-Lipschitz we obtain the same result for V_t. □

C.3 Value of $\mathcal{B}_t(V_{t+1})$ when $V_{t+1} = \min_{\alpha\in\Gamma_{t+1}} \langle \alpha\ , b\rangle$

Assume that $V_{t+1} : b \mapsto \min_{\alpha\in\Gamma_{t+1}} \langle \alpha\ , b\rangle$ where $\Gamma_{t+1} \subset \mathbb{R}^{|\mathbb{X}|}$. Then we obtain that

$$
\begin{aligned}
\mathcal{B}_t(V_{t+1})(b) &= \min_{u\in\mathbb{U}_t} \Big(bL_t^u + \sum_{o\in\mathbb{O}_{t+1}} (b_tM_t^{u,o}\mathbf{1})V_{t+1}\Big(\frac{bM_t^{u,o}}{bM_t^{u,o}\mathbf{1}}\Big)\Big) && (48)\\
&= \min_{u\in\mathbb{U}_t} \Big(bL_t^u + \sum_{o\in\mathbb{O}_{t+1}} (bM_t^{u,o}\mathbf{1}) \min_{\alpha\in\Gamma_{t+1}} \Big(\frac{bM_t^{u,o}\alpha}{bM_t^{u,o}\mathbf{1}}\Big)\Big) && (49)\\
&= \min_{u\in\mathbb{U}_t} \Big(bL_t^u + \sum_{o\in\mathbb{O}_{t+1}} bM_t^{u,o}\alpha^\sharp(u,o)\Big) && (\text{with } \alpha^\sharp(u,o) = \arg\min\nolimits_{\alpha\in\Gamma_{t+1}} \tfrac{bM_t^{u,o}\alpha}{bM_t^{u,o}\mathbf{1}})\\
&= \min_{u\in\mathbb{U}_t} b\Big(L_t^u + \sum_{o\in\mathbb{O}_{t+1}} M_t^{u,o}\alpha^\sharp(u,o)\Big) && (50)\\
&= \min_{\alpha\in\Gamma_t} \langle \alpha\ , b\rangle\ , && (51)
\end{aligned}
$$

with $\Gamma_t = \big\{L_t^u + \sum_{o\in\mathbb{O}_{t+1}} M_t^{u,o}\alpha^\sharp(u,o) \,\big|\, u \in \mathbb{U}_t \text{ and } \alpha^\sharp(u,o) = \arg\min_{\alpha\in\Gamma_{t+1}} \frac{bM_t^{u,o}\alpha}{bM_t^{u,o}\mathbf{1}}\big\}$. We therefore obtain that the Bellman value function at time t has the same form as the Bellman value function at time $t+1$.

We are in a context where the Bellman function that is to to be computed is polyhedral concave with a huge polyhedron. It is thus tempting to use our algorithm with polyhedral concave upper approximations and sup of quadratic or Lipschitz mappings as lower approximations.

The *Problem-child trajectory* technique is used in POMDP algorithms as an heuristic but without a convergence proof as far as we have investigated.

C.4 A lower bound of $\mathcal{B}_t(V_{t+1})$

We consider a special case where $V_{t+1} : \mathbb{X} \to \mathbb{R}$ is given by $V_{t+1}(b) = \left\langle b \, , \widehat{V}_{t+1} \right\rangle$ and we compute $\mathcal{B}_t(V_{t+1})$ as follows

$$\begin{aligned}
\mathcal{B}_t(V_{t+1})(b) &= \min_{u \in \mathbb{U}_t} \Big(bL_t^u + \sum_{o \in \mathbb{O}_{t+1}} bM_t^{u,o} \widehat{V}_{t+1} \Big) \\
&= \min_{u \in \mathbb{U}_t} \Big(bL_t^u + \sum_{o \in \mathbb{O}_{t+1}, x \in \mathbb{X}_t, x' \in \mathbb{X}_{t+1}} Q_{t+1}(o \,|\, x', u) P_t^u(x, x') b(x) \widehat{V}_{t+1}(x') \Big) \\
&= \min_{u \in \mathbb{U}_t} \Big(bL_t^u + \sum_{x \in \mathbb{X}_t, x' \in \mathbb{X}_{t+1}} P_t^u(x, x') b(x) \widehat{V}_{t+1}(x') \Big) \qquad \big(\textstyle\sum_o Q_{t+1}(o \,|\, x', u) = 1\big) \\
&\geq \sum_{x \in \mathbb{X}_t} b(x) \min_{u \in \mathbb{U}_t} \Big(L_t(u, x) + \sum_{x' \in \mathbb{X}_{t+1}} P_t^u(x, x') \widehat{V}_{t+1}(x') \Big) \\
&= \sum_{x \in \mathbb{X}_t} b(x) \widehat{V}_t(x) = b\widehat{V}_t \ ,
\end{aligned}$$

with

$$\widehat{V}_t(x) = \min_{u \in \mathbb{U}_t} \Big(L_t(u, x) + \sum_{x' \in \mathbb{X}_{t+1}} P_t^u(x, x') \widehat{V}_{t+1}(x') \Big) \ . \tag{52}$$

Using the fact that at time T we have that $V_T = \left\langle b \, , \widehat{V}_T \right\rangle$ with $\widehat{V}_T = K$ we obtain that for all $t \in [\![0, T]\!]$ $V_t \geq \left\langle b \, , \widehat{V}_t \right\rangle$ where $\widehat{V}_t$ is the Value function of the fully observed Bellman equation associated to the POMDP.

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