Asymptotic analysis of some point processes

Aurélien Vasseur

ParisTech
INSTITUT DES SCIENCES ET TECHNOLOGIES
PARIS INSTITUTE OF TECHNOLOGY

TELECOM ParisTech

2017-ENST-0062

EDITE - ED 130

Doctorat ParisTech

T H È S E

pour obtenir le grade de docteur délivré par

TELECOM ParisTech

Spécialité « Informatique et Réseaux »

présentée et soutenue publiquement par

Aurélien VASSEUR

le 1er décembre 2017

Analyse asymptotique de processus ponctuels

Directeurs de thèse : **Laurent DECREUSEFOND**
Frédéric CHAZAL

Jury

M. Jean-Christophe BRETON, Professeur, Université de Rennes 1, Rennes, France	Rapporteur
M. Nicolas PRIVAULT, Professeur, Nanyang Technological University, Singapour	Rapporteur
Mme. Hermine BIERMÉ, Professeur, Université de Poitiers, Poitiers, France	Examinateur
M. Pierre CALKA, Professeur, Université de Rouen, Rouen, France	Examinateur
Mme. Laure COUTIN, Professeur, Université Paul Sabatier, Toulouse, France	Examinateur
M. Frédéric CHAZAL, Directeur de Recherche, INRIA, Saclay, France	Directeur de thèse
M. Laurent DECREUSEFOND, Professeur, Télécom ParisTech, Paris, France	Directeur de thèse

T H È S E

TELECOM ParisTech
école de l'Institut Mines-Télécom - membre de ParisTech
46 rue Barrault 75013 Paris - (+33) 1 45 81 77 77 - www.telecom-paristech.fr

Résumé

La méthode de Stein constitue une des principales techniques pour la résolution de certains problèmes d'approximation en théorie des probabilités. Dans ce manuscrit, nous l'appliquons au contexte des processus ponctuels.

La première partie de ces investigations se concentre sur le processus ponctuel de Poisson. Sa propriété caractéristique d'indépendance fournit le moyen d'expliquer intuitivement pourquoi une suite de processus ponctuels de moins en moins répulsive peut converger vers un tel processus ponctuel. Ceci nous amène plus généralement à démontrer des résultats de convergence pour des suites de processus ponctuels construites à partir d'opérations telles que la superposition, l'amincissement ou l'homothétie. L'utilisation d'une distance sur les processus ponctuels, appelée distance de Kantorovich-Rubinstein, permet en outre l'obtention de taux de convergence.

La seconde partie est centrée sur une classe de processus ponctuels avec beaucoup d'attractivité, appelés processus ponctuels α-stables. Leur structure basée sur un processus ponctuel de Poisson nous permet d'élargir à ces processus la méthode utilisée précédemment et de proposer de nouveaux résultats, via certaines propriétés que nous établissons sur ces processus ponctuels.

Mots-clés : Processus ponctuel, géométrie stochastique, méthode de Stein, intensité de Papangelou, processus ponctuel de Poisson, stabilité, convergence.

Abstract

Stein's method constitutes one of the main techniques to solve some approximation problems in probability theory. In this manuscript, we apply it in the context of point processes.

The first part of these investigations focuses on the Poisson point process. Its characteristic independence property provides a way to explain intuitively why a sequence of point processes becoming less and less repulsive can converge to such a point process. More generally, this leads to show some convergence results for some sequences of point processes built by several operations such as superposition, thinning and rescaling. The use of a distance on point processes, the so-called Kantorovich-Rubinstein distance, enables moreover the getting of some convergence rates.

The second part is centered on a class of point processes with important attractiveness, called discrete α-stable point processes. Their structure based on a Poisson point process gives us a way to enlarge to these point processes the method used previously and to propose new results, via some properties that we state on these point processes.

Keywords: Point process, stochastic geometry, Stein's method, Papangelou intensity, Poisson point process, stability, convergence.

Contents

Remerciements

J'adresse ici mes sincères et chaleureux remerciements à tous ceux qui, chacun à sa façon, m'ont accompagné durant ces trois années de doctorat.

Tout d'abord, un grand merci à mes directeurs de thèse Laurent Decreusefond et Frédéric Chazal, je leur suis très reconnaissant de m'avoir mis dans d'excellentes conditions de travail et leur dois d'avoir ressenti cette thèse comme une expérience aussi enrichissante qu'agréable.

Merci aux membres du jury, Jean-Christophe Breton, Nicolas Privault, Hermine Biermé, Pierre Calka et Laure Coutin, dont les suggestions pertinentes constituent des contributions non négligeables à l'amélioration du manuscrit ; merci en particulier à Pierre Calka, qui m'a initié avec enthousiasme à la théorie des processus ponctuels pendant ma deuxième année de master à l'université de Rouen et incité à poursuivre en thèse dans ce domaine.

Merci à mes collègues et coauteurs Philippe Martins, Anaïs Vergne et Jean-Sébastien Gomez, qui ont su généreusement me faire profiter de leurs expériences et connaissances en matière de recherche, enseignement et étude des réseaux.

Cette thèse fut l'occasion d'un stage d'un mois à l'université technologique de Nanyang à Singapour sous la direction de Dusit Niyato, que je remercie, ainsi que Ian Flint, pour l'accueil qui m'a été fait et les échanges que nous avons pu avoir.

Merci à Bruno Portier, mon directeur de mémoire en deuxième année de master, pour tous ses encouragements.

Je remercie chacun des chercheurs - docteurs ou doctorants - rencontrés à Télécom ParisTech ou au fil des séminaires et conférences, avec qui les discussions furent toujours bénéfiques.

Affectueuses pensées pour tous mes proches, famille et amis (en particulier ceux de l'association des paralysés de France), dont la présence bienveillante m'a permis d'avancer jusqu'à la soutenance avec confiance, sérénité et, bien sûr, en musique.

Merci, surtout, à ma chère Fiona.

Note about funding

This research was partially supported by Labex DigiCosme (project ANR-11-LABEX-0045-DIGICOSME) operated by ANR as part of the program "Investissement d'Avenir" Idex Paris-Saclay (ANR-11-IDEX-0003-02).

Chapter 1

Introduction

1.1 Background

Point processes are formally seen as random locally finite subsets of points and provide a powerful mathematical tool with some applications in many areas, such as forestry [71], astronomy [4], epidemiology [34], telecommunications and precisely wireless networks [5], and more generally in each field where the repartition of some particles need to be analyzed in a mathematical way. The choice of a category of point processes to model this repartition strongly depends if the interactions between particles induce attractiveness, with some clusters of points, or repulsiveness, with some space between particles.

1.1.1 Characterization

To describe the distribution of a point process, several characterizations are available. Among them, the Laplace functional offers the advantage to obtain some information on point processes built by superposition or other transformations. However, this tool is not really intuitive and it may be useful to characterize a point process by rather considering other functionals such as its Janossy function and correlation function.

If $x_1, \ldots, x_n$ are n particles in our state space, the Janossy function j is defined in such a way that $j(x_1, \ldots, x_n)$ represents intuitively the probability of finding exactly n particles at the locations $x_1, \ldots, x_n$, while the correlation function ρ is such that $\rho(x_1, \ldots, x_n)$ represents the probability of finding at least n particles at $x_1, \ldots, x_n$, with maybe other particles in other locations.

Correlation function also provides a way to specify the repulsiveness or attractiveness of a point process, which will be considered as repulsive (respectively attractive) as soon as, for any x, y,

$$\rho(x, y) \leq \rho(x)\rho(y) \text{ (resp. } \rho(x, y) \geq \rho(x)\rho(y)). \tag{1.1}$$

An other typical functional provides both a way to characterize a point process and an intuitive interpretation: introduced in 1974 by Papangelou [57], the so-called Papangelou intensity c is such that $c(x, \phi)$ represents the probability of finding a particle in the location x given that there is a particle located at each point of the configuration (or locally finite subset) ϕ. In particular, this leads to consider the variations of this quantity when the configuration ϕ increases: if $\omega \subset \phi$ implies that

$$c(x,\phi) \leq c(x,\omega) \text{ (resp. } c(x,\phi) \geq c(x,\omega)) \tag{1.2}$$

then it rather signals repulsiveness (resp. attractiveness).

The Poisson point process may be characterized as the only point process with no interactions between its particles, that is, without repulsiveness or attractiveness. For this point process, each compact subset has a Poisson-distributed number of particles and the respective numbers of particles in two disjoint compact subsets are independent. It verifies the equality in (1.1) and (1.2) and may be in this sense considered as the "zero" point process between repulsive and attractive point processes.

1.1.2 Transformations

A way to transform a point process into a new point process with less interactions between its particles (in other words to reduce its level of repulsiveness or attractiveness) is to use operations which insert some independence. We will focus specifically on two transformations: independent superposition and independent thinning. We can note that there are some inter-dependencies in a superposition at most only between particles from the same term of this superposition, while the thinning operation keep or delete independently each particle. Thus, it becomes clear that, under suitable assumptions, a sequence of point processes built in such ways is susceptible to converge to a point process without interactions, that is, a Poisson point process.

1.1.3 Some classical point processes

Poisson point processes may also be included in larger classes of point processes, that will be called Poisson-like point processes. Among them, Cox point processes [17] are defined as Poisson point processes conditionally to a random intensity and may provide a useful tool to model attractive repartition of particles. We can also cite purely random point processes [51] where a random number of points are drawn independently according to a fixed probability measure, and conditional Poisson point processes, which include hardcore conditional Poisson point processes [40].

Gibbs point processes are repulsive point processes, especially in the sense given by (1.2), and were introduced in the field of statistical physics [60]. The repulsiveness of a Gibbs point process appears naturally in the expression of its total potential energy U, defined as

$$U(x_1,\ldots,x_n) = \sum_{r=1}^{n} \sum_{1\leq i_1<\cdots<i_r\leq n} \Psi_r(x_{i_1},\ldots,x_{i_r}),$$

where Ψ_r quantifies the degree of repulsion between r given particles.

An other useful model for the repartition of particles with some repulsion (called fermion particles in the literature) appears with determinantal point processes, introduced by Macchi in 1975 [50], and whose mathematical structure were studied in details by Soshnikov

[68], Shirai and Takahashi [66], then Hough et al. [41]. As mentioned above, the repulsive behavior of the particles from such point processes may be intuitively interpreted by observing the correlation function and the Papangelou intensity. The correlation function ρ of a determinantal point process is defined as

$$\rho(x_1, \dots, x_n) = \det(K(x_i, x_j))_{1 \leq i,j \leq n}, \tag{1.3}$$

where K is the kernel of a functional operator, from which we deduce as expected the inequation about repulsiveness in (1.1). Moreover, Georgii and Yoo provide an explicit expression for the Papangelou intensity c of a determinantal point process [35] and show in particular the inequation given by (1.2) for repulsiveness.

It was actually shown [66] that determinantal point processes may be included in a wider class of point processes, called α-determinantal and permanantal point processes, where determinant is replaced by α-determinant defined as

$$\det_\alpha A = \sum_{\sigma \in \Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.4}$$

where the coefficient α provides an indication on the repulsive or attractive nature of the point process. More precisely, its particles exhibit repulsiveness as soon as $\alpha < 0$ (in particular $\alpha = -1$ corresponds to determinantal point process) and attractiveness when $\alpha > 0$. The attractive point processes of this last category provide a model in statistical physics for the repartition of boson particles. The case $\alpha = 0$ leads to the Poisson point process, which consolidates the idea expressed previously of being represented as a "zero" point process. The reader may also consult Decreusefond et al. [22] for a survey and Lavancier et al. [46] for statistical inference on determinantal point processes.

The Ginibre point process is a key example of determinantal point process on the complex space $\mathbb{C}$, and was introduced in 1965 by Ginibre [36]. It may be interpreted as a point process with Gaussian repulsions between its particles. If $\beta \in (0,1]$, a β-Ginibre point process is built by combining a thinning with parameter β and a rescaling with parameter β. When β tends to 0, a β-Ginibre point process is close to a Poisson point process. In this sense, a Poisson point process may be considered as a β-Ginibre point process with $\beta = 0$. The simulation of β-Ginibre point processes is investigated in [23]. Among results on this topic, let mention the article of Goldman about its Palm measure and Voronoi tessellation [37], its link with random matrices [47] and some applications to wireless networks [30, 38].

Stable distributions were introduced in 1924 by Lévy [49]. A random variable X is said to be strictly α-stable (or more succinctly StαS) if, for any $t \in [0,1]$, it verifies

$$t^{\frac{1}{\alpha}} X^{(1)} + (1-t)^{\frac{1}{\alpha}} X^{(2)} \overset{\mathcal{D}}{=} X, \tag{1.5}$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X and $\overset{\mathcal{D}}{=}$ denotes the equality between probability distributions. Such mathematical objects exist for $\alpha \in (0,2]$ and include Gaussian distributions, Cauchy distributions and Lévy distribution. On this topic, we refer to surveys from Samorodnitsky and Taqqu [61] or more recently Nolan [55].

A discrete version of stability was introduced in 1979 by Steutel and Van Harn [70] for non-negative integer-valued random variables by replacing the real multiplication in (1.5) by the random operation $\circ$ which defines $t \circ n$ for any $n \in \mathbb{N}$ and $t \in [0,1]$ as a random variable whose distribution is binomial with parameters n and t. So-called discrete α-stable (or DαS) distributions exist for $\alpha \in (0,1]$ and it is shown that they have two representations: they may be seen respectively as a Poisson distribution conditionally to a StαS random intensity, and as a random sum of independent so-called Sibuya distributions whose number of terms has a Poisson distribution. The case $\alpha = 1$ corresponds to the Poisson distributions, which are the only discrete stable distributions with a finite expectation. Sibuya distributions were introduced in 1979 by Sibuya [67] and also depend on $\alpha \in (0,1]$. These elements are developed in the works of Christoph and Schreiber [14, 15] and Devroye [31, 32].

The notion of stability was generalized in 2008 by Davydov et al. to random elements with values in a commutative semi-group [19] and then in 2011 to random measures and point processes [20]. The definition of strict stability for random variables is expanded in a natural way to random measures, while stability for point processes is an extension of the concept of stability for discrete random variables, where $\circ$ is the thinning operation on point processes. If $\alpha = 1$, a DαS point process is a Poisson point process. For $\alpha \neq 1$, a DαS point process has representations which are similar to those obtained for the case of discrete random variables: it may be considered, on one hand as a Cox point process with StαS random intensity, on the other hand as a cluster Poisson point process, whose daughter point processes are Sibuya point processes, that are purely random point processes with a Sibuya number of points. When $\alpha \neq 1$, the number of points of each cluster has an infinite expectation which implies both that such a point process has a high level of attractiveness, and that its intensity measure is not finite. For this last reason, a DαS is characterized by the intensity measure of the Poisson point process defined on the space of probability measures, called spectral measure. Moreover, by considering its cluster representation, it appears that a DαS point process may be seen as a marked Poisson point process, where the marks are Sibuya point processes.

1.1.4 Convergence

Let present now the different modes of convergence which will be used thereafter. A central question in the field of optimal transport is to determine for which coupling between two random elements X and Y the value of a given cost function Δ is minimal. If X and Y are some point processes and the cost function Δ is a distance on the configuration space, the optimal transport cost Δ^*, also called Kantorovich-Rubinstein distance, between the probability distributions $\mathbb{P}_X$ and $\mathbb{P}_Y$ of X and Y is defined as

$$\Delta^*(\mathbb{P}_X, \mathbb{P}_Y) := \inf_{\mathbf{C} \in \Sigma(\mathbb{P}_X, \mathbb{P}_Y)} \int_{N_{\mathbb{X}} \times N_{\mathbb{X}}} \Delta(\omega_1, \omega_2) \mathbf{C}(\mathrm{d}(\omega_1, \omega_2)) \tag{1.6}$$

where $\Sigma(\mathbb{P}_X, \mathbb{P}_Y)$ denotes the set of probability measures on $\mathbb{M} \times \mathbb{M}$ with first marginal $\mathbb{P}_X$ and second marginal $\mathbb{P}_Y$. In this case, there is at least one coupling $\mathbf{C} \in \Sigma(\mathbb{P}_X, \mathbb{P}_Y)$ for which the infimum is attained [73]. This distance associated to the total variation distance provides a strong topology on point processes since it is strictly finer than for the total variation distance [25], and it is shown by Decreusefond et al. that this Kantorovich-Rubinstein distance between finite Poisson point processes is bounded by the total variation distance between its intensity measures [24]. It is also included in the larger class of so-called Wasserstein distances, where a L^p-distance appears in (1.6). Wasserstein distances were investigated in [9, 21, 45] for Poisson point processes, and more recently in [29] by Del Moral and Tugaut for Kalman-Bucy filters, and in the field of persistent homology in [12] where Chazal et al. compare this distances with so-called bottleneck, Hausdorff and Gromov-Hausdorff distances. Other examples of metrics on point processes are proposed in [64] and [63].

The most common type of convergence used on point processes is convergence in law, that is, convergence for the Laplace functionals. It provides a strictly weaker topology than total variation distance and its associated Kantorovich-Rubinstein distance. However, such a topology is metrizable and it is shown [42] that the distance Δ_P, called here Polish distance, between two measures ν_1 and ν_2 given by

$$\Delta_P(\nu_1, \nu_2) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{|\langle f_k, \nu_1\rangle - \langle f_k, \nu_2\rangle|}{1 + |\langle f_k, \nu_1\rangle - \langle f_k, \nu_2\rangle|},$$

where $(f_k)_{k\in\mathbb{N}}$ is an appropriate sequence of functions, defines a metric for this topology and then provides a way to precise some convergence rates.

1.1.5 A glimpse at Stein's method

A fruitful way to get some approximations in probability theory is drawn from the Stein's method, introduced in 1972 by Stein [69] in order to give a convergence speed for the central limit theorem. In this method, we consider a given random object X on a space $\mathbb{X}$ (a Gaussian variable in the original problem) from which we wish to get an approximation, and a functional operator L such that, for any random element Y of $\mathbb{X}$,

$$\mathbb{E}[LF(Y)] = 0 \text{ for a large class of functions } F \iff Y = X.$$

The aim is then to solve the so-called Stein's equation, that is, for any test function $F : \mathbb{X} \to \mathbb{R}$, to find a function $H_F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ such that, for any $x \in \mathbb{X}$,

$$LH_F(x) = \mathbb{E}[F(X)] - F(x).$$

Typical uses of this method include its adaptation for Poisson distribution by Chen [13], the works of Barbour et al. on Poisson point process approximation [7, 8, 9], those of Peccati et al. [56, 45] which combine this method with Malliavin calculus, its application to infinite-dimensional Gaussian approximation by Shih [65], to total variation between Gibbs point processes by Schumacher and Stucki [62], to Brownian approximation by Coutin and Decreusefond [16] and for the Poisson point process to U-statistics by Decreusefond et al. [25].

1.2 Contributions

The primary motivation of this work was the following. Consider the locations of base stations, i.e. antennas, of the mobile network in Paris. If we have a look at the global process of all base stations of all operators and for all operating frequencies, we obtain the left picture of Figure 1.1. It turns out to be compatible with the null hypothesis of being a Poisson point process. However, if we look at the positions of base stations deployed by one operator, in one frequency band, we get a picture similar to the right picture of Figure 1.1. It was shown in [38] that this deployement is statistically compatible with a β-Ginibre point process.

Figure 1.1: On the left, positions of all base stations in Paris. On the right, locations of base stations for one frequency band.

When superposing a large number of independent processes with internal repulsion but a few points, it is intuitively clear that the resulting process does not exhibit strong interdependencies between its atoms and should thus resemble a Poisson point process. This is the intuition we wanted to quantify by determining how fast does the convergence hold.

The main elements from the point process theory described in the previous background are presented more formally in Chapter 2, excluding the parts on stability and Stein's method, which will be detailed further. The contribution of this chapter is a proof to state that the Kantorovich-Rubinstein associated to the discrete distance equals the total variation distance (Theorem 2.2.10).

In Chapter 3, we apply Stein's method to finite Poisson point processes and deduce some convergence results by using the Papangelou intensities.

After giving a short description of the Stein's method in Section 3.1, we present in more details in Section 3.2 its application to a finite Poisson point process. According to the so-called generator approach, we build a Markov process, called Glauber process, associated to this Poisson point process, we deduce its semi-group, infinitesimal generator and gradient for which we state some useful properties. We state in Section 3.3 the so-called Stein-Dirichlet representation formula and obtain an upper bound for the Kantorovich-Rubinstein distance associated to the total variation distance between a finite Poisson point process and an other finite point process.

Since this bound is the L_1-distance between their respective Papangelou intensities, we give in Section 3.4 the elements concerning Papangelou intensities which will be necessary to state some convergence results in the next sections. More precisely, we propose definitions of repulsiveness and weakly repulsiveness and settle some properties relative to repulsive point processes, finite point processes, transformations and classical point processes.

From all these preliminary results, we deduce some convergence rates when considering Kantorovich-Rubinstein distance between Poisson or Cox point processes and other point processes, which are Poisson-like point processes in Section 3.5 and repulsive point processes in Section 3.6.

In Section 3.7, we focus on a theorem from Kallenberg [42] which states that, under general suitable assumptions, a sequence of point processes built by thinning converges in law to a Cox point process. The application of the previous method provides a convergence rate for this result.

In Chapter 4, taking account that DαS point processes have a cluster Poisson structure, we adapt to them the scheme adopted for the Poisson point process.

In Section 4.1, we recall the main elements of the theory of DαS point processes from [20] and then propose in Section 4.2 new results relative to Papangelou intensities of DαS point processes and generalizations of the Mecke formula. Although the Poisson point process plays a key role in the cluster representation of a DαS point process, the importance of the Sibuya point processes may be put forward by the following property, stated in Section 4.3: a discrete α-stable sum (in a sense which has to be precised) of Sibuya point processes with exponent β is discrete $\alpha\beta$-stable. An equivalent property for strictly stable random measures is also stated: a random measure which is StαS conditionally to a strictly β-stable random spectral measure is strictly $\alpha\beta$-stable.

In Section 4.4, the Stein's method is applied to DαS point processes, according to the Poisson structure given by the cluster representation. We describe the Glauber process endowed with its semi-group and infinitesimal generator and propose two different definitions of gradient. Similarly to the Poisson point process, we show ergodicity and state a new Stein-Dirichlet representation formula.

Still taking care to adapt the previous scheme, we introduce in Section 4.5 α-Papangelou intensities. We provide the expression of the α-Papangelou intensity of a DαS point process, the link with Papangelou intensities and settle some formulas relative to the superposition and thinning of point processes.

The elements settled in the previous sections provide sufficient tools in order to state some convergence results, organized as follows: Section 4.6 is dedicated to the results from which α takes different values, while results for a fixed α are given in Section 4.7. Due to the fact that these point processes have a number of points with infinite expectation, these convergence results may only be verified for total variation distance.

The main result of Section 4.6 is that this distance taken between a DαS and a DβS point process with the same spectral measure has a convergence rate given by $1-\frac{\alpha}{\beta}$ when $\alpha < \beta$.

In Section 4.7, we exhibit two versions of the Kallenberg's theorem in the context of DαS point processes, which provides a way to approximate a StαS random measure with DαS point processes. We also bound the distance respectively between two DαS point processes with different spectral measures and between an appropriate superposition of point processes and a DαS point process.

In Appendix A, we expose the results given in [38], where the β-Ginibre point processes are analyzed as a model for the repartition of the base stations in a wireless network.

In Appendix B, a French summary of this work is given.

Other contributions are subject to several publications: [26, 10, 27, 28]. An application of DαS point processes to a jamming model is also in preparation [48].

All along this manuscript, definitions, theorems, lemmas and corollaries which appear in green are essentially new results or at least given with some new proofs.

In brief, our contributions are:

- A new bound on the distance between point processes based on the Papangelou intensity.
- Numerous applications of this result to point processes with correlations, some of them highly useful for the understanding of wireless telecommunication systems.
- The development of a Malliavin calculus for DαS point processes. Diverse applications of this framework to the convergence rate of superposition and thinning of such processes.

Publications

- Gomez J.S., Vasseur A., Vergne A., Martins P., Decreusefond L., Chen W., *A case study on regularity in cellular network deployment*, In: IEEE Wireless Communications Letters, 4(4):421–424, 2015.
- Decreusefond L., Vasseur A., *Asymptotics of superposition of point processes*, In: Nielsen F., Barbaresco F. (eds), Geometric Science of Information, Lecture Notes in Computer Science, 9389, Springer, Paris, 2015.
- Bardenet R., Lavancier F., Mary X., Vasseur A., *On a few statistical applications of determinantal point processes*, In: Coeurjolly J.F., Leclercq-Samson A. (eds), ESAIM: proceedings and surveys, Vol. 60, p. 180-202, 2017.
- Decreusefond L., Vasseur A., Zuyev S., *Asymptotics of discrete stable point processes*, Conference SPA 2017, Moscou, 2017.
- Decreusefond L., Vasseur A., *Asymptotics of some point processes transformations*, Full paper in preparation, 2017.
- Lu X., Vasseur A., Flint I., Niyato D., *Application of discrete stable point processes to a jamming model*, Full paper in preparation, 2017.

Chapter 2

Preliminaries

In this chapter, we recall some basic definitions and properties from the point process theory. In Section 2.1, we fix the mathematical notations and focus on functions characterizing point processes. In Section 2.2, we recall some useful notions about convergence on point processes and also propose a proof to establish that the Kantorovich-Rubinstein distance associated to the discrete distance equals the total variation distance. The main transformations of point processes and their properties are recalled in Section 2.3. In Section 2.4, we present the Poisson-based point processes which are used in the following chapters, and Section 2.5 focuses on the class of α-determinantal/permanantal point processes.

2.1 Generalities on point processes

In this work, we use classical mathematical notations. In particular, $\mathbb{N}$ denotes the space of positive integers, $\mathbb{N}_0$ the space of non-negative integers, $\mathbb{R}$ the space of real numbers and $\mathbb{C}$ the space of complex numbers.

We consider a locally compact metric space $\mathbb{X}$ endowed with its Borel tribe $\mathcal{X}$, a (not necessarily diffuse) Radon measure ℓ on $\mathbb{X}$ and its distance $\Delta_{\mathbb{X}}$. The family of relatively compact Borel sets is denoted by $\mathcal{X}_0$. A distance Δ on $\mathbb{X}$ will be denoted $\Delta_{|\mathbb{X}}$ if necessary.

The set of bounded measurable functions from $\mathbb{X}$ to $\mathbb{R}_+$ with compact support is denoted $\mathcal{B}_+(\mathbb{X})$. If f is a function from $\mathbb{X}$ to $\mathbb{C}$, then $\|f\|_\infty$ designs the supremum of the set $\{|f(x)| : x \in \mathbb{X}\}$. For $p \in [1,+\infty)$, $L^p(\mathbb{X},\ell)$ denotes the space of functions $f : \mathbb{X} \to \mathbb{C}$ such that $|f|^p$ is integrable with respect to ℓ. The space of continuous functions from $\mathbb{X}$ to $\mathbb{R}$ (respectively $\mathbb{C}$) with compact support is denoted $\mathcal{C}_K(\mathbb{X},\mathbb{R})$ (respectively $\mathcal{C}_K(\mathbb{X},\mathbb{C})$). The integral of an integrable function f with respect to ℓ will be more simply written $\int_{\mathbb{X}} f(x)\mathrm{d}x$ when there is no ambiguity.

The space of measures on $(\mathbb{X},\mathcal{X})$ will be denoted $\mathbb{M}$, $\mathbb{M}_R$ is the space of Radon measures on $\mathbb{X}$ and $\mathbb{M}_1$ the family of all probability measures on $\mathbb{X}$. The space of measures on $\mathbb{M}$, the space of Radon measures on $\mathbb{M}$ and the space of probability measures on $\mathbb{M}$ will be respectively denoted by $\mathbb{M}'$, $\mathbb{M}'_R$ and $\mathbb{M}'_1$. For any $x \in \mathbb{X}$, δ_x denotes the Dirac measure centered on x. For any $A \in \mathcal{X}$, $\ell(A)$ may also be denoted $|A|$. For any $A \subset \mathbb{X}$, $\mathbf{1}_A$ designs the indicator function of the subset A of $\mathbb{X}$. If $\varphi \in \mathbb{M}$ and m is a function on $\mathbb{X}$ integrable with respect to φ, then $\langle m, \varphi\rangle$ designs the integral of m with respect to φ, the measure ν with density m with respect to φ is denoted $m\varphi$. In this case, m is denoted $\frac{\mathrm{d}\nu}{\mathrm{d}\varphi}$ and the fact that ν is absolutely continuous with respect to φ is denoted $\nu \ll \varphi$.

For any random element X of $\mathbb{X}$, $\mathbb{P}_X$ designs the probability distribution of X. If F is a function from $\mathbb{X}$ to $\mathbb{R}$ integrable with respect to $\mathbb{P}_X$, then the expectation of $F(X)$ is denoted $\mathbb{E}_X[F(X)]$ or more simply $\mathbb{E}[F(X)]$. If Δ is a distance on $\mathbb{M}_1$ and X_1, X_2 two random elements of $\mathbb{X}$ with respective distributions $\mathbb{P}_1, \mathbb{P}_2$, then we will also write $\Delta(X_1, X_2)$ instead of $\Delta(\mathbb{P}_1, \mathbb{P}_2)$.

The following notions concern point process theory and come essentially from [18].

Definition 2.1.1 (Counting measure - Configuration).
*A **counting measure** ξ on $\mathbb{X}$ is a measure on $\mathbb{X}$ such that, for any $A \in \mathcal{X}_0$,*

$$\xi(A) \in \mathbb{N}_0.$$

*A **configuration** (respectively **finite configuration**) on $\mathbb{X}$ is a locally finite (respectively finite) counting measure on $\mathbb{X}$.*

The space of configurations on $\mathbb{X}$ is denoted $N_{\mathbb{X}}$ and $\widehat{N}_{\mathbb{X}}$ designs the space of finite configurations on $\mathbb{X}$. We endow $N_{\mathbb{X}}$ with $\mathcal{N}_{\mathbb{X}}$ defined as the smallest σ-algebra on $N_{\mathbb{X}}$ such that $\phi \in N_{\mathbb{X}} \mapsto \phi(A)$ is measurable for any $A \in \mathcal{X}_0$. The restriction of $\mathcal{N}_{\mathbb{X}}$ to $\widehat{N}_{\mathbb{X}}$ is denoted $\widehat{\mathcal{N}}_{\mathbb{X}}$.

For any $\omega, \phi \in N_{\mathbb{X}}$ and $x \in \mathbb{X}$, we will write $x \in \omega$ if x is charged by the measure ω, $\omega \subset \phi$ if $\omega \leq \phi$, $\omega\phi$ instead of $\omega + \phi$, ωx instead of $\omega + \delta_x$, $\omega \cap \phi$ instead of $\min(\omega, \phi)$, $\phi \setminus \omega$ with a similar meaning, and $\omega(\mathbb{X})$ will also be denoted $|\omega|$.

Let be a function $F : N_{\mathbb{X}} \to \mathbb{R}$. If F is integrable with respect to ω, then the integral $\int_{\mathbb{X}} F(x)\omega(\mathrm{d}x)$ will be often denoted $\sum_{x\in\omega} F(x)$. For any $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathbb{X}$, $\{x_1, \ldots, x_k\}$ also denotes the configuration ω defined, for any $A \in \mathcal{X}$, by

$$\omega(A) = \#\left\{x \in A \;:\; \exists i \in \{1, \ldots, k\}, x = x_i\right\}$$

where, for any finite set B, $\#B$ designs the number of elements in B. By a slight abuse of notation, we will also denote $F(x_1, \ldots, x_k)$ instead of $F(\{x_1, \ldots, x_k\})$.

Definition 2.1.2 (Point process - Intensity measure).
*A **point process** Φ on $\mathbb{X}$ is a random configuration on $\mathbb{X}$. Its **intensity measure** is the measure M on $(\mathbb{X}, \mathcal{X})$ defined, for any $A \in \mathcal{X}$, by*

$$M(A) = \mathbb{E}[\Phi(A)].$$

The classical functionals which are presented below provide different ways to characterize a point process distribution.

Definition 2.1.3 (Laplace and probability generating functionals).
*Let Φ be a random measure on $\mathbb{X}$. Its **Laplace functional** $\mathcal{L}_\Phi : \mathcal{B}_+(\mathbb{X}) \to \mathbb{R}_+$ is given*

for any $f \in \mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_\Phi(f) = \mathbb{E}\Big[\exp\Big(-\int_\mathbb{X} f(x)\Phi(\mathrm{d}x)\Big)\Big].$$

*Its **probability generating functional** (p.g.fl.) G_Φ is given for any function u from $\mathbb{X}$ to $(0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:*

$$G_\Phi(u) = \mathcal{L}_\Phi(-\log u) = \mathbb{E}\Big[\exp\Big(\int_\mathbb{X} \log u(x)\Phi(\mathrm{d}x)\Big)\Big] = \mathbb{E}\Big[\prod_{x\in\Phi} u(x)\Big].$$

Definition 2.1.4 (Janossy measure).
*Let Φ be a finite point process on $\mathbb{X}$. Its **Janossy measure** J is given for any $A \in \widehat{\mathcal{N}}_\mathbb{X}$ by:*

$$\mathbb{P}(\Phi \in A) = \sum_{k=0}^{+\infty} \frac{1}{k!} J(A^{(k)}),$$

where, for any $k \in \mathbb{N}_0$,

$$A^{(k)} = \{\phi \in A : |\phi| = k\}.$$

Definition 2.1.5 (Janossy function).
*Let Φ be a finite point process on $\mathbb{X}$ with Janossy measure J. Its **Janossy function** (with respect to ℓ) $j : \widehat{N}_\mathbb{X} \to \mathbb{R}_+$ is defined, if it exists, for any measurable function $u : \widehat{N}_\mathbb{X} \to \mathbb{R}_+$ by:*

$$\mathbb{E}[u(\Phi)] = \sum_{k=0}^{+\infty} \frac{1}{k!} \int_{\mathbb{X}^k} u(x_1,\dots,x_k) j(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k.$$

It also verifies, for any $A \in \widehat{\mathcal{N}}_\mathbb{X}$ and $k \in \mathbb{N}_0$,

$$J(A^{(k)}) = \int_{A^k} j(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k.$$

Definition 2.1.6 (Correlation function).
*Let Φ be a (not necessarily finite) point process on $\mathbb{X}$. Its **correlation function** (with respect to ℓ) $\rho : \widehat{N}_\mathbb{X} \to \mathbb{R}_+$ is defined, if it exists, for any measurable function $u : \widehat{N}_\mathbb{X} \to \mathbb{R}_+$ by:*

$$\mathbb{E}\Big[\sum_{\substack{\alpha\in\widehat{N}_{\mathbb{X}}\\ \alpha\subset\Phi}} u(\alpha)\Big] = \sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{X}^k} u(x_1,\dots,x_k)\rho(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k.$$

Theorem 2.1.7 (Link between correlation and Janossy functions).
Let Φ be a finite point process on $\mathbb{X}$. Its Janossy function j and correlation function ρ verify the two following identities for any $n\in\mathbb{N}$ and $x_1,\dots,x_n\in\mathbb{X}$:

$$j(x_1,\dots,x_n) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\int_{\mathbb{X}^k}\rho(x_1,\dots,x_n,y_1,\dots,y_k)\mathrm{d}y_1\dots\mathrm{d}y_k;$$

$$\rho(x_1,\dots,x_n) = \sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{X}^k} j(x_1,\dots,x_n,y_1,\dots,y_k)\mathrm{d}y_1\dots\mathrm{d}y_k.$$

Theorem 2.1.8 (Properties of the Janossy and correlation functions).
Let Φ be a finite point process on $\mathbb{X}$. Its Janossy function j and correlation function ρ verify:

- $j(\varnothing) = \mathbb{P}(|\Phi| = 0)$;
- $\rho(\varnothing) = 1$;
- $\displaystyle\sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{X}^k} j(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k = 1$;
- $\displaystyle\sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{X}^k}\rho(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k = \mathbb{E}[2^{|\Phi|}]$.

Theorem 2.1.9 (Condition to be a correlation function).
Let $\rho : \widehat{N}_{\mathbb{X}} \to \mathbb{R}_+$ be a function with finite total integrals

$$I_k = \int_{\mathbb{X}^k}\rho(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k$$

where $k\in\mathbb{N}$, and suppose that for some $\epsilon > 0$ the series $\sum_{k=1}^{+\infty} I_k z^k$ is convergent for $|z| < 1+\epsilon$. Then, a necessary and sufficient condition for ρ to be the correlation function of a finite point process is that for any $n\in\mathbb{N}$ and any $x_1,\dots,x_n\in\mathbb{X}$,

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\int_{\mathbb{X}^k}\rho(x_1,\dots,x_n,y_1,\dots,y_k)\mathrm{d}y_1\dots\mathrm{d}y_k \geq 0.$$

Theorem 2.1.10 (Condition to be a Janossy function).
Let be a function $j : \widehat{N}_{\mathbb{X}} \to \mathbb{R}_+$. Then a necessary and sufficient condition for j to be the Janossy function of a finite point process is given by:

$$\sum_{k=0}^{+\infty} \frac{1}{k!} \int_{\mathbb{X}^k} j(x_1, \ldots, x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k = 1.$$

Definition 2.1.11 (Reduced Campbell measure).
The ***reduced Campbell measure*** *of a point process Φ on $\mathbb{X}$ is the measure C on the product space $(\mathbb{X} \times N_{\mathbb{X}}, \mathcal{X} \otimes \mathcal{N}_{\mathbb{X}})$ defined for any $A \in \mathcal{X} \otimes \mathcal{N}_{\mathbb{X}}$ by*

$$C(A) = \mathbb{E}\Big[\sum_{x \in \Phi} \mathbf{1}_A(x, \Phi \setminus x)\Big].$$

Definition 2.1.12 (Palm measure).
Let Φ be a point process on $\mathbb{X}$. The family $(\mathbb{P}_{\Phi}^x)_{x \in \mathbb{X}}$ of probability measures on $N_{\mathbb{X}}$ is defined, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$, by

$$\mathbb{E}\Big[\sum_{x \in \Phi} u(x, \Phi \setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}^x[u(x, \Phi)] \ell(\mathrm{d}x),$$

where $\mathbb{E}^x$ denotes the expectation associated to the probability measure $\mathbb{P}_{\Phi}^x$. For any $x \in \mathbb{X}$, the probability measure $\mathbb{P}_{\Phi}^x$ is called the ***Palm measure given*** *x of Φ.*

Definition 2.1.13 (Papangelou intensity).
Let Φ be a point process on $\mathbb{X}$ with reduced Campbell measure C such that $C \ll \ell \otimes \mathbb{P}_{\Phi}$. Any Radon-Nikodym density c of C relative to $\ell \otimes \mathbb{P}_{\Phi}$ is then called (a version of) the ***Papangelou intensity*** *of Φ. More explicitly c is a Papangelou intensity of Φ if, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,*

$$\mathbb{E}\Big[\sum_{x \in \Phi} u(x, \Phi \setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[c(x, \Phi) u(x, \Phi)] \ell(\mathrm{d}x).$$

Theorem 2.1.14 (Link between Palm measure and Papangelou intensity).
Let Φ be a point process on $\mathbb{X}$ with reduced Campbell measure C such that $C \ll \ell \otimes \mathbb{P}_{\Phi}$. Then, there exists a version of the Papangelou intensity c of Φ such that, for any $x \in \mathbb{X}$, its Palm measure $\mathbb{P}_{\Phi}^x$ given x verifies,

$$\mathbb{P}^x_\Phi(\mathrm{d}\phi) = c(x,\phi)\mathbb{P}_\Phi(\mathrm{d}\phi).$$

Theorem 2.1.15 (**Correlation function of the Palm measure**).
Let Φ be a point process on $\mathbb{X}$ with correlation function ρ. Then, for any $x \in \mathbb{X}$, its Palm measure $\mathbb{P}^x$ given x has a correlation function ρ^x given, for any $\phi \in \widehat{N_\mathbb{X}}$, by

$$\rho^x(\phi) = \frac{\rho(\phi x)}{\rho(x)}\mathbf{1}_{\{\rho(x)\neq 0\}}.$$

Theorem 2.1.16 (**Link between Papangelou intensity and Janossy function**).
If Φ is a finite point process on $\mathbb{X}$ with Janossy function j such that $\{j = 0\}$ is an increasing set, then its Papangelou intensity c is given for any $x \in \mathbb{X}$ and $\phi \in N_\mathbb{X}$ by:

$$c(x,\phi) = \frac{j(x\phi)}{j(\phi)}\mathbf{1}_{\{j(\phi)\neq 0\}}.$$

Theorem 2.1.17 (**Link between correlation function and Papangelou intensity**).
Let Φ be a finite point process on $\mathbb{X}$ with correlation function ρ and Papangelou intensity c. Then, for any $x \in \mathbb{X}$,

$$\mathbb{E}[c(x,\Phi)] = \rho(x).$$

The notion of coupling may also be useful when covering finite point processes.

Definition 2.1.18 (**Coupling - Coupling event**).
Let Φ_1, Φ_2 be two finite point processes. We say that $(\widehat{\Phi}_1, \widehat{\Phi}_2)$ is a ***coupling*** *of (Φ_1, Φ_2) if:*

$$\widehat{\Phi}_1 \overset{\mathcal{D}}{=} \Phi_1 \text{ and } \widehat{\Phi}_2 \overset{\mathcal{D}}{=} \Phi_2.$$

We say that an event C is a ***coupling event*** *associated to $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ if it verifies:*

$$C \subset \{\widehat{\Phi}_1 = \widehat{\Phi}_2\}.$$

Definition 2.1.19 (**Maximal coupling - Maximal coupling event**).
Let Φ_1, Φ_2 be two finite point processes. Let C be a coupling event associated to a coupling $(\widehat{\Phi}_1, \widehat{\Phi}_2)$ of (Φ_1, Φ_2). We say that such a coupling is ***maximal*** *and that C is a* ***maximal coupling event*** *if, for any coupling $(\widetilde{\Phi}_1, \widetilde{\Phi}_2)$ and any coupling event B*

associated to $(\widetilde{\Phi}_1, \widetilde{\Phi}_2)$,

$$\mathbb{P}(B) \leq \mathbb{P}(C).$$

2.2 Convergence

The different modes of convergence considered further are presented in this section.

Definition 2.2.1 (Vague convergence on $\mathbb{M}$).
Let be a sequence $(M_n)_{n\in\mathbb{N}} \subset \mathbb{M}$. We say that (M_n) ***converges vaguely*** *to a measure $M \in \mathbb{M}$ if, for any positive function $f \in C_K(\mathbb{X}, \mathbb{R})$,*

$$\lim_{n\to+\infty} \int_{\mathbb{X}} f(x) M_n(\mathrm{d}x) = \int_{\mathbb{X}} f(x) M(\mathrm{d}x).$$

A classical notion of convergence on the space of point processes is the following.

Definition 2.2.2 (Convergence in law for point processes).
Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of point processes on $\mathbb{X}$. We say that (Φ_n) ***converges in law*** *to a point process Φ on $\mathbb{X}$ if, for any bounded and continuous (for the vague topology) function $F : N_{\mathbb{X}} \to \mathbb{R}$,*

$$\lim_{n\to+\infty} \mathbb{E}[F(\Phi_n)] = \mathbb{E}[F(\Phi)].$$

Definition 2.2.3 (1-Lipschitz function).
Let $\mathbb{L}$ be a subset of $\mathbb{M}$ and let Δ be a distance on $\mathbb{L}$. We say that a function $h : \mathbb{L} \to \mathbb{R}$ is ***1-Lipschitz*** *according to Δ if for any $\nu_1, \nu_2 \in \mathbb{L}$,*

$$|h(\nu_1) - h(\nu_2)| \leq \Delta(\nu_1, \nu_2),$$

and denote by $Lip_1(\mathbb{L}, \Delta)$ the set of all these maps which are measurable.

Definition 2.2.4 (Discrete distance).
We call ***discrete distance*** *the distance Δ_D on $\mathbb{M}$ defined for any $\nu_1, \nu_2 \in \mathbb{M}$ by:*

$$\Delta_D(\nu_1, \nu_2) := \mathbf{1}_{\{\nu_1 \neq \nu_2\}}.$$

Definition 2.2.5 (Total variation distance).

We call ***total variation distance*** *the distance* Δ_{TV} *on* $\mathbb{M}$ *defined for any* $\nu_1, \nu_2 \in \mathbb{M}$ *by:*

$$\Delta_{TV}(\nu_1, \nu_2) := \sup_{\substack{A \in \mathcal{X} \\ \nu_1(A), \nu_2(A) < \infty}} |\nu_1(A) - \nu_2(A)|.$$

Remark 2.2.6 (Total variation distance).

Note that, for any $\nu_1, \nu_2 \in \mathbb{M}$,

$$\Delta_{TV}(\nu_1, \nu_2) = \int_{\mathbb{X}} \left| \frac{\mathrm{d}\nu_1(x)}{\mathrm{d}(\nu_1 + \nu_2)} - \frac{\mathrm{d}\nu_2(x)}{\mathrm{d}(\nu_1 + \nu_2)} \right| (\nu_1 + \nu_2)(\mathrm{d}x).$$

In particular, if $\nu_1, \nu_2 \in \widehat{N}_{\mathbb{X}}$,

$$\Delta_{TV}(\nu_1, \nu_2) = |\nu_1 \setminus \nu_2| + |\nu_2 \setminus \nu_1|.$$

Since $N_{\mathbb{X}}$ is Polish, it is known that the topology for convergence in law is metrizable. We now describe this corresponding distance.

Definition 2.2.7 (Polish distance).

Let $f = (f_k)_{k \in \mathbb{N}} \subset \mathcal{C}_K(\mathbb{X})$ *generating* $\mathcal{X}$. *We call* ***Polish distance*** *(associated to* f*) the distance* Δ_P *on* $\mathbb{M}$ *defined for any* $\nu_1, \nu_2 \in \mathbb{M}$ *by:*

$$\Delta_P(\nu_1, \nu_2) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \Psi(|\langle f_k, \nu_1 \rangle - \langle f_k, \nu_2 \rangle|),$$

where for any $x \in \mathbb{R}_+$,

$$\Psi(x) = \frac{x}{1+x}.$$

For this last definition, we can assume without loss of generality that the Polish distance on $\mathbb{M}'$ is chosen such that $f \subset \mathcal{C}_K(\mathbb{M}) \cap \mathrm{Lip}_1(\Delta_{TV})$.

Definition 2.2.8 (Kantorovich-Rubinstein distance).

Let Δ *be a distance on* $\mathbb{M}$. *The* ***Kantorovich-Rubinstein distance*** Δ^* *associated to* Δ *between two probability measures* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *on* $\mathbb{M}$ *is defined as*

$$\Delta^*(\mathbb{P}_1, \mathbb{P}_2) := \inf_{\mathbf{C} \in \Sigma(\mathbb{P}_1, \mathbb{P}_2)} \int_{N_{\mathbb{X}} \times N_{\mathbb{X}}} \Delta(\omega_1, \omega_2) \mathbf{C}(\mathrm{d}(\omega_1, \omega_2))$$

where $\Sigma(\mathbb{P}_1,\mathbb{P}_2)$ denotes the set of probability measures on $\mathbb{M}\times\mathbb{M}$ with first marginal $\mathbb{P}_1$ and second marginal $\mathbb{P}_2$.

In Definition 2.2.8, Δ^* may be seen as the optimal transportation cost for the cost function $\Delta(\cdot,\cdot)$. Moreover, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are concentrated on $N_{\mathbb{X}}$, then there is at least one coupling $\mathbf{C}\in\Sigma(\mathbb{P}_1,\mathbb{P}_2)$ for which the infimum is attained, and the Kantorovich duality theorem (see e.g. [73]) says that this minimum equals

$$\Delta^*(\mathbb{P}_1,\mathbb{P}_2)=\sup\left|\int_{N_{\mathbb{X}}}F(\omega)\mathbb{P}_1(\mathrm{d}\omega)-\int_{N_{\mathbb{X}}}F(\omega)\mathbb{P}_2(\mathrm{d}\omega)\right|,$$

where the supremum is over all $F\in\mathrm{Lip}_1(N_{\mathbb{X}},\Delta)$ that are integrable with respect to $\mathbb{P}_1$ and $\mathbb{P}_2$.

We now adapt the method of maximal coupling used in [72] for random variables to show that, for point processes, the Kantorovich-Rubinstein distance associated to discrete distance is the total variation distance.

Lemma 2.2.9 (Construction of a maximal coupling).

Let Φ_1,Φ_2 be two finite point processes with respective Janossy measures J_1,J_2. Put

$$c:=\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1\wedge J_2)\big((\widehat{N}_{\mathbb{X}})^{(k)}\big),$$

where the measure $J_1\wedge J_2$ is defined for any $E\in\widehat{\mathcal{N}}_{\mathbb{X}}$ by:

$$(J_1\wedge J_2)(E)=\inf_{\substack{B\in\widehat{\mathcal{N}}_{\mathbb{X}}\\ B\subset E}}(J_1(B)+J_2(E\setminus B)).$$

Consider the coupling $(\widehat{\Phi}_1,\widehat{\Phi}_2)$ built as follows.
If $c=0$, take $\widehat{\Phi}_1,\widehat{\Phi}_2$ independent and $C=\emptyset$.
If $c=1$, take $\widehat{\Phi}_1=\widehat{\Phi}_2$ and $C=\Omega$.
If $0<c<1$, let I,V,W_1,W_2 be independent random variables such that, for any $i\in\{1,2\}$,

- $\mathbb{P}(I=1)=c=1-\mathbb{P}(I=0)$;
- *V is a point process with Janossy measure* $\frac{1}{c}(J_1\wedge J_2)$;
- *W_i is a point process with Janossy measure* $\frac{1}{1-c}(J_i-J_1\wedge J_2)$.

For any $i\in\{1,2\}$, one defines

$$\begin{aligned}\widehat{\Phi}_i &= V \text{ if } I=1;\\ &= W_i \text{ if } I=0.\end{aligned}$$

Put $C=\{I=1\}$. Then $(\widehat{\Phi}_1,\widehat{\Phi}_2)$ is a maximal coupling of (Φ_1,Φ_2), C is a maximal coupling event and

$$\mathbb{P}(C)=\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1\wedge J_2)\big((\widehat{N}_{\mathbb{X}})^{(k)}\big).$$

Proof. One one hand, let show that, for any coupling event B associated to a coupling of (Φ_1,Φ_2), one has:

$$\mathbb{P}(B)\le\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1\wedge J_2)\big((\widehat{N}_{\mathbb{X}})^{(k)}\big).$$

For any $A\in\widehat{\mathcal{N}}_{\mathbb{X}}$, one has $A=\cup_{k=0}^{+\infty}A^{(k)}$ where, for all $k\in\mathbb{N}_0$, $A^{(k)}=\{\phi\in A:|\phi|=k\}$, then for any $i\in\{1,2\}$,

$$\mathbb{P}(\Phi_i\in A)=\sum_{k=0}^{+\infty}\frac{1}{k!}J_i(A^{(k)}).$$

Then, for any $F\in\widehat{\mathcal{N}}_{\mathbb{X}}$,

$$\mathbb{P}(\widehat{\Phi}_1\in A,B)=\sum_{k=0}^{+\infty}\mathbb{P}(\widehat{\Phi}_1\in(A\cap F)^{(k)},B)+\mathbb{P}(\widehat{\Phi}_1\in(A\cap F^c)^{(k)},B),$$

and, since B is a coupling event associated to a coupling of (Φ_1,Φ_2),

$$\mathbb{P}(\widehat{\Phi}_1\in A,B)=\sum_{k=0}^{+\infty}\mathbb{P}(\widehat{\Phi}_1\in(A\cap F)^{(k)},B)+\mathbb{P}(\widehat{\Phi}_2\in(A\cap F^c)^{(k)},B).$$

Then, the definition of the Janossy measure yields

$$\mathbb{P}(\widehat{\Phi}_1\in A,B)\le\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1(F^{(k)})+J_2((F^c)^{(k)})).$$

Thus,

$$\mathbb{P}(\widehat{\Phi}_1\in A,B)\le\inf_{F\in\widehat{\mathcal{N}}_{\mathbb{X}}}\Big(\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1(F^{(k)})+J_2((F^c)^{(k)}))\Big)$$

and the definition of $J_1\wedge J_2$ provides that

$$\mathbb{P}(\widehat{\Phi}_1\in A,B)\le\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1\wedge J_2)(A^{(k)}).$$

Then, taking $A=\widehat{N}_{\mathbb{X}}$, one has

$$\mathbb{P}(B)\le\sum_{k=0}^{+\infty}\frac{1}{k!}(J_1\wedge J_2)\big((\widehat{N}_{\mathbb{X}})^{(k)}\big).$$

On the other hand, for any $i \in \{1,2\}$, one has:

$$\mathbb{P}(\widehat{\Phi}_i \in A) = \mathbb{P}(\widehat{\Phi}_i \in A \mid I=1)\mathbb{P}(I=1) + \mathbb{P}(\widehat{\Phi}_i \in A \mid I=0)\mathbb{P}(I=0).$$

Then, by hypothesis on V, W_i and I,

$$\begin{aligned}\mathbb{P}(\widehat{\Phi}_i \in A) &= \mathbb{P}(V \in A)c + \mathbb{P}(W_i \in A)(1-c)\\ &= c\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{1}{c}(J_1 \wedge J_2)(A^{(k)})\\ &\quad +(1-c)\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{1}{1-c}(J_i - J_1 \wedge J_2)(A^{(k)}),\end{aligned}$$

and then

$$\mathbb{P}(\widehat{\Phi}_i \in A) = \sum_{k=0}^{+\infty}\frac{1}{k!}J_i(A^{(k)}) = \mathbb{P}(\Phi_i \in A).$$

Moreover, C is a coupling event and $\mathbb{P}(C) = \mathbb{P}(I=1) = c$, which ends to show the lemma. □

Note that if Φ_1, Φ_2 are two finite point processes with respective Janossy measures J_1, J_2 and Janossy functions j_1, j_2 with respect to ℓ, one can observe that the measure $J_1 \wedge J_2$ admits the density (with respect to ℓ) $j_1 \wedge j_2$ defined by $j_1 \wedge j_2 = \min(j_1, j_2)$.

Moreover, for any $k \in \mathbb{N}_0$ and any $x_1, \ldots, x_k \in \mathbb{X}$,

$$\sup_{A \in \widehat{\mathcal{N}}_{\mathbb{X}}}\big(J_1(A^{(k)}) - J_2(A^{(k)})\big) = \int_{\mathbb{X}^k}(j_1 - j_2)^+(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k.$$

Theorem 2.2.10 (Maximal coupling and distances).
Let Φ_1, Φ_2 be two finite point processes and $(\widehat{\Phi}_1, \widehat{\Phi}_2)$ a maximal coupling of (Φ_1, Φ_2). Then,

$$\Delta_{TV}(\Phi_1, \Phi_2) = \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) = \Delta_D^*(\Phi_1, \Phi_2).$$

Proof. Recall now that $\Delta_{TV}(\Phi_1, \Phi_2) = \sup_{A \in \mathcal{N}(\mathbb{X})}|\mathbb{P}(\Phi_1 \in A) - \mathbb{P}(\Phi_2 \in A)|$.

For any $A \in \mathcal{N}(\mathbb{X})$,

$$\mathbb{P}(\Phi_1 \in A) - \mathbb{P}(\Phi_2 \in A) = \sum_{k=0}^{+\infty}\frac{1}{k!}\big(J_1(A^{(k)}) - J_2(A^{(k)})\big).$$

Then, since, for any $k \in \mathbb{N}_0$,

$$\sup_{A \in \widehat{\mathcal{N}}_{\mathbb{X}}}\big(J_1(A^{(k)}) - J_2(A^{(k)})\big) = \sup_{B_k \subset \widehat{N}_{\mathbb{X}}^{(k)}}\big(J_1(B_k) - J_2(B_k)\big),$$

and since $(\widehat{N}_{\mathbb{X}}^{(k)})_{k\in\mathbb{N}_0}$ is a sequence of pairwise disjoint sets, one has

$$\sup_{A\in\widehat{\mathcal{N}}(\mathbb{X})} (\mathbb{P}(\Phi_1 \in A) - \mathbb{P}(\Phi_2 \in A)) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big).$$

In the same way,

$$\sup_{A\in\widehat{\mathcal{N}}(\mathbb{X})} (\mathbb{P}(\Phi_2 \in A) - \mathbb{P}(\Phi_1 \in A)) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) - J_1(A^{(k)})\big).$$

Let show now that the difference between the two last equations is null:

$$\begin{aligned}
\sum_{k=0}^{+\infty} \frac{1}{k!}\Big(\sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) - \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) - J_1(A^{(k)})\big)\Big) = \\
= \sum_{k=0}^{+\infty} \frac{1}{k!}\Big(\inf_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) - \inf_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) - J_1(A^{(k)})\big)\Big) \\
= \sum_{k=0}^{+\infty} \frac{1}{k!}\Big(\inf_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) + J_2(\widehat{N}_{\mathbb{X}}^{(k)} \setminus A^{(k)}) - J_2(\widehat{N}_{\mathbb{X}}^{(k)})\big) \\
- \inf_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) + J_1(\widehat{N}_{\mathbb{X}}^{(k)} \setminus A^{(k)}) - J_1(\widehat{N}_{\mathbb{X}}^{(k)})\big)\Big)
\end{aligned}$$

and then, from the definition of $J_1 \wedge J_2$,

$$\begin{aligned}
& \sum_{k=0}^{+\infty} \frac{1}{k!}\Big(\sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) - \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) - J_1(A^{(k)})\big)\Big) = \\
= & \sum_{k=0}^{+\infty} \frac{1}{k!}\Big(\big((J_1 \wedge J_2)(\widehat{N}_{\mathbb{X}}^{(k)}) - J_2(\widehat{N}_{\mathbb{X}}^{(k)})\big) - \big((J_1 \wedge J_2)(\widehat{N}_{\mathbb{X}}^{(k)}) - J_1(\widehat{N}_{\mathbb{X}}^{(k)})\big)\Big) \\
= & \sum_{k=0}^{+\infty} \frac{1}{k!} J_1(\widehat{N}_{\mathbb{X}}^{(k)}) - \sum_{k=0}^{+\infty} \frac{1}{k!} J_2(\widehat{N}_{\mathbb{X}}^{(k)}).
\end{aligned}$$

Since

$$\sum_{k=0}^{+\infty} \frac{1}{k!} J_1(\widehat{N}_{\mathbb{X}}^{(k)}) = \sum_{k=0}^{+\infty} \frac{1}{k!} J_2(\widehat{N}_{\mathbb{X}}^{(k)}) = 1,$$

it follows that

$$\sum_{k=0}^{+\infty} \frac{1}{k!}\Big(\sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) - \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) - J_1(A^{(k)})\big)\Big) = 0.$$

Hence,

$$\Delta_{TV}(\Phi_1,\Phi_2) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big).$$

This implies that

$$\Delta_{TV}(\Phi_1,\Phi_2) - \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) - (1-c),$$

where c is given by Lemma 2.2.9, and then

$$\Delta_{TV}(\Phi_1,\Phi_2) - \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) = \sum_{k=0}^{+\infty} \frac{1}{k!} \Big(\sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) + (J_1 \wedge J_2)(\widehat{N}_{\mathbb{X}}^{(k)})\Big) - 1.$$

The expression of $J_1 \wedge J_2$ yields

$$\Delta_{TV}(\Phi_1,\Phi_2) - \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) = \sum_{k=0}^{+\infty} \frac{1}{k!} \Big(\sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) + \inf_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) + J_1(\widehat{N}_{\mathbb{X}}^{(k)} \backslash A^{(k)})\big)\Big) - 1,$$

and then

$$\begin{aligned} \Delta_{TV}(\Phi_1,\Phi_2) - \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) &= \sum_{k=0}^{+\infty} \frac{1}{k!} \Big(\sup_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_1(A^{(k)}) - J_2(A^{(k)})\big) \\ &\quad + \inf_{A\in\widehat{\mathcal{N}}_{\mathbb{X}}} \big(J_2(A^{(k)}) - J_1(A^{(k)})\big) + J_1(\widehat{N}_{\mathbb{X}}^{(k)})\Big) - 1 \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} J_1(\widehat{N}_{\mathbb{X}}^{(k)}) - 1. \end{aligned}$$

Thus, since $\sum_{k=0}^{+\infty} \frac{1}{k!} J_1(\widehat{N}_{\mathbb{X}}^{(k)}) = 1$, it follows that

$$\Delta_{TV}(\Phi_1,\Phi_2) - \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) = 0.$$

Then $\Delta_{TV}(\Phi_1,\Phi_2) = \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2)$. The second equation is obtained from the two expressions of the Kantorovich-Rubinstein distance since, on one hand,

$$\begin{aligned} \Delta_D^*(\Phi_1,\Phi_2) &= \inf_{\mathbf{C}\in\Sigma(\mathbb{P}_1,\mathbb{P}_2)} \int_{N_{\mathbb{X}}\times N_{\mathbb{X}}} \Delta_D(\omega_1,\omega_2)\mathbf{C}(\mathrm{d}(\omega_1,\omega_2)) \\ &\leq \int_{N_{\mathbb{X}}\times N_{\mathbb{X}}} \Delta_D(\omega_1,\omega_2)\mathbb{P}_{(\widehat{\Phi}_1,\widehat{\Phi}_2)} \\ &= \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) \end{aligned}$$

and, on the other hand,

$$\begin{aligned}\Delta_D^*(\Phi_1,\Phi_2) &= \sup_{\substack{h\in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}},\Delta_D)\\ h \text{ integrable}}} \left| \int_{N_{\mathbb{X}}} h(\omega)\mathbb{P}_{\Phi_1}(\mathrm{d}\omega) - \int_{N_{\mathbb{X}}} h(\omega)\mathbb{P}_{\Phi_2}(\mathrm{d}\omega)\right| \\ &\geq \sup_{A\in\mathcal{N}_{\mathbb{X}}} \left| \int_{N_{\mathbb{X}}} \mathbf{1}_A(\omega)\mathbb{P}_{\Phi_1}(\mathrm{d}\omega) - \int_{N_{\mathbb{X}}} \mathbf{1}_A(\omega)\mathbb{P}_{\Phi_2}(\mathrm{d}\omega)\right| \\ &= \Delta_{TV}(\Phi_1,\Phi_2),\end{aligned}$$

since, for any $A\in\mathcal{N}_{\mathbb{X}}$, $\mathbf{1}_A\in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}},\Delta_D)$. The proof is thus complete. □

We now compare the different topologies induced by the formerly defined distances.

Theorem 2.2.11 (Link between topologies).
For any distance Δ on the space of point processes on $\mathbb{X}$, let $\mathcal{T}(\Delta)$ be the topology associated to the corresponding metric space. Then:

- $\Delta_{P|\mathbb{M}_1'}$ *and* $\Delta_{P|N_{\mathbb{X}}}^*$ *provide both a metric for the convergence in law;*
- $\mathcal{T}(\Delta_{P|\mathbb{M}_1'}) = \mathcal{T}(\Delta_{P|N_{\mathbb{X}}}^*) \subsetneq \mathcal{T}(\Delta_{TV|\mathbb{M}_1'}) \subsetneq \mathcal{T}(\Delta_{TV|N_{\mathbb{X}}}^*)$;
- $\Delta_{TV|\mathbb{M}_1'} = \Delta_{D|N_{\mathbb{X}}}^* \leq \Delta_{TV|N_{\mathbb{X}}}^*$;
- $\Delta_{P|\mathbb{M}_1'} \leq \Delta_{TV|N_{\mathbb{X}}}^*$.

2.3 Transformations of point processes

The following results are for the most of them a part of the folklore. We mention them for the sake of self-completeness.

Definition 2.3.1 (Reduction to a compact subset).
Let Φ be a point process on $\mathbb{X}$ and Λ a compact subset of $\mathbb{X}$. The (finite) point process on $\mathbb{X}$ defined by $\Phi_{|\Lambda} := \Phi\cap\Lambda$ is called the ***reduction to*** Λ *of* Φ.

Theorem 2.3.2 (Correlation function of a reduced point process).
Let Φ be a point process on $\mathbb{X}$ with correlation function ρ, Λ a compact subset of $\mathbb{X}$ and $\Phi_{|\Lambda}$ the reduction of Φ to Λ. Then, its correlation function ρ_Λ is given for any $\phi\in\widehat{N}_{\mathbb{X}}$ by

$$\rho_\Lambda(\phi) = \rho(\phi)\mathbf{1}_{\{\phi\subset\Lambda\}}.$$

Definition 2.3.3 (**Superposition**).

*Let $\Phi_1,\dots,\Phi_n$ ($n \in \mathbb{N}$) be independent point processes on $\mathbb{X}$. The point process $\sum_{i=1}^n \Phi_i$ on $\mathbb{X}$ is called the **(independent) superposition** of $\Phi_1,\dots,\Phi_n$.*

Theorem 2.3.4 (**Laplace functional of a superposition**).

Let $\Phi_1,\dots,\Phi_n$ ($n \in \mathbb{N}$) be independent point processes on $\mathbb{X}$ and Φ their independent superposition. Then, its Laplace functional is given for any $f \in \mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_\Phi(f) = \prod_{k=1}^n \mathcal{L}_{\Phi_k}(f);$$

and its p.g.fl. is given for any $u : \mathbb{X} \to (0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \prod_{k=1}^n G_{\Phi_k}(u).$$

Theorem 2.3.5 (**Correlation and Janossy functions of a superposition**).

Let $\Phi_1,\dots,\Phi_n$ ($n \in \mathbb{N}$) be independent point processes on $\mathbb{X}$ with respective correlation functions $\rho_1,\dots,\rho_n$ and Janossy functions $j_1,\dots,j_n$. Let Φ be their independent superposition. Then, its correlation function ρ is given for any $x_1,\dots,x_k \in \mathbb{X}$ by

$$\rho(x_1,\dots,x_k) = \sum_{k_1+\dots+k_n=k} \binom{k}{k_1,\dots,k_n} \prod_{i=1}^n \rho_i(x_{k_1+\dots+k_{i-1}},\dots,x_{k_1+\dots+k_i});$$

and its Janossy function j is given for any $x_1,\dots,x_k \in \mathbb{X}$ by

$$j(x_1,\dots,x_k) = \sum_{k_1+\dots+k_n=k} \binom{k}{k_1,\dots,k_n} \prod_{i=1}^n j_i(x_{k_1+\dots+k_{i-1}},\dots,x_{k_1+\dots+k_i}).$$

Definition 2.3.6 (**Thinning**).

*Let Φ be a point process on $\mathbb{X}$ and β be a function from $\mathbb{X}$ to $[0,1]$. The point process on $\mathbb{X}$ denoted $\beta \circ \Phi$ and built by keeping with probability $\beta(x)$ and deleting with probability $1-\beta(x)$ each point x of Φ independently is called the **β-thinning** of Φ.*

Remark 2.3.7 (**Randomization of a point process**).

Consider a probability kernel ν from $\mathbb{X}$ to a locally compact metric space $\mathbb{Y}$. Following [43], a ν-randomization of an arbitrary point process Φ may be built as follows: for *any $\phi \in N_{\mathbb{X}}$, let $\tilde{\phi}$ be the point process on $\mathbb{X} \times \mathbb{Y}$ defined by:*

$$\tilde{\phi} = \{(x, Y_x) \ : \ x \in \phi\},$$

where $(Y_x)_{x \in \phi}$ is a family of independent random elements of $\mathbb{Y}$ such that, for any $x \in \phi$, $\nu(x, \cdot)$ is the probability distribution of Y_x. Denoting μ_ϕ the probability distribution of $\tilde{\phi}$, we may define the ν-randomization of Φ as the point process $\nu \star \Phi$ such that

$$\mathbb{P}(\nu \star \Phi \in \cdot \mid \Phi) = \mu_\Phi \text{ almost surely (a.s.).}$$

For the case when $\mathbb{Y} = \{0, 1\}$ and $\nu(x, \{1\}) = \beta(x)$ with $\beta : \mathbb{X} \to [0, 1]$, the point process $\nu \star \Phi(\cdot \times \{1\})$ is exactly the β-thinning of Φ.

Theorem 2.3.8 (Laplace functional of a thinned point process).
Let Φ be a point process on $\mathbb{X}$, let β be a function from $\mathbb{X}$ to $[0, 1]$ and $\beta \circ \Phi$ the β-thinning of Φ. Then, its Laplace functional is given for any $f \in \mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_{\beta \circ \Phi}(f) = G_\Phi(1 - \beta(1 - e^{-f}));$$

and its p.g.fl. is given for any $u : \mathbb{X} \to (0, 1]$ such that $1 - u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_{\beta \circ \Phi}(u) = G_\Phi(1 - \beta(1 - u)).$$

Theorem 2.3.9 (Correlation and Janossy functions of a thinned point process).
Let Φ be a point process on $\mathbb{X}$ with correlation function ρ and Janossy function j, let β be a function from $\mathbb{X}$ to $[0, 1]$ and $\beta \circ \Phi$ the β-thinning of Φ. Then, its correlation function ρ_β is given for any $x_1, \dots, x_k \in \mathbb{X}$ by

$$\rho_\beta(x_1, \dots, x_k) = \rho(x_1, \dots, x_k) \prod_{i=1}^{k} \beta(x_i);$$

its Janossy function j_β is given for any $x_1, \dots, x_k \in \mathbb{X}$ by

$$j_\beta(x_1, \dots, x_k) = \prod_{i=1}^{k} \beta(x_i) \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\mathbb{X}^n} j(x_1, \dots, x_k, y_1, \dots, y_n) \prod_{l=1}^{n} (1 - \beta(y_l)) \mathrm{d}y_1 \dots \mathrm{d}y_n.$$

Definition 2.3.10 (Rescaling).
*Let Φ be a point process on $\mathbb{R}^d$ and ϵ be a positive real number. The point process $\Phi^{(\epsilon)}$ on $\mathbb{R}^d$ defined as $\Phi^{(\epsilon)} = \{y \in \mathbb{R}^d \mid \exists x \in \Phi, y = \epsilon^{\frac{1}{d}} x\}$ is called the ϵ-**rescaling** of Φ.*

Theorem 2.3.11 (Laplace functional of a rescaled point process).
Let Φ be a point process on $\mathbb{R}^d$, let ϵ be a positive real number and $\Phi^{(\epsilon)}$ the ϵ-rescaling of Φ. Then, its Laplace functional is given for any $f \in \mathcal{B}_+(\mathbb{R}^d)$ by:

$$\mathcal{L}_{\Phi^{(\epsilon)}(f)} = \mathcal{L}_\Phi(f(\epsilon^{\frac{1}{d}}\cdot));$$

and its p.g.fl. is given for any $u : \mathbb{R}^d \to (0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{R}^d)$ by:

$$G_{\Phi^{(\epsilon)}}(u) = G_\Phi(u(\epsilon^{\frac{1}{d}}\cdot)).$$

Theorem 2.3.12 (Correlation and Janossy functions of a rescaled point process).
Let Φ be a point process on $\mathbb{R}^d$ with correlation function ρ and Janossy function j, let ϵ be a positive real number and $\Phi^{(\epsilon)}$ the ϵ-rescaling of Φ. Then, its correlation function $\rho^{(\epsilon)}$ is given for any $x_1, \dots, x_k \in \mathbb{R}^d$ by

$$\rho^{(\epsilon)}(x_1, \dots, x_k) = \frac{1}{\epsilon^k}\rho(\epsilon^{-\frac{1}{d}}x_1, \dots, \epsilon^{-\frac{1}{d}}x_k);$$

and its Janossy function $j^{(\epsilon)}$ is given for any $x_1, \dots, x_k \in \mathbb{R}^d$ by

$$j^{(\epsilon)}(x_1, \dots, x_k) = \frac{1}{\epsilon^k}j(\epsilon^{-\frac{1}{d}}x_1, \dots, \epsilon^{-\frac{1}{d}}x_k).$$

2.4 Poisson-based point processes

We recall here the definitions and some basic properties of some Poisson-based point processes, in particular their Laplace functionals, Janossy functions and correlation functions.

Definition 2.4.1 (Binomial point process - Bernoulli point process).
Let μ be a probability measure on $\mathbb{X}$ and $N \in \mathbb{N}_0$. A ***binomial point process*** *with parameter N and supported by μ has exactly N points drawn independently according to μ. In particular, a binomial point process with parameter $N = 1$ and supported by μ is called a* ***Bernoulli point process*** *supported by μ.*

Definition 2.4.2 (Poisson point process - Homogeneous Poisson point process).
Let M be a Radon measure on $\mathbb{X}$. The ***Poisson point process*** *(PPP) Φ with intensity measure M is defined as the unique point process on $\mathbb{X}$ with intensity measure M such that, for any disjoint relatively compact subsets Λ_1, Λ_2, the random variables $\Phi(\Lambda_1)$ and $\Phi(\Lambda_2)$ are independent.*
Moreover, if there exists $\lambda \in [0, +\infty)$ such that $M(\mathrm{d}x) = \lambda\ell(\mathrm{d}x)$, then Φ is said to be ***homogeneous*** *with intensity λ.*

Theorem 2.4.3 (**Characterization of a Poisson point process**).

A Poisson point process on $\mathbb{X}$ with finite intensity measure M may be defined as a finite point process Φ on $\mathbb{X}$ such that its total number of points N has a Poisson distribution with parameter $M(\mathbb{X})$ and, conditionally to N, Φ is a binomial point process on $\mathbb{X}$ with parameter N and supported by $\frac{M(\cdot)}{M(\mathbb{X})}$.

Theorem 2.4.4 (**Laplace functional of a Poisson point process**).

Let Φ be a Poisson point process on $\mathbb{X}$ with intensity measure M. Then, its Laplace functional is given for any $f \in \mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_\Phi(f) = \exp\Big\{-\int_{\mathbb{X}} 1 - e^{-f(x)} M(\mathrm{d}x)\Big\};$$

and its p.g.fl. is given for any $u : \mathbb{X} \to (0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \exp\Big\{-\int_{\mathbb{X}} 1 - u(x) M(\mathrm{d}x)\Big\}.$$

Theorem 2.4.5 (**Correlation and Janossy functions of a Poisson point process**).

Let Φ be a Poisson point process with intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$. Then its correlation function ρ is given for any $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$\rho(\phi) = \prod_{x\in\phi} m(x);$$

and, if $M(\mathbb{X}) < +\infty$, its Janossy function j is given for any $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$j(\phi) = e^{-M(\mathbb{X})} \prod_{x\in\phi} m(x).$$

Theorem 2.4.6 (**Mecke formula for a Poisson point process**).

Let Φ be a point process on $\mathbb{X}$ with intensity measure M. If Φ is a Poisson point process, then, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x, \Phi \setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[u(x,\Phi)] M(\mathrm{d}x).$$

Conversely, if M is locally finite and Φ such that, for any $x \in \mathbb{X}$, $\Phi(\{x\}) \leq 1$ a.s., then Φ is a Poisson point process with intensity measure M.

Theorem 2.4.7 (Transformations of PPPs - Invariance property).

Let $n \in \mathbb{N}$ and $\Phi_1, \dots, \Phi_n$ be n Poisson point processes with respective intensity measures $M_1, \dots, M_n$. Then, their independent superposition is a Poisson point process with intensity measure $M_1 + \cdots + M_n$. Moreover, if $\beta \in [0,1]$ and Φ is a Poisson point process with intensity measure M, then $\beta \circ \Phi$ is a Poisson point process with intensity measure βM.

In particular, a Poisson point process Φ verifies the following invariance property: for any $t \in [0,1]$,

$$t \circ \Phi^{(1)} + (1-t) \circ \Phi^{(2)} \overset{\mathcal{D}}{=} \Phi,$$

where $\Phi^{(1)}$ and $\Phi^{(2)}$ are independent copies of Φ.

Definition 2.4.8 (Purely random point process).

Let $(p_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}_+$ be a sequence such that $\sum_{n=0}^{+\infty} p_n = 1$ and let μ be a probability measure on $\mathbb{X}$. A ***purely random point process*** *(PRPP) supported by μ and (p_n) is a finite point process Φ on $\mathbb{X}$ such that its number of points N in $\mathbb{X}$ verifies for any $n \in \mathbb{N}_0$*

$$\mathbb{P}(N = n) = p_n;$$

and, conditionally to N, Φ is a binomial point process on $\mathbb{X}$ with parameter N and supported by μ.

Theorem 2.4.9 (Laplace functional of a purely random point process).

Let Φ be a purely random point process on $\mathbb{X}$ supported by a probability measure μ and a distribution $(p_n)_{n\in\mathbb{N}_0}$. Then, its Laplace functional is given for any $f \in \mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_\Phi(f) = \sum_{k=0}^{+\infty} p_k \Big(\int_{\mathbb{X}} e^{-f(x)} \mu(\mathrm{d}x) \Big)^k;$$

and its p.g.fl. is given for any $u : \mathbb{X} \to (0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \sum_{k=0}^{+\infty} p_k \Big(\int_{\mathbb{X}} u(x)\mu(\mathrm{d}x) \Big)^k = g\Big(\int_{\mathbb{X}} u(x)\mu(\mathrm{d}x) \Big),$$

where g is the probability generating function of the distribution $(p_n)_{n\in\mathbb{N}_0}$.

Theorem 2.4.10 (Correlation and Janossy functions of a PRPP).

Let Φ be a purely random point process on $\mathbb{X}$ supported by a probability measure $\mu(\mathrm{d}x) = q(x)\mathrm{d}x$ and a distribution $(p_n)_{n\in\mathbb{N}_0}$ such that $p_n \neq 0$ for any $n \in \mathbb{N}_0$. Then its correlation function ρ is given for any $n \in \mathbb{N}_0$ and any $x_1, \dots, x_n \in \mathbb{X}$ by

$$\rho(x_1,\dots,x_n) = q(x_1)\dots q(x_n)\sum_{k=0}^{+\infty}\frac{(n+k)!}{k!}p_{n+k};$$

and its Janossy function j is given for any $n\in\mathbb{N}_0$ and any $x_1,\dots,x_n\in\mathbb{X}$ by

$$j(x_1,\dots,x_n) = p_n n! q(x_1)\dots q(x_n).$$

Theorem 2.4.11 (Properties of a purely random point process).
Let Φ be a purely random point process on $\mathbb{X}$ supported by a probability measure μ and a distribution $(p_n)_{n\in\mathbb{N}_0}$. Then, for any measurable function $F: N_{\mathbb{X}}\to\mathbb{R}_+$,

$$\mathbb{E}[F(\Phi)] = \sum_{n=0}^{+\infty}p_n\int_{\mathbb{X}^n}F(x_1,\dots,x_n)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n).$$

If $p_n\neq 0$ for any $n\in\mathbb{N}_0$, then, for any measurable function $u:\mathbb{X}\times N_{\mathbb{X}}\to\mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{x\in\Phi}u(x,\Phi\setminus x)\Big] = \int_{\mathbb{X}}\mathbb{E}\Big[(|\Phi|+1)\frac{p_{|\Phi|+1}}{p_{|\Phi|}}u(x,\Phi)\Big]\mu(\mathrm{d}x).$$

In particular, a binomial point process is a purely random point process with a determinist number of points, and a finite Poisson point process is a purely random point process whose number of points has a Poisson distribution.

Definition 2.4.12 (Cox point process).
Let M be a random measure on $\mathbb{X}$. A ***Cox point process*** *directed by M is a point process Φ such that, conditionally to M, Φ is a Poisson point process with intensity measure M.*

Theorem 2.4.13 (Laplace functional of a Cox point process).
Let Φ be a Cox point process directed by a random measure M. Then, its Laplace functional is given for any $f\in\mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_\Phi(f) = \mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{X}}1-e^{-f(x)}M(\mathrm{d}x)\Big\}\Big] = \mathcal{L}_M(1-e^{-f});$$

and its p.g.fl. is given for any $u:\mathbb{X}\to(0,1]$ such that $1-u\in\mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{X}}1-u(x)M(\mathrm{d}x)\Big\}\Big] = \mathcal{L}_M(1-u).$$

Theorem 2.4.14 (Correlation and Janossy functions of a Cox point process).
Let Φ be a Cox point process directed by a random measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$. Then its correlation function ρ is given for any $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$\rho(\phi) = \mathbb{E}_m\Big[\prod_{x\in\phi} m(x)\Big];$$

and, if $M(\mathbb{X}) < +\infty$ a.s., its Janossy function j is given for any $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$j(\phi) = \mathbb{E}_m\Big[e^{-\int_{\mathbb{X}} m(x)\mathrm{d}x} \prod_{x\in\phi} m(x)\Big].$$

Definition 2.4.15 (Conditional Poisson point process).
Let $C \in \mathcal{N}_{\mathbb{X}}$ and Φ be a Poisson point process with intensity measure M. Let $(\Phi^{(n)})_{n\in\mathbb{N}}$ be a sequence of independent copies of Φ. Let Φ_C be the point process defined as:

$$\Phi_C := \Phi^{(n)} \text{ if } \Phi^{(n)} \in C \text{ and, } \forall i \in \{1,\dots,n-1\},\ \Phi^{(i)} \notin C.$$

We say that Φ_C is the ***conditional Poisson point process*** *associated to Φ with intensity measure M and condition C.*

Definition 2.4.16 (Hardcore Poisson point process).
Let Φ be a Poisson point process on the metric space $(\mathbb{X}, \Delta_{\mathbb{X}})$ with finite intensity measure M, let $R > 0$ and let $C_R \in \mathcal{N}_{\mathbb{X}}$ be the set defined as:

$$C_R := \{\phi \in \widehat{N}_{\mathbb{X}} : \forall x, y \in \phi, x \neq y \implies \Delta_{\mathbb{X}}(x,y) \geq R\}.$$

We say that the conditional Poisson point process $\Phi_R := \Phi_{C_R}$ associated to Φ with parameter measure M and condition C_R is a ***hardcore (conditional) Poisson point process*** *with parameter R.*

Definition 2.4.17 (Bounded Poisson point process).
Let Φ be a Poisson point process on $\mathbb{X}$ with finite intensity measure M, let $N \in \mathbb{N}_0$ and let $C_N \in \mathcal{N}_{\mathbb{X}}$ be the set defined as:

$$C_N := \{\phi \in \widehat{N}_{\mathbb{X}} : \phi(\mathbb{X}) \leq N\}.$$

We say that the conditional Poisson point process $\Phi_N := \Phi_{C_N}$ associated to Φ with parameter measure M and condition C_N is a ***bounded (conditional) Poisson point process*** *with parameter N.*

Theorem 2.4.18 (Laplace functional of a conditional PPP).

Let Φ_C be a conditional Poisson point process with parameter measure M and conditional set C. If M is finite, then its Laplace functional is given for any $f \in \mathcal{B}_+(\mathbb{X})$ by:

$$\mathcal{L}_{\Phi_C}(f) = \frac{e^{-M(\mathbb{X})}}{p_C} \sum_{k=0}^{+\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \left(e^{-\sum_{i=1}^k f(x_i)}\right) \mathbf{1}_C(x_1, \ldots, x_k) M(\mathrm{d}x_1) \ldots M(\mathrm{d}x_k);$$

and its p.g.fl. is given for any $u : \mathbb{X} \to (0,1]$ such that $1 - u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_{\Phi_C}(u) = \frac{e^{-M(\mathbb{X})}}{p_C} \sum_{k=0}^{+\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \left(\prod_{i=1}^k u(x_i)\right) \mathbf{1}_C(x_1, \ldots, x_k) M(\mathrm{d}x_1) \ldots M(\mathrm{d}x_k),$$

where $p_C = \mathbb{P}(\Phi \in C)$ and Φ is the Poisson point process associated to Φ_C.

Theorem 2.4.19 (Correlation and Janossy functions of a conditional PPP).

Let Φ_C be a conditional Poisson point process with intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$ and conditional set C. Then its correlation function ρ is given, if C is decreasing, for any $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$\rho(\phi) = \frac{1}{p_C} \prod_{x \in \phi} m(x) \mathbf{1}_C(\phi);$$

and, if $M(\mathbb{X}) < +\infty$ a.s., its Janossy function j is given for any $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$j(\phi) = \frac{e^{-M(\mathbb{X})}}{p_C} \prod_{x \in \phi} m(x) \mathbf{1}_C(\phi),$$

where $p_C = \mathbb{P}(\Phi \in C)$ and Φ is the Poisson point process associated to Φ_C.

Definition 2.4.20 (Gibbs point process).

*A point process Φ on $\mathbb{X}$ is said to be a **Gibbs point process** with temperature parameter $\theta > 0$ and total potential energy*

$$U(x_1, \ldots, x_n) = \sum_{r=1}^{n} \sum_{1 \le i_1 < \cdots < i_r \le n} \Psi_r(x_{i_1}, \ldots, x_{i_r}),$$

*where $\Psi_r : \mathbb{X} \to \mathbb{R}_+$ is a measurable and symmetric function, called r^{th}**-order interaction potential**, if its Janossy function j is given for any $\phi \in \widehat{N}_{\mathbb{X}}$ by*

$$j(\phi) = C(\theta)e^{-\theta U(\phi)},$$

for some partition function $C(\theta) > 0$.

2.5 α-determinantal/permanantal point processes

The following definitions and properties on determinantal and permanantal point processes are extracted from [11].

Definition 2.5.1 (Integral operator - Hilbert-Schmidt and symmetric operators).
*A map $T : L^2(\mathbb{X}, \ell) \to L^2(\mathbb{X}, \ell)$ is said to be an **integral operator** whenever there exists a measurable function, called the **kernel** of T and still denoted by T, such that*

$$Tf(x) = \int_{\mathbb{X}} T(x, y)f(y)\ell(\mathrm{d}y).$$

In particular,

$$\int_{\mathbb{X}} |Tf(x)|^2 \ell(\mathrm{d}x) < +\infty$$

or equivalently

$$\int_{\mathbb{X}^2} |T(x, y)|^2 \ell(\mathrm{d}x)\ell(\mathrm{d}y) < \ +\infty$$

*and the operator T is said to be **Hilbert-Schmidt**.*
*The operator T is said to be **symmetric** if, for any $f, g \in L^2(\mathbb{X}, \ell)$,*

$$\langle Tf, g\rangle = \langle f, Tg\rangle,$$

or equivalently, for any $x, y \in \mathbb{X}$,

$$T(y, x) = \overline{T(x, y)}.$$

Definition 2.5.2 (Trace-class function).
*Let $T : L^2(\mathbb{X}, \ell) \to L^2(\mathbb{X}, \ell)$ be a bounded function. The function T is said to be **trace-class** whenever for one complete orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of $L^2(\mathbb{X}, \ell)$,*

$$\sum_{n=1}^{+\infty} |\langle Th_n, h_n\rangle| < +\infty,$$

*and the **trace** of T is then defined by:*

$$tr(T) = \sum_{n=1}^{+\infty} \langle Th_n, h_n \rangle.$$

Definition 2.5.3 (Fredholm determinant).
Let T be a trace-class operator. The ***Fredholm determinant*** *of $I+T$ is defined by*

$$\mathrm{Det}(I+T) = \exp\Big(\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} tr(T^n) \Big),$$

where I stands for the identity operator.

Definition 2.5.4 (α-determinant).
For a square matrix $A = (a_{ij})_{1 \le i,j \le n}$ of size $n \times n$, the ***α-determinant*** $\det_\alpha A$ *is defined by:*

$$\det_\alpha A = \sum_{\sigma \in \Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation is taken over the symmetric group Σ_n, the set of all permutations of $\{1, 2, \dots, n\}$ and $\nu(\sigma)$ is the number of cycles in the permutation σ.

Theorem 2.5.5 (Link between Fredholm determinant and α-determinant).
For a class integral operator T, if $\|\alpha T\| < 1$, we have:

$$\mathrm{Det}(I - \alpha T)^{-\frac{1}{\alpha}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} \det_\alpha (T(x_i, x_j))_{1 \le i,j \le n} \ell(\mathrm{d}x_1) \dots \ell(\mathrm{d}x_n).$$

If $\alpha \in \{-\frac{1}{m}; m \in \mathbb{N}\}$, this is true without the condition $\|\alpha T\| < 1$.

From now and all along this presentation of determinantal point processes, we suppose that the map K is an Hilbert-Schmidt operator from $L^2(\mathbb{X}, \ell)$ into $L^2(\mathbb{X}, \ell)$ which satisfies the following conditions:

- K is a bounded symmetric integral operator on $L^2(\mathbb{X}, \ell)$, with kernel $K(\cdot, \cdot)$, i.e., for any $x \in \mathbb{X}$,

$$Kf(x) = \int_{\mathbb{X}} K(x,y) f(y) \ell(\mathrm{d}y).$$

- The spectrum of K is included in $[0, 1)$.

- The map K is locally trace-class, i.e., for all compact $\Lambda \subset \mathbb{X}$, the restriction $K_\Lambda = P_\Lambda K P_\Lambda$ of K to $L^2(\Lambda, \ell_{|\Lambda})$ is trace-class.

Definition 2.5.6 (**α-determinantal/permanantal point process**).
*Let $\alpha \in \{\frac{2}{m} : m \in \mathbb{N}\} \cup \{-\frac{1}{m} : m \in \mathbb{N}\} \cup \{0\}$. A point process Φ is said to be an **α-determinantal/permanantal point process** (α-DPPP) with kernel K if its Laplace functional is given, for any $f \in \mathcal{B}_+(\mathbb{X})$, by:*

$$\mathcal{L}_\Phi(f) = Det\big(I + \alpha\sqrt{1-e^{-f}}\, K\, \sqrt{1-e^{-f}}\big)^{-\frac{1}{\alpha}}.$$

Its p.g.fl. is given for any $u : \mathbb{X} \to (0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = Det\big(I + \alpha\sqrt{1-u}\, K\, \sqrt{1-u}\big)^{-\frac{1}{\alpha}},$$

where for any $g \in L^2(\mathbb{X}, \ell)$ and any $x \in \mathbb{X}$,

$$\sqrt{1-e^{-f}}\, K\, \sqrt{1-e^{-f}} g(x) := \sqrt{1-e^{-f(x)}} \int_{\mathbb{X}} K(x,y) g(y) \sqrt{1-e^{-f(y)}} \ell(\mathrm{d}y).$$

*Such a point process is called for $\alpha > 0$ an **α-permanantal point process**, for $\alpha < 0$ an **α-determinantal point process** (α-DPP) and for $\alpha = -1$ more simply a **determinantal point process**. For $\alpha = 0$, the point process is a Poisson point process with intensity measure $K(x,x)\ell(\mathrm{d}x)$.*

Theorem 2.5.7 (**Spectral decomposition**).
Let $K : L^2(\mathbb{X}, \ell) \to L^2(\mathbb{X}, \ell)$ be a symmetric and Hilbert-Schmidt integral operator with kernel $K(\cdot,\cdot)$. Then, there exists a complete orthonormal basis $(h_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{X}, \ell)$ and a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0 such that, for any $x \in \mathbb{X}$,

$$Kf(x) = \sum_{n=1}^{+\infty} \lambda_n \langle f, h_n \rangle h_n(x),$$

or equivalently, for any $x, y \in \mathbb{X}$,

$$K(x,y) = \sum_{n=1}^{+\infty} \lambda_n h_n(x) h_n(y).$$

Since K is Hilbert-Schmidt,

$$\sum_{n=1}^{+\infty} \lambda_n^2 < +\infty,$$

and, if K is trace-class, then

$$\sum_{n=1}^{+\infty} \lambda_n < +\infty.$$

Moreover, if for any $n \in \mathbb{N}$, $1+\alpha\lambda_n \neq 0$, then the integral operator $J = (I+\alpha K)^{-1}K$ has a kernel J given for any $x, y \in \mathbb{X}$ by:

$$J(x,y) = \sum_{n=1}^{+\infty} \frac{\lambda_n}{1+\alpha\lambda_n} h_n(x) h_n(y).$$

Theorem 2.5.8 (Correlation and Janossy functions of an α-DPPP).

Let Φ be an α-DPPP with kernel K. Then its correlation function ρ is given for any $\phi \in \widehat{N_{\mathbb{X}}}$ by

$$\rho(\phi) = \mathrm{det}_\alpha K(\phi, \phi);$$

and, if Φ is finite, its Janossy function j is given for any $\phi \in \widehat{N_{\mathbb{X}}}$ by

$$j(\phi) = \mathrm{Det}(I+\alpha K)\mathrm{det}_\alpha J(\phi,\phi),$$

with first determinant understood as a Fredholm determinant and where J is the kernel of the integral operator $J = (I+\alpha K)^{-1}K$.

Theorem 2.5.9 (Reduction of an α-DPPP).

Let Φ be an α-DPPP with kernel K and let Λ be a compact subset of $\mathbb{X}$. Then the reduction to Λ of Φ is also an α-DPPP whose kernel is given, for any $x, y \in \mathbb{X}$, by:

$$K_\Lambda(x,y) = K(x,y)\boldsymbol{1}_{\Lambda\times\Lambda}(x,y).$$

Theorem 2.5.10 (Superposition of copies of a DPP).

For any $n \in \mathbb{N}$, a $(-1/n)$-determinantal point process with kernel K is the independent superposition of n determinantal point processes with kernel $\frac{1}{n}K$.

Moreover, such a sequence of point processes converges in law to a Poisson point process with intensity measure $K(x,x)\ell(\mathrm{d}x)$.

Theorem 2.5.11 (Thinning of an α-DPPP).

Let Φ be an α-DPPP with kernel K and let β be a real number of $[0,1]$. Then the β-thinning of Φ is also an α-DPPP whose kernel is given, for any $x, y \in \mathbb{X}$, by:

$$K_\beta(x,y) = \beta K(x,y).$$

Theorem 2.5.12 (**Rescaling of an α-DPPP**).

Let Φ be an α-DPPP on $\mathbb{R}^d$ with kernel K and let ϵ be a positive real number. Then the ϵ-rescaling of Φ is also an α-DPPP whose kernel is given, for any $x, y \in \mathbb{R}^d$, by:

$$K_\epsilon(x, y) = \frac{1}{\epsilon} K(\epsilon^{-\frac{1}{d}} x, \epsilon^{-\frac{1}{d}} y).$$

Definition 2.5.13 (**Ginibre point process - β-Ginibre point process**).

*The **Ginibre point process** (GPP) with intensity $\rho = \frac{\gamma}{\pi}$ (with $\gamma > 0$) is a determinantal point process on $\mathbb{C}$ whose kernel K_γ is given for any $x, y \in \mathbb{C}$ by:*

$$K_\gamma(x, y) = \frac{\gamma}{\pi} e^{-\frac{\gamma}{2}(|x|^2+|y|^2)} e^{\gamma x \overline{y}}.$$

*If β is a real number between 0 and 1, the β-**Ginibre point process** (β-GPP) with intensity $\rho = \frac{\gamma}{\pi}$ is a determinantal point process on $\mathbb{C}$ whose kernel $K_{\gamma,\beta}$ is given for any $x, y \in \mathbb{C}$ by:*

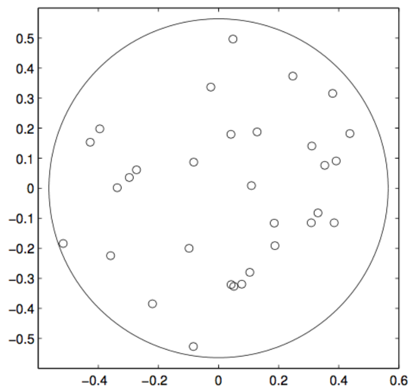
$$K_{\gamma,\beta}(x, y) = \frac{\gamma}{\pi} e^{-\frac{\gamma}{2\beta}(|x|^2+|y|^2)} e^{\frac{\gamma}{\beta} x \overline{y}}.$$

A β-Ginibre point process may be built by combining two operations on a Ginibre point process: a thinning with parameter β (one keeps each point independently with probability β) then a rescaling with parameter $\sqrt{\beta}$, such that we keep the same intensity. Hence, the parameter β provides an information concerning the degree of repulsiveness of the point process: the smaller β is, the less repulsive the β-Ginibre point process is. Note that such a point process is not defined for $\beta > 1$.

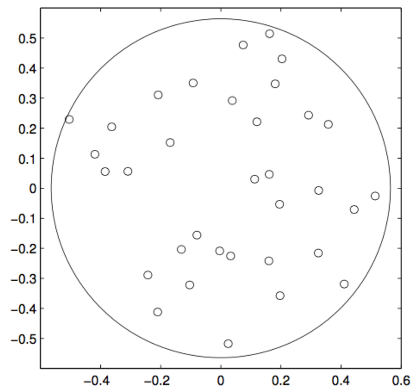
Simulations 2.5.14 (**β-Ginibre point processes**).

Some realizations of a Poisson point process and β-Ginibre point processes, reduced to a ball, for different values of β are given in Figure 2.1.

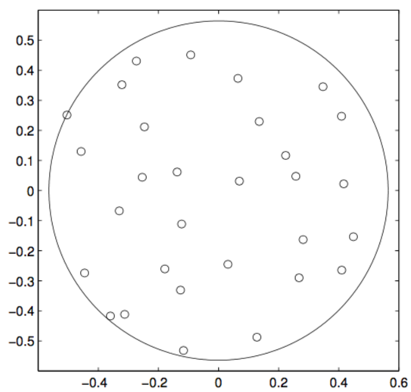
The method used for these simulations is given in [23]. One can observe that the repulsiveness between the particles is weaker and weaker as β decreases and almost null as β tend to 0.



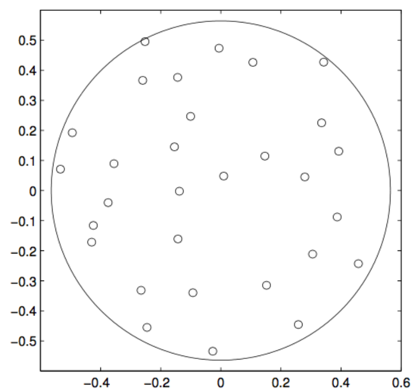
(a) PPP



(b) ¼-GPP



(c) ¾-GPP



(d) 1-GPP

Figure 2.1: Realizations of a Poisson point process and β-Ginibre point processes for $\beta \in \{\frac{1}{4}, \frac{3}{4}, 1\}$.

Chapter 3

Stein's method, Papangelou intensity and applications

In this chapter, we apply Stein's method to finite Poisson point processes and deduce some convergence results by using the Papangelou intensities. In Section 3.1, we roughly describe the Stein's method applied to a finite Poisson point process. In Section 3.2 we associate to a Glauber process its semi-group, infinitesimal generator and gradient, and state their properties. We deduce in Section 3.3 a representation formula and then an upper bound for the distance between a finite Poisson point process and an other finite point process (Theorem 3.3.2). In Section 3.4, Papangelou intensities are investigated. After proposing a definition of repulsiveness, we give some properties relative to repulsive point processes, finite point processes, transformations and classical point processes. In Section 3.5, we apply the upper bound given by Theorem 3.3.2 in order to establish some convergence results on Poisson point processes, Cox point processes, purely random point processes and conditional Poisson point processes. From this upper bound we settle some similar results on repulsive point processes in Section 3.6: it concerns the distance between a Poisson point process and respectively a superposition, a thinned and rescaled determinantal point process and a Gibbs point process. In Section 3.7, still from Theorem 3.3.2, we provide a convergence speed according to the Polish distance for a result settled by Kallenberg on thinned point processes.

3.1 A general Stein principle

This section aims to present Stein's method applied to a finite Poisson point process. We use the Stein's principle and the construction given in [25], but our proofs are sometimes different, highlighting the properties of the thinning operation and the invariance of the Poisson process distribution (Theorem 2.4.7): for any Poisson point process Φ and any $t \in [0,1]$,

$$t \circ \Phi^{(1)} + (1-t) \circ \Phi^{(2)} \stackrel{\mathcal{D}}{=} \Phi,$$

where $\Phi^{(1)}$ and $\Phi^{(2)}$ are independent copies of Φ.

The first step of Stein's method consists in characterizing the target object, here a finite Poisson point process. The way is to consider a functional operator L which, at a formal level, satisfies for a finite point process Φ the identity

$$\mathbb{E}[LF(\Phi)] = 0 \text{ for a large class of functions } F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$$

if and only if, Φ is a Poisson point process with finite intensity measure M.

The second step is to solve the so-called Stein's equation, that is to find, for any test function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$, a function $H_F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ such that, for any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$LH_F(\phi) = \mathbb{E}[F(\zeta)] - F(\phi),$$

where ζ is a Poisson point process with finite intensity measure M.

We use the so-called generator approach (see [59] for a survey) also applied in [65, 16] and based on theory of spatial birth-and-death processes [58]. In our case, L is built as an infinitesimal generator of a Markov process, also called Glauber process, with the distribution of ζ as its invariant distribution. If $(P_t)_{t\geq 0}$ is the semi-group associated to the Glauber process, one can show that, for any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$LH_F(\phi) = \int_0^{+\infty} LP_sF(\phi)\mathrm{d}s,$$

which leads to the so-called Stein-Dirichlet representation formula:

$$\mathbb{E}[F(\zeta)] - F(\phi) = \int_0^{+\infty} LP_sF(\phi)\mathrm{d}s,$$

from which we can deduce an upper bound for $\Delta^*_{TV}(\zeta, \Phi)$, where Φ is a finite point process. All these elements are exposed with more details in which follows.

3.2 Semi-group, gradient, infinitesimal generator

Definition 3.2.1 (Semi-group).

*For any $t \in \mathbb{R}_+$, let P_t be an operator on the space of measurable and bounded functions $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$. We say that the family $(P_t)_{t\geq 0}$ is a **semi-group** on $\widehat{N}_{\mathbb{X}}$ if, for any $t, s \in \mathbb{R}_+$,*

$$P_{t+s} = P_t \circ P_s.$$

Definition 3.2.2 (Infinitesimal generator).

*Let $(P_t)_{t\geq 0}$ be a semi-group on $\widehat{N}_{\mathbb{X}}$. Its **infinitesimal generator** L is defined for any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$ such that $t \mapsto P_tF(\phi)$ is derivable in 0 by:*

$$LF(\phi) = \left.\frac{\mathrm{d}P_tF(\phi)}{\mathrm{d}t}\right|_{t=0}.$$

Definition 3.2.3 (Glauber process for a Poisson point process).
Let ζ be a Poisson point process with finite intensity measure M. The ***Glauber process*** $(G_t)_{t\geq 0}$ *associated to ζ is defined for any $t \in \mathbb{R}_+$ and $\phi \in \widehat{N}_{\mathbb{X}}$ by:*

$$G_t(\phi) = e^{-t} \circ \phi + (1-e^{-t}) \circ \zeta.$$

For any $t \in \mathbb{R}_+$, the operator P_t is defined for any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$ by:

$$P_t F(\phi) = \mathbb{E}[F(G_t(\phi))] = \mathbb{E}[F(e^{-t} \circ \phi + (1-e^{-t}) \circ \zeta)].$$

Its dynamics can be described as follows: imagine a homogeneous Poisson process ζ_b on $\mathbb{R}_+$ with intensity $M(\mathbb{X})$. The jump times of ζ_b determine the birth times of the particles in ζ, placed in $\mathbb{X}$ according to the distribution $\frac{M(\cdot)}{M(\mathbb{X})}$. The lifetime of each particle is exponentially distributed with parameter 1.

Theorem 3.2.4 (Semi-group).
The family $(P_t)_{t\geq 0}$ given by Definition 3.2.3 is a semi-group.

Proof. For any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$, since thinning is associative,

$$P_s(P_t F)(\phi) = \int_{\widehat{N}_{\mathbb{X}}} \int_{\widehat{N}_{\mathbb{X}}} F(e^{-(t+s)} \circ \phi + e^{-s} \circ (1-e^{-t}) \circ \psi + (1-e^{-s}) \circ \eta) \mathbb{P}_\zeta(\mathrm{d}\psi) \mathbb{P}_\zeta(\mathrm{d}\eta).$$

Furthermore, since

$$e^{-s}(1-e^{-t}) + (1-e^{-s}) = 1 - e^{-(t+s)},$$

by the invariance property of the Poisson point process distribution (Theorem 2.4.7), we deduce that

$$e^{-s} \circ (1-e^{-t}) \circ \zeta^{(1)} + (1-e^{-s}) \circ \zeta^{(2)} \overset{\mathcal{D}}{=} (1-e^{-(t+s)}) \circ \zeta,$$

where $\zeta^{(1)}$ and $\zeta^{(2)}$ are independent copies of ζ.
Hence,

$$P_s(P_t F)(\phi) = \int_{\widehat{N}_{\mathbb{X}}} F(e^{-(t+s)} \circ \phi + (1-e^{-(t+s)}) \circ \eta) \mathbb{P}_\zeta(\mathrm{d}\eta)$$

and the proof is thus complete. □

Note that, in the previous proof, we only use associativity of thinning and the invariance property of a Poisson point process distribution.

Definition 3.2.5 (**Gradient in direction $x \in \mathbb{X}$**).

For any $x \in \mathbb{X}$, the ***gradient*** *D_x in direction x is defined, for any measurable function $F : N_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in N_{\mathbb{X}}$, by:*

$$D_x F(\phi) = F(\phi + x) - F(\phi).$$

Theorem 3.2.6 (**Closability**).

Let ζ be a Poisson point process on $\mathbb{X}$ with intensity measure M. Let $F, G : N_{\mathbb{X}} \to \mathbb{R}$ be two measurable and bounded functions. If $F(\phi) = G(\phi)$ $\mathbb{P}_\zeta(\mathrm{d}\phi)$-a.s., then

$$D_x F(\phi) = D_x G(\phi) \ (M \otimes \mathbb{P}_\zeta)(\mathrm{d}x, \mathrm{d}\phi)\text{-a.s.}.$$

Proof. By the Mecke formula applied to ζ, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\begin{aligned}
\int_{\mathbb{X}} \mathbb{E}[D_x F(\zeta) u(x,\zeta)] M(\mathrm{d}x) &= \mathbb{E}\Big[F(\zeta) \sum_{x \in \zeta} u(x, \zeta \setminus x)\Big] - \int_{\mathbb{X}} \mathbb{E}[F(\zeta) u(x,\zeta)] M(\mathrm{d}x) \\
&= \mathbb{E}\Big[F(\zeta)\Big(\sum_{x \in \zeta} u(x, \zeta \setminus x) - \int_{\mathbb{X}} u(x,\zeta) M(\mathrm{d}x)\Big)\Big].
\end{aligned}$$

Hence, if $F(\phi) = 0$ $\mathbb{P}_\Phi(\mathrm{d}\zeta)$-a.s., then $D_x F(\phi) = 0$ $(M \otimes \mathbb{P}_\zeta)(\mathrm{d}x, \mathrm{d}\phi)$-a.s., as expected. □

Theorem 3.2.7 (**Infinitesimal generator**).

Let $(P_t)_{t \geq 0}$ be the semi-group associated to a Poisson point process ζ with finite intensity measure M. Then, its infinitesimal generator L is given for any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$ by:

$$LF(\phi) = \int_{\mathbb{X}} D_x F(\phi) M(\mathrm{d}x) + \sum_{y \in \phi} (F(\phi \setminus y) - F(\phi)).$$

Moreover, a point process Φ is a Poisson point process with intensity measure M if and only if, for any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$,

$$\mathbb{E}[LF(\Phi)] = 0.$$

Proof. For any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\begin{aligned}
\left.\frac{\mathrm{d}P_t F(\phi)}{\mathrm{d}t}\right|_{t=0} &= \lim_{t \to 0} \frac{1}{t}(P_t F(\phi) - P_0 F(\phi)) \\
&= \lim_{t \to 0} \frac{1}{t}(\mathbb{E}[F(e^{-t} \circ \phi + (1 - e^{-t}) \circ \zeta)] - F(\phi)),
\end{aligned}$$

and, for any $t > 0$,

$$\begin{aligned}\mathbb{E}[F(e^{-t}\circ\phi+(1-e^{-t})\circ\zeta)] &= p_{00}(t)F(\phi)+\sum_{x\in\phi}p_{01}^{(x)}(t)F(\phi\setminus x)\\ &\quad +p_{10}(t)\int_{\mathbb{X}}F(\phi+x)\frac{M(\mathrm{d}x)}{M(\mathbb{X})}+R(t),\end{aligned}$$

where for any $x\in\phi$,

$$\begin{aligned}p_{00}(t) &= \mathbb{P}(e^{-t}\circ\phi=\phi,(1-e^{-t})\circ\zeta=\emptyset)\\ &= \mathbb{P}(e^{-t}\circ\phi=\phi)\mathbb{P}((1-e^{-t})\circ\zeta=\emptyset)\\ &= e^{-t|\phi|}e^{-(1-e^{-t})M(\mathbb{X})},\end{aligned}$$

where the computation of the second factor is deduced from the fact that $(1-e^{-t})\circ\zeta$ is a Poisson point process with intensity measure $(1-e^{-t})M$,

$$\begin{aligned}p_{01}^{(x)}(t) &= \mathbb{P}(\phi\setminus(e^{-t}\circ\phi)=x,(1-e^{-t})\circ\zeta=\emptyset)\\ &= \mathbb{P}(\phi\setminus(e^{-t}\circ\phi)=x)\mathbb{P}((1-e^{-t})\circ\zeta=\emptyset)\\ &= (1-e^{-t})e^{-t(|\phi|-1)}e^{-(1-e^{-t})M(\mathbb{X})},\end{aligned}$$

$$\begin{aligned}p_{10}(t) &= \mathbb{P}(e^{-t}\circ\phi=\phi,|(1-e^{-t})\circ\zeta|=1)\\ &= \mathbb{P}(e^{-t}\circ\phi=\phi)\mathbb{P}(|(1-e^{-t})\circ\zeta|=1)\\ &= e^{-t|\phi|}(1-e^{-t})M(\mathbb{X})e^{-(1-e^{-t})M(\mathbb{X})},\end{aligned}$$

$$R(t)=\mathbb{E}[F(e^{-t}\circ\phi+(1-e^{-t})\circ\zeta)\mathbf{1}_{|\phi\setminus(e^{-t}\circ\phi)|+|(1-e^{-t})\circ\zeta|\geq 2}].$$

Then,

$$\begin{aligned}&\frac{1}{t}\Big(\mathbb{E}[F(e^{-t}\circ\phi+(1-e^{-t})\circ\zeta)]-F(\phi)\Big)=\\ &=\frac{1}{t}\Big(\sum_{x\in\phi}p_{01}^{(x)}(t)(F(\phi\setminus x)-F(\phi))\\ &\qquad +p_{10}(t)\int_{\mathbb{X}}F(\phi+x)-F(\phi)\frac{M(\mathrm{d}x)}{M(\mathbb{X})}-p_{\infty}(t)F(\phi)+R(t)\Big)\\ &=\sum_{x\in\phi}\frac{p_{01}^{(x)}(t)}{t}(F(\phi\setminus x)-F(\phi))\\ &\qquad +\frac{p_{10}(t)}{t}\int_{\mathbb{X}}F(\phi+x)-F(\phi)\frac{M(\mathrm{d}x)}{M(\mathbb{X})}-\frac{p_{\infty}(t)}{t}F(\phi)+\frac{R(t)}{t},\end{aligned}$$

where

$$
\begin{aligned}
p_\infty(t) &= \mathbb{P}(|\phi \setminus (e^{-t}\circ\phi)| + |(1-e^{-t})\circ\zeta| \geq 2) \\
&= 1 - \Big(p_{00}(t) + \sum_{x\in\phi} p_{01}^{(x)}(t) + p_{10}(t)\Big).
\end{aligned}
$$

Since for any $x \in \phi$,

$$
\lim_{t\to 0} \frac{p_{01}^{(x)}(t)}{t} = 1,\ \lim_{t\to 0} \frac{p_{10}(t)}{t} = M(\mathbb{X}) \text{ and } \lim_{t\to 0} \frac{1-p_{00}(t)}{t} = |\phi| + M(\mathbb{X}),
$$

we get that

$$
\lim_{t\to 0} \frac{p_\infty(t)}{t} = 0,
$$

then by boundedness of F that

$$
\lim_{t\to 0} \frac{R(t)}{t} = 0,
$$

hence the first result. The second result is a consequence of the Mecke formula for a Poisson point process, given by Theorem 2.4.6.

□

Lemma 3.2.8 (Commutation relation).
Let ζ be a Poisson point process with finite intensity measure M and $(P_t)_{t\geq 0}$ its semi-group. Then, for any $t \in \mathbb{R}_+$, any $x \in \mathbb{X}$, any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$
D_x P_t F(\phi) = e^{-t} P_t D_x F(\phi).
$$

Proof. For any $t \in \mathbb{R}_+$, any $x \in \mathbb{X}$, any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$, from the definitions of D_x and P_t,

$$
\begin{aligned}
D_x P_t F(\phi) &= P_t F(\phi + x) - P_t F(\phi) \\
&= \mathbb{E}[F(e^{-t}\circ(\phi + x) + (1-e^{-t})\circ\zeta) - F(e^{-t}\circ\phi + (1-e^{-t})\circ\zeta)].
\end{aligned}
$$

Hence, since thinning is distributive with respect to sum,

$$
D_x P_t F(\phi) = \mathbb{E}[F(e^{-t}\circ\phi + e^{-t}\circ x + (1-e^{-t})\circ\zeta) - F(e^{-t}\circ\phi + (1-e^{-t})\circ\zeta)],
$$

and then, since $\mathbb{P}(e^{-t}\circ x = x) = 1 - \mathbb{P}(e^{-t}\circ x = \emptyset) = e^{-t}$, it follows that

$$
D_x P_t F(\phi) = e^{-t} P_t D_x F(\phi).
$$

The proof is thus complete. □

Lemma 3.2.9 (Ergodicity).
Let ζ be a Poisson point process with finite intensity measure M and $(P_t)_{t\geq 0}$ its semi-group. For any $F \in Lip_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\lim_{t\to+\infty} P_t F(\phi) = \mathbb{E}[F(\zeta)].$$

Proof. For any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$, $t \in \mathbb{R}_+$ and $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\begin{aligned}
&\left|P_t F(\phi) - \mathbb{E}[F(\zeta)]\right| \leq \left|P_t F(\phi) - P_t F(\varnothing)\right| + \left|P_t F(\varnothing) - \mathbb{E}[F(\zeta)]\right| \\
&= \left|\mathbb{E}[F(e^{-t}\circ\phi + (1-e^{-t})\circ\zeta)] - \mathbb{E}[F((1-e^{-t})\circ\zeta)]\right| \\
&\qquad\qquad\qquad\qquad + \left|\mathbb{E}[F((1-e^{-t})\circ\zeta)] - \mathbb{E}[F(\zeta)]\right|.
\end{aligned}$$

On one hand, since $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$,

$$\begin{aligned}
\left|\mathbb{E}[F(e^{-t}\circ\phi + (1-e^{-t})\circ\zeta)] - \mathbb{E}[F((1-e^{-t})\circ\zeta)]\right| &\leq \mathbb{E}[\Delta_{TV}(e^{-t}\circ\phi, \varnothing)] \\
&= \mathbb{E}[|e^{-t}\circ\phi|],
\end{aligned}$$

and, since $|e^{-t}\circ\phi|$ has a binomial distribution with parameters $|\phi|$ and e^{-t},

$$\left|\mathbb{E}[F(e^{-t}\circ\phi + (1-e^{-t})\circ\zeta)] - \mathbb{E}[F((1-e^{-t})\circ\zeta)]\right| \leq e^{-t}|\phi|.$$

On the other hand,

$$\mathbb{E}[F(\zeta)] = \mathbb{E}[F((1-e^{-t})\circ\zeta + e^{-t}\circ\zeta)],$$

then, since $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$,

$$\begin{aligned}
\left|\mathbb{E}[F((1-e^{-t})\circ\zeta)] - \mathbb{E}[F(\zeta)]\right| &\leq \mathbb{E}[\Delta_{TV}(e^{-t}\circ\zeta, \varnothing)] \\
&= \mathbb{E}[|e^{-t}\circ\zeta|],
\end{aligned}$$

and, since $|e^{-t}\circ\zeta|$ has a Poisson distribution with parameter $e^{-t}M(\mathbb{X})$,

$$\left|\mathbb{E}[F(e^{-t}\circ\phi + (1-e^{-t})\circ\zeta)] - \mathbb{E}[F((1-e^{-t})\circ\zeta)]\right| \leq e^{-t}M(\mathbb{X}),$$

which concludes this proof. □

3.3 Representation formula and consequences

Theorem 3.3.1 (**Stein-Dirichlet representation formula**).

Let ζ be a Poisson point process with finite intensity measure M, $(P_t)_{t\geq 0}$ its semi-group and L its infinitesimal generator. For any $F \in Lip_1(\widehat{N_{\mathbb{X}}}, \Delta_{TV})$ and any $\phi \in \widehat{N_{\mathbb{X}}}$,

$$\mathbb{E}[F(\zeta)] - F(\phi) = \int_0^{+\infty} LP_sF(\phi)\mathrm{d}s.$$

Proof. For any $F \in \mathrm{Lip}_1(\widehat{N_{\mathbb{X}}}, \Delta_{TV})$ and any $\phi \in \widehat{N_{\mathbb{X}}}$, from the definition of L,

$$\int_0^{+\infty} LP_sF(\phi)\mathrm{d}s = \int_0^{+\infty} \left(\frac{\mathrm{d}P_t(P_sF)}{\mathrm{d}t}\right)\Big|_{t=0}(\phi)\mathrm{d}s.$$

Hence, since $(P_t)_{t\geq 0}$ is a semi-group,

$$\int_0^{+\infty} LP_sF(\phi)\mathrm{d}s = \int_0^{+\infty} \left(\frac{\mathrm{d}P_{t+s}F}{\mathrm{d}t}\right)\Big|_{t=0}(\phi)\mathrm{d}s$$

and it yields

$$\begin{aligned}\int_0^{+\infty} LP_sF(\phi)\mathrm{d}s &= \int_0^{+\infty} \frac{\mathrm{d}P_sF}{\mathrm{d}s}(\phi)\mathrm{d}s \\ &= \lim_{s\to+\infty} P_sF(\phi) - P_0F(\phi).\end{aligned}$$

Then, by Lemma 3.2.9,

$$\int_0^{+\infty} LP_sF(\phi)\mathrm{d}s = \mathbb{E}[F(\zeta)] - F(\phi).$$

The proof is thus complete. □

In [8], Barbour and Brown apply Stein's method to Poisson point process. They deduce an upper bound for the total variation distance between a finite Poisson point process and an other finite point process using Palm measure. Our approach, which leads to the following fundamental theorem, focuses on Papangelou intensity rather than Palm measure - a functional rather than a probability measure - in order to give an easier way to perform calculations.

Theorem 3.3.2 (**Upper-bound theorem**).

Let ζ be a Poisson point process on $\mathbb{X}$ with finite intensity measure $M(\mathrm{d}x) = m(x)\ell(\mathrm{d}x)$ and Φ a second finite point process on $\mathbb{X}$ with Papangelou intensity c. Then,

$$\Delta_{TV}^*(\Phi, \zeta) \leq \int_{\mathbb{X}} \mathbb{E}[|m(x) - c(x, \Phi)|]\ell(\mathrm{d}x).$$

Proof. For any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$, by Theorem 3.3.1,

$$\mathbb{E}[F(\zeta)] - \mathbb{E}[F(\Phi)] = \mathbb{E}\Big[\int_0^{+\infty} LP_s F(\Phi)\mathrm{d}s\Big].$$

Then, from the expression of the generator L,

$$\mathbb{E}[F(\zeta)] - \mathbb{E}[F(\Phi)] = \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{X}} D_x P_s F(\Phi) M(\mathrm{d}x)\Big] - \mathbb{E}\Big[\sum_{y \in \Phi} P_s F(\Phi) - P_s F(\Phi \setminus y)\Big]\mathrm{d}s$$

and then, by the definition of the Papangelou intensity,

$$\begin{aligned} \mathbb{E}[F(\zeta)] - \mathbb{E}[F(\Phi)] &= \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{X}} D_x P_s F(\Phi) m(x)\mathrm{d}x\Big] - \mathbb{E}\Big[\int_{\mathbb{X}} c(x,\Phi) D_x P_s F(\Phi)\mathrm{d}x\Big]\mathrm{d}s \\ &= \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{X}} D_x P_s F(\Phi)(m(x) - c(x,\Phi))\mathrm{d}x\Big]\mathrm{d}s. \end{aligned}$$

Thus, by Lemma 3.2.8,

$$\mathbb{E}[F(\zeta)] - \mathbb{E}[F(\Phi)] = \int_0^{+\infty} e^{-s}\mathbb{E}\Big[\int_{\mathbb{X}} P_s D_x F(\Phi)(m(x) - c(x,\Phi))\mathrm{d}x\Big]\mathrm{d}s$$

and then, since $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$ and $\|P_s\| \le 1$,

$$\begin{aligned} \big|\mathbb{E}[F(\zeta)] - \mathbb{E}[F(\Phi)]\big| &\le \int_0^{+\infty} e^{-s}\mathbb{E}\Big[\int_{\mathbb{X}} |D_x F(\Phi)||m(x) - c(x,\Phi)|\mathrm{d}x\Big]\mathrm{d}s \\ &\le \int_{\mathbb{X}} \mathbb{E}[|m(x) - c(x,\Phi)|]\mathrm{d}x. \end{aligned}$$

The proof is thus complete. □

3.4 Papangelou intensity and repulsiveness

Following Georgii and Yoo [35], we define repulsiveness according to the Papangelou intensity.

Definition 3.4.1 (Repulsiveness - Weak repulsiveness). *A point process Φ on $\mathbb{X}$ with a version c of its Papangelou intensity is said to be* ***repulsive*** *(according to c) if, for any $\omega, \phi \in N_{\mathbb{X}}$ such that $\omega \subset \phi$ and any $x \in \mathbb{X}$,*

$$c(x,\phi) \le c(x,\omega),$$

*and **weakly repulsive** (according to c) if, for any $\phi \in N_{\mathbb{X}}$ and any $x \in \mathbb{X}$,*

$$c(x,\phi) \leq c(x,\varnothing).$$

Definition 3.4.2 (Increasing and decreasing functions and subsets).
*A function $f : N_{\mathbb{X}} \to \mathbb{R}$ is said to be **increasing** (resp. **decreasing**) if, for any $\phi_1, \phi_2 \in N_{\mathbb{X}}$,*

$$(\phi_1 \subset \phi_2) \implies (f(\phi_1) \leq f(\phi_2)) \text{ (resp. } (\phi_1 \subset \phi_2) \implies (f(\phi_1) \geq f(\phi_2))).$$

*A subset A of $N_{\mathbb{X}}$ is said to be **increasing** (resp. **decreasing**) if $\mathbf{1}_A$ is increasing (resp. decreasing), that is, if for any $\phi_1 \in A$ and $\phi_2 \in N_{\mathbb{X}}$,*

$$(\phi_1 \subset \phi_2) \implies (\phi_2 \in A) \text{ (resp. } (\phi_2 \subset \phi_1) \implies (\phi_2 \in A)).$$

The following lemmas though very elementary are the key to the next results.

Lemma 3.4.3 (A first property of the Papangelou intensity).
If Φ is a finite and weakly repulsive point process on $\mathbb{X}$ with Papangelou intensity c and void probability p_0, then for any $x \in \mathbb{X}$,

$$|c(x,\varnothing) - \rho(x)| \leq (1-p_0)c(x,\varnothing).$$

Proof. On one hand, by Theorem 2.1.17, for any $x \in \mathbb{X}$, $p_0\rho(x) = p_0\mathbb{E}[c(x,\Phi)]$, then, since Φ is repulsive, $p_0\rho(x) \leq p_0 c(x,\varnothing)$. On the other hand, still by Theorem 2.1.17, for any $x \in \mathbb{X}$,

$$\rho(x) = \mathbb{E}[c(x,\Phi)] \geq p_0 c(x,\varnothing)$$

and it follows from both last inequalities that

$$|p_0 c(x,\varnothing) - p_0\rho(x)| \leq (1-p_0)p_0 c(x,\varnothing),$$

hence, the result. □

Lemma 3.4.4 (A second property of the Papangelou intensity).
If Φ is a finite and weakly repulsive point process on $\mathbb{X}$ with Papangelou intensity c, then for any $x \in \mathbb{X}$,

$$\mathbb{E}[|c(x,\Phi) - \rho(x)|] \leq 2(c(x,\varnothing) - \rho(x)).$$

Proof. For any $x \in \mathbb{X}$, by the triangle inequality,

$$\mathbb{E}[|c(x,\Phi)-\rho(x)|] \le \mathbb{E}[|c(x,\Phi)-c(x,\varnothing)|]+\mathbb{E}[|c(x,\varnothing)-\rho(x)|].$$

Since Φ is weakly repulsive and since, in this case, $\rho(x) = \mathbb{E}[c(x,\Phi)] \le c(x,\varnothing)$, we deduce that

$$\mathbb{E}[|c(x,\Phi)-\rho(x)|] \le \mathbb{E}[c(x,\varnothing)-c(x,\Phi)]+c(x,\varnothing)-\rho(x).$$

Hence, still since $\rho(x) = \mathbb{E}[c(x,\Phi)]$,

$$\mathbb{E}[|c(x,\Phi)-\rho(x)|] \le 2(c(x,\varnothing)-\rho(x)).$$

The proof is thus complete. □

Lemma 3.4.5 (**A third property of the Papangelou intensity**).
Let Φ be a finite point process on $\mathbb{X}$ with Papangelou intensity c. Then,

$$\mathbb{P}(|\Phi|=1) = \mathbb{P}(|\Phi|=0)\int_{\mathbb{X}} c(x,\varnothing)\mathrm{d}x.$$

Proof. This equation is deduced by applying the formula which defines the Papangelou intensity (Definition 2.1.13) for $u : \mathbb{X}\times N_{\mathbb{X}} \to \mathbb{R}_+$ given for any $x\in\mathbb{X}$ and $\phi \in N_{\mathbb{X}}$ by

$$u(x,\phi) = \mathbf{1}_{\{\phi=\varnothing\}},$$

which concludes the proof. □

We now show how to compute the Papangelou intensity for different transformations of point processes.

Theorem 3.4.6 (**Papangelou intensity of a reduced point process**).
Let Φ be a point process on $\mathbb{X}$ with Papangelou intensity c, Λ a compact subset of $\mathbb{X}$ and $\Phi_{|\Lambda}$ the reduction of Φ to Λ. Then, its Papangelou intensity c_Λ verifies for any $x\in\mathbb{X}$

$$c_\Lambda(x,\Phi_{|\Lambda}) = c(x,\Phi)\mathbf{1}_{\{x\in\Lambda\}} \text{ a.s.}.$$

Proof. For any measurable function $u : \widehat{N}_{\mathbb{X}} \to \mathbb{R}_+$, by definition of $\Phi_{|\Lambda}$,

$$\mathbb{E}\Big[\sum_{x\in\Phi_{|\Lambda}} u(x,\Phi_{|\Lambda}\setminus x)\Big] = \mathbb{E}\Big[\sum_{x\in\Phi} u(x,(\Phi\setminus x)\cap\Lambda)\mathbf{1}_{x\in\Lambda}\Big].$$

Then, by the definition of the Papangelou intensity,

$$\mathbb{E}\Big[\sum_{x\in\Phi_{|\Lambda}} u(x,\Phi_{|\Lambda}\setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}\Big[c(x,\Phi)u(x,\Phi\cap\Lambda)\mathbf{1}_{x\in\Lambda}\Big]\mathrm{d}x,$$

and the expected result is derived. □

Theorem 3.4.7 (Papangelou intensity of a superposition).

Let $\Phi_1, \dots, \Phi_n$ ($n \in \mathbb{N}$) be independent point processes on $\mathbb{X}$ with respective Papangelou intensities $c_1, \dots, c_n$ and Φ their independent superposition. Then, its Papangelou intensity c verifies for any $x \in \mathbb{X}$

$$c\Big(x, \sum_{i=1}^{n} \Phi_i\Big) = \sum_{i=1}^{n} c_i(x, \Phi_i) \text{ a.s.}.$$

Proof. Denoting $k_{[n]} = k_1 + \cdots + k_n$, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{y \in \Phi_1 + \cdots + \Phi_n} u\Big(y, \sum_{i=1}^{n} \Phi_i \setminus \{y\}\Big)\Big] = \sum_{i=1}^{n} \mathbb{E}\Big[\sum_{y \in \Phi_i} u\Big(y, \sum_{i=1}^{n} \Phi_i \setminus \{y\}\Big)\Big].$$

Then, applying the definition of the Papangelou intensity for each Φ_i,

$$\mathbb{E}\Big[\sum_{y \in \Phi_1 + \cdots + \Phi_n} u\Big(y, \sum_{i=1}^{n} \Phi_i \setminus \{y\}\Big)\Big] = \sum_{i=1}^{n} \mathbb{E}\Big[\int_E u(y, \sum_{i=1}^{n} \Phi_i) c_i(y, \Phi_i) \mathrm{d}y\Big],$$

from which we deduce that

$$\mathbb{E}\Big[\sum_{y \in \Phi_1 + \cdots + \Phi_n} u\Big(y, \sum_{i=1}^{n} \Phi_i \setminus \{y\}\Big)\Big] = \mathbb{E}\Big[\int_E u\Big(y, \sum_{i=1}^{n} \Phi_i\Big) \sum_{i=1}^{n} c_i(y, \Phi_i) \mathrm{d}y\Big],$$

which yields the identity verified by the Papangelou intensities. □

The right hand side in Theorem 3.4.7 is not truly the Papangelou intensity of $\sum_{i=1}^{n} \Phi_i$ since it is not $(\sum_{i=1}^{n} \Phi_i)$-measurable but this ersatz is sufficient for our purposes.

Corollary 3.4.8 (Superposition and repulsiveness).

Let $\Phi_1, \dots, \Phi_n$ ($n \in \mathbb{N}$) be independent and weakly repulsive point processes on $\mathbb{X}$. Then their independent superposition is also weakly repulsive.

Proof. For any $i \in \{1, \dots, n\}$, let c_i be a version of the Papangelou intensity of Φ_i such that Φ_i is weakly repulsive according to c_i. Then, by Theorem 3.4.7, one can find a version c of the Papangelou intensity of the superposition verifying, for any $x \in \mathbb{X}$:

$$c(x, \varnothing) = \sum_{i=1}^{n} c_i(x, \varnothing) \geq \sum_{i=1}^{n} c_i(x, \Phi_i) = c(x, \sum_{i=1}^{n} \Phi_i) \text{ a.s.},$$

from which we conclude the proof. □

Theorem 3.4.9 (Papangelou intensity of a thinned point process).

Let Φ be a point process on $\mathbb{X}$, let β be a function from $\mathbb{X}$ to $[0,1]$ and $\beta \circ \Phi$ the β-thinning of Φ. Then, its Papangelou intensity c_β verifies, for any $x \in \mathbb{X}$,

$$c_\beta(x,\beta\circ\Phi) = \beta(x)\mathbb{E}[c(x,\Phi)\mid\beta\circ\Phi]\ a.s..$$

Proof. For any measurable function $u:\mathbb{X}\times N_{\mathbb{X}}\to\mathbb{R}_+$, one has:

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} u(x,\beta\circ\Phi\setminus x)\Big] &= \mathbb{E}\Big[\sum_{x\in\Phi} u(x,\beta\circ\Phi\setminus x)\mathbf{1}_{x\in\beta\circ\Phi}\Big]\\ &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi} u(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\mathbf{1}_{\tau=\beta\circ\Phi}\Big],\end{aligned}$$

then, conditioning with respect to Φ,

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} u(x,\beta\circ\Phi\setminus x)\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi} u(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\mathbf{1}_{\tau=\beta\circ\Phi}\mid\Phi\Big]\Big]\\ &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi}\mathbb{P}(\tau=\beta\circ\Phi\mid\Phi)u(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\Big].\end{aligned}$$

Since, for any $\tau\subset\phi$, $\mathbb{P}(\tau=\beta\circ\phi)=\Big(\prod_{x\in\tau}\beta(x)\Big)\Big(\prod_{x\in\phi\setminus\tau}\big(1-\beta(x)\big)\Big)$, one gets:

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} u(x,\beta\circ\Phi\setminus x)\Big] &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi}\Big(\prod_{y\in\tau}\beta(y)\Big)\Big(\prod_{y\in\Phi\setminus\tau}\big(1-\beta(y)\big)\Big)u(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\Big]\\ &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi\setminus x}\beta(x)\Big(\prod_{y\in\tau}\beta(y)\Big)\Big(\prod_{y\in(\Phi\setminus x)\setminus\tau}\big(1-\beta(y)\big)\Big)u(x,\tau)\Big].\end{aligned}$$

Then, from the definition of the Papangelou intensity,

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} u(x,\beta\circ\Phi\setminus x)\Big]=\\ &=\int_{\mathbb{X}}\mathbb{E}\Big[c(x,\Phi)\sum_{\tau\subset\Phi}\beta(x)\Big(\prod_{y\in\tau}\beta(y)\Big)\Big(\prod_{y\in\Phi\setminus\tau}\big(1-\beta(y)\big)\Big)u(x,\tau)\Big]\mathrm{d}x.\end{aligned}$$

The previous arguments yield

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} u(x,\beta\circ\Phi\setminus x)\Big] &= \int_{\mathbb{X}}\mathbb{E}\Big[\beta(x)c(x,\Phi)\sum_{\tau\subset\Phi}\mathbb{P}(\beta\circ\Phi=\tau\mid\Phi)u(x,\tau)\Big]\mathrm{d}x\\ &= \int_{\mathbb{X}}\mathbb{E}\Big[\beta(x)c(x,\Phi)\sum_{\tau\subset\Phi}\mathbf{1}_{\beta\circ\Phi=\tau}u(x,\tau)\Big]\mathrm{d}x\\ &= \int_{\mathbb{X}}\mathbb{E}[\beta(x)c(x,\Phi)u(x,\beta\circ\Phi)]\mathrm{d}x,\end{aligned}$$

hence, the result. □

Theorem 3.4.10 (**Papangelou intensity of a rescaled point process**).

Let Φ be a point process on $\mathbb{R}^d$ with Papangelou intensity c, let ϵ be a positive real number and $\Phi^{(\epsilon)}$ the ϵ-rescaling of Φ. Then, its Papangelou intensity $c^{(\epsilon)}$ is given for any $x \in \mathbb{R}^d$ and $\phi \in \widehat{N}_{\mathbb{R}^d}$ by

$$c^{(\epsilon)}(x,\phi) = \frac{1}{\epsilon} c(\epsilon^{-\frac{1}{d}} x, \epsilon^{-\frac{1}{d}} \phi).$$

Proof. By Theorem 2.1.16,

$$c^{(\epsilon)}(x,\phi) = \frac{j^{(\epsilon)}(x\phi)}{j^{(\epsilon)}(\phi)} \mathbf{1}_{j^{(\epsilon)}(\phi)\neq 0},$$

where $j^{(\epsilon)}$ is the Janossy function of $\Phi^{(\epsilon)}$ whose expression is given by Theorem 2.3.12, and then the expected result is deduced, still by Theorem 2.1.16. □

Let us now give an expression of the Papangelou intensity for classical point processes. The following result is a direct consequence of the Mecke formula for a Poisson point process (Theorem 2.4.6) and is also mentioned in [35].

Theorem 3.4.11 (**Papangelou intensity of a Poisson point process**).

Let Φ be a Poisson point process with intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$. Then, its Papangelou intensity c is given for any $x \in \mathbb{X}$ and $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$c(x,\phi) = m(x).$$

Theorem 3.4.12 (**Papangelou intensity of a purely random point process**).

Let Φ be a purely random point process on $\mathbb{X}$ supported by a distribution $(p_n)_{n\in\mathbb{N}_0}$ such that $p_n \neq 0$ for any $n \in \mathbb{N}_0$, and a probability measure $\mu(\mathrm{d}x) = q(x)\ell(\mathrm{d}x)$. Then its Papangelou intensity c is given for any $n \in \mathbb{N}_0$ and any $x, x_1, \ldots, x_n \in \mathbb{X}$ by

$$c(x, \{x_1, \ldots, x_n\}) = (n+1)\frac{p_{n+1}}{p_n} q(x).$$

Moreover, Φ is repulsive if and only if, for any $n \in \mathbb{N}_0$,

$$(n+1)p_{n+1}^2 \geq (n+2)p_n p_{n+2}$$

and weakly repulsive if and only if, for any $n \in \mathbb{N}_0$,

$$p_0(n+1)p_{n+1} \leq p_n p_1.$$

Proof. The Janossy function of a purely random point process is given by Theorem 2.4.10 and then we deduce the expression of the Papangelou intensity from Theorem 2.1.16, which provides the link between Janossy function and Papangelou intensity. In particular, this implies that Φ is repulsive if and only if, for any $n \in \mathbb{N}_0$ and any $x \in \mathbb{X}$,

$$(n+2)\frac{p_{n+2}}{p_{n+1}}q(x) \leq (n+1)\frac{p_{n+1}}{p_n}q(x),$$

which is equivalent to the expected assertion, and that Φ is weakly repulsive if and only if, for any $n \in \mathbb{N}_0$ and any $x \in \mathbb{X}$,

$$(n+1)\frac{p_{n+1}}{p_n}q(x) \leq \frac{p_1}{p_0}q(x),$$

hence, the result. □

Theorem 3.4.13 (Papangelou intensity of a conditional Poisson point process). *Let Φ be a conditional Poisson point process with intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$ and conditional set C. Then its Papangelou intensity c is given for any $n \in \mathbb{N}_0$ and any $x, x_1, \ldots, x_n \in \mathbb{X}$ by*

$$c(x, \{x_1, \ldots, x_n\}) = m(x)\mathbf{1}_{\{x_1,\ldots,x_n,x\}\in C}\mathbf{1}_{\{x_1,\ldots,x_n\}\in C}.$$

Moreover, if C is decreasing, then Φ is repulsive.

Proof. A version of the Papangelou intensity is deduced from the Janossy function by Theorem 2.1.16 and the Janossy function of a conditional Poisson point process is given by Theorem 2.4.19, which provides the expected expression. As a consequence, Φ is repulsive if and only if, for any $x, x_1, \ldots, x_n, x_{n+1} \in \mathbb{X}$,

$$m(x)\mathbf{1}_{\{x_1,\ldots,x_{n+1},x\}\in C}\mathbf{1}_{\{x_1,\ldots,x_{n+1}\}\in C} \leq m(x)\mathbf{1}_{\{x_1,\ldots,x_n,x\}\in C}\mathbf{1}_{\{x_1,\ldots,x_n\}\in C}.$$

Hence, if C is decreasing, then this last hypothesis is verified, and Φ is repulsive. □

Theorem 3.4.14 (Papangelou intensity of a Gibbs point process). *Let Φ be a Gibbs point process with temperature parameter $\theta > 0$ and total potential energy*

$$U(x_1, \ldots, x_n) = \sum_{r=1}^{n} \sum_{1\leq i_1<\cdots<i_r\leq n} \Psi_r(x_{i_1}, \ldots, x_{i_r}),$$

where Ψ_r is the r^{th}-order interaction potential. Then its Papangelou intensity c is given for any $x \in \mathbb{X}$ and $\phi \in \widehat{N}_{\mathbb{X}}$ by

$$c(x, \phi) = e^{-\theta(U(x\phi)-U(\phi))}.$$

Moreover, Φ is repulsive.

Proof. The expression of the Papangelou intensity is deduced from the definition of a Gibbs point process and from Theorem 2.1.16. In order to show that Φ is repulsive, one can observe that, for any $x, x_1, \ldots, x_n, x_{n+1} \in \mathbb{X}$,

$$\big(U(x_1,\dots,x_n,x_{n+1},x)-U(x_1,\dots,x_n,x_{n+1})\big)-\big(U(x_1,\dots,x_n,x)-U(x_1,\dots,x_n)\big)=$$
$$=\Big(\Psi_1(x)+\sum_{r=1}^{n+2}\sum_{1\le i_1<\dots<i_{r-1}\le n+1}\Psi_r(x_{i_1},\dots,x_{i_{r-1}},x)\Big)$$
$$-\Big(\Psi_1(x)+\sum_{r=1}^{n+1}\sum_{1\le i_1<\dots<i_{r-1}\le n}\Psi_r(x_{i_1},\dots,x_{i_{r-1}},x)\Big)$$
$$=\Psi_{n+2}(x_1,\dots,x_{n+1},x)+\sum_{r=2}^{n+1}\sum_{1\le i_1<\dots<i_{r-2}\le n}\Psi_r(x_{i_1},\dots,x_{i_{r-2}},x_{n+1},x)$$
$$\ge 0.$$

The proof is thus complete. □

The following result is given in [35].

Theorem 3.4.15 (Papangelou intensity of an α-DPPP).
Let Φ be an α-DPPP with kernel K and associated kernel J. Then its Papangelou intensity c is given for any $x\in\mathbb{X}$ and $\phi\in\widehat{N}_{\mathbb{X}}$ by

$$c(x,\phi)=\frac{\det_\alpha J(x\phi,x\phi)}{\det_\alpha J(\phi,\phi)}.$$

Moreover, if $\alpha=-1$, then Φ is repulsive.

3.5 Application to Poisson-like point processes

Let us now apply the upper bound given in Theorem 3.3.2 for Poisson-like point processes. In particular, we give the two following results for respectively finite Poisson and Cox point processes, which have already been shown in [24].

Theorem 3.5.1 (Application to a Poisson point process).
Let ζ_1,ζ_2 be two Poisson point processes on $\mathbb{X}$ with respective intensity measures M_1 and M_2. Then,

$$\Delta^*_{TV}(\zeta_1,\zeta_2)\le\Delta_{TV}(M_1,M_2).$$

Proof. For any $i\in\{1,2\}$, the Papangelou intensity of ζ_i with respect to M_1+M_2 is given by $\frac{\mathrm{d}M_i}{\mathrm{d}(M_1+M_2)}$. The result is deduced by combining Theorem 3.3.2 and Remark 2.2.6. □

Theorem 3.5.2 (**Application to a Cox point process**).
Let Γ_1, Γ_2 be two Cox point processes on $\mathbb{X}$ directed by respective almost surely finite random measures M_1 and M_2. Then,

$$\Delta^*_{TV}(\Gamma_1, \Gamma_2) \leq \Delta^*_{TV}(M_1, M_2).$$

Proof. Using the notations of Definition 2.2.8,

$$\begin{aligned}\Delta^*_{TV}(\Gamma_1, \Gamma_2) &= \inf_{\mathbf{C}\in\Sigma(\mathbb{P}_{\Gamma_1},\mathbb{P}_{\Gamma_2})} \int_{N_{\mathbb{X}}\times N_{\mathbb{X}}} \Delta_{TV}(\omega_1, \omega_2)\mathbf{C}(\mathrm{d}(\omega_1, \omega_2)) \\ &\leq \inf_{\mathbf{C}\in\Sigma(\mathbb{P}_{M_1},\mathbb{P}_{M_2})} \int_{\mathbb{M}\times\mathbb{M}} \Delta^*_{TV}(\zeta_{\varphi_1}, \zeta_{\varphi_2})\mathbf{C}(\mathrm{d}(\varphi_1, \varphi_2)).\end{aligned}$$

By Theorem 3.5.1, it follows as expected that

$$\Delta^*_{TV}(\Gamma_1, \Gamma_2) \leq \inf_{\mathbf{C}\in\Sigma(\mathbb{P}_{M_1},\mathbb{P}_{M_2})} \int_{\mathbb{M}\times\mathbb{M}} \Delta_{TV}(\varphi_1, \varphi_2)\mathbf{C}(\mathrm{d}(\varphi_1, \varphi_2)),$$

from which we conclude the proof. □

Remark 3.5.3 (**Application to a Cox point process**).
The topology used in Theorem 3.5.2 may be too strong. In this case, it is relevant to mention that a similar result can be obtained for $\Delta_{TV}(= \Delta^*_D)$ instead of Δ^*_{TV}. Indeed, since trivially, for any Poisson point processes ζ_1 and ζ_2 with respective intensity measures M_1 and M_2,

$$\Delta^*_D(\zeta_1, \zeta_2) \leq \Delta_D(M_1, M_2),$$

it follows, by adapting the proof of Theorem 3.5.2, that

$$\Delta_{TV}(\Gamma_1, \Gamma_2) \leq \Delta^*_D(M_1, M_2).$$

Theorem 3.5.4 (**Application to a purely random point process**).
Let M be a finite measure on $\mathbb{X}$ such that $M(\mathrm{d}x) = m(x)\mathrm{d}x$ and $\mu \in \mathbb{M}_1$ such that $\mu(\mathrm{d}x) = \frac{m(x)}{M(\mathbb{X})}\mathrm{d}x$. Let Φ be a purely random point process on $\mathbb{X}$ supported by μ and the distribution $(p_n)_{n\in\mathbb{N}_0}$ such that $p_n \neq 0$ for any $n \in \mathbb{N}_0$. Then,

$$\Delta^*_{TV}(\Phi, \zeta_M) \leq \sum_{n=0}^{+\infty} \left|(n+1)p_{n+1} - M(\mathbb{X})p_n\right|,$$

where ζ_M is the Poisson point process on $\mathbb{X}$ with intensity measure M.

Proof. The point process Φ has a Papangelou intensity c given for any $x, x_1, \ldots, x_n \in \mathbb{X}$ by:

$$c(x,\{x_1,\ldots,x_n\}) = \frac{n+1}{M(\mathbb{X})}\frac{p_{n+1}}{p_n}m(x).$$

Then, by Theorem 3.3.2,

$$\Delta^*_{TV}(\Phi,\zeta_M) \le \int_{\mathbb{X}} \sum_{n=0}^{+\infty} p_n \left| \frac{n+1}{M(\mathbb{X})}\frac{p_{n+1}}{p_n}m(x) - m(x) \right| \mathrm{d}x,$$

and then

$$\Delta^*_{TV}(\Phi,\zeta_M) \le \sum_{n=0}^{+\infty} \left|(n+1)p_{n+1} - M(\mathbb{X})p_n\right|.$$

The proof is thus complete. □

Theorem 3.5.5 (**Application to a conditional Poisson point process**).
Let Φ be a Poisson point process with finite intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$. Let Φ_C be the conditional Poisson point process associated to Φ with intensity measure M and condition $C \in \mathcal{N}_{\mathbb{X}}$. Then,

$$\Delta^*_{TV}(\Phi_C,\Phi) \le \int_{\mathbb{X}} m(x)\mathbb{P}(\Phi_C x \notin C)\mathrm{d}x.$$

Proof. By Theorem 3.3.2 and from the expression of the Papangelou intensity of Φ_C (Theorem 3.4.13),

$$\begin{aligned}\Delta^*_{TV}(\Phi_C,\Phi) &\le \int_{\mathbb{X}} \mathbb{E}[|m(x) - m(x)\mathbf{1}_C(\Phi_C x)\mathbf{1}_C(\Phi_C)|]\mathrm{d}x \\ &= \int_{\mathbb{X}} m(x)\mathbb{P}(\Phi_C x \notin C)\mathrm{d}x,\end{aligned}$$

since $\Phi_C \in C$ almost surely. □

Corollary 3.5.6 (**Application to a hardcore Poisson point process**).
Let Φ be a Poisson point process with finite intensity λ restricted to a relatively compact subset Λ of $\mathbb{X} = \mathbb{R}^d$. Let Φ_R be the hardcore Poisson point process associated to Φ with parameter measure M and parameter $R > 0$. Then,

$$\Delta^*_{TV}(\Phi_R,\Phi) \le \frac{\lambda^2|\Lambda|}{p_R}V_d(R)$$

where

$$p_R = \mathbb{P}(\forall x,y \in \Phi, x \ne y \implies \Delta_{\mathbb{X}}(x,y) \ge R)$$

and

$$V_d(R) = \frac{\pi^{\frac{d}{2}} R^d}{\Gamma(\frac{d}{2})}.$$

Proof. By Theorem 3.5.5,

$$\Delta^*_{TV}(\Phi_R, \Phi) \leq \lambda \int_{\mathbb{X}} \mathbb{P}(\Phi_R x \notin C_R)\mathrm{d}x,$$

then, by Theorem 2.4.19,

$$\Delta^*_{TV}(\Phi_R, \Phi) \leq e^{-\lambda|\Lambda|} \frac{\lambda}{p_R} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \int_{\Lambda^{k+1}} \mathbf{1}_{C_R^c}(\{x_1, \dots, x_k, x\}) \mathbf{1}_{C_R}(\{x_1, \dots, x_k\}) \mathrm{d}x_1 \dots \mathrm{d}x_k \mathrm{d}x,$$

and then, since $\mathbf{1}_{C_R} \leq 1$ and $\mathbf{1}_{C_R^c} = 1 - \mathbf{1}_{C_R}$,

$$\Delta^*_{TV}(\Phi_R, \Phi) \leq e^{-\lambda|\Lambda|} \frac{\lambda}{p_R} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \int_{\Lambda^{k+1}} (1 - \mathbf{1}_{C_R}(\{x_1, \dots, x_k, x\})) \mathrm{d}x_1 \dots \mathrm{d}x_k \mathrm{d}x.$$

Moreover, since $\mathbf{1}_{C_R}(\{x_1, \dots, x_k, x\}) \geq \prod_{i=1}^{k} \mathbf{1}_{\Delta_{\mathbb{X}}(x_i, x) \geq R}$, one has

$$\begin{aligned} \Delta^*_{TV}(\Phi_R, \Phi) &\leq e^{-\lambda|\Lambda|} \frac{\lambda}{p_R} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \int_{\Lambda^{k+1}} (1 - \prod_{i=1}^{k} \mathbf{1}_{\Delta_{\mathbb{X}}(x_i, x) \geq R}) \mathrm{d}x_1 \dots \mathrm{d}x_k \mathrm{d}x \\ &= e^{-\lambda|\Lambda|} \frac{\lambda}{p_R} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \int_{\Lambda^k} |\Lambda| - \Big(\int_{\Lambda} \prod_{i=1}^{k} \mathbf{1}_{\Delta_{\mathbb{X}}(x_i, x) \geq R} \mathrm{d}x \Big) \mathrm{d}x_1 \dots \mathrm{d}x_k, \end{aligned}$$

and then, since $V_d(R)$ is the volume of a ball of $\mathbb{R}^d$ with radius R,

$$\begin{aligned} \Delta^*_{TV}(\Phi_R, \Phi) &\leq e^{-\lambda|\Lambda|} \frac{\lambda}{p_R} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \int_{\Lambda^k} |\Lambda| - (|\Lambda| - kV_d(R)) \mathrm{d}x_1 \dots \mathrm{d}x_k \\ &= \frac{\lambda^2 |\Lambda|}{p_R} V_d(R). \end{aligned}$$

Hence, the expected result. □

Corollary 3.5.7 (**Application to a bounded Poisson point process**). *Let Φ be a Poisson point process with finite intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$. Let Φ_N be the bounded Poisson point process associated to Φ with parameter measure M and parameter $N \in \mathbb{N}_0$. Then,*

$$\Delta^*_{TV}(\Phi_N,\Phi) \le \frac{e^{-M(\mathbb{X})}}{p_N}\frac{(M(\mathbb{X}))^{N+1}}{N!}$$

where $p_N = \mathbb{P}(\Phi(\mathbb{X}) \le N)$.

Proof. By Theorem 3.5.5,

$$\Delta^*_{TV}(\Phi_N,\Phi) \le \int_{\mathbb{X}} \mathbb{P}(\Phi_N x \notin C_N) m(x)\mathrm{d}x,$$

then, since by Theorem 2.4.19, for any $x \in \mathbb{X}$,

$$\mathbb{P}(\Phi_N x \notin C_N) = \frac{e^{-M(\mathbb{X})}}{p_N}\sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{X}^{k+1}} \mathbf{1}_{C_N^c}(\{x_1,\dots,x_k,x\})\mathbf{1}_{C_N}(\{x_1,\dots,x_k\})m(x_1)\dots m(x_k)\mathrm{d}x_1\dots\mathrm{d}x_k,$$

it yields

$$\begin{aligned}\Delta^*_{TV}(\Phi_N,\Phi) &\le \frac{e^{-M(\mathbb{X})}}{p_N}\frac{1}{N!}\int_{\mathbb{X}^{N+1}} m(x_1)\dots m(x_N)m(x)\mathrm{d}x_1\dots\mathrm{d}x_N\mathrm{d}x \\ &= \frac{e^{-M(\mathbb{X})}}{p_N}\frac{1}{N!}(M(\mathbb{X}))^{N+1}.\end{aligned}$$

The proof is thus complete. □

3.6 Application to weakly repulsive point processes

In this section, we apply Theorem 3.3.2 to weakly repulsive point processes.

Theorem 3.6.1 (Application to a superposition).
For any $n \in \mathbb{N}$, let Φ_n the superposition of n independent, finite and weakly repulsive point processes $\Phi_{n,1},\dots,\Phi_{n,n}$, with respective correlation functions $\rho_{n,1},\dots,\rho_{n,n}$ and let ζ_M be a Poisson point process with intensity measure $M(\mathrm{d}x) = m(x)\ell(\mathrm{d}x)$. Then,

$$\Delta^*_{TV}(\Phi_n,\zeta_M) \le R_n + 2n\Big(\max_{i\in\{1,\dots,n\}}\int_{\mathbb{X}}\rho_{n,i}(x)\ell(\mathrm{d}x)\Big)^2,$$

where

$$R_n := \int_{\mathbb{X}}\Big|\sum_{i=1}^{n}\rho_{n,i}(x) - m(x)\Big|\ell(\mathrm{d}x).$$

Proof. For any $k \in \mathbb{N}_0$, we use the notation $p_{n,i,k} := \mathbb{P}(|\Phi_{n,i}| = k)$. By Theorem 3.3.2,

$$\Delta_{TV}^*(\Phi_n, \zeta_M) \le \int_{\mathbb{X}} \mathbb{E}[|c_n(x,\Phi_n) - m(x)|]dx.$$

Then, by Theorem 3.4.7, $\Delta_{TV}^*(\Phi_n, \zeta_M) \le R_n + \sum_{i=1}^{n} A_{n,i}$, where

$$\begin{aligned}
A_{n,i} &= \int_{\mathbb{X}} \mathbb{E}[|c_{n,i}(x,\Phi_{n,i}) - \rho_{n,i}(x)|]\ell(dx) \\
&= \sum_{k\ge 0} \int_{\mathbb{X}} \mathbb{E}[|c_{n,i}(x,\Phi_{n,i}) - \rho_{n,i}(x)|\mathbf{1}_{\{|\Phi_{n,i}|=k\}}]\ell(dx) \\
&= B_{n,i} + C_{n,i}
\end{aligned}$$

with

$$B_{n,i} = p_{n,i,0} \int_{\mathbb{X}} |c_{n,i}(x,\varnothing) - \rho_{n,i}(x)|\ell(dx),$$

$$C_{n,i} = \sum_{k\ge 1} \int_{\mathbb{X}} \mathbb{E}[|c_{n,i}(x,\Phi_{n,i}) - \rho_{n,i}(x)|\mathbf{1}_{\{|\Phi_{n,i}|=k\}}]\ell(dx).$$

By Lemma 3.4.5,

$$p_{n,i,0} \int_{\mathbb{X}} c_{n,i}(x,\varnothing)\ell(dx) = p_{n,i,1} \le (1 - p_{n,i,0})$$

and by Lemma 3.4.3 we get

$$B_{n,i} \le (1 - p_{n,i,0})^2.$$

Since $c_{n,i}(x,\Phi_{n,i}) \le c_{n,i}(x,\varnothing)$ and $\rho_{n,i}(x) \le c_{n,i}(x,\varnothing)$, we also have

$$C_{n,i} \le \sum_{k\ge 1} p_{n,i,k} \int_{\mathbb{X}} c_{n,i}(x,\varnothing)\ell(dx) = (1 - p_{n,i,0}) \int_{\mathbb{X}} c_{n,i}(x,\varnothing)\ell(dx) \le (1 - p_{n,i,0})^2,$$

and then we get

$$A_{n,i} \le 2(1 - p_{n,i,0})^2 \le 2\Big(\int_{\mathbb{X}} \rho_{n,i}(x)\ell(dx)\Big)^2$$

where the second inequation is derived from the Markov inequality. Hence,

$$\Delta_{TV}^*(\Phi_n, \zeta_M) \le R_n + 2n\Big(\max_{i\in\{1,\dots,n\}} \int_{\mathbb{X}} \rho_{n,i}(x)\ell(dx)\Big)^2,$$

from which we conclude the proof. □

Remark 3.6.2 (**Application to a superposition**).

Under the assumptions and notations of Theorem 3.6.1, and supposing moreover that there exists a real constant C such that for any $n \in \mathbb{N}$,

$$\max_{i\in\{1,\ldots,n\}} \int_{\mathbb{X}} \rho_{n,i}(x)\ell(\mathrm{d}x) \leq \frac{C}{n},$$

one has for any $n \in \mathbb{N}$,

$$\Delta^*_{TV}(\Phi_n, \zeta_M) \leq R_n + \frac{2C^2}{n}.$$

Corollary 3.6.3 (**Application to a** $(-1/n)$**-DPP**).

Let $n \in \mathbb{N}$, Φ_n be a finite $(-1/n)$-determinantal point process with kernel K and ζ be a Poisson point process with intensity measure $K(x,x)\mathrm{d}x$. Then,

$$\Delta^*_{TV}(\Phi_n, \zeta) \leq \frac{2}{n}\Big(\int_{\mathbb{X}} K(x,x)\mathrm{d}x\Big)^2.$$

Proof. By Theorem 2.5.10, Φ_n is the independent superposition of n determinantal point processes with kernel $\frac{1}{n}K$. By Remark 3.6.2, for any $n \in \mathbb{N}$,

$$\Delta^*_{TV}(\Phi_n, \zeta_M) \leq R_n + \frac{2C^2}{n},$$

where

$$R_n = \int_{\mathbb{X}} \Big| \sum_{i=1}^{n} \frac{1}{n} K(x,x) - K(x,x) \Big| \ell(\mathrm{d}x) = 0$$

and

$$C = \int_{\mathbb{X}} K(x,x)\mathrm{d}x,$$

from which we can conclude.

□

Corollary 3.6.4 (**Application to i.i.d. random variables**).

Let h be a probability density function on $[0,1]$ such that $h(0_+) := \lim_{x\to 0_+} h(x) \in \mathbb{R}$, and Λ be a compact subset of $\mathbb{R}_+$. For any $n \in \mathbb{N}$, assuming that $X_{n,1},\ldots,X_{n,n}$ are n independent and identically distributed (i.i.d.) random variables with probability density function $h_n = \frac{1}{n}h(\frac{1}{n}\cdot)$, the point process Φ_n defined as $\Phi_n = \{X_{n,1},\ldots,X_{n,n}\} \cap \Lambda$ verifies the following inequality:

$$\Delta^*_{TV}(\Phi_n,\zeta) \leq \int_\Lambda \left| h\Big(\frac{1}{n}x\Big) - h(0_+)\right| \mathrm{d}x + \frac{2}{n}\Big(\int_\Lambda h\Big(\frac{1}{n}x\Big)\mathrm{d}x\Big)^2$$

where ζ is the homogeneous Poisson point process with intensity $h(0_+)$ reduced to Λ.

Proof. The result is obtained by applying Theorem 3.6.1 to $(\Phi_{n,i})_{1\leq i\leq n}$ such that for each $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$, $\Phi_{n,i} = \{X_{n,i}\} \cap \Lambda$. $\square$

Theorem 3.6.5 (**Application to a thinned superposition**).
Let Φ be a point process on a compact subset Λ of $\mathbb{X}$ with Papangelou intensity c and intensity measure $M(\mathrm{d}x) = m(x)\mathrm{d}x$. Let ζ be a Poisson point process with intensity measure M. For any $n \in \mathbb{N}$, the point process Φ_n is defined by:

$$\Phi_n = \sum_{k=1}^{n} \frac{1}{n} \circ \Phi^{(k)},$$

where $\Phi^{(1)},\ldots,\Phi^{(n)}$ are n independent copies of Φ. If there exists an integrable function $K : \Lambda \to \mathbb{R}_+$ such that, for any $x \in \Lambda$, $\mathbb{V}[c(x,\Phi)] \leq K(x)$, then

$$\Delta^*_{TV}(\Phi_n,\zeta) \leq \frac{1}{\sqrt{n}} \int_\Lambda \sqrt{K(x)}\mathrm{d}x.$$

Proof. By Theorem 3.3.2, one has for any $n \in \mathbb{N}$,

$$\Delta^*_{TV}(\Phi_n,\zeta) \leq \int_\Lambda \mathbb{E}\Big[\Big|c_n(x,\Phi_n) - m(x)\Big|\Big]\mathrm{d}x,$$

where c_n is the Papangelou intensity of Φ_n. Combining Theorem 3.4.7 for the Papangelou intensity of an independent superposition and Theorem 3.4.9 for the Papangelou intensity of a thinning, it follows that

$$\Delta^*_{TV}(\Phi_n,\zeta) \leq \int_\Lambda \mathbb{E}\Big[\Big|\sum_{k=1}^{n} \frac{1}{n}\mathbb{E}\big[c(x,\Phi^{(k)}) \mid \frac{1}{n}\circ\Phi^{(k)}\big] - m(x)\Big|\Big]\mathrm{d}x.$$

Hence, by Jensen's inequality,

$$\Delta^*_{TV}(\Phi_n,\zeta) \leq \int_\Lambda \sqrt{\mathbb{V}\Big[\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\big[c(x,\Phi^{(k)}) \mid \frac{1}{n}\circ\Phi^{(k)}\big]\Big]}\mathrm{d}x,$$

and, by some variance properties,

$$\begin{aligned}
\Delta^*_{TV}(\Phi_n,\zeta) &\leq \frac{1}{\sqrt{n}} \int_\Lambda \sqrt{\mathbb{V}[\mathbb{E}[c(x,\Phi) \mid \frac{1}{n}\circ\Phi]]}\mathrm{d}x \\
&\leq \frac{1}{\sqrt{n}} \int_\Lambda \sqrt{\mathbb{V}[c(x,\Phi)]}\mathrm{d}x.
\end{aligned}$$

By hypothesis, for any $x \in \Lambda$, $\mathbb{V}[c(x,\Phi)] \leq K(x)$ and one deduces the expected result. $\square$

Theorem 3.6.6 (**Application to a thinned and rescaled DPP**).

Let K be the kernel of a stationary determinantal point process Φ on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}$, Λ be a compact subset of $\mathbb{R}^d$, $\beta \in (0,1)$ and $\zeta_{\Lambda,\lambda}$ designs the homogeneous Poisson point process with intensity λ reduced to Λ. Let $\Phi_{\Lambda,\beta}$ be the point process on $\mathbb{R}^d$ obtained by combining a β-thinning with a β-rescaling on the point process Φ that one reduces to Λ. Then,

$$\Delta^*_{TV}(\Phi_{\Lambda,\beta},\zeta_{\Lambda,\lambda}) \leq \frac{2\beta}{1-\beta}\lambda|\Lambda|.$$

Proof. The family of determinantal point processes is stable with respect to several transformations: the reduction to a compact set, the thinning and the rescaling. Their corresponding kernels are respectively provided by Theorems 2.5.9, 2.5.11 and 2.5.12. Combining these expressions, it follows that $\Phi_{\Lambda,\beta}$ is the determinantal point process with kernel $K_{\Lambda,\beta}$ defined by

$$K_{\Lambda,\beta} : (x,y) \in \mathbb{X}\times\mathbb{X} \mapsto K\Big(\frac{x}{\beta^{\frac{1}{d}}},\frac{y}{\beta^{\frac{1}{d}}}\Big)\mathbf{1}_{\Lambda\times\Lambda}(x,y).$$

By Theorem 2.5.7, there exists a complete orthonormal basis $(h_j,\ j\in\mathbb{N})$ of $L^2(\mathbb{X},\ell;\mathbb{C})$ and a sequence $(\lambda_j,\ j\in\mathbb{N}) \subset [0,1]^{\mathbb{N}}$ such that for any $x,y\in\mathbb{X}$,

$$K(x,y) = \sum_{j=1}^{+\infty}\lambda_j h_j(x)h_j(y).$$

Then, for any $x,y\in\mathbb{X}$,

$$\begin{aligned} K_{\Lambda,\beta}(x,y) &= K\Big(\frac{x}{\beta^{\frac{1}{d}}},\frac{y}{\beta^{\frac{1}{d}}}\Big)\mathbf{1}_{\Lambda\times\Lambda}(x,y) \\ &= \sum_{j=1}^{+\infty}\lambda_j h_j\Big(\frac{x}{\beta^{\frac{1}{d}}}\Big)\mathbf{1}_\Lambda(x)h_j\Big(\frac{y}{\beta^{\frac{1}{d}}}\Big)\mathbf{1}_\Lambda(y) \\ &= \sum_{j=1}^{+\infty}\lambda_{\Lambda,\beta,j}h_{\Lambda,\beta,j}(x)h_{\Lambda,\beta,j}(y), \end{aligned}$$

where, for any $j\in\mathbb{N}$ and any $x\in\mathbb{X}$,

$$Z^2_{\Lambda,\beta,j} = \int_{\beta^{-\frac{1}{d}}\Lambda}|h_j(y)|^2\mathrm{d}y,$$

$$h_{\Lambda,\beta,j}(x) = \frac{1}{\sqrt{\beta}}Z^{-1}_{\Lambda,\beta,j}h_j\Big(\frac{x}{\beta^{\frac{1}{d}}}\Big)\mathbf{1}_\Lambda(x),$$

$$\lambda_{\Lambda,\beta,j} = \lambda_j \beta Z^2_{\Lambda,\beta,j}.$$

By Theorem 2.5.7, since, for any $j \in \mathbb{N}$, $\lambda_{\Lambda,\beta,j} < 1$, one can associate to $K_{\Lambda,\beta}$ the kernel $J_{\Lambda,\beta}$ such that for any $x, y \in \mathbb{R}^d$,

$$J_{\Lambda,\beta}(x,y) = \sum_{j=1}^{+\infty} \frac{\lambda_{\Lambda,\beta,j}}{1-\lambda_{\Lambda,\beta,j}} h_{\Lambda,\beta,j}(x) h_{\Lambda,\beta,j}(y),$$

and, by Theorem 3.4.15, for any $x \in \mathbb{R}^d$,

$$J_{\Lambda,\beta}(x,x) = c_{\Lambda,\beta}(x,\varnothing).$$

In particular, still by Theorem 3.4.15, $\Phi_{\Lambda,\beta}$ is a weakly repulsive point process, then, by Lemma 3.4.4, for any $x \in \Lambda$ and $\phi \in N_\Lambda$,

$$\mathbb{E}[|c_{\Lambda,\beta}(x,\phi) - \lambda|] \leq 2(c_{\Lambda,\beta}(x,\varnothing) - \lambda).$$

Then, by Theorem 3.3.2,

$$\Delta^*_{TV}(\Phi_{\Lambda,\beta}, \zeta_{\Lambda,\lambda}) \leq 2 \int_\Lambda \left(c_{\Lambda,\beta}(x,\varnothing) - \lambda\right) \mathrm{d}x.$$

By previous identities, one has

$$\int_\Lambda c_{\Lambda,\beta}(x,\varnothing)\mathrm{d}x = \sum_{j=1}^{+\infty} \frac{\lambda_{\Lambda,\beta,j}}{1-\lambda_{\Lambda,\beta,j}}.$$

Then, noting that

$$\int_\Lambda \lambda \mathrm{d}x = \int_\Lambda K\Big(\frac{x}{\beta^{\frac{1}{d}}}, \frac{x}{\beta^{\frac{1}{d}}}\Big)\mathrm{d}x = \int_\Lambda K_{\Lambda,\beta}(x,x)\mathrm{d}x = \int_\Lambda \sum_{j=1}^{+\infty} \lambda_{\Lambda,\beta,j} h^2_{\Lambda,\beta,j}(x)\mathrm{d}x = \sum_{j=1}^{+\infty} \lambda_{\Lambda,\beta,j},$$

one obtains

$$\Delta^*_{TV}(\Phi_{\Lambda,\beta}, \zeta_{\Lambda,\lambda}) \leq 2 \sum_{j=1}^{+\infty} \frac{\lambda_{\Lambda,\beta,j}}{1-\lambda_{\Lambda,\beta,j}} - \lambda_{\Lambda,\beta,j} = 2 \sum_{j=1}^{+\infty} \frac{\lambda^2_{\Lambda,\beta,j}}{1-\lambda_{\Lambda,\beta,j}},$$

and, using for any $j \in \mathbb{N}$ the expression of $\lambda_{\Lambda,\beta,j}$,

$$\Delta^*_{TV}(\Phi_{\Lambda,\beta}, \zeta_{\Lambda,\lambda}) \leq 2 \sum_{j=1}^{+\infty} \frac{\lambda_j^2 \beta^2 Z^4_{\Lambda,\beta,j}}{1-\lambda_j \beta Z^2_{\Lambda,\beta,j}}.$$

Since $\lambda_j \leq 1$ and $Z^2_{\Lambda,\beta,j} \leq 1$, it follows that

$$\Delta^*_{TV}(\Phi_{\Lambda,\beta}, \zeta_{\Lambda,\lambda}) \leq 2 \frac{\beta^2}{1-\beta} \sum_{j=1}^{+\infty} \lambda_j Z^2_{\Lambda,\beta,j},$$

and the computation of the right hand side provides:

$$\begin{aligned} 2\frac{\beta^2}{1-\beta}\sum_{j=1}^{+\infty}\lambda_j Z_{\Lambda,\beta,j}^2 &= 2\frac{\beta^2}{1-\beta}\int_{\beta^{-\frac{1}{d}}\Lambda}\sum_{j=1}^{+\infty}\lambda_j|h_j(x)|^2\mathrm{d}x \\ &= 2\frac{\beta^2}{1-\beta}\int_{\beta^{-\frac{1}{d}}\Lambda}\lambda\mathrm{d}x \\ &= 2\frac{\beta}{1-\beta}\lambda|\Lambda|, \end{aligned}$$

which concludes the proof. □

The application to Gibbs point processes given in the following only focuses on pairwise Gibbs point processes, that is such that, for any $r \in \mathbb{N}\setminus\{1,2\}$, $\Psi_r \equiv 0$.

Theorem 3.6.7 (Application to a Gibbs point process).
Let $\epsilon \in \mathbb{R}_+$ and Φ be a Gibbs point process on $\mathbb{X}$ with temperature parameter $\theta > 0$, partition function $C(\theta)$ and total potential energy

$$U(x_1,\ldots,x_k) = \sum_{i=1}^{k}\Psi_1(x_i) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\Psi_2(x_i,x_j),$$

such that $\int_{\mathbb{X}} e^{-\theta\Psi_1(x)}\mathrm{d}x < +\infty$, $\Psi_2 \geq 0$ *and* $\|\Psi_2\|_\infty \leq \epsilon$.

Then,

$$\Delta_{TV}^*(\Phi,\zeta_M) \leq (M(\mathbb{X}))^2\theta\epsilon,$$

where ζ_M is the Poisson point process on $\mathbb{X}$ with intensity measure $M(\mathrm{d}x) = e^{-\theta\Psi_1(x)}\mathrm{d}x$.

Proof. By Theorem 3.4.14, the point process Φ has a Papangelou intensity c given for any $x, x_1, \ldots, x_k \in \mathbb{X}$ by:

$$c(x,\{x_1,\ldots,x_k\}) = e^{-\theta(\Psi_1(x)+\sum_{i=1}^k\Psi_2(x,x_i))}.$$

Then,

$$|c(x,\{x_1,\ldots,x_k\}) - e^{-\theta\Psi_1(x)}| = e^{-\theta\Psi_1(x)}|e^{-\theta\sum_{i=1}^k\Psi_2(x,x_i)} - 1| \leq e^{-\theta\Psi_1(x)}(1-e^{-\theta k\epsilon}),$$

and, since for any $x \geq 0$, $1-e^{-x} \leq x$, one gets

$$1-\mathbb{E}[e^{-\theta|\Phi|\epsilon}] \leq \mathbb{E}[\theta|\Phi|\epsilon] = \theta\epsilon\mathbb{E}[|\Phi|].$$

Moreover, by Theorem 2.1.17,

$$\mathbb{E}[|\Phi|] = \int_{\mathbb{X}} \mathbb{E}[c(x,\Phi)]\mathrm{d}x = \int_{\mathbb{X}} \mathbb{E}[e^{-\theta(\Psi_1(x)+\sum_{y\in\Phi}\Psi_2(x,y))}]\mathrm{d}x,$$

and, since $\Psi_2 \geq 0$, it follows that

$$\mathbb{E}[|\Phi|] \leq \int_{\mathbb{X}} e^{-\theta\Psi_1(x)}\mathrm{d}x = M(\mathbb{X}).$$

As a consequence, by Theorem 3.3.2,

$$\begin{aligned}
\Delta^*_{TV}(\Phi,\zeta_M) &\leq \int_{\mathbb{X}} \mathbb{E}[|c(x,\Phi) - e^{-\theta\Psi_1(x)}|]\mathrm{d}x \\
&= (M(\mathbb{X}))^2\theta\epsilon.
\end{aligned}$$

This proof is thus complete. □

3.7 Extension of a Kallenberg's theorem

The aim of this section is to provide a convergence speed for the following theorem, from Kallenberg (Theorem 14.19 in [43]).

Theorem 3.7.1 (Kallenberg's theorem).
Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of point processes on $\mathbb{X}$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{X}$ to $[0,1)$ such that $(p_n)_{n\in\mathbb{N}}$ tends to 0 uniformly. Let M be a random measure on $\mathbb{X}$ and Γ_M be a Cox point process directed by M. Then,

$$p_n\Phi_n \xrightarrow[n\to+\infty]{law} M \iff p_n \circ \Phi_n \xrightarrow[n\to+\infty]{law} \Gamma_M.$$

Lemma 3.7.2 (Polish distance between Cox point processes).
Let M_1, M_2 be random measures on $\mathbb{X}$ and $\Gamma_{M_1}, \Gamma_{M_2}$ be Cox point processes directed by M_1, M_2 respectively. Then,

$$\Delta_P(\Gamma_{M_1}, \Gamma_{M_2}) = \overline{\Delta}_P(M_1, M_2),$$

with $\overline{\Delta}_P$ denoting the Polish distance on $\mathbb{M}'_1$ associated to $g = (g_k)_{k\in\mathbb{N}}$ defined, for any $k \in \mathbb{N}$ and any $\varphi \in \mathbb{M}$, by $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, where ζ_φ is a Poisson point process on $\mathbb{X}$ with intensity measure φ.

Proof. This equation is directly deduced from the definition of the Polish distance Δ_P. □

Lemma 3.7.3 (Papangelou intensity of a thinned configuration).
Let be $\varphi \in N_{\mathbb{X}}$ and a measurable function $p : \mathbb{X} \to [0,1)$. Then, a version c of the Papangelou intensity of $p \circ \varphi$ with respect to the measure $p(x)\varphi(\mathrm{d}x)$ is provided for any $x \in \mathbb{X}$ and any $\eta \in N_{\mathbb{X}}$ by

$$c(x,\eta) = \mathbf{1}_{\{x \in \varphi \setminus \eta\}} \frac{1}{1-p(x)}.$$

Proof. The aim is to get a measurable function $c : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$ verifying, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{x \in p\circ\varphi} u(x,(p\circ\varphi)\setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[c(x,p\circ\varphi)u(x,p\circ\varphi)]p(x)\varphi(\mathrm{d}x).$$

On one hand, let us compute the left hand side:

$$\begin{aligned}\mathbb{E}\Big[\sum_{x \in p\circ\varphi} u(x,(p\circ\varphi)\setminus x)\Big] &= \mathbb{E}\Big[\sum_{\eta\subset\varphi} \mathbf{1}_{\{\eta = p\circ\varphi\}} \sum_{x\in\eta} u(x,\eta\setminus x)\Big] \\ &= \sum_{\eta\subset\varphi} \mathbb{P}(\eta = p\circ\varphi) \sum_{x\in\eta} u(x,\eta\setminus x).\end{aligned}$$

Then, since for any $\eta \subset \varphi$,

$$\mathbb{P}(\eta = p\circ\varphi) = \Big(\prod_{t\in\eta} p(t)\Big)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big),$$

it follows that

$$\mathbb{E}\Big[\sum_{x \in p\circ\varphi} u(x,(p\circ\varphi)\setminus x)\Big] = \sum_{\eta\subset\varphi}\Big(\prod_{t\in\eta} p(t)\Big)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big)\sum_{x\in\eta} u(x,\eta\setminus x).$$

Hence,

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x \in p\circ\varphi} u(x,(p\circ\varphi)\setminus x)\Big] = \\ &\qquad = \sum_{x\in\varphi}\sum_{\substack{\eta\subset\varphi \\ x\in\eta}}\Big(\prod_{t\in\eta} p(t)\Big)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big)u(x,\eta\setminus x) \\ &\qquad = \sum_{x\in\varphi}\sum_{\eta\subset\varphi\setminus\{x\}}\Big(\prod_{t\in\eta} p(t)\Big)p(x)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big)\frac{1}{1-p(x)}u(x,\eta),\end{aligned}$$

and finally

$$\mathbb{E}\Big[\sum_{x\in p\circ\varphi} u(x,(p\circ\varphi)\setminus x)\Big] = \sum_{x\in\varphi}\sum_{\eta\subset\varphi} \mathbf{1}_{\{x\notin\eta\}}\Big(\prod_{t\in\eta} p(t)\Big)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big)\frac{p(x)}{1-p(x)}u(x,\eta).$$

On the other hand, for a given measurable function $c : \mathbb{X}\times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\begin{aligned}\int_{\mathbb{X}} \mathbb{E}[c(x,p\circ\varphi)u(x,p\circ\varphi)]p(x)\varphi(\mathrm{d}x) = \\ &= \int_{\mathbb{X}} \mathbb{E}\Big[\sum_{\eta\subset\varphi}\mathbf{1}_{\{\eta=p\circ\varphi\}}c(x,\eta)u(x,\eta)\Big]p(x)\varphi(\mathrm{d}x) \\ &= \int_{\mathbb{X}}\sum_{\eta\subset\varphi}\mathbb{P}(\eta=p\circ\varphi)c(x,\eta)u(x,\eta)p(x)\varphi(\mathrm{d}x) \\ &= \sum_{x\in\varphi}p(x)\sum_{\eta\subset\varphi}\Big(\prod_{t\in\eta}p(t)\Big)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big)c(x,\eta)u(x,\eta),\end{aligned}$$

where the last equality is obtained by the expression of $\mathbb{P}(\eta = p\circ\varphi)$ as above. One deduces that

$$\begin{aligned}\int_{\mathbb{X}} \mathbb{E}[c(x,p\circ\varphi)u(x,p\circ\varphi)]p(x)\varphi(\mathrm{d}x) = \\ &= \sum_{x\in\varphi}\sum_{\eta\subset\varphi}\Big(\prod_{t\in\eta}p(t)\Big)\Big(\prod_{s\in\varphi\setminus\eta}(1-p(s))\Big)p(x)c(x,\eta)u(x,\eta),\end{aligned}$$

and the result is got by identification. □

Lemma 3.7.4 (Distance between thinned and Cox point processes). *Let Φ be a point process on $\mathbb{X}$ and p be a function from $\mathbb{X}$ to [0,1). Let $\Gamma_{p\Phi}$ be a Cox point process directed by $p\Phi$. Then,*

$$\Delta^*_{TV}(\mathbb{P}_{p\circ\Phi},\mathbb{P}_{\Gamma_{p\Phi}}) \le 2\mathbb{E}\Big[\sum_{x\in\Phi}p^2(x)\Big].$$

Proof. For any $\varphi \in N_{\mathbb{X}}$, let $\zeta_{p\varphi}$ be a Poisson point process with intensity measure $p\varphi$. By Theorem 3.3.2,

$$\Delta^*_{TV}(\mathbb{P}_{p\circ\varphi},\mathbb{P}_{\zeta_{p\varphi}}) \le \int_{\mathbb{X}}\mathbb{E}[|c(x,p\circ\varphi)-1|]p(x)\varphi(\mathrm{d}x),$$

where c is a version of the Papangelou intensity of $p\circ\varphi$ with respect to $p\varphi$. An expression of c is given by Lemma 3.7.3, and it follows that

$$\Delta^*_{TV}(\mathbb{P}_{p\circ\varphi},\mathbb{P}_{\zeta_{p\varphi}}) \le \int_{\mathbb{X}}\mathbb{E}\Big[\Big|\mathbf{1}_{\{x\in\varphi\setminus p\circ\varphi\}}\frac{1}{1-p(x)}-1\Big|\Big]p(x)\varphi(\mathrm{d}x).$$

Hence, the computation of the right hand side in the last inequality aims to obtain that

$$\begin{aligned}
\Delta^*_{TV}(\mathbb{P}_{p\circ\varphi}, \mathbb{P}_{\zeta_{p\varphi}}) &\leq \int_{\mathbb{X}} \Big(\mathbb{E}\Big[\mathbf{1}_{\{x\in p\circ\varphi\}} \Big| \mathbf{1}_{\{x\in\varphi\setminus p\circ\varphi\}} \frac{1}{1-p(x)} - 1 \Big| \Big] \\
&\quad + \mathbb{E}\Big[\mathbf{1}_{\{x\notin p\circ\varphi\}} \Big| \mathbf{1}_{\{x\in\varphi\setminus p\circ\varphi\}} \frac{1}{1-p(x)} - 1 \Big| \Big] \Big) p(x)\varphi(\mathrm{d}x) \\
&= \int_{\mathbb{X}} \Big(p(x) + (1-p(x))\big(\frac{1}{1-p(x)} - 1\big) \Big) p(x)\varphi(\mathrm{d}x) \\
&= 2\sum_{x\in\varphi} p^2(x),
\end{aligned}$$

and we can deduce that

$$\Delta^*_{TV}(\mathbb{P}_{p\circ\Phi}, \mathbb{P}_{\Gamma_{p\Phi}}) \leq 2\mathbb{E}\Big[\sum_{x\in\Phi} p^2(x) \Big],$$

from which we can conclude. □

Theorem 3.7.5 (Extension of Theorem 3.7.1).
Let Φ be a point process on $\mathbb{X}$ and let p be a measurable function from $\mathbb{X}$ to $[0,1)$. Let M be a random measure on $\mathbb{X}$ and Γ_M be a Cox point process directed by M. Then,

$$\Delta_P(p\circ\Phi, \Gamma_M) \leq 2\mathbb{E}\Big[\sum_{x\in\Phi} p^2(x) \Big] + \overline{\Delta}_P(p\Phi, M),$$

with $\overline{\Delta}_P$ denoting the Polish distance on $\mathbb{M}'_1$ associated to $g = (g_k)_{k\in\mathbb{N}}$ defined, for any $n\in\mathbb{N}$ and any $\varphi\in\mathbb{M}$, by $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, where ζ_φ is a Poisson point process on $\mathbb{X}$ with intensity measure φ.

Proof. By the triangle inequality,

$$\Delta_P(\Gamma_M, p\circ\Phi) \leq \Delta_P(\Gamma_M, \Gamma_{p\Phi}) + \Delta_P(\Gamma_{p\Phi}, p\circ\Phi).$$

One one hand, by Lemma 3.7.2,

$$\Delta_P(\Gamma_M, \Gamma_{p\Phi}) = \overline{\Delta}_P(M, p\Phi).$$

On the other hand, since $(f_k)_{k\in\mathbb{N}} \subset \mathrm{Lip}_1(\Delta_{TV})$,

$$\Delta_P(\Gamma_{p\Phi}, p\circ\Phi) \leq \Delta^*_{TV}(\Gamma_{p\Phi}, p\circ\Phi),$$

and then, by Lemma 3.7.4,

$$\Delta_P(\Gamma_{p\Phi}, p\circ\Phi) \leq 2\mathbb{E}\Big[\sum_{x\in\Phi} p^2(x) \Big],$$

which concludes the proof. □

Under the assumptions of Theorem 3.7.5, it is actually possible to show that

$$\Delta^*_{TV}(p \circ \Phi, \Gamma_M) \le 2\mathbb{E}\Big[\sum_{x \in \Phi} p^2(x)\Big] + \Delta^*_{TV}(p\Phi, M),$$

where $\Delta \in \{\Delta_D, \Delta_{TV}\}$. However, the random measure $p\Phi$ has almost surely a discrete support, and this implies that we cannot suppose that the quantity $\Delta^*_{TV}(p\Phi, M)$ is close to 0 in the general case, in particular when M admits almost surely a density with respect to the measure ℓ. That is the reason why we choose to use the Polish distance instead of a stronger distance for this last convergence result.

Chapter 4

Discrete α-stable point processes

In this chapter, we focus on discrete α-stable point processes and adapt to them the Stein's method used for Poisson point processes in Chapter 3.

After recalling in Section 4.1 the main results existing on the theory of discrete α-stable point processes, we give in Section 4.2 an expression for the Papangelou intensity of a discrete α-stable point process, and settle three generalizations of the Mecke formula. We provide in Section 4.3 a way to link discrete α-stable point processes - respectively strictly α-stable random measures - for different values of α. In Section 4.4, the Stein's method is applied to discrete α-stable point processes. We build the Glauber process, settle some properties of its corresponding semi-group, infinitesimal generator and gradients, and give a representation formula similar to that given for the Poisson point process. In Section 4.5, we introduce α-Papangelou intensities and establish some properties: we give the expression of the α-Papangelou intensity of a discrete α-stable point process, establish the link with Papangelou intensities and provide some results concerning usual transformations. In Section 4.6, some convergence results are settled when α varies. We provide some upper bounds for total variation distance between a discrete α-stable point process and a Poisson point process, and more generally between discrete α-stable point processes whose exponents α are different. In Section 4.7, we give some convergence results when α is fixed. Two adaptations of the Kallenberg's theorem (Section 3.7) are settled, and we deduce a way to approximate a strictly α-stable random measure with point processes. Finally, we bound the distance between discrete α-stable point processes with different spectral measures and between a superposition and a discrete α-stable point process.

4.1 Generalities on discrete α-stable point processes

All along this and following sections, $\alpha \in (0,1]$. We recall some basic definitions and properties from the theory of stable random measures and point processes, which mainly arise from [20].

Definition 4.1.1 (Strictly α-stable random measure).
A random measure ξ on $\mathbb{X}$ is said to be ***strictly α-stable*** *(StαS or StS) if, for*

any $t \in [0,1]$,

$$t^{\frac{1}{\alpha}}\xi^{(1)} + (1-t)^{\frac{1}{\alpha}}\xi^{(2)} \stackrel{\mathcal{D}}{=} \xi,$$

where $\xi^{(1)}$ *and* $\xi^{(2)}$ *are independent copies of the random measure* ξ.

Definition 4.1.2 (Lévy measure - Homogeneous measure of order $-\alpha$).
A Radon measure Λ *on* $\mathbb{M}_R \setminus \{0\}$ *is a* ***Lévy measure*** *if, for any* $h \in \mathcal{B}_+(\mathbb{X})$,

$$\int_{\mathbb{M}_R\setminus\{0\}} (1 - e^{-\langle h,\mu\rangle})\Lambda(\mathrm{d}\mu) < +\infty,$$

and Λ *is* ***homogeneous of order*** $-\alpha$ *if, for any measurable* $A \subset \mathbb{M}_R \setminus \{0\}$ *and any* $t > 0$,

$$\Lambda(tA) = t^{-\alpha}\Lambda(A).$$

Theorem 4.1.3 (Laplace functional of a strictly α-stable random measure).
A locally finite random measure ξ *is StαS if and only if* ξ *is deterministic in the case* $\alpha = 1$ *and in the case* $\alpha \in (0,1)$ *if and only if its Laplace functional is given for any* $h \in \mathcal{B}_+(\mathbb{X})$ *by:*

$$\mathcal{L}_\xi(h) = \exp\Big\{-\int_{\mathbb{M}_R\setminus\{0\}} (1 - e^{-\langle h,\mu\rangle})\Lambda(\mathrm{d}\mu)\Big\},$$

where Λ *is a Lévy measure and homogeneous of order* $-\alpha$.

Definition 4.1.4 (Spectral set).
If μ *is a finite non-negative measure, it is possible to normalize it by dividing it by its total mass and so arriving at a probability measure. The normalization procedure can be extended to all locally finite measures as follows. Let* $(B_n)_{n\in\mathbb{N}}$ *be a fixed countable base of the topology on* $\mathbb{X}$ *that consists of relatively compact sets. Append* $B_0 = \mathbb{X}$ *to this base. For each* $\mu \in \mathbb{M}_R \setminus \{0\}$ *consider the sequence of its values* $(\mu(B_n))_{n\in\mathbb{N}_0}$*, possibly starting with infinity, but otherwise finite. Let* $i(\mu)$ *be the smallest non-negative integer* i *for which* $0 < \mu(B_i) < +\infty$*; in particular,* $i(\mu) = 0$ *for a finite measure. The set* $\mathbb{S}$*, called* ***spectral set****, defined by*

$$\mathbb{S} = \{\mu \in \mathbb{M}_R : \mu(B_{i(\mu)}) = 1\}$$

is measurable, since

$$\mathbb{S} = \mathbb{M}_1 \cup \bigcup_{n=1}^{+\infty}\{\mu \in \mathbb{M}_R : \mu(B_0) = +\infty, \mu(B_1) = \cdots = \mu(B_{n-1}) = 0, \mu(B_n) = 1\}.$$

Note that $\mathbb{S} \cap \{\mu : \mu(\mathbb{X}) < +\infty\} = \mathbb{M}_1$. Furthermore, every $\mu \in \mathbb{M}_R \setminus \{0\}$ can be uniquely associated with the pair $(\hat{\mu}, \mu(B_{i(\mu)})) \in \mathbb{S} \times \mathbb{R}_+$, so that $\mu = \mu(B_{i(\mu)})\hat{\mu}$. It is straightforward to check that the mapping $\mu \mapsto (\hat{\mu}, \mu(B_{i(\mu)}))$ is measurable. Hence we have the following polar decomposition: $\mathbb{M}_R = \mathbb{S} \times \mathbb{R}_+$.

The spectral set on $\mathbb{M}$ is denoted $\mathbb{S}'$.

Definition 4.1.5 (Spectral measure).

Let Λ be a Lévy measure and homogeneous of order $-\alpha$ on $\mathbb{M}_R \setminus \{0\}$. Let σ be the measure on $\mathbb{M}$ defined, for any measurable subset A of $\mathbb{S}$, by

$$\sigma(A) = \Gamma(1-\alpha)\Lambda(\{t\mu \ : \ \mu \in A, \ t \geq 1\}),$$

*where Γ denotes the gamma function. The measure σ is called the **spectral measure** associated to Λ.*

Note that if σ is a measure on $\mathbb{S}$, then there exists a Lévy measure Λ on $\mathbb{M}_R \setminus \{0\}$ which is homogeneous of order $-\alpha$ such that σ is the spectral measure associated to Λ if, for any $B \in \mathcal{X}_0$,

$$\int_{\mathbb{S}} \mu(B)^{\alpha} \sigma(\mathrm{d}\mu) < +\infty.$$

Theorem 4.1.6 (Laplace functional of a strictly α-stable random measure).

Let ξ be a StαS random measure with Lévy measure Λ given by Theorem 4.1.3. Then, for any $h \in \mathcal{B}_+(\mathbb{X})$,

$$\mathcal{L}_{\xi}(h) = \exp\left\{ -\int_{\mathbb{S}} \langle h, \mu \rangle^{\alpha} \sigma(\mathrm{d}\mu) \right\},$$

where σ is the spectral measure associated to Λ.

Furthermore,

- *ξ is a.s. finite if and only if its Lévy measure Λ (resp., spectral measure σ) is supported by finite measures and $\sigma(\mathbb{S}) = \sigma(\mathbb{M}_1)$ is finite.*
- *The Laplace functional given by Theorem 4.1.3 defines a non-random measure if and only if $\alpha = 1$. In this case $\xi = \int_{\mathbb{S}} \mu(\cdot)\sigma(\mathrm{d}\mu)$.*

Theorem 4.1.7 (Strictly α-stable random measure with $\sigma = c\delta_{\mu}$).

Let ξ be a StαS random measure with spectral measure $\sigma = c\delta_{\mu}$, where $c \in \mathbb{R}_+$ and $\mu \in \mathbb{S}$. Then,

$$\xi = c^{\frac{1}{\alpha}} X_\alpha \mu,$$

where X_α is a positive strictly α-stable random variable with Laplace functional given, for any $t \in \mathbb{R}_+$, by

$$\mathbb{E}[e^{-tX_\alpha}] = e^{-t^\alpha}.$$

Definition 4.1.8 (Discrete α-stable point process).
*A point process Φ is said to be **discrete α-stable** (DαS or DS) if, for any $t \in [0,1]$,*

$$t^{\frac{1}{\alpha}} \circ \Phi^{(1)} + (1-t)^{\frac{1}{\alpha}} \circ \Phi^{(2)} \overset{\mathcal{D}}{=} \Phi,$$

where $\Phi^{(1)}$ and $\Phi^{(2)}$ are independent copies of the point process Φ.

Theorem 4.1.9 (Cox representation of a DαS point process).
A point process Φ is DαS if and only if it is a Cox point process with a StαS intensity measure ξ. Its p.g.fl. is then given for any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \mathcal{L}_\xi(1-u) = \exp\Big\{ - \int_{\mathbb{M}_R \setminus \{0\}} (1 - e^{-\langle 1-u, \mu \rangle}) \Lambda(\mathrm{d}\mu) \Big\}.$$

In this case, Φ is a Poisson point process if and only if $\alpha = 1$, and then its intensity measure is $\int_{\mathbb{S}} \mu(\cdot)\sigma(\mathrm{d}\mu)$.

Corollary 4.1.10 (Probability generating functional of a DαS point process).
A point process Φ is DαS if and only if its p.g.fl. is given for any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \exp\Big\{ - \int_{\mathbb{S}} \langle 1-u, \mu \rangle^\alpha \sigma(\mathrm{d}\mu) \Big\},$$

for some locally finite spectral measure σ on $\mathbb{S}$ such that, for any $B \in \mathcal{X}_0$,

$$\int_{\mathbb{S}} \mu(B)^\alpha \sigma(\mathrm{d}\mu) < +\infty.$$

Corollary 4.1.11 (Superposition and thinning of DαS point processes).
Let $\Phi^{(1)}, \dots, \Phi^{(n)}$ be independent copies of the DαS point process Φ with spectral measure σ and StαS random intensity ξ and let $t \in [0,1]$. Then,

- $\sum_{k=1}^{n} \Phi^{(k)}$ *is a DαS point process with spectral measure $n\sigma$ and StαS random intensity $n\xi$;*

- *$t \circ \Phi$ is a DαS point process with spectral measure $t^{\alpha}\sigma$ and StαS random intensity $t^{\alpha}\xi$.*

In particular, $\frac{1}{n^{\frac{1}{\alpha}}} \circ \sum_{k=1}^{n} \Phi^{(k)}$ *is distributed as* Φ.

Definition 4.1.12 (Sibuya distribution).
We say that a random variable Y on $\mathbb{N}$ has the ***Sibuya distribution*** *with exponent α if its probability generating function is given for any $s \in (0,1]$ by:*

$$\mathbb{E}[s^Y] = 1 - (1-s)^{\alpha}.$$

Definition 4.1.13 (t-scaled Sibuya distribution).
Let $t \in [0,1]$. We say that a random variable Y on $\mathbb{N}_0$ has the ***t-scaled Sibuya distribution*** *with exponent α if its probability generating function is given for any $s \in (0,1]$ by:*

$$\mathbb{E}[s^Y] = 1 - t(1-s)^{\alpha}.$$

Theorem 4.1.14 (t-scaled Sibuya distribution).
If a random variable Y has a t-scaled Sibuya distribution, then

$$\mathbb{P}(Y=0) = 1-t, \ \mathbb{P}(Y=1) = \alpha t$$

and, for any $n \in \mathbb{N}$ such that $n \geq 2$,

$$\mathbb{P}(Y=n) = (1-\alpha)\Big(1-\frac{\alpha}{2}\Big)\dots\Big(1-\frac{\alpha}{n-1}\Big)\frac{\alpha}{n}t.$$

Remark 4.1.15 (t-scaled Sibuya distribution).
If Y has a t-scaled Sibuya distribution with exponent α such that $\alpha \neq 1$, then, for any $n \in \mathbb{N}$,

$$\frac{\mathbb{P}(Y=n+1)}{\mathbb{P}(Y=n)} = \frac{n-\alpha}{n+1}.$$

If $\alpha = 1$, then Y has a Bernoulli distribution with parameter t and, in particular, if Y has a Sibuya distribution with exponent 1, then $Y = 1$ a.s..

A characterization of the Sibuya distribution is provided in [32] and given by the next theorem.

Theorem 4.1.16 (**Generation of a Sibuya distribution**).
Let $\alpha \in (0,1)$. Let E, G, H be independent random variables such that E has an exponential distribution and G, H have gamma distributions with respective parameters α and $1-\alpha$. Let P be a random variable such that, conditionally to E, G, H, P has a Poisson distribution with parameter $\frac{EH}{G}$. Then, $1 + P$ has a Sibuya distribution with exponent α.

From [15], we get the following result.

Theorem 4.1.17 (**Sum of a Sibuya random number of Sibuya random variables**).
Let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of i.i.d. Sibuya random variables with exponent α and Y be a Sibuya random variable with exponent β and independent of $(X^{(n)})_{n\in\mathbb{N}}$. Then, the random variable Z defined by $Z = \sum_{n=1}^{Y} X^{(n)}$ is a Sibuya random variable with exponent $\alpha\beta$.

Let us now give the definition of a Sibuya point process, introduced in [20].

Definition 4.1.18 (**Sibuya point process**).
*Let μ be a probability distribution on $\mathbb{X}$. A point process Υ on $\mathbb{X}$ is called a **Sibuya point process** with exponent α and parameter measure μ if its p.g.fl. is given for any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:*

$$G_\Upsilon(u) = 1 - \langle 1-u, \mu\rangle^\alpha.$$

In which follows, Υ_μ^α (or simply Υ_μ when there is no ambiguity) denotes unless otherwise specified a Sibuya point process with exponent α and parameter measure μ.

A Sibuya point process on $\mathbb{X}$ with exponent 1 and parameter measure $\mu \in \mathbb{M}_1$ has exactly one point, which is placed on $\mathbb{X}$ according to μ.

Remark 4.1.19 (**Thinning of a Sibuya point process**).
If $t \in [0,1]$ and Υ is a Sibuya point process with exponent α and parameter measure μ, then the t-thinning of Υ has a p.g.fl. defined for any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_{t\circ\Upsilon}(u) = 1 - \langle 1-u, t\mu\rangle^{\alpha},$$

and then we can extend the previous definition for the case where the measure parameter μ is such that $\mu(\mathbb{X}) < 1$ (denoted $\mu \in \mathbb{M}_1^*$).

Definition 4.1.20 (Sibuya point process with parameter measure in $\mathbb{M}_1^*$).
*Let $\mu \in \mathbb{M}_1^*$. A point process Υ on $\mathbb{X}$ is called the **Sibuya point process** with exponent α and parameter measure μ if its p.g.fl. is given for any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:*

$$G_{\Upsilon}(u) = 1 - \langle 1-u, \mu\rangle^{\alpha}.$$

Theorem 4.1.21 (Sibuya point process seen as purely random).
A Sibuya point process Υ on $\mathbb{X}$ with exponent α and parameter measure $\mu \in \mathbb{M}_1$ is a purely random point process supported by the measure μ and the Sibuya distribution with exponent α and parameter measure μ.

Theorem 4.1.22 (Thinning of a Sibuya point process).
Let $t \in [0,1]$ and Y be a random variable with a Sibuya distribution with exponent α. Then, the random variable $t \circ Y$ has a t^{α}-scaled Sibuya distribution.

More generally, if Υ_{μ} is a Sibuya point process with exponent α and parameter measure μ, then $t \circ \Upsilon_{\mu}$ is a Sibuya point process with exponent α and parameter measure $t\mu$.

Moreover, if Z a random variable with Bernoulli distribution with parameter t^{α}, then the point process Φ_t defined by $\Phi_t = \mathbf{1}_{\{Z=1\}}\Upsilon_{\mu}$ is a Sibuya point process with exponent α and parameter measure $t\mu$.

Proof. The discrete random variable $t \circ Y$ has a probability generating function given, for any $s \in (0,1]$, by:

$$\begin{aligned} \mathbb{E}[s^{t\circ Y}] &= \mathbb{E}[(1-t(1-s))^{Y}] \\ &= 1-(1-[1-t(1-s)])^{\alpha} \\ &= 1-t^{\alpha}(1-s)^{\alpha}. \end{aligned}$$

The second result is obtained by adapting this proof to a Sibuya point process. □

The equivalent of Theorem 4.1.17 for Sibuya point processes is stated in the following theorem.

Theorem 4.1.23 (**Sum of a Sibuya random number of Sibuya point processes**).

Let $(\Upsilon^{(n)})_{n\in\mathbb{N}}$ be a sequence of i.i.d. Sibuya random variables with exponent α and parameter measure $\mu \in \mathbb{M}_1^*$, and Y be a Sibuya random variable with exponent β and independent of $(\Upsilon^{(n)})_{n\in\mathbb{N}}$. Then, the point process Ψ defined by

$$\Psi = \sum_{n=1}^{Y} \Upsilon^{(n)}$$

is a Sibuya point process with exponent $\alpha\beta$ and parameter measure μ.

Proof. For any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$,

$$\begin{aligned} G_\Psi(u) &= \mathbb{E}\Big[\mathbb{E}\Big[\prod_{x\in\sum_{n=1}^{Y}\Upsilon^{(n)}} u(x) \;\Big|\; Y\Big]\Big] \\ &= \mathbb{E}[(1-\langle 1-u,\mu\rangle^{\alpha})^{Y}]. \end{aligned}$$

Then, since $\mathbb{E}[s^Y] = 1-(1-s)^{\beta}$,

$$\begin{aligned} G_\Psi(u) &= 1-(1-(1-\langle 1-u,\mu\rangle^{\alpha}))^{\beta} \\ &= 1-\langle 1-u,\mu\rangle^{\alpha\beta}. \end{aligned}$$

This proof is thus complete. □

Theorem 4.1.24 (**Cluster representation of a DαS point process**).

A DαS point process Φ with the Lévy measure supported by finite measures (equivalently, with a spectral measure σ supported by $\mathbb{M}_1$) can be represented as a cluster process with a Poisson center process on $\mathbb{M}_1$ driven by intensity measure σ and daughter processes being Sibuya point processes Υ_μ with exponent α and parameter measure $\mu \in \mathbb{M}_1$. Put another way, the point process Φ may be represented as:

$$\Phi = \sum_{\mu\in\zeta} \Upsilon_\mu,$$

where ζ is a Poisson point process on $\mathbb{M}_1$ with intensity measure σ. Its p.g.fl. is given for any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$ by:

$$G_\Phi(u) = \exp\Big\{\int_{\mathbb{M}_1} (G_{\Upsilon_\mu}(u)-1)\sigma(\mathrm{d}\mu)\Big\}.$$

Simulations 4.1.25 (**DαS point processes**).

Some realizations of discrete α-stable point processes for different values of α are *given in Figure 4.1. In each case, the spectral measure is supported by Gaussian distributions with a fixed variance parameter. A Sibuya number of points is then drawn independently according to each Gaussian measure. A way to simulate a random variable with a Sibuya distribution is given by Theorem 4.1.16. From the choice of Gaussian measures we can clearly observe the cluster representation of a discrete α-stable point process. The points from each cluster are represented with a different color. Note that the number of points in the clusters increases as α decreases and that a realization of a discrete α-stable point process is close to the realization of a Poisson point process as α tends to 1.*

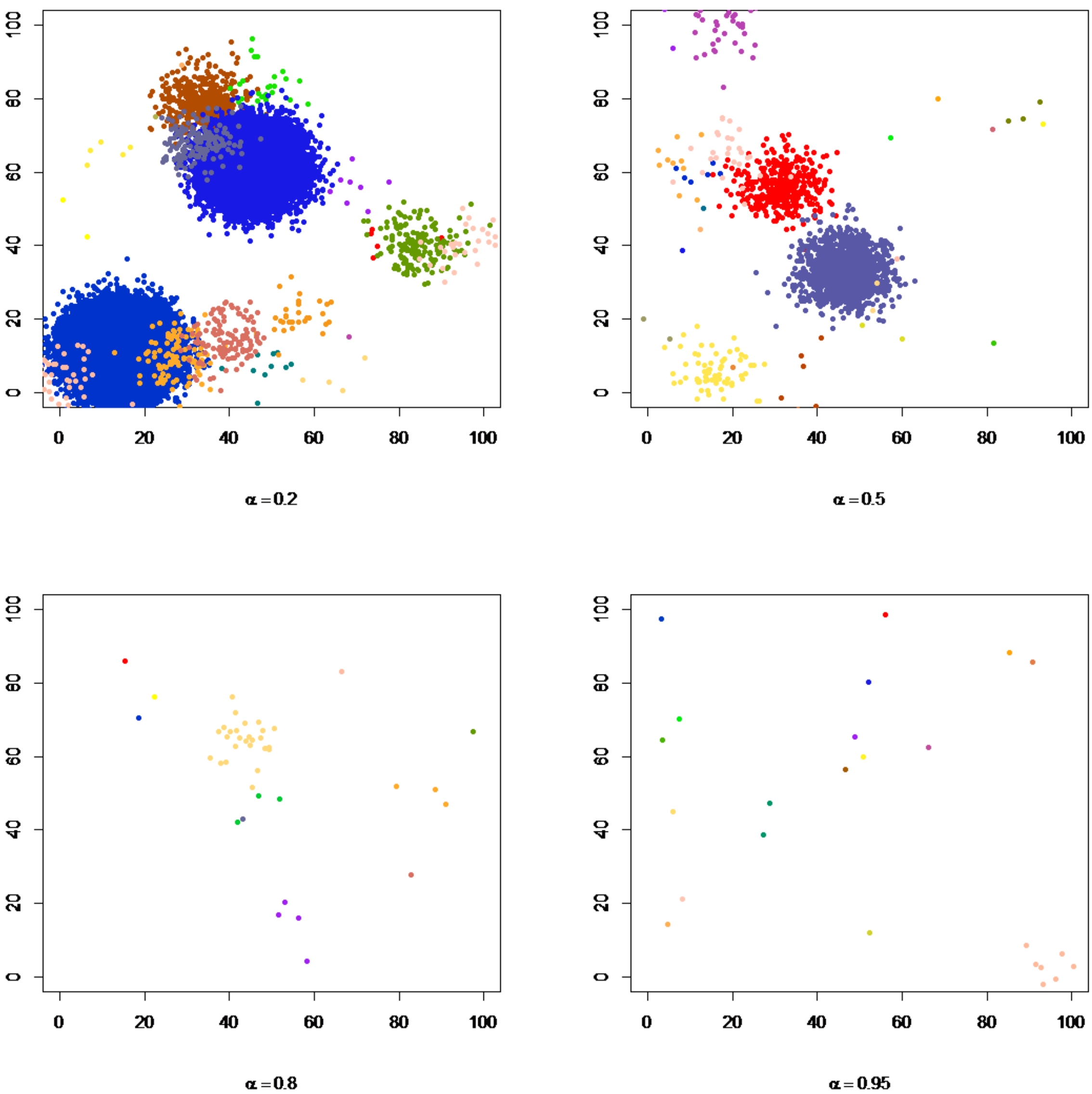


Figure 4.1: Realizations of DαS point processes for $\alpha \in \{0.2, 0.5, 0.8, 0.95\}$.

4.2 Papangelou intensity and Mecke formulas

This section is dedicated to some new results. First of them relate to Sibuya point processes.

Theorem 4.2.1 (Papangelou intensity of a t-scaled Sibuya point process).
Let $t \in [0,1)$ and Υ_t be the t-scaled Sibuya point process on $\mathbb{X}$ with exponent α and parameter measure $\mu \in \mathbb{M}_1$ such that $\mu(\mathrm{d}x) = q_\mu(x)\ell(\mathrm{d}x)$. Then, a version c_t of its Papangelou intensity with respect to ℓ is given for any $x \in \mathbb{X}$ and $\phi \in N_{\mathbb{X}} \setminus \{\varnothing\}$ by:

$$c_t(x,\phi) = (|\phi| - \alpha)q_\mu(x) \text{ and } c_t(x,\varnothing) = \frac{\alpha t}{1-t} q_\mu(x).$$

Proof. Since by Remark 4.1.15, for any $n \in \mathbb{N}$,

$$\frac{\mathbb{P}(|\Upsilon_t| = n+1)}{\mathbb{P}(|\Upsilon_t| = n)} = \frac{n-\alpha}{n+1},$$

and since Υ_t is a purely random point process (Theorem 4.1.21), the result is a direct consequence of Theorem 3.4.12. □

The Papangelou intensity c_t of a t-scaled Sibuya point process on $\mathbb{X}$ with exponent α verifies, for any $\omega, \phi \in \widehat{N}_{\mathbb{X}}$ such that $\omega \subset \phi$ and any $x \in \mathbb{X}$,

$$c_t(x,\omega) \le c_t(x,\phi),$$

if and only if $\alpha + t \le 1$. Let note that, in this case, by adapting our definition of repulsiveness to attractiveness, Υ_t may be seen as an attractive point process.

Theorem 4.2.2 (Mecke formula for a Sibuya point process).
Let Υ be a Sibuya point process on $\mathbb{X}$ with exponent α and parameter measure $\mu \in \mathbb{M}_1$. Then for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x, \Upsilon \setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[(|\Upsilon| - \alpha)u(x,\Upsilon)]\mu(\mathrm{d}x) + \alpha \int_{\mathbb{X}} u(x,\varnothing)\mu(\mathrm{d}x).$$

Proof. For any $n \in \mathbb{N}$, let us denote $p_n := \mathbb{P}(|\Upsilon| = n)$. Applying the first equation of Theorem 2.4.11, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\begin{aligned}
\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x,\Upsilon\setminus x)\Big] &= \sum_{n=0}^{+\infty} p_n \int_{\mathbb{X}^n} \sum_{i=1}^{n} u(x_i, \{x_1,\dots,x_n\}\setminus x_i)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n) \\
&= \sum_{n=1}^{+\infty} p_n n \int_{\mathbb{X}^n} u(x,\{x_1,\dots,x_{n-1}\})\mu(\mathrm{d}x)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{n-1}).
\end{aligned}$$

Then, by a change of variable in the sum,

$$\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x,\Upsilon\setminus x)\Big] = \int_{\mathbb{X}} \sum_{n=0}^{+\infty} p_{n+1}(n+1)\int_{\mathbb{X}^n} u(x,\{x_1,\dots,x_n\})\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n)\mu(\mathrm{d}x),$$

and then

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x,\Upsilon\setminus x)\Big] &= \int_{\mathbb{X}} \sum_{n=1}^{+\infty} p_n \int_{\mathbb{X}^n} (n+1)\frac{p_{n+1}}{p_n} u(x,\{x_1,\dots,x_n\})\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n)\mu(\mathrm{d}x) \\ &\quad + p_1\int_{\mathbb{X}} u(x,\varnothing)\mu(\mathrm{d}x).\end{aligned}$$

Still by Theorem 2.4.11, it follows that

$$\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x,\Upsilon\setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}\Big[(|\Upsilon|+1)\frac{p_{|\Upsilon|+1}}{p_{|\Upsilon|}} u(x,\Upsilon)\Big]\mu(\mathrm{d}x) + p_1\int_{\mathbb{X}} u(x,\varnothing)\mu(\mathrm{d}x).$$

Hence, by Remark 4.1.15,

$$\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x,\Upsilon\setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[(|\Upsilon|-\alpha)u(x,\Upsilon)]\mu(\mathrm{d}x) + \alpha\int_{\mathbb{X}} u(x,\varnothing)\mu(\mathrm{d}x),$$

hence, the result. □

Note that an example of sufficient condition on u to obtain an equation between finite values in Theorem 4.2.2 is given, for any $x\in\mathbb{X}$ and $\phi\in N_{\mathbb{X}}$, by

$$|u(x,\phi)| \le \frac{1}{|\phi|+1}.$$

From this last theorem, we deduce the following result.

Theorem 4.2.3 (Papangelou intensity of a DαS point process).
Let $\Phi = \sum_{\mu\in\zeta}\Upsilon_\mu$ be a DαS point process such that its spectral measure σ is supported by $\{\mu\in\mathbb{M}_1 : \mu \ll \ell\}$. Then, its Papangelou intensity c with respect to ℓ verifies, for any $x\in\mathbb{X}$,

$$c(x,\Phi) = \sum_{\mu\in\zeta}(|\Upsilon_\mu|-\alpha)q_\mu(x) + \alpha\int_{\mathbb{M}_1} q_\mu(x)\sigma(\mathrm{d}\mu)\ a.s.$$

where for any $\mu\in\mathbb{M}_1$ such that $\mu\ll\ell$, $q_\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\ell}$.

Proof. For any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\begin{aligned}
&\int_{\mathbb{X}} \mathbb{E}\Big[\Big(\sum_{\mu\in\zeta}(|\Upsilon_\mu| - \alpha)q_\mu(x) + \alpha \int_{\mathbb{M}_1} q_\mu(x)\sigma(\mathrm{d}\mu)\Big)u(x,\Phi)\Big]\ell(\mathrm{d}x) = \\
= \ &\int_{\mathbb{X}} \mathbb{E}\Big[\sum_{\mu\in\zeta}(|\Upsilon_\mu| - \alpha)q_\mu(x)u(x,\Phi)\Big]\ell(\mathrm{d}x) + \alpha \int_{\mathbb{X}}\int_{\mathbb{M}_1} q_\mu(x)\sigma(\mathrm{d}\mu)\mathbb{E}[u(x,\Phi)]\ell(\mathrm{d}x).
\end{aligned}$$

Consider the first term of this sum. Conditioning with respect to ζ,

$$\int_{\mathbb{X}} \mathbb{E}\Big[\sum_{\mu\in\zeta}(|\Upsilon_\mu| - \alpha)q_\mu(x)u(x,\Phi)\Big]\ell(\mathrm{d}x) = \mathbb{E}\Big[\sum_{\mu\in\zeta}\int_{\mathbb{X}} \mathbb{E}\Big[(|\Upsilon_\mu| - \alpha)u\Big(x, \sum_{\nu\in\zeta}\Upsilon_\nu\Big) \,\Big|\, \zeta\Big]q_\mu(x)\ell(\mathrm{d}x)\Big],$$

and then, by Theorem 4.2.2,

$$\begin{aligned}
&\int_{\mathbb{X}} \mathbb{E}\Big[\sum_{\mu\in\zeta}(|\Upsilon_\mu| - \alpha)q_\mu(x)u(x,\Phi)\Big]\ell(\mathrm{d}x) = \\
= \ &\mathbb{E}\Big[\sum_{\mu\in\zeta}\mathbb{E}\Big[\sum_{x\in\Upsilon_\mu} u\Big(x, \Big(\sum_{\nu\in\zeta}\Upsilon_\nu\Big)\setminus x\Big) \,\Big|\, \zeta\Big]\Big] - \alpha\int_{\mathbb{X}} u\Big(x, \sum_{\nu\in\zeta\setminus\mu}\Upsilon_\nu\Big)q_\mu(x)\ell(\mathrm{d}x)\Big].
\end{aligned}$$

Applying the Mecke formula for the Poisson point process ζ in the second term of this last expression yields, as expected,

$$\begin{aligned}
&\int_{\mathbb{X}} \mathbb{E}\Big[\sum_{\mu\in\zeta}(|\Upsilon_\mu| - \alpha)q_\mu(x)u(x,\Phi)\Big]\ell(\mathrm{d}x) = \\
= \ &\mathbb{E}\Big[\sum_{x\in\Phi} u(x, \Phi\setminus x)\Big] - \alpha\int_{\mathbb{X}}\int_{\mathbb{M}_1} \mathbb{E}[u(x,\Phi)]q_\mu(x)\sigma(\mathrm{d}\mu)\ell(\mathrm{d}x).
\end{aligned}$$

The proof is thus complete. □

We propose now to adapt the Mecke formula, known for Poisson point processes, to DαS point processes.

Theorem 4.2.4 (First Mecke formula for a DαS point process).
Let $\Phi = \sum_{\mu\in\zeta}\Upsilon_\mu$ be a DαS point process such that its spectral measure σ is supported by $\mathbb{M}_1$. Then, for any measurable function $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x,\Phi\setminus x)\Big] = \int_{\mathbb{M}_1}\int_{\mathbb{X}} \mathbb{E}[(|\Upsilon_\mu|-\alpha)u(x,\Phi+\Upsilon_\mu)]\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu)$$
$$+\alpha\int_{\mathbb{M}_1}\int_{\mathbb{X}} \mathbb{E}[u(x,\Phi)]\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu).$$

Proof. For any measurable function $u : \mathbb{X}\times N_{\mathbb{X}} \to \mathbb{R}_+$, the cluster representation of the DαS Φ yields

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x,\Phi\setminus x)\Big] = \mathbb{E}\Big[\sum_{\mu\in\zeta}\sum_{x\in\Upsilon_\mu} u\Big(x,\sum_{\nu\in\zeta}\Upsilon_\nu\setminus x\Big)\Big].$$

Thus, from the Mecke formula for the Poisson point process ζ,

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x,\Phi\setminus x)\Big] = \int_{\mathbb{M}_1}\mathbb{E}\Big[\sum_{x\in\Upsilon_\mu} u\Big(x,\sum_{\nu\in\zeta+\mu}\Upsilon_\nu\setminus x\Big)\Big]\sigma(\mathrm{d}\mu),$$

and the first formula is deduced by Theorem 4.2.2. This formula may be derived by applying the Mecke formula for a Poisson point process on the first term of the right hand side of the first formula. □

Theorem 4.2.5 (Second Mecke formula for a DαS point process).
Let $\Phi = \sum_{\mu\in\zeta}\Upsilon_\mu$ be a DαS point process such that its spectral measure σ is supported by $\mathbb{M}_1$. Then, for any measurable function $u : \mathbb{X}\times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x,\Phi\setminus x)\Big] = \mathbb{E}\Big[\sum_{\mu\in\zeta}(|\Upsilon_\mu|-\alpha)\int_{\mathbb{X}} u(x,\Phi)\mu(\mathrm{d}x)\Big]+\alpha\int_{\mathbb{M}_1}\int_{\mathbb{X}}\mathbb{E}[u(x,\Phi)]\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu).$$

Proof. This formula may be derived by applying the Mecke formula for a Poisson point process on the first term of the right hand side of the formula given by Theorem 4.2.4. □

Theorem 4.2.6 (Third Mecke formula for a DαS point process).
Let Φ be a DαS point process on $\mathbb{X}$ with finite spectral measure σ supported by $\mathbb{M}_1$. Then, for any measurable function $u : N_{\mathbb{X}} \to \mathbb{R}_+$ such that, for any $\phi\in N_{\mathbb{X}}$ and $\mu\in\mathbb{M}_1^*$, $D_\mu u(\phi) < +\infty$,

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] = \alpha\int_{\mathbb{M}_1}\mathbb{E}[D_\mu u(\Phi)]\sigma(\mathrm{d}\mu).$$

Proof. For $u(\phi) = \mathbf{1}_{\{\phi(K)=0\}}$, $\phi\in N_{\mathbb{X}}$ and K a relatively compact subset, on one hand,

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big]=-\mathbb{P}(\Phi(K)=1).$$

On the other hand, from the definition of D_μ and the expression of u,

$$\alpha\int_{\mathbb{M}_1}\mathbb{E}[D_\mu u(\Phi)]\sigma(\mathrm{d}\mu)=\alpha\int_{\mathbb{M}_1}\mathbb{E}[\mathbf{1}_{\{(\Phi+\Upsilon_\mu)(K)=0\}}-\mathbf{1}_{\{\Phi(K)=0\}}]\sigma(\mathrm{d}\mu).$$

Then,

$$\begin{aligned}\alpha\int_{\mathbb{M}_1}\mathbb{E}[D_\mu u(\Phi)]\sigma(\mathrm{d}\mu) &= -\alpha\int_{\mathbb{M}_1}\mathbb{E}[\mathbf{1}_{\{\Phi(K)=0\}}\mathbf{1}_{\{\Upsilon_\mu(K)\neq 0\}}]\sigma(\mathrm{d}\mu)\\ &= -\alpha\int_{\mathbb{M}_1}\mathbb{P}(\Phi(K)=0)\mathbb{P}(\Upsilon_\mu(K)\neq 0)\sigma(\mathrm{d}\mu),\end{aligned}$$

and, since, for any $\mu\in\mathbb{M}_1$,

$$\mathbb{P}(\Upsilon_\mu(K)\neq 0)=\mu^\alpha(K),$$

it yields

$$\alpha\int_{\mathbb{M}_1}\mathbb{E}[D_\mu u(\Phi)]\sigma(\mathrm{d}\mu)=-\alpha\mathbb{P}(\Phi(K)=0)\int_{\mathbb{M}_1}\mu^\alpha(K)\sigma(\mathrm{d}\mu).$$

Furthermore, on one hand, since $\Phi=\sum_{\mu\in\zeta}\Upsilon_\mu$,

$$\mathbb{P}(\Phi(K)=0)=\sum_{n=0}^{+\infty}\mathbb{P}(|\zeta|=n)\mathbb{P}\Big(\sum_{\mu\in\zeta}\Upsilon_\mu(K)=0\;\Big|\;|\zeta|=n\Big).$$

Then, since

$$\mathbb{P}(|\zeta|=n)=e^{-\sigma(\mathbb{M}_1)}\frac{\sigma(\mathbb{M}_1)^n}{n!}$$

and

$$\mathbb{P}\Big(\sum_{\mu\in\zeta}\Upsilon_\mu(K)=0\;\Big|\;|\zeta|=n\Big)=\Big(\int_{\mathbb{M}_1}\mathbb{P}(\Upsilon_\mu(K)=0)\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}\Big)^n,$$

it follows that

$$\mathbb{P}(\Phi(K)=0)=\sum_{n=0}^{+\infty}e^{-\sigma(\mathbb{M}_1)}\frac{\sigma(\mathbb{M}_1)^n}{n!}\Big(\int_{\mathbb{M}_1}\mathbb{P}(\Upsilon_\mu(K)=0)\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}\Big)^n.$$

Then,

$$\mathbb{P}(\Phi(K)=0)=e^{-\sigma(\mathbb{M}_1)}\sum_{n=0}^{+\infty}\frac{1}{n!}\Big(\int_{\mathbb{M}_1}(1-\mu^\alpha(K))\sigma(\mathrm{d}\mu)\Big)^n$$

and finally

$$\mathbb{P}(\Phi(K)=0)=\exp\Big\{-\int_{\mathbb{M}_1}\mu^\alpha(K)\sigma(\mathrm{d}\mu)\Big\}.$$

On the other hand,

$$\mathbb{P}(\Phi(K)=1)=\sum_{n=1}^{+\infty}\mathbb{P}(|\zeta|=n)\mathbb{P}\Big(\sum_{\mu\in\zeta}\Upsilon_\mu(K)=1\;\Big|\;|\zeta|=n\Big).$$

Since

$$\mathbb{P}\Big(\sum_{\mu\in\zeta}\Upsilon_\mu(K)=1\;\Big|\;|\zeta|=n\Big)=n\Big(\int_{\mathbb{M}_1}\mathbb{P}(\Upsilon_\mu(K)=0)\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}\Big)^{n-1}\int_{\mathbb{M}_1}\mathbb{P}(\Upsilon_\mu(K)=1)\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}$$

and

$$\mathbb{P}(|\zeta|=n)=e^{-\sigma(\mathbb{M}_1)}\frac{\sigma(\mathbb{M}_1)^n}{n!},$$

one deduces

$$\mathbb{P}(\Phi(K)=1)=\sum_{n=1}^{+\infty}e^{-\sigma(\mathbb{M}_1)}\frac{\sigma(\mathbb{M}_1)^n}{n!}n\Big(\int_{\mathbb{M}_1}\mathbb{P}(\Upsilon_\mu(K)=0)\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}\Big)^{n-1}\int_{\mathbb{M}_1}\mathbb{P}(\Upsilon_\mu(K)=1)\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}$$

and then, since, for any $\mu\in\mathbb{M}_1$,

$$\mathbb{P}(\Upsilon_\mu(K)=1)=\alpha\mu^\alpha(K),$$

it follows that

$$\begin{aligned}\mathbb{P}(\Phi(K)=1) &= \sum_{n=0}^{+\infty}e^{-\sigma(\mathbb{M}_1)}\frac{1}{n!}\Big(\int_{\mathbb{M}_1}(1-\mu^\alpha(K))\sigma(\mathrm{d}\mu)\Big)^n\alpha\int_{\mathbb{M}_1}\mu^\alpha(K)\sigma(\mathrm{d}\mu)\\ &= \alpha\int_{\mathbb{M}_1}\mu^\alpha(K)\sigma(\mathrm{d}\mu)\mathbb{P}(\Phi(K)=0).\end{aligned}$$

Thus, the equation is true for $u=\mathbf{1}_{\{\cdot\cap K=0\}}$.

By Dynkin's $\pi-\lambda$ theorem, since $\big\{\{\Phi(K)=0\},K\in\mathcal{X}_0\big\}$ is a π-system, the equation is also true for $u=\mathbf{1}_A$, where $A\in\mathcal{N}_{\mathbb{X}}$, then by linearity for u simple positive function, then using Monotone property for u measurable positive, which concludes the proof. □

Consider the case $\alpha=1$. A Sibuya point process Υ and a t-scaled Sibuya point process Υ_t are respectively a Bernoulli point process and the t-thinning of a Bernoulli point process, then Theorem 4.2.1 states simply that, for $t\neq 1$, the Papangelou intensity c_t of Υ_t with respect to ℓ may be given, for any $x\in\mathbb{X}$ and $\phi\in N_{\mathbb{X}}$, by:

$$c_t(x,\phi) = \frac{t}{1-t} q_\mu(x) \mathbf{1}_{\{\phi=\varnothing\}},$$

and Theorem 4.2.2 is trivial. In Theorem 4.2.3, Φ is a Poisson point process with intensity measure $q_\mu(x)\sigma(\mathrm{d}\mu)\ell(\mathrm{d}x)$ and its Papangelou intensity c with respect to ℓ verifies, for any $x \in \mathbb{X}$,

$$c(x,\Phi) = \int_{\mathbb{M}_1} q_\mu(x)\sigma(\mathrm{d}\mu) \text{ a.s.},$$

as expected. Theorems 4.2.4, 4.2.5 and 4.2.6 state the Mecke formula for a Poisson point process.

4.3 Link between α-stability and β-stability

In the two following theorems, we explore the link between stable random measures and point processes with the same spectral measure, but different exponents. These results provide interesting tools to understand, at least in an intuitive way, the respective structures of StαS random measures and DαS point processes.

Theorem 4.3.1 (Link between StαS and StβS random measures). *Let $\alpha, \beta, \gamma \in (0,1]$ such that $\alpha = \beta\gamma$. Let σ be a locally finite measure on $\mathbb{S}$ such that, for any $B \in \mathcal{X}_0$,*

$$\int_{\mathbb{S}} \mu(B)^\alpha \sigma(\mathrm{d}\mu) < +\infty.$$

Let $\iota : \mu \in \mathbb{S} \mapsto \delta_\mu \in \mathbb{S}'$ and ξ' be a StγS random measure on $\mathbb{M}_1$ with spectral measure $\sigma' = \sigma \circ \iota^{-1}$. Then, a random measure ξ such that, conditionally to ξ', ξ is a StβS random measure with spectral measure ξ', is a StαS random measure on $\mathbb{X}$ with spectral measure σ.

Proof. For any $h \in \mathcal{B}_+(\mathbb{X})$, let recall that

$$\mathcal{L}_\xi(h) = \mathbb{E}[\exp\{-\langle h, \xi\rangle\}].$$

Then, conditioning with respect to ξ',

$$\begin{aligned} \mathcal{L}_\xi(h) &= \mathbb{E}[\mathbb{E}[\exp\{-\langle h, \xi\rangle\} \mid \xi']] \\ &= \mathbb{E}[\exp\{-\langle h, \nu\rangle^\beta \xi'(\mathrm{d}\nu)\}] \end{aligned}$$

and then

$$\mathcal{L}_\xi(h) = \mathcal{L}_{\xi'}(\langle h, \nu\rangle^\beta).$$

Moreover, since ξ' is StγS with spectral measure σ', this implies that

$$\mathcal{L}_\xi(h) = \exp\Big\{-\int_{\mathbb{S}'} \langle\langle h, \nu\rangle^\beta, \mu'\rangle^\gamma \sigma'(\mathrm{d}\mu')\Big\},$$

and then, using that $\sigma' = \sigma \circ \iota^{-1}$,

$$\mathcal{L}_\xi(h) = \exp\Big\{-\int_{\mathbb{S}} \langle\langle h, \nu\rangle^\beta, \delta_\mu\rangle^\gamma \sigma(\mathrm{d}\mu)\Big\}.$$

Finally,

$$\begin{aligned}\mathcal{L}_\xi(h) &= \exp\Big\{-\int_{\mathbb{S}} (\langle h, \mu\rangle^\beta)^\gamma \sigma(\mathrm{d}\mu)\Big\} \\ &= \exp\Big\{-\int_{\mathbb{S}} \langle h, \mu\rangle^\alpha \sigma(\mathrm{d}\mu)\Big\},\end{aligned}$$

which provides the expected result. □

Theorem 4.3.2 (Link between DαS and DβS point processes).
Let $\alpha, \beta, \gamma \in (0,1]$ such that $\alpha = \beta\gamma$. Let σ be a locally finite measure on $\mathbb{S}$ such that, for any $B \in \mathcal{X}_0$,

$$\int_{\mathbb{S}} \mu(B)^\alpha \sigma(\mathrm{d}\mu) < +\infty.$$

Let $\iota : \mu \in \mathbb{S} \mapsto \delta_\mu \in \mathbb{S}'$ and Φ' be a DγS point process on $\mathbb{M}_1$ with spectral measure $\sigma' = \sigma \circ \iota^{-1}$. Then, the point process $\Phi = \sum_{\mu\in\Phi'} \Upsilon_\mu^\beta$ is a DαS point process on $\mathbb{X}$ with spectral measure σ.

Proof. For any function $u : \mathbb{X} \to (0,1]$ such that $1-u \in \mathcal{B}_+(\mathbb{X})$, since $\Phi = \sum_{\mu\in\Phi'} \Upsilon_\mu^\beta$,

$$G_\Phi(u) = G_{\Phi'}(G_{\Upsilon_\mu^\beta}(u)),$$

and the respective expressions of the probability generating functions provide

$$G_\Phi(u) = \exp\Big\{-\int_{\mathbb{S}'} \langle\langle 1-u, \mu\rangle^\beta, \mu'\rangle^\gamma \sigma'(\mathrm{d}\mu')\Big\}.$$

Then, since $\sigma' = \sigma \circ \iota^{-1}$,

$$G_\Phi(u) = \exp\Big\{-\int_{\mathbb{S}} \langle\langle 1-u, \mu\rangle^\beta, \delta_\nu\rangle^\gamma \sigma(\mathrm{d}\nu)\Big\}$$

and finally

$$
\begin{aligned}
G_\Phi(u) &= \exp\Big\{-\int_{\mathbb{S}} \langle 1-u,\nu\rangle^{\beta\gamma}\sigma(\mathrm{d}\nu)\Big\}\\
&= \exp\Big\{-\int_{\mathbb{S}} \langle 1-u,\nu\rangle^{\alpha}\sigma(\mathrm{d}\nu)\Big\},
\end{aligned}
$$

hence, the result. □

4.4 Stein's method for DαS point processes

In this section, the Stein's method is investigated for finite DαS point processes. Since a DαS point process Φ has a Poisson cluster representation $\sum_{\mu\in\zeta}\Upsilon_\mu$, it may be identified as the image of a projection P on $N_{\mathbb{X}}$ of a marked Poisson point process $\widetilde{\Phi}$ on $\mathbb{M}_1\times\widehat{N_{\mathbb{X}}}$, defined in such a way that the mark of each $\mu\in\zeta$ is a Sibuya point process Υ_μ, that is,

$$
\widetilde{\Phi} = \sum_{\mu\in\zeta}\delta_{(\mu,\Upsilon_\mu)}.
$$

This application P is defined as $P = S\circ Q$, where Q is the projection from $N_{\mathbb{M}_1\times\widehat{N_{\mathbb{X}}}}$ to $N_{\widehat{N_{\mathbb{X}}}}$ and the application $S : N_{\widehat{N_{\mathbb{X}}}}\to N_{\mathbb{X}}$ is defined, for any $\{\phi_1,\dots,\phi_n\}\in N_{\widehat{N_{\mathbb{X}}}}$, by

$$
S(\{\phi_1,\dots,\phi_n\}) = \phi_1+\dots+\phi_n.
$$

It may be summarized by the following scheme:

$$
\begin{array}{ccc}
N_{\mathbb{M}_1\times\widehat{N_{\mathbb{X}}}} & \longrightarrow & N_{\mathbb{X}}\\[2ex]
\displaystyle\sum_{\mu\in\zeta}\delta_{(\mu,\Upsilon_\mu)} & \xrightarrow{P} & \displaystyle\sum_{\mu\in\zeta}\Upsilon_\mu.
\end{array}
$$

From this approach, we may deduce a version of the Stein's method which easily adapts the one used for a finite Poisson point process. However, the Poisson point process ζ is not observed and the corresponding gradient, which would be defined, for any measurable function $F : N_{\mathbb{M}_1\times N_{\mathbb{X}}}\to\mathbb{R}$, any $\phi\in N_{\mathbb{M}_1\times N_{\mathbb{X}}}$, $\mu\in\mathbb{M}_1$ and $\eta\in N_{\mathbb{X}}$, as

$$
D_{(\mu,\eta)}F(\phi) = F(\phi+(\mu,\eta)) - F(\phi)
$$

cannot be computed in practical cases.

Intuitively, if we wish to keep the same principle for the definition of discrete gradient, we have to find something to add to $\sum_{\mu\in\zeta}\Upsilon_\mu$. This added term has to be similar (in some way) to a Sibuya point process Υ_μ. Our idea is then to focus on the random quantity $F(\phi+\Upsilon_\mu)-F(\phi)$, where $\phi\in N_{\mathbb{X}}$.

We begin these investigations by stating the following lemma.

Lemma 4.4.1 (Thinning and Sibuya point processes).
Let Ψ be a point process on $\mathbb{M}_1$. Then, for any $t \in [0,1]$,

$$t \circ \sum_{\mu \in \Psi} \Upsilon_\mu \overset{\mathcal{D}}{=} \sum_{\mu \in \Psi} t \circ \Upsilon_\mu \overset{\mathcal{D}}{=} \sum_{\mu \in t^\alpha \circ \Psi} \Upsilon_\mu,$$

where, for any $\mu \in \mathbb{M}_1^*$, Υ_μ is a Sibuya point process with exponent α and parameter measure μ.

Proof. Let $\Phi := \sum_{\mu \in \Psi} \Upsilon_\mu$. For any positive function u such that $1-u \in \mathcal{B}_+(\mathbb{X})$, for any $t \in [0,1]$, on one hand, by Theorem 2.3.8,

$$G_{t\circ\Phi}(u) = G_\Phi[1-t(1-u)] = G_\Psi[1-t^\alpha\langle 1-u, \mu\rangle^\alpha],$$

and we can deduce that

$$G_{t\circ\Phi}(u) = G_\Psi[G_{t\circ\Upsilon_\mu}(u)].$$

On the other hand,

$$G_{t\circ\Phi}(u) = G_\Psi[1-t^\alpha(1-(1-\langle 1-u,\mu\rangle^\alpha))] = G_\Psi[1-t^\alpha(1-G_{\Upsilon_\mu}[u])]$$

and then

$$G_{t\circ\Phi}(u) = G_{t^\alpha\circ\Psi}[G_{\Upsilon_\mu}[u]],$$

which concludes the proof. $\square$

Definition 4.4.2 (Glauber process for a DαS point process).
*Let Φ be a DαS point process with finite spectral measure σ supported by $\mathbb{M}_1$. The **Glauber process** $(G_t)_{t\geq 0}$ associated to Φ is defined for any $t \in \mathbb{R}_+$ and $\phi \in \widehat{N}_\mathbb{X}$ by:*

$$G_t(\phi) = e^{-\frac{t}{\alpha}} \circ \phi + (1-e^{-t})^{\frac{1}{\alpha}} \circ \Phi.$$

For any $t \in \mathbb{R}_+$, the operator P_t is defined for any measurable and bounded function $F : \widehat{N}_\mathbb{X} \to \mathbb{R}$ and any $\phi \in \widehat{N}_\mathbb{X}$ by:

$$P_tF(\phi) = \mathbb{E}[F(G_t(\phi))] = \mathbb{E}[F(e^{-\frac{t}{\alpha}} \circ \phi + (1-e^{-t})^{\frac{1}{\alpha}} \circ \Phi)].$$

Since $\Phi = \sum_{\mu\in\zeta} \Upsilon_\mu$ where ζ is a Poisson point process with intensity measure σ on $\mathbb{M}_1$ and according to Lemma 4.4.1, one has, for any $t \in \mathbb{R}_+$, any measurable and bounded function $F : \widehat{N}_\mathbb{X} \to \mathbb{R}$ and any $\phi \in \widehat{N}_\mathbb{X}$,

$$P_tF(\phi) = \mathbb{E}[F(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu)].$$

Its dynamics can be described as follows: imagine a homogeneous Poisson process ζ_b on $\mathbb{R}_+$ with intensity $\sigma(\mathbb{M}_1)$. The jump times of ζ_b determine the birth times of the probability measures in ζ, placed in $\mathbb{M}_1$ according to the distribution $\frac{\sigma(\cdot)}{\sigma(\mathbb{M}_1)}$. For each new measure $\mu \in \mathbb{M}_1$, a Sibuya point process Υ_μ is placed in $\mathbb{X}$. The lifetime of each Sibuya point process is exponentially distributed with parameter 1 (and its particles dead simultaneously). The lifetime of each particle from ϕ is exponentially distributed with parameter $\frac{1}{\alpha}$ and these lifetimes are independent.

If $\alpha = 1$, the Glauber process associated to the DαS point process with spectral measure σ defined in this section corresponds to the Glauber process associated to the Poisson point process with intensity measure $\mu(\mathrm{d}x)\sigma(d\mu)$, built in Section 3.1.

Theorem 4.4.3 (Semi-group).
The family $(P_t)_{t\geq 0}$ given by Definition 4.4.2 is a semi-group.

Proof. For any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$, any $\phi \in \widehat{N}_{\mathbb{X}}$ and any $s, t \in \mathbb{R}_+$, since thinning is associative,

$$P_s(P_tF)(\phi) = \int_{\widehat{N}_{\mathbb{X}}} \int_{\widehat{N}_{\mathbb{X}}} F(e^{-\frac{t+s}{\alpha}} \circ \phi + e^{-\frac{s}{\alpha}} \circ (1-e^{-t})^{1/\alpha} \circ \psi + (1-e^{-s})^{1/\alpha} \circ \eta)\mathbb{P}_\Phi(\mathrm{d}\psi)\mathbb{P}_\Phi(\mathrm{d}\eta).$$

Moreover, since

$$(e^{-\frac{s}{\alpha}}(1-e^{-t})^{1/\alpha})^\alpha + ((1-e^{-s})^{1/\alpha})^\alpha = 1 - e^{-(t+s)},$$

we deduce from the definition of a DαS point process (Definition 4.1.8) that

$$e^{-\frac{s}{\alpha}} \circ (1-e^{-t})^{1/\alpha} \circ \Phi^{(1)} + (1-e^{-s})^{1/\alpha} \circ \Phi^{(2)} \stackrel{\mathcal{D}}{=} (1-e^{-(t+s)})^{1/\alpha}\Phi,$$

where $\Phi^{(1)}$ and $\Phi^{(2)}$ are independent copies of Φ, and it yields

$$P_s(P_tF)(\phi) = \int_{\widehat{N}_{\mathbb{X}}} F(e^{-\frac{t+s}{\alpha}} \circ \phi + (1-e^{-(t+s)})^{1/\alpha} \circ \psi)\mathbb{P}_\Phi(\mathrm{d}\psi),$$

from which we can conclude. □

In a similar way to the proof given for the semi-group associated to a finite Poisson point process (Theorem 3.2.4), the previous proof only uses associativity of thinning and the invariance property of a DαS point process distribution given by its own definition.

The creation of a new Sibuya point process in the Glauber process is induced by the creation of its corresponding probability measure. This leads to consider the two following definitions of gradient.

Definition 4.4.4 (Gradient in direction $\omega \in N_{\mathbb{X}}$).
*For any $\omega \in N_{\mathbb{X}}$, the **gradient** D_ω in direction ω is defined, for any measurable function $F : N_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in N_{\mathbb{X}}$, by:*

$$D_\omega F(\phi) = F(\phi + \omega) - F(\phi).$$

Definition 4.4.5 (Gradient in direction $\mu \in \mathbb{M}_1^*$).
For any $\mu \in \mathbb{M}_1^*$, the ***gradient*** *D_μ in direction μ with exponent α is defined, for any measurable and bounded function $F : N_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in N_{\mathbb{X}}$, by:*

$$D_\mu^\alpha F(\phi) = \mathbb{E}_{\Upsilon_\mu}[D_{\Upsilon_\mu} F(\phi)] = \mathbb{E}_{\Upsilon_\mu}[F(\phi + \Upsilon_\mu) - F(\phi)],$$

where for any $\mu \in \mathbb{M}_1^*$, Υ_μ is a Sibuya point process with exponent α and parameter measure μ. We denote D_μ instead of D_μ^α when there is no ambiguity.

By identifying $x \in \mathbb{X}$ with the configuration $\{x\} \in N_{\mathbb{X}}$, one observes that the application $x \in \mathbb{X} \mapsto D_x$ defined in Section 3.1 is the restriction to $\{\omega \in N_{\mathbb{X}} : |\omega| = 1\}$ of the application $\omega \in N_{\mathbb{X}} \mapsto D_\omega$. Moreover, considering the case $\alpha = 1$, one has, for any $\mu \in \mathbb{M}_1^*$, any measurable and bounded function $F : N_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in N_{\mathbb{X}}$,

$$D_\mu^1 F(\phi) = \int_{\mathbb{X}} D_x F(\phi) \mu(\mathrm{d}x) = \int_{\mathbb{X}} \big(F(\phi + x) - F(\phi)\big) \mu(\mathrm{d}x).$$

Theorem 4.4.6 (Closability).
Let Φ be a DαS point process on $\mathbb{X}$ with spectral measure σ supported by $\mathbb{M}_1$. Let $F, G : N_{\mathbb{X}} \to \mathbb{R}$ be two measurable and bounded functions. If $F(\phi) = G(\phi)$ $\mathbb{P}_\Phi(\mathrm{d}\phi)$-a.s., then

$$D_\mu F(\phi) = D_\mu G(\phi) \ (\sigma \otimes \mathbb{P}_\Phi)(\mathrm{d}\mu, \mathrm{d}\phi)\text{-a.s.}.$$

Proof. Let $\Phi = \sum_{\mu \in \zeta} \Upsilon_\mu$ where ζ is a Poisson point process on $\mathbb{M}_1$ with intensity measure σ. By the Mecke formula applied to ζ, one has, for any measurable function $u : \mathbb{M}_1 \times N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\begin{aligned} \int_{\mathbb{M}_1} \mathbb{E}[D_\mu F(\Phi) u(\mu, \Phi)] \sigma(\mathrm{d}\mu) &= \mathbb{E}\Big[F(\Phi) \sum_{\mu \in \zeta} u\Big(\mu, \sum_{\nu \in \zeta \setminus \mu} \Upsilon_\nu\Big)\Big] - \int_{\mathbb{M}_1} \mathbb{E}[F(\Phi) u(\mu, \Phi)] \sigma(\mathrm{d}\mu) \\ &= \mathbb{E}\Big[F(\Phi)\Big(\sum_{\mu \in \zeta} u\Big(\mu, \sum_{\nu \in \zeta \setminus \mu} \Upsilon_\nu\Big) - \int_{\mathbb{M}_1} u(\mu, \Phi) \sigma(\mathrm{d}\mu)\Big)\Big]. \end{aligned}$$

Hence, if $F(\phi) = 0$ $\mathbb{P}_\Phi(\mathrm{d}\phi)$-a.s., then $D_\mu F(\phi) = 0$ $(\sigma \otimes \mathbb{P}_\Phi)(\mathrm{d}\mu, \mathrm{d}\phi)$-a.s., as expected. □

Theorem 4.4.7 (Property of the gradient in direction $\mu \in \mathbb{M}_1^*$).
Let Φ a DαS point process. Let $t \in [0, 1]$ and $\mu \in \mathbb{M}_1^*$. Then, for any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$,

$$D_{t\mu} F = t^\alpha D_\mu F.$$

Proof. For any measure $\mu \in \mathbb{M}_1$, any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any

$\phi \in \widehat{N}_{\mathbb{X}}$, by Theorem 4.1.22,

$$\begin{aligned} D_{t\mu}F(\phi) &= \mathbb{E}[F(\phi+\Upsilon_{t\mu})-F(\phi)] \\ &= t^{\alpha}\mathbb{E}[F(\phi+\Upsilon_{\mu})-F(\phi)] \\ &= t^{\alpha}D_{\mu}F(\phi). \end{aligned}$$

One can then deduce the result for the case where $\mu \in \mathbb{M}_1^*$. □

Theorem 4.4.8 (Infinitesimal generator).
Let Φ a DαS point process with finite spectral measure σ supported by $\mathbb{M}_1$ and $(P_t)_{t\geq 0}$ its semi-group. Then, its infinitesimal generator L is given for any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$ by:

$$LF(\phi) = \int_{\mathbb{M}_1} D_{\mu}F(\phi)\sigma(\mathrm{d}\mu) + \frac{1}{\alpha}\sum_{y\in\phi}(F(\phi\setminus y)-F(\phi)).$$

Proof. For any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\begin{aligned} \left.\frac{\mathrm{d}P_tF(\phi)}{\mathrm{d}t}\right|_{t=0} &= \lim_{t\to 0}\frac{1}{t}\big(P_tF(\phi)-P_0F(\phi)\big) \\ &= \lim_{t\to 0}\frac{1}{t}\Big(\mathbb{E}[F(e^{-\frac{t}{\alpha}}\circ\phi + \sum_{\mu\in(1-e^{-t})\circ\zeta}\Upsilon_{\mu})]-F(\phi)\Big), \end{aligned}$$

and, for any $t > 0$,

$$\begin{aligned} \mathbb{E}\Big[F\Big(e^{-\frac{t}{\alpha}}\circ\phi + \sum_{\mu\in(1-e^{-t})\circ\zeta}\Upsilon_{\mu}\Big)\Big] &= p_{00}(t)F(\phi)+\sum_{x\in\phi}p_{01}^{(x)}(t)F(\phi\setminus x) \\ &\quad + p_{10}(t)\int_{\mathbb{M}_1}\mathbb{E}[F(\phi+\Upsilon_{\mu})]\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)}+R(t), \end{aligned}$$

where for any $x \in \phi$,

$$\begin{aligned} p_{00}(t) &= \mathbb{P}(e^{-\frac{t}{\alpha}}\circ\phi=\phi,(1-e^{-t})\circ\zeta=\varnothing) \\ &= \mathbb{P}(e^{-\frac{t}{\alpha}}\circ\phi=\phi)\mathbb{P}((1-e^{-t})\circ\zeta=\varnothing) \\ &= e^{-\frac{t|\phi|}{\alpha}}e^{-(1-e^{-t})\sigma(\mathbb{M}_1)}, \end{aligned}$$

$$\begin{aligned} p_{01}^{(x)}(t) &= \mathbb{P}(\phi\setminus(e^{-\frac{t}{\alpha}}\circ\phi)=x,(1-e^{-t})\circ\zeta=\varnothing) \\ &= \mathbb{P}(\phi\setminus(e^{-\frac{t}{\alpha}}\circ\phi)=x)\mathbb{P}((1-e^{-t})\circ\zeta=\varnothing) \\ &= (1-e^{-\frac{t}{\alpha}})e^{-\frac{t}{\alpha}(|\phi|-1)}e^{-(1-e^{-t})\sigma(\mathbb{M}_1)}, \end{aligned}$$

$$\begin{aligned} p_{10}(t) &= \mathbb{P}(e^{-\frac{t}{\alpha}} \circ \phi = \phi, |(1-e^{-t}) \circ \zeta| = 1) \\ &= \mathbb{P}(e^{-\frac{t}{\alpha}} \circ \phi = \phi)\mathbb{P}(|(1-e^{-t}) \circ \zeta| = 1) \\ &= e^{-\frac{t|\phi|}{\alpha}}(1-e^{-t})\sigma(\mathbb{M}_1)e^{-(1-e^{-t})\sigma(\mathbb{M}_1)}, \end{aligned}$$

$$R(t) = \mathbb{E}\Big[F(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu)\mathbf{1}_{|\phi \setminus (e^{-\frac{t}{\alpha}} \circ \phi)| + |(1-e^{-t})\circ\zeta| \geq 2}\Big].$$

Then,

$$\begin{aligned} &\frac{1}{t}\Big(\mathbb{E}[F(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu)] - F(\phi)\Big) = \\ &= \frac{1}{t}\Big(\sum_{x\in\phi} p_{01}^{(x)}(t)(F(\phi \setminus x) - F(\phi)) \\ &\qquad + p_{10}(t)\int_{\mathbb{M}_1} \mathbb{E}[F(\phi + \Upsilon_\mu) - F(\phi)]\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)} - p_\infty(t)F(\phi) + R(t)\Big) \\ &= \sum_{x\in\phi} \frac{p_{01}^{(x)}(t)}{t}\big(F(\phi \setminus x) - F(\phi)\big) \\ &\qquad + \frac{p_{10}(t)}{t}\int_{\mathbb{M}_1} \mathbb{E}[F(\phi + \Upsilon_\mu) - F(\phi)]\frac{\sigma(\mathrm{d}\mu)}{\sigma(\mathbb{M}_1)} - \frac{p_\infty(t)}{t}F(\phi) + \frac{R(t)}{t}, \end{aligned}$$

where

$$\begin{aligned} p_\infty(t) &= \mathbb{P}(|\phi \setminus (e^{-\frac{t}{\alpha}} \circ \phi)| + |(1-e^{-t}) \circ \zeta| \geq 2) \\ &= 1 - \Big(p_{00}(t) + \sum_{x\in\phi} p_{01}^{(x)}(t) + p_{10}(t)\Big). \end{aligned}$$

Since for any $x \in \phi$,

$$\lim_{t\to 0} \frac{p_{01}^{(x)}(t)}{t} = \frac{1}{\alpha},\ \lim_{t\to 0} \frac{p_{10}(t)}{t} = \sigma(\mathbb{M}_1) \text{ and } \lim_{t\to 0} \frac{1-p_{00}(t)}{t} = \frac{|\phi|}{\alpha} + \sigma(\mathbb{M}_1),$$

we get that

$$\lim_{t\to 0} \frac{p_\infty(t)}{t} = 0,$$

then by boundedness of F that

$$\lim_{t\to 0} \frac{R(t)}{t} = 0,$$

hence the result. □

Lemma 4.4.9 (Commutation relation).
Let Φ a DαS point process with finite spectral measure σ supported by $\mathbb{M}_1$, $(P_t)_{t\geq 0}$ its semi-group. Then, for any $t \in \mathbb{R}_+$, any $\mu \in \mathbb{M}_1^*$, any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$D_\mu P_t F(\phi) = P_t D_{e^{-\frac{t}{\alpha}}\mu} F(\phi) = e^{-t} P_t D_\mu F(\phi),$$

and for any $x \in \mathbb{X}$,

$$D_x P_t F(\phi) = e^{-\frac{t}{\alpha}} P_t D_x F(\phi).$$

Proof. For any $t \in \mathbb{R}_+$, any $\mu \in \mathbb{M}_1^*$, any measurable and bounded function $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ and any $\phi \in \widehat{N}_{\mathbb{X}}$, using the definitions of D_μ and P_t,

$$\begin{aligned} D_\mu P_t F(\phi) &= \int_{\widehat{N}_{\mathbb{X}}} P_t F(\phi + \upsilon) - P_t F(\phi) \mathbb{P}_{\Upsilon_\mu}(\mathrm{d}\upsilon) \\ &= \int_{\widehat{N}_{\mathbb{X}}} \mathbb{E}\Big[F\Big(e^{-\frac{t}{\alpha}} \circ (\phi + \upsilon) + \sum_{\nu \in (1-e^{-t})\circ\zeta} \Upsilon_\nu \Big) - F\Big(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\nu \in (1-e^{-t})\circ\zeta} \Upsilon_\nu \Big) \Big] \mathbb{P}_{\Upsilon_\mu}(\mathrm{d}\upsilon). \end{aligned}$$

Hence, since thinning is distributive with respect to sum,

$$\begin{aligned} D_\mu P_t F(\phi) = \int_{\widehat{N}_{\mathbb{X}}} \mathbb{E}\Big[F\Big(e^{-\frac{t}{\alpha}} \circ \phi + e^{-\frac{t}{\alpha}} \circ \upsilon + \sum_{\nu \in (1-e^{-t})\circ\zeta} \Upsilon_\nu \Big) \\ - F\Big(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\nu \in (1-e^{-t})\circ\zeta} \Upsilon_\nu \Big) \Big] \mathbb{P}_{\Upsilon_\mu}(\mathrm{d}\upsilon), \end{aligned}$$

and then, still using the definitions of D_μ and P_t,

$$D_\mu P_t F(\phi) \quad = \quad P_t D_{e^{-\frac{t}{\alpha}}\mu} F(\phi) = e^{-t} P_t D_\mu F(\phi),$$

where the last equality is deduced from Theorem 4.4.7. The second part of the lemma is shown in a similar way by replacing D_μ by D_x. □

Lemma 4.4.10 (Ergodicity).
Let Φ a DαS point process with finite spectral measure σ supported by $\mathbb{M}_1$ and $(P_t)_{t\geq 0}$ its semi-group. For any $F \in Lip_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\lim_{t \to +\infty} P_t F(\phi) = \mathbb{E}[F(\Phi)].$$

Proof. For any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$, $t \in \mathbb{R}_+$ and $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\begin{aligned}
&\left|P_t F(\phi) - \mathbb{E}[F(\Phi)]\right| \le \left|P_t F(\phi) - P_t F(\varnothing)\right| + \left|P_t F(\varnothing) - \mathbb{E}[F(\Phi)]\right| \\
&= \left|\mathbb{E}\Big[F\Big(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big] - \mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big]\right| \\
&\qquad\qquad\qquad\qquad + \left|\mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big] - \mathbb{E}[F(\Phi)]\right|.
\end{aligned}$$

On one hand, since $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$,

$$\begin{aligned}
\left|\mathbb{E}\Big[F\Big(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big] - \mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big]\right| &\le \mathbb{E}[\Delta_D(e^{-\frac{t}{\alpha}} \circ \phi, \varnothing)] \\
&= \mathbb{P}(|e^{-\frac{t}{\alpha}} \circ \phi| \neq 0),
\end{aligned}$$

and, since $|e^{-\frac{t}{\alpha}} \circ \phi|$ has a binomial distribution with parameters $|\phi|$ and $e^{-\frac{t}{\alpha}}$,

$$\left|\mathbb{E}\Big[F\Big(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big] - \mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big]\right| \le 1 - (1 - e^{-\frac{t}{\alpha}})^{|\phi|}.$$

On the other hand,

$$\mathbb{E}[F(\Phi)] = \mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu + \sum_{\mu \in e^{-t}\circ\zeta} \Upsilon_\mu\Big)\Big],$$

then, since $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$,

$$\begin{aligned}
\left|\mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big] - \mathbb{E}[F(\Phi)]\right| &\le \mathbb{E}\Big[\Delta_D\Big(\sum_{\mu \in e^{-t}\circ\zeta} \Upsilon_\mu, \varnothing\Big)\Big] \\
&= \mathbb{P}(|e^{-t} \circ \zeta| \neq 0),
\end{aligned}$$

and, since $|e^{-t} \circ \zeta|$ has a Poisson distribution with parameter $e^{-t}\sigma(\mathbb{M}_1)$,

$$\left|\mathbb{E}\Big[F\Big(\sum_{\mu \in (1-e^{-t})\circ\zeta} \Upsilon_\mu\Big)\Big] - \mathbb{E}[F(\Phi)]\right| \le 1 - e^{-e^{-t}\sigma(\mathbb{M}_1)},$$

which concludes this proof. □

Theorem 4.4.11 (Stein-Dirichlet representation formula).
Let Φ a DαS point process with finite spectral measure σ supported by $\mathbb{M}_1$, $(P_t)_{t\ge 0}$ its semi-group and L its infinitesimal generator. For any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\mathbb{E}[F(\Phi)] - F(\phi) = \int_0^{+\infty} LP_s F(\phi)\mathrm{d}s.$$

Proof. Since $(P_t)_{t\geq 0}$ is a semi-group, we get as in the proof of Theorem 3.3.1 that, for any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$ and any $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\int_0^{+\infty} LP_s F(\phi)\mathrm{d}s = \lim_{s\to+\infty} P_s F(\phi) - P_0 F(\phi).$$

Then, by Lemma 4.4.10,

$$\int_0^{+\infty} LP_s F(\phi)\mathrm{d}s = \mathbb{E}[F(\Phi)] - F(\phi).$$

The proof is thus complete. □

4.5 α-Papangelou intensity

The α-Papangelou intensity adapts the definition of Papangelou intensity to DαS point processes.

Definition 4.5.1 (α**-Papangelou intensity**).
*Let Φ be a point process on $\mathbb{X}$ and λ a Radon measure on $\mathbb{M}_1$. The application $c_\alpha : \mathbb{M}_1 \times N_{\mathbb{X}} \to \mathbb{R}_+$ is called (a version of) the α-**Papangelou intensity** of Φ with respect to λ if, for any measurable function $u : N_{\mathbb{X}} \to \mathbb{R}_+$,*

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi) - u(\Phi \setminus y))\Big] = \alpha\mathbb{E}\Big[\int_{\mathbb{M}_1} c_\alpha(\mu, \Phi) D_\mu u(\Phi)\lambda(\mathrm{d}\mu)\Big].$$

Theorem 4.5.2 (α**-Papangelou intensity of a DαS point process**).
Let Φ be a DαS point process such that its finite spectral measure σ is absolutely continuous with respect to a Radon measure λ on $\mathbb{M}_1$.

Then, the application $c_\alpha : \mathbb{M}_1 \times N_{\mathbb{X}} \to \mathbb{R}_+$ defined for any $(\mu, \phi) \in \mathbb{M}_1 \times N_{\mathbb{X}}$ by

$$c_\alpha(\mu, \phi) = \frac{\mathrm{d}\sigma(\mu)}{\mathrm{d}\lambda}$$

is the α-Papangelou intensity of Φ with respect to λ.

Proof. This result is a direct consequence of Theorem 4.2.6. □

Theorem 4.5.3 (Papangelou intensity and α-Papangelou intensity).
Let $\Phi = \sum_{\mu\in\zeta}\Upsilon_\mu$ be a point process on $\mathbb{X}$ where ζ is a point process on $\mathbb{M}_1$ with Papangelou intensity c with respect to a Radon measure λ and such that the Sibuya point processes of the sum are independent. Then, the α-Papangelou intensity c_α of Φ with respect to λ verifies, for any $\mu \in \mathbb{M}_1$,

$$c_\alpha(\mu,\Phi) = c(\mu,\zeta) \text{ a.s.}.$$

Proof. For $u(\phi) = \mathbf{1}_{\{\phi(K)=0\}}$, $\phi \in N_{\mathbb{X}}$ and K a relatively compact subset, on one hand, one notes that

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] = -\mathbb{E}[\mathbf{1}_{\{\Phi(K)=1\}}].$$

Thus,

$$\begin{aligned}\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] &= -\mathbb{E}[\mathbf{1}_{\{\sum_{\nu\in\zeta}\Upsilon_\nu(K)=1\}}] \\ &= -\mathbb{E}\Big[\sum_{\mu\in\zeta}\mathbf{1}_{\{\Upsilon_\mu(K)=1\}}\mathbf{1}_{\{\sum_{\nu\in\zeta\setminus\mu}\Upsilon_\nu(K)=0\}}\Big],\end{aligned}$$

and, conditioning with respect to ζ,

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] = -\mathbb{E}\Big[\sum_{\mu\in\zeta}\mathbb{P}(\Upsilon_\mu(K)=1\mid\zeta)\prod_{\nu\in\zeta\setminus\mu}\mathbb{P}(\Upsilon_\nu(K)=0\mid\zeta)\Big],$$

where, almost surely, for any $\mu, \nu \in \zeta$,

$$\mathbb{P}(\Upsilon_\mu(K)=1\mid\zeta) = \alpha\mu^\alpha(K)$$

and

$$\mathbb{P}(\Upsilon_\nu(K)=0\mid\zeta) = (1-\nu^\alpha(K)).$$

Then,

$$\begin{aligned}\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] &= -\mathbb{E}\Big[\sum_{\mu\in\zeta}\alpha\mu^\alpha(K)\prod_{\nu\in\zeta\setminus\mu}(1-\nu^\alpha(K))\Big] \\ &= -\alpha\mathbb{E}\Big[\sum_{\mu\in\zeta}\mu^\alpha(K)\prod_{\nu\in\zeta\setminus\mu}(1-\nu^\alpha(K))\Big].\end{aligned}$$

On the other hand, using the expressions of D_μ and u,

$$\alpha\int_{\mathbb{M}_1}\mathbb{E}[c(\mu,\zeta)D_\mu u(\Phi)]\lambda(\mathrm{d}\mu) = \alpha\int_{\mathbb{M}_1}\mathbb{E}\big[c(\mu,\zeta)\big(\mathbf{1}_{\{(\Phi+\Upsilon_\mu)(K)=0\}}-\mathbf{1}_{\{\Phi(K)=0\}}\big)\big]\lambda(\mathrm{d}\mu).$$

Then,

$$\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) D_\mu u(\Phi)] \lambda(\mathrm{d}\mu) = -\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) \mathbf{1}_{\{\Phi(K)=0\}} \mathbf{1}_{\{\Upsilon_\mu(K)\neq 0\}}] \lambda(\mathrm{d}\mu),$$

and, conditioning with respect to ζ,

$$\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) D_\mu u(\Phi)] \lambda(\mathrm{d}\mu) = -\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) \mathbb{P}(\Phi(K)=0 \mid \zeta) \mathbb{P}(\Upsilon_\mu(K)\neq 0)] \lambda(\mathrm{d}\mu).$$

Hence, since, almost surely, for any $\mu \in \mathbb{M}_1$,

$$\mathbb{P}(\Phi(K)=0 \mid \zeta) = \prod_{\nu\in\zeta} \mathbb{P}(\Upsilon_\nu(K)=0 \mid \zeta) = \prod_{\nu\in\zeta} (1-\nu^\alpha(K))$$

and

$$\mathbb{P}(\Upsilon_\mu(K)\neq 0) = \mu^\alpha(K),$$

it follows that

$$\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) D_\mu u(\Phi)] \lambda(\mathrm{d}\mu) = -\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) \mu^\alpha(K) \prod_{\nu\in\zeta} (1-\nu^\alpha(K))] \lambda(\mathrm{d}\mu).$$

Then, by the definition of the Papangelou intensity,

$$\alpha \int_{\mathbb{M}_1} \mathbb{E}[c(\mu,\zeta) D_\mu u(\Phi)] \lambda(\mathrm{d}\mu) = -\alpha \mathbb{E}\Big[\sum_{\mu\in\zeta} \mu^\alpha(K) \prod_{\nu\in\zeta\setminus\mu} (1-\nu^\alpha(K))\Big],$$

and then the equation is true for $u = \mathbf{1}_{\{\cdot\cap K=\varnothing\}}$.

By Dynkin's $\pi-\lambda$ theorem, since $\big\{\{\Phi(K)=0\}, K \text{ relatively compact subset of } \mathbb{X}\big\}$ is a π-system, the equation is also true for $u=\mathbf{1}_A$, where $A\in\mathcal{N}_{\mathbb{X}}$, then by linearity for u simple positive function, then using Monotone property for u measurable positive, which concludes the proof. □

Theorem 4.5.4 (**α-Papangelou intensity of a superposition**).

Let $\Phi_1,\ldots,\Phi_n$ ($n\in\mathbb{N}$) be independent point processes on $\mathbb{X}$ with respective α-Papangelou intensities $c_{\alpha,1},\ldots,c_{\alpha,n}$ and Φ their independent superposition. Then, the α-Papangelou c_α of Φ verifies for any $\mu\in\mathbb{M}_1$

$$c_\alpha(\mu,\Phi) = \sum_{k=1}^{n} c_{\alpha,k}(\mu,\Phi_k) \text{ a.s..}$$

Proof. For any measurable $u : N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] = \sum_{k=1}^{n}\mathbb{E}\Big[\sum_{y\in\Phi_k}(u(\Phi)-u(\Phi\setminus y))\Big]$$
$$= \sum_{k=1}^{n}\int_{N_{\mathbb{X}}}\dots\int_{N_{\mathbb{X}}}\sum_{y\in\phi_k}(u(\phi_1\dots\phi_n)-u(\phi_1\dots\phi_n\setminus y))\mathbb{P}_{\Phi_1}(\phi_1)\dots\mathbb{P}_{\Phi_n}(\phi_n).$$

Then, applying the definition of the Papangelou intensity for each Φ_k ($k\in\{1,\dots,n\}$),

$$\begin{aligned}\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big] &= \alpha\sum_{k=1}^{n}\mathbb{E}\Big[\int_{\mathbb{M}_1}c_{\alpha,k}(\mu,\Phi_k)D_\mu u(\Phi)\lambda(\mathrm{d}\mu)\Big]\\ &= \alpha\mathbb{E}\Big[\sum_{k=1}^{n}\int_{\mathbb{M}_1}c_{\alpha,k}(\mu,\Phi_k)D_\mu u(\Phi)\lambda(\mathrm{d}\mu)\Big],\end{aligned}$$

from which we can conclude. □

Theorem 4.5.5 (**α-Papangelou intensity of a thinned point process**).
Let Φ be a point process on $\mathbb{X}$, let β be a function from $\mathbb{X}$ to $[0,1]$ and $\beta\circ\Phi$ the β-thinning of Φ. Then, its α-Papangelou intensity $c_{\alpha,\beta}$ verifies for any $\mu\in\mathbb{M}_1$

$$c_{\alpha,\beta}(\mu,\beta\circ\Phi) = \beta(x)\mathbb{E}[c_\alpha(\mu,\Phi)\mid\beta\circ\Phi]\ a.s..$$

Proof. For any measurable function $u : \mathbb{X}\times N_{\mathbb{X}} \to \mathbb{R}_+$, we denote, for any $\varphi,\phi\in N_{\mathbb{X}}$, $U(\varphi,\phi) = u(\varphi\phi)-u(\phi)$. It follows that

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi}U(x,\beta\circ\Phi\setminus x)\Big] &= \mathbb{E}\Big[\sum_{x\in\Phi}U(x,\beta\circ\Phi\setminus x)\mathbf{1}_{x\in\beta\circ\Phi}\Big]\\ &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi}U(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\mathbf{1}_{\tau=\beta\circ\Phi}\Big],\end{aligned}$$

thus, conditioning with respect to Φ,

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi}U(x,\beta\circ\Phi\setminus x)\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi}\mathbb{E}[U(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\mathbf{1}_{\tau=\beta\circ\Phi}\mid\Phi]\Big]\Big]\\ &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi}\mathbb{P}(\tau=\beta\circ\Phi\mid\Phi)U(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\Big].\end{aligned}$$

Since, for any $\tau\subset\phi$, $\mathbb{P}(\tau=\beta\circ\phi) = \Big(\prod_{x\in\tau}\beta(x)\Big)\Big(\prod_{x\in\phi\setminus\tau}(1-\beta(x))\Big)$, one gets:

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} U(x,\beta\circ\Phi\setminus x)\Big] &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi}\Big(\prod_{y\in\tau}\beta(y)\Big)\Big(\prod_{y\in\phi\setminus\tau}(1-\beta(y))\Big)U(x,\tau\setminus x)\mathbf{1}_{x\in\tau}\Big] \\ &= \mathbb{E}\Big[\sum_{x\in\Phi}\sum_{\tau\subset\Phi\setminus x}\beta(x)\Big(\prod_{y\in\tau}\beta(y)\Big)\Big(\prod_{y\in(\Phi\setminus x)\setminus\tau}(1-\beta(y))\Big)U(x,\tau)\Big].\end{aligned}$$

Then, by using the definition of the α-Papangelou intensity,

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} U(x,\beta\circ\Phi\setminus x)\Big] = \\ &= \alpha\int_{\mathbb{M}_1}\int_{N_{\mathbb{X}}} c_\alpha(\mu,\phi) \\ &\qquad \int_{N_{\mathbb{X}}}\sum_{\tau\subset\phi}\beta(x)\Big(\prod_{y\in\tau}\beta(y)\Big)\Big(\prod_{y\in\phi\setminus\tau}(1-\beta(y))\Big)U(v,\tau)\mathbb{P}_{\Upsilon_\mu}(\mathrm{d}v)\mathbb{P}_\Phi(\mathrm{d}\phi)\lambda(\mathrm{d}\mu).\end{aligned}$$

The previous arguments yield

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} U(x,\beta\circ\Phi\setminus x)\Big] = \\ &= \alpha\beta(x)\int_{\mathbb{M}_1}\int_{N_{\mathbb{X}}} c_\alpha(\mu,\phi)\int_{N_{\mathbb{X}}}\sum_{\tau\subset\phi}\mathbb{P}(\beta\circ\phi=\tau\mid\phi)U(v,\tau)\mathbb{P}_{\Upsilon_\mu}(\mathrm{d}v)\mathbb{P}_\Phi(\mathrm{d}\phi)\lambda(\mathrm{d}\mu)\end{aligned}$$

and then that

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x\in\beta\circ\Phi} U(x,\beta\circ\Phi\setminus x)\Big] = \\ &= \alpha\beta(x)\int_{\mathbb{M}_1}\int_{N_{\mathbb{X}}} c_\alpha(\mu,\phi)\sum_{\tau\subset\phi}\mathbf{1}_{\beta\circ\phi=\tau}\int_{N_{\mathbb{X}}} U(v,\tau)\mathbb{P}_{\Upsilon_\mu}(\mathrm{d}v)\mathbb{P}_\Phi(\mathrm{d}\phi)\lambda(\mathrm{d}\mu) \\ &= \alpha\beta(x)\int_{\mathbb{M}_1}\int_{N_{\mathbb{X}}} c_\alpha(\mu,\phi)\int_{N_{\mathbb{X}}} U(v,\phi)\mathbb{P}_{\Upsilon_\mu}(\mathrm{d}v)\mathbb{P}_{\beta\circ\Phi}(\mathrm{d}\phi)\lambda(\mathrm{d}\mu),\end{aligned}$$

from which we conclude the proof. □

4.6 α-dependent convergence results

In this section, convergence results are given for Δ_{TV} instead of Δ^*_{TV}. Indeed, this last distance is too strong to be used between discrete stable point processes, since the number of points of such point processes has an infinite expectation. In particular, the upper bound provided by Theorem 3.5.2 for Cox point processes - the Kantorovich-Rubinstein distance between two StS measures - is infinite in this case.

Theorem 4.6.1 (**Distance between discrete stable point processes**).
For any $i \in \{1,2\}$, let Φ_i be a Dα_iS point process with almost surely finite Stα_iS random intensity ξ_i. Then,

$$\Delta_{TV}(\Phi_1, \Phi_2) \leq \Delta_D^*(\xi_1, \xi_2).$$

Proof. For any $i \in \{1,2\}$, by Theorem 4.1.9, Φ_i is a Cox point process directed by ξ_i. The result follows by Remark 3.5.3. □

Lemma 4.6.2 (**Distance between a DαS point process and a PPP**).
Let Φ_α be a DαS point process with finite spectral measure σ and Φ be a Poisson point process on $\mathbb{X}$ with intensity measure $\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu)$. Then,

$$\Delta_{TV}(\Phi_\alpha, \Phi) \leq (1-\alpha)C(\alpha)\sigma(\mathbb{M}_1),$$

where, for any $a \in (0,1]$,

$$C(a) = 2\min\Big(\Big\{1+a, 1+2\frac{e^{-2}}{a}\Big\}\Big) \leq 2\Big(1+\frac{\sqrt{2}}{e}\Big).$$

Proof. Let recall that, by Theorem 2.2.10, $\Delta_{TV} = \Delta_D^*$. Let $F \in \mathrm{Lip}_1(\widehat{N_{\mathbb{X}}}, \Delta_D)$. By Theorem 4.4.11,

$$\mathbb{E}[F(\Phi_\alpha)] - \mathbb{E}[F(\Phi)] = \mathbb{E}\Big[\int_0^{+\infty} LP_sF(\Phi)\mathrm{d}s\Big],$$

where L is the infinitesimal generator associated to Φ_α and $(P_t)_{t\geq 0}$ its semi-group. Then, by Theorem 4.4.8,

$$\mathbb{E}[F(\Phi_\alpha)] - \mathbb{E}[F(\Phi)] = \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{M}_1} D_\mu P_sF(\Phi)\sigma(\mathrm{d}\mu)\Big] + \frac{1}{\alpha}\mathbb{E}\Big[\sum_{y\in\Phi} P_sF(\Phi\setminus y) - P_sF(\Phi)\Big]\mathrm{d}s.$$

By the Mecke formula applied to the Poisson point process Φ, one has for any $s \in \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{y\in\Phi} P_sF(\Phi\setminus y) - P_sF(\Phi)\Big] = -\mathbb{E}\Big[\int_{\mathbb{M}_1}\int_{\mathbb{X}} D_xP_sF(\Phi)\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu)\Big],$$

then, by Theorem 4.4.7 and Lemma 4.4.9,

$$\mathbb{E}[F(\Phi_\alpha)]-\mathbb{E}[F(\Phi)]=\int_0^{+\infty}\mathbb{E}\Big[\int_{\mathbb{M}_1}e^{-s}P_sD_\mu F(\Phi)\sigma(\mathrm{d}\mu)\Big]$$
$$-\frac{1}{\alpha}\mathbb{E}\Big[\int_{\mathbb{M}_1}e^{-\frac{s}{\alpha}}\int_{\mathbb{X}}P_sD_xF(\Phi)\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu)\Big]\mathrm{d}s$$

and then

$$\mathbb{E}[F(\Phi_\alpha)]-\mathbb{E}[F(\Phi)]=\int_0^{+\infty}\int_{\mathbb{M}_1}\mathbb{E}\Big[P_s\Big(e^{-s}D_\mu F(\cdot)-\frac{e^{-\frac{s}{\alpha}}}{\alpha}\int_{\mathbb{X}}D_xF(\cdot)\mu(\mathrm{d}x)\Big)(\Phi)\Big]\sigma(\mathrm{d}\mu)\mathrm{d}s.$$

Thus, by a change of variable in the integral, it follows that

$$\mathbb{E}[F(\Phi_\alpha)]-\mathbb{E}[F(\Phi)]=\int_0^{+\infty}\int_{\mathbb{M}_1}\mathbb{E}\Big[P_s\Big(\alpha e^{-\alpha s}D_\mu F(\cdot)-e^{-s}\int_{\mathbb{X}}D_xF(\cdot)\mu(\mathrm{d}x)\Big)(\Phi)\Big]\sigma(\mathrm{d}\mu)\mathrm{d}s.$$

For any $\phi\in N_{\mathbb{X}}$, $s\in\mathbb{R}_+$ and $\mu\in\mathbb{M}_1$,

$$\begin{aligned}&\Big|\alpha e^{-\alpha s}D_\mu F(\phi)-e^{-s}\int_{\mathbb{X}}D_xF(\phi)\mu(\mathrm{d}x)\Big|=\\&=\Big|\alpha e^{-\alpha s}\mathbb{E}[(F(\phi+\Upsilon_\mu)-F(\phi))\mathbf{1}_{\{|\Upsilon_\mu|=1\}}]+\alpha e^{-\alpha s}\mathbb{E}\Big[(F(\phi+\Upsilon_\mu)-F(\phi))\mathbf{1}_{\{|\Upsilon_\mu|\geq 2\}}\Big]\\&\qquad-e^{-s}\int_{\mathbb{X}}(F(\phi+x)-F(\phi))\mu(\mathrm{d}x)\Big|\\&=\Big|(\alpha^2e^{-\alpha s}-e^{-s})\int_{\mathbb{X}}(F(\phi+x)-F(\phi))\mu(\mathrm{d}x)+\alpha e^{-\alpha s}\mathbb{E}\Big[(F(\phi+\Upsilon_\mu)-F(\phi))\mathbf{1}_{\{|\Upsilon_\mu|\geq 2\}}\Big]\Big|,\end{aligned}$$

using that

$$\begin{aligned}\mathbb{E}\Big[(F(\phi+\Upsilon_\mu)-F(\phi))\mathbf{1}_{\{|\Upsilon_\mu|=1\}}\Big]&=\mathbb{P}(|\Upsilon_\mu|=1)\int_{\mathbb{X}}\Big(F(\phi+x)-F(\phi)\Big)\mu(\mathrm{d}x)\\&=\alpha\int_{\mathbb{X}}\Big(F(\phi+x)-F(\phi)\Big)\mu(\mathrm{d}x).\end{aligned}$$

Hence, since $F\in\mathrm{Lip}_1(\widehat{N}_{\mathbb{X}},\Delta_D)$ and $\mathbb{P}(|\Upsilon_\mu|\geq 2)=1-\alpha$,

$$\Big|\alpha e^{-\alpha s}D_\mu F(\phi)-e^{-s}\int_{\mathbb{X}}(F(\phi+x)-F(\phi))\mu(\mathrm{d}x)\Big|\leq\Big|\alpha^2e^{-\alpha s}-e^{-s}\Big|+\alpha e^{-\alpha s}(1-\alpha).$$

Then, since $\|P_s\| \leq 1$, one has:

$$\begin{aligned}\Delta_{TV}(\Phi, \Phi_\alpha) &\leq \Big(\int_0^{+\infty} |\alpha^2 e^{-\alpha s} - e^{-s}| \mathrm{d}s + (1-\alpha)\int_0^{+\infty} \alpha e^{-\alpha s} \mathrm{d}s\Big)\sigma(\mathbb{M}_1) \\ &= \Big(\int_0^{+\infty} |\alpha^2 e^{-\alpha s} - e^{-s}| \mathrm{d}s + 1 - \alpha\Big)\sigma(\mathbb{M}_1).\end{aligned}$$

Let g be the function defined for any $s \in \mathbb{R}_+$ by $g(s) = \alpha^2 e^{-\alpha s} - e^{-s}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s) = e^{-s}(1 - \alpha^3 e^{(1-\alpha)s}),$$

we get that g is strictly increasing on $[0, \frac{3\ln\alpha}{\alpha-1}]$ and strictly decreasing on $[\frac{3\ln\alpha}{\alpha-1}, +\infty)$. Moreover, $g(s) = 0 \iff s = \frac{2\ln\alpha}{\alpha-1}$, then g is negative on $[0, \frac{2\ln\alpha}{\alpha-1}]$ and positive on $[\frac{2\ln(\alpha)}{\alpha-1}, +\infty)$. Thus,

$$\begin{aligned}\int_0^{+\infty} |\alpha^2 e^{-\alpha s} - e^{-s}| \mathrm{d}s &= \int_0^{\frac{2\ln\alpha}{\alpha-1}} (e^{-s} - \alpha^2 e^{-\alpha s}) \mathrm{d}s + \int_{\frac{2\ln\alpha}{\alpha-1}}^{+\infty} (\alpha^2 e^{-\alpha s} - e^{-s}) \mathrm{d}s \\ &= [\alpha e^{-\alpha s} - e^{-s}]_0^{\frac{2\ln\alpha}{\alpha-1}} + [e^{-s} - \alpha e^{-\alpha s}]_{\frac{2\ln\alpha}{\alpha-1}}^{+\infty} \\ &= (1-\alpha)\Big(\frac{2}{\alpha} e^{\frac{2\ln\alpha}{1-\alpha}} + 1\Big),\end{aligned}$$

hence, since $\ln\alpha \leq \alpha - 1$, the result is obtained. □

Theorem 4.6.3 (Distance between DS point processes with a fixed measure σ). *Let $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. Let Φ_α and Φ_β be respectively DαS and DβS point processes, with the same finite spectral measure σ. Then,*

$$\Delta_{TV}(\Phi_\alpha, \Phi_\beta) \leq \Big(1 - \frac{\alpha}{\beta}\Big) C\Big(\frac{\alpha}{\beta}\Big)\sigma(\mathbb{M}_1),$$

where, for any $a \in (0,1]$,

$$C(a) = 2\min\Big(\Big\{1 + a, 1 + 2\frac{e^{-2}}{a}\Big\}\Big) \leq 2\Big(1 + \frac{\sqrt{2}}{e}\Big).$$

Proof. Let $\gamma = \dfrac{\alpha}{\beta}$. By Theorem 4.3.2, for any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$,

$$\mathbb{E}[F(\Phi_\alpha)] - \mathbb{E}[F(\Phi_\beta)] = \mathbb{E}\Big[F\Big(\sum_{\mu \in \Psi'_\gamma} \Upsilon_\mu^\beta\Big)\Big] - \mathbb{E}\Big[F\Big(\sum_{\mu \in \zeta'} \Upsilon_\mu^\beta\Big)\Big],$$

where Ψ'_γ is a DγS point process on $\mathbb{M}_1$ with spectral measure $\sigma' = \sigma \circ \iota^{-1}$ with

$$\iota : \mu \in \mathbb{S} \mapsto \delta_\mu \in \mathbb{S}'$$

and ζ' is a Poisson point process on $\mathbb{M}_1$ with intensity measure σ. Then,

$$\mathbb{E}[F(\Phi_\alpha)] - \mathbb{E}[F(\Phi_\beta)] = \mathbb{E}[G(\Psi'_\gamma)] - \mathbb{E}[G(\zeta')],$$

where, for any $\phi' \in N_{\mathbb{M}_1}$,

$$G(\phi') = \mathbb{E}\Big[F\Big(\sum_{\mu\in\phi'} \Upsilon_\mu^\beta\Big)\Big].$$

Moreover, for any $\phi', \omega' \in N_{\mathbb{M}_1}$,

$$|G(\phi') - G(\omega')| \le \mathbb{E}\Big[\Big|F\Big(\sum_{\mu\in\phi'} \Upsilon_\mu^\beta)\Big) - F\Big(\sum_{\mu\in\omega'} \Upsilon_\mu^\beta)\Big)\Big|\Big] \le 1,$$

then $G \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$, and then, by Lemma 4.6.2,

$$\Delta_{TV}(\Phi_\alpha, \Phi_\beta) \le \Delta^*_{D_{|\mathbb{M}'_1}}(\Psi'_\gamma, \zeta') \le C(\gamma)(1-\gamma)\sigma'(\mathbb{M}'_1).$$

Hence, since $\sigma'(\mathbb{M}'_1) = \sigma(\mathbb{M}_1)$, the expected result is obtained. □

Theorem 4.6.4 (Distance between a DαS point process and a PPP: case $\sigma = c\delta_\mu$). *Let Φ_α be a DαS point process with spectral measure $\sigma = c\delta_\mu$, where $c \in \mathbb{R}_+$ and $\mu \in \mathbb{M}_1$. Let Φ be a Poisson point process with intensity measure $c\mu$. Then, for any $\epsilon \in \mathbb{R}_+$,*

$$\Delta_{TV}(\Phi_\alpha, \Phi) \le c^{\frac{1}{\alpha}}\epsilon + |c^{\frac{1}{\alpha}} - c| + \mathbb{P}(|X_\alpha - 1| \ge \epsilon),$$

where X_α is a positive strictly α-stable random variable with Laplace functional given, for any $t \in \mathbb{R}_+$, by

$$\mathbb{E}[e^{-tX_\alpha}] = e^{-t^\alpha}.$$

Proof. For any $\epsilon \in \mathbb{R}_+$ and $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$, by the triangle inequality,

$$\Big|\mathbb{E}[F(\Phi_\alpha)] - \mathbb{E}[F(\Phi)]\Big| \le \Big|\mathbb{E}\Big[(F(\Phi_\alpha) - F(\Phi))\mathbf{1}_{\{|X_\alpha - 1|\le\epsilon\}}\Big]\Big| + \Big|\mathbb{E}\Big[(F(\Phi_\alpha) - F(\Phi))\mathbf{1}_{\{|X_\alpha - 1|\ge\epsilon\}}\Big]\Big|.$$

On one hand, conditioning with respect to X_α,

$$\begin{aligned}\Big|\mathbb{E}\Big[(F(\Phi_\alpha) - F(\Phi))\mathbf{1}_{\{|X_\alpha - 1|\le\epsilon\}}\Big]\Big| &= \Big|\mathbb{E}\Big[\mathbf{1}_{\{|X_\alpha - 1|\le\epsilon\}}\mathbb{E}[(F(\Phi_\alpha) - F(\Phi))\Big|X_\alpha]\Big]\Big| \\ &\le \mathbb{E}\Big[\mathbf{1}_{\{|X_\alpha - 1|\le\epsilon\}}\Big|\mathbb{E}\Big[(F(\Phi_\alpha) - F(\Phi))\Big|X_\alpha\Big]\Big|\Big].\end{aligned}$$

Furthermore, by Theorem 4.1.7, Φ_α is a Cox point process with random intensity $X_\alpha c^{\frac{1}{\alpha}}\mu$, then, by Theorem 3.3.2,

$$\begin{aligned}\left|\mathbb{E}\left[(F(\Phi_\alpha)-F(\Phi))\middle|X_\alpha\right]\right| &\leq \int_{\mathbb{X}}\left|X_\alpha c^{\frac{1}{\alpha}}-c\right|\mu(\mathrm{d}x)\\ &= |X_\alpha c^{\frac{1}{\alpha}}-c|\\ &\leq c^{\frac{1}{\alpha}}|X_\alpha-1|+|c^{\frac{1}{\alpha}}-c| \text{ a.s.,}\end{aligned}$$

and then

$$\left|\mathbb{E}\left[(F(\Phi_\alpha)-F(\Phi))\mathbf{1}_{\{|X_\alpha-1|\leq\epsilon\}}\right]\right| \leq c^{\frac{1}{\alpha}}\epsilon+|c^{\frac{1}{\alpha}}-c|.$$

On the other hand, since $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$,

$$\left|\mathbb{E}\left[(F(\Phi_\alpha)-F(\Phi))\mathbf{1}_{\{|X_\alpha-1|\geq\epsilon\}}\right]\right| \leq \mathbb{P}(|X_\alpha-1|\geq\epsilon),$$

and it yields the expected result. □

Let compare the upper bounds obtained in Lemma 4.6.2 and Theorem 4.6.4 for the DαS point process Φ_α with spectral measure $\sigma = \delta_\mu$ and the Poisson point process Φ with intensity measure μ. By Lemma 4.6.2,

$$\Delta_{TV}(\Phi_\alpha, \Phi) \leq (1-\alpha)C(\alpha) \leq 4(1-\alpha),$$

then it seems to be relevant to consider the quantity $p_\alpha(\epsilon) := \mathbb{P}(|X_\alpha - 1| \geq \epsilon)$ for $\epsilon = 4(1-\alpha)$ and when α is close to 1. Since we have no analytic expression for the cumulative distribution function of X_α, we only give an approximation of $p_\alpha(4(1-\alpha))$ using the Monte Carlo method. We simulate a sample of $N = 10^6$ realizations of X_α for $\alpha \in [0.990, 1)$ using the method proposed by Theorem 1.19 of [55] and observe in Figure 4.3 that $p_\alpha(4(1-\alpha))$ tends to 1 - instead of 0 - when α tends to 1. Hence, it seems that the upper bound is better in Lemma 4.6.2 than in Theorem 4.6.4.

However, the same method used for $\epsilon = \sqrt{1-\alpha}$ allows to observe (Figure 4.3) that $p_\alpha(\sqrt{1-\alpha})$ seems to be bounded by $\sqrt{1-\alpha}$ and then to conjecture that

$$\Delta_{TV}(\Phi_\alpha, \Phi) \leq 2\sqrt{1-\alpha}.$$

Theorem 4.6.5 (Polish distance between DS point processes).

For any $i \in \{1,2\}$, let Φ_i be a Dα_iS point process with St(α_i)S random intensity ξ_i. Then,

$$\Delta_P(\Phi_1, \Phi_2) = \overline{\Delta}_P(\xi_1, \xi_2),$$

with $\overline{\Delta}_P$ denoting the Polish distance on $\mathbb{M}_1'$ associated to $g = (g_k)_{k\in\mathbb{N}}$ defined, for any $k \in \mathbb{N}$ and any $\varphi \in \mathbb{M}$, by $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, where ζ_φ is a Poisson point process on $\mathbb{X}$ with intensity measure φ.

Proof. Since, for any $i \in \{1,2\}$, Φ_i is a Cox point process directed by ξ_i, this equality is a direct consequence of Lemma 3.7.2. □

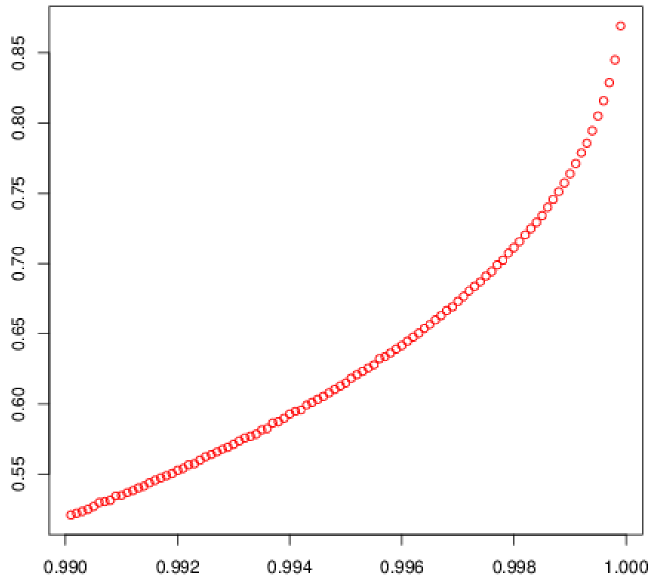


Figure 4.2: Approximation obtained by the Monte Carlo method of the graph of the application $\alpha \in [0.990, 1) \mapsto p_\alpha(4(1-\alpha))$.

4.7 Convergence results for thinnings and superpositions

In this section, convergence results are also given for Δ_{TV} (more explanations are given previously at the beginning of Section 4.6).

Theorem 4.7.1 (Kallenberg's theorem applied to a DαS point process).
Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of point processes on $\mathbb{X}$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{X}$ to $[0,1)$ such that $(p_n)_{n\in\mathbb{N}}$ tends to 0 uniformly. Let ξ be a StαS random measure and Ψ be a DαS point process, with the same finite spectral measure σ. Then,

$$p_n\Phi_n \xrightarrow[n\to+\infty]{law} \xi \iff p_n \circ \Phi_n \xrightarrow[n\to+\infty]{law} \Psi.$$

Moreover, for any $n \in \mathbb{N}$,

$$\Delta_P(p_n \circ \Phi_n, \Psi) \leq 2\mathbb{E}\Big[\sum_{x\in\Phi_n} p_n^2(x)\Big] + \overline{\Delta}_P(p_n\Phi_n, \xi),$$

with $\overline{\Delta}_P$ denoting the Polish distance on $\mathbb{M}_1'$ associated to $g = (g_k)_{k\in\mathbb{N}}$ defined, for any $k \in \mathbb{N}$ and any $\varphi \in \mathbb{M}$, by $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, where ζ_φ is a Poisson point process on $\mathbb{X}$ with intensity measure φ.

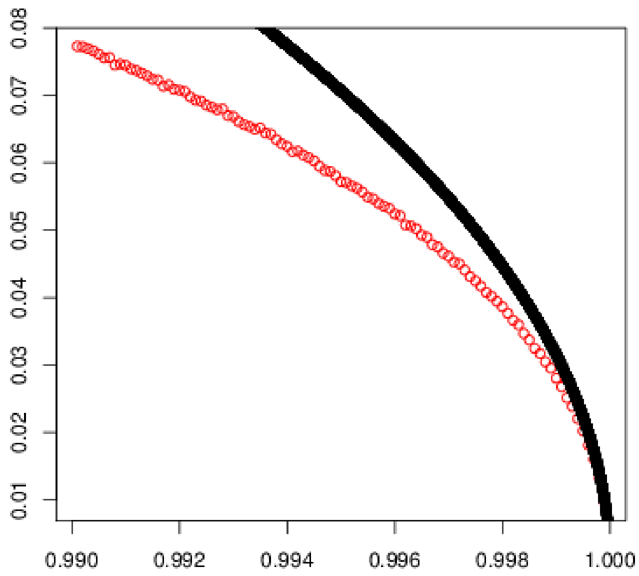


Figure 4.3: In red, approximation obtained by the Monte Carlo method of the graph of the application $\alpha \in [0.990, 1) \mapsto p_\alpha(\sqrt{1-\alpha})$. In black, graph of $\alpha \in [0.990, 1) \mapsto \sqrt{1-\alpha}$.

Proof. Since, by Theorem 4.1.9, Ψ is a Cox point process directed by ξ, both assertions are derived respectively from Theorem 3.7.1 and Theorem 3.7.5. $\square$

Corollary 4.7.2 (Approximation of a StαS random measure).
Let Φ be a DαS point process with StαS intensity measure ξ. Then the sequence (ξ_n) of random measures defined for any $n \in \mathbb{N}$ by:

$$\xi_n = \frac{1}{n^{\frac{1}{\alpha}}} \sum_{k=1}^{n} \Phi^{(k)},$$

where $\Phi^{(1)}, \ldots, \Phi^{(n)}$ are n independent copies of Φ, converges in law to ξ.

Proof. By Corollary 4.1.11, $\frac{1}{n^{\frac{1}{\alpha}}} \circ \sum_{k=1}^{n} \Phi^{(k)}$ has the same distribution as Φ. The expected result is obtained by Theorem 4.7.1 applied for $p_n = \frac{1}{n^{\frac{1}{\alpha}}}$. $\square$

Theorem 4.7.3 (Kallenberg's theorem for sums of Sibuya point processes).
Let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence of point processes on $\mathbb{M}_1$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{X}$ to $[0,1)$ tending uniformly to 0. For any $n \in \mathbb{N}$, let $\Phi_n = \sum_{\mu\in\Psi_n} \Upsilon_\mu$ where, conditionally to Ψ_n, $(\Upsilon_\mu)_{\mu\in\mathbb{M}_1}$ is a family of independent Sibuya point processes

with exponent α. Let σ be a random almost surely finite element of $\mathbb{M}$ and Γ be a Cox point process directed by σ. If $p_n^\alpha \Psi_n \xrightarrow[n\to+\infty]{law} \sigma$, then:

$$p_n \circ \Phi_n \xrightarrow[n\to+\infty]{law} \sum_{\mu\in\Gamma} \Upsilon_\mu.$$

Proof. This assertion is deduced by adapting the proof of Lemma 4.4.1 for a p_n-thinning ($n \in \mathbb{N}$) and from Theorem 3.7.1. □

Theorem 4.7.4 (Upper-bound theorem for a DαS point process).
Let Ψ be a DαS point process on $\mathbb{X}$ with finite spectral measure $\sigma(\mathrm{d}\mu) = m(\mu)\lambda(\mathrm{d}\mu)$ (with $\lambda \in \mathbb{M}'_R$) and Φ a second finite point process on $\mathbb{X}$ with α-Papangelou intensity c_α with respect to λ. Then,

$$\Delta_{TV}(\Phi,\Psi) \le \int_{\mathbb{M}_1} \mathbb{E}[|m(\mu) - c_\alpha(\mu,\Phi)|]\lambda(\mathrm{d}\mu).$$

Proof. For any $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_D)$, by Theorem 4.4.11,

$$\mathbb{E}[F(\Phi)] - \mathbb{E}[F(\Psi)] = \mathbb{E}\Big[\int_0^{+\infty} LP_sF(\Phi)\mathrm{d}s\Big].$$

Then, using the expression of the generator L,

$$\mathbb{E}[F(\Phi)] - \mathbb{E}[F(\Psi)] = \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{M}_1} D_\mu P_s F(\Phi)\sigma(\mathrm{d}\mu)\Big] - \frac{1}{\alpha}\mathbb{E}\Big[\sum_{y\in\Phi} P_sF(\Phi) - P_sF(\Phi\setminus y)\Big]\mathrm{d}s,$$

and then, by the definition of the α-Papangelou intensity,

$$\begin{aligned}\mathbb{E}[F(\Phi)] - \mathbb{E}[F(\Psi)] &= \\ &= \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{M}_1} D_\mu P_s F(\Phi) m(\mu)\lambda(\mathrm{d}\mu)\Big] - \mathbb{E}\Big[\int_{\mathbb{M}_1} c_\alpha(\mu,\Phi) D_\mu P_s F(\Phi)\lambda(\mathrm{d}\mu)\Big]\mathrm{d}s \\ &= \int_0^{+\infty} \mathbb{E}\Big[\int_{\mathbb{M}_1} D_\mu P_s F(\Phi)(m(\mu) - c_\alpha(\mu,\Phi))\lambda(\mathrm{d}\mu)\Big]\mathrm{d}s.\end{aligned}$$

Hence, by Lemma 4.4.9,

$$\mathbb{E}[F(\Phi)] - \mathbb{E}[F(\Psi)] = \int_0^{+\infty} e^{-s}\mathbb{E}\Big[\int_{\mathbb{M}_1} P_s D_\mu F(\Phi)(m(\mu) - c_\alpha(\mu,\Phi))\lambda(\mathrm{d}\mu)\Big]\mathrm{d}s,$$

then, since $F \in \mathrm{Lip}_1(\widehat{N_{\mathbb{X}}}, \Delta_D)$ and $\|P_s\| \leq 1$,

$$\begin{aligned}\left|\mathbb{E}[F(\Phi)] - \mathbb{E}[F(\Psi)]\right| &\leq \int_0^{+\infty} e^{-s} \mathbb{E}\Big[\int_{\mathbb{M}_1} |D_\mu F(\Phi)||m(\mu) - c_\alpha(\mu, \Phi)|\lambda(\mathrm{d}\mu)\Big]\mathrm{d}s \\ &\leq \int_{\mathbb{M}_1} \mathbb{E}[|m(\mu) - c_\alpha(\mu, \Phi)|]\lambda(\mathrm{d}\mu).\end{aligned}$$

The proof is thus complete. □

Corollary 4.7.5 (Distance between DαS point processes).
Let Φ_1 and Φ_2 be two DαS point processes with respective finite spectral measures σ_1 and σ_2. Then,

$$\Delta_{TV}(\Phi_1, \Phi_2) \leq \Delta_{TV}(\sigma_1, \sigma_2).$$

Proof. By Theorem 4.5.2, for any $i \in \{1, 2\}$, the α-Papangelou intensity of Φ_i with respect to $\sigma_1 + \sigma_2$ equals the density of σ_i with respect to $\sigma_1 + \sigma_2$. From the upper bound provided by Theorem 4.7.4, it follows that

$$\Delta_{TV}(\Phi_1, \Phi_2) \leq \int_{\mathbb{M}_1} \left|\frac{\mathrm{d}\sigma_1(\mu)}{\mathrm{d}(\sigma_1 + \sigma_2)} - \frac{\mathrm{d}\sigma_2(\mu)}{\mathrm{d}(\sigma_1 + \sigma_2)}\right| (\sigma_1 + \sigma_2)(\mathrm{d}\mu).$$

As mentioned in Remark 2.2.6, the right hand side equals $\Delta_{TV}(\sigma_1, \sigma_2)$, as expected. □

Corollary 4.7.6 (Application to a superposition).
Let Φ be a finite point process on $\mathbb{X}$ with α-Papangelou intensity c_α with respect to a Radon measure λ on $\mathbb{M}_1$ such that, for any $\mu \in \mathbb{M}_1$, $m(\mu) := \mathbb{E}[c_\alpha(\mu, \Phi)] \in \mathbb{R}_+$. Let Ψ a DαS point process on $\mathbb{X}$ with finite spectral measure $\sigma(\mathrm{d}\mu) = m(\mu)\lambda(\mathrm{d}\mu)$. For any $n \in \mathbb{N}$, the point process Φ_n is defined by:

$$\Phi_n = \sum_{k=1}^{n} \frac{1}{n^{\frac{1}{\alpha}}} \circ \Phi^{(k)},$$

where $\Phi^{(1)}, \dots, \Phi^{(n)}$ are n independent copies of Φ. If there exists an integrable (with respect to λ) function $K : \mathbb{M}_1 \to \mathbb{R}_+$ such that, for any $\mu \in \mathbb{M}_1$, $\mathbb{V}[c_\alpha(\mu, \Phi)] \leq K(\mu)$, then

$$\Delta_{TV}(\Phi_n, \Psi) \leq \frac{1}{n^{\frac{1}{\alpha} - \frac{1}{2}}} \int_{\mathbb{M}_1} \sqrt{K(\mu)}\lambda(\mathrm{d}\mu).$$

Proof. By Theorem 4.7.4, one has for any $n \in \mathbb{N}$,

$$\Delta_{TV}(\Phi_n, \Psi) \leq \int_{\mathbb{M}_1} \mathbb{E}\Big[\Big|c_{\alpha,n}(x, \Phi_n) - m(x)\Big|\Big]\lambda(\mathrm{d}\mu),$$

where $c_{\alpha,n}$ is the α-Papangelou intensity of Φ_n. Then, by combining Theorems 4.5.4 and 4.5.5,

$$\Delta_{TV}(\Phi_n,\Psi) \leq \int_{\mathbb{M}_1} \mathbb{E}\Big[\Big|\sum_{k=1}^{n} \frac{1}{n^{\frac{1}{\alpha}}} \mathbb{E}\big[c_\alpha(\mu,\Phi^{(k)}) \mid \frac{1}{n^{1/\alpha}} \circ \Phi^{(k)}\big] - m(\mu)\Big|\Big] \lambda(\mathrm{d}\mu)$$

and Jensen's inequality yields

$$\Delta_{TV}(\Phi_n,\Psi) \leq \int_{\mathbb{M}_1} \sqrt{\mathbb{V}\Big[\frac{1}{n^{\frac{1}{\alpha}}} \sum_{k=1}^{n} \mathbb{E}\big[c_\alpha(\mu,\Phi^{(k)}) \mid \frac{1}{n^{1/\alpha}} \circ \Phi^{(k)}\big]\Big]} \lambda(\mathrm{d}\mu).$$

Thus, by variance properties,

$$\begin{aligned}\Delta_{TV}(\Phi_n,\Psi) &\leq \frac{1}{n^{\frac{1}{\alpha}-\frac{1}{2}}} \int_{\mathbb{M}_1} \sqrt{\mathbb{V}\Big[\mathbb{E}\big[c_\alpha(\mu,\Phi) \mid \frac{1}{n^{1/\alpha}} \circ \Phi\big]\Big]} \lambda(\mathrm{d}\mu) \\ &\leq \frac{1}{n^{\frac{1}{\alpha}-\frac{1}{2}}} \int_{\mathbb{M}_1} \sqrt{\mathbb{V}[c_\alpha(\mu,\Phi)]} \lambda(\mathrm{d}\mu),\end{aligned}$$

hence, the result. $\square$

Appendix A

A case study on regularity in cellular network deployment

A.1 Introduction

Statistical models of transmitters locations aim to provide tools to understand real network deployment. For telecommunication companies, the a priori knowledge of the distribution of the antenna locations helps to predict and manage the costs of a network deployment. Such models also provide mathematical tractable methods to estimate the coverage probability of a given network. These results would also interest telecommunication regulators and public health authorities, since electromagnetic exposure has become a worldwide issue.

The first model introduced in radio networks was the regular hexagonal deterministic network. Although the regular lattice of cells gives an approximation of the cellular concept, it fails to catch the proper reality of network deployment. It proves also to be an optimistic bound in terms of interference estimation [1]. The random nature of the parameters involved in defining a proper coverage strategy makes it difficult to use a deterministic and regular model. Stochastic geometry ideas, especially about random point processes - i.e. Poisson Point Processes, Matérn hardcore point processes, Ginibre Point Process and β-Ginibre Point Process - were then widely explored in the wireless communication literature. Pioneer works in this field were realized by Baccelli et al. on PPP [5]. Many results were then derived, such as the coverage probability in respect of the signal-to-interference-plus-noise ratio (SINR). Last developments of PPP models also include modeling of heterogeneous (k-tier) networks [33] and modeling of wireless network signals [44]. However, positions of the base stations in a PPP deployed network are uncorrelated with one another. Therefore clusters of points may occur. Mean inter-site distance of such configurations is thus smaller than what happens in reality. As a result, PPP models generate more interference than that of a real network. The articles of Andrews et al. [1] and Nakata et al. [54] show that the PPP provide with the most pessimistic prediction of outage probability compared with other repulsive models.

Spatial correlations between base stations locations exist, since they have to be separated from one another to maximize coverage and minimize inter-site interferences. To take into account these effects, repulsive (or regular) models were introduced in the literature. A simple approach is to transform a PPP into a repulsive point process by thinning. Such processes are called Matérn hardcore point processes. Interferences for such deployed networks were investigated in [39] but hardcore models proved to be difficult to manipulate since the outage probability can not be analytically deduced. Softcore processes then rose community's interest. Among them, GPP and β-GPP (two determinantal point processes) were investigated in the wireless communication field. They were at first introduced by Shirai et al. [66] in quantum physics to model fermion interactions. Works of Miyoshi et al. [52] and Deng et al. [30] have derived coverage probability in respect of the SINR for both GPP and β-GPP models.

In this work, we show that base station distribution for an operator and for a technology can be fitted with a β-GPP distribution in the Paris area. The distribution of all base stations of all operators can be fitted with a PPP. Our main contribution lies in the theoretical justification of this phenomenon. We prove that the independent superposition of different β-GPPs converges in distribution to a PPP. Finally we draw conclusions on the coverage-capacity trade-off made by different operators. Qualitative results are derived from the inferred values of β and the intensity ρ. The function ρ can give information on the dimensioning strategy adopted by the operator, while β gives insights on the coverage.

Other existing papers on antenna deployment models mainly consider the computation of the SINR and coverage probability for a wide set of point processes. We are instead interested in validating the β-GPP model and the PPP superposition model with real data on a dense urban area. Such a case study is made possible because French frequency regulator (ANFR) provides location in an open access database [2].

In Section A.2, we introduce the convergence in distribution theorem for an independent superposition of β-GPPs. In Section A.3, we give the method used to fit the β-GPP model with the actual data. A qualitative interpretation of the deployment strategies is then realized from inferred β and ρ.

A.2 Theoretical model

In this section, we introduce the β-GPP convergence theorem. Definitions of correlation function, determinantal point process and β-Ginibre point process are given in Chapter 2. Let recall the following convergence theorem from Kallenberg [42].

Theorem A.2.1 (Characterization for the convergence in law).
Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of point processes on $\mathbb{X}$ and Φ a point process on $\mathbb{X}$. Then $(\Phi_n)_{n\in\mathbb{N}}$ converges in law to Φ if and only if, for any relatively compact subset A of $\mathbb{X}$,

(i) $\lim_{n\to+\infty} \mathbb{P}(\Phi_n(A)=0) = \mathbb{P}(\Phi(A)=0)$;

(ii) $\liminf_{n\to+\infty} \mathbb{P}(\Phi_n(A)>1) \leq \mathbb{P}(\Phi(A)>1)$;

(iii) $\lim_{t\to+\infty} \limsup_{n\to+\infty} \mathbb{P}(\Phi_n(A)>t) = 0$.

One of the main novelties in this work is the study of the independent superposition of multiple β-GPPs. We give the key convergence theorem for the β-GPPs.

Theorem A.2.2 (**Convergence in law of a superposition of β-GPPs**)**.**
Let $n \in \mathbb{N}$ and Φ_n be the independent superposition of n point processes $\Phi_{n,1}, \dots, \Phi_{n,n}$, such that, for any $i \in \{1, ..., n\}$, $\Phi_{n,i}$ is a $\beta_{n,i}$-Ginibre point process with intensity $\frac{\gamma_i}{n\pi}$, where $\beta_{n,i} \in (0,1]$ and $\gamma_i \in (0, +\infty)$. Let suppose that:

- *the sequence $(\gamma_k)_{k \in \mathbb{N}^*}$ is bounded;*
- $\displaystyle\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \gamma_i = \gamma$, *with $\gamma \in [0, +\infty)$.*

Then, $(\Phi_n)_{n \in \mathbb{N}}$ converges in law to a homogeneous Poisson point process Φ with intensity $\dfrac{\gamma}{\pi}$.

Proof. Theorem A.2.2 is achieved if $(\Phi_n)_{n \in \mathbb{N}}$ and Φ verify all conditions of Theorem A.2.1. Let A be a relatively compact subset of $\mathbb{C}$. By Markov inequality, for any $n \in \mathbb{N}$ and any $t \geq 1$,

$$\mathbb{P}(\Phi_n(A) > t) \leq \mathbb{E}[\Phi_n(A)].$$

Then, since, for any $n \in \mathbb{N}$,

$$\mathbb{E}[\Phi_n(A)] = \sum_{i=1}^{n} \mathbb{E}[\Phi_{n,i}(A)] = \frac{|A|}{n\pi} \sum_{i=1}^{n} \gamma_i,$$

and since by hypothesis $(\frac{1}{n} \sum_{i=1}^{n} \gamma_i)_{n \in \mathbb{N}}$ is convergent and then bounded, it derives that the sequence $(\mathbb{E}[\Phi_n(A)])_{n \in \mathbb{N}}$ is also bounded and then condition (iii) is satisfied.
Since Φ is a homogeneous Poisson point process with intensity $\frac{\gamma}{\pi}$, one has

$$\mathbb{P}(\Phi(A) = 0) = e^{-|A|\frac{\gamma}{\pi}}$$

and

$$\mathbb{P}(\Phi(A) \leq 1) = \left(1 + |A|\frac{\gamma}{\pi}\right) e^{-|A|\frac{\gamma}{\pi}}.$$

We have yet to calculate the left-hand side of both inequalities (i) and (ii). By Proposition 3 in [37], since $\Phi_{n,i}$ is a determinantal point process with kernel $K_{n,i}$ given for any $x, y \in \mathbb{C}$ by

$$K_{n,i}(x,y) = \frac{\gamma_i}{n\pi} e^{-\frac{\gamma_i}{2n\beta_{n,i}}(|x|^2+|y|^2)} e^{\frac{\gamma_i}{n\beta_{n,i}} x\overline{y}},$$

it derives that

$$\mathbb{P}(\Phi_{n,i}(A) = 0) = 1 + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p} \int_{A^p} \det[K_{n,i}](v_1, ..., v_p) \ell(\mathrm{d}v_1) \dots \ell(\mathrm{d}v_p)$$

and

$$\mathbb{P}(\Phi_{n,i}(A)=1)=\mathbb{P}(\Phi_{n,i}(A)=0)\int_A R_{A,n,i}(z)\ell(\mathrm{d}z)$$

where ℓ designs the Lebesgue measure on $\mathbb{C}$ and, for any $z\in\mathbb{C}$,

$$R_{A,n,i}(z)=K_{n,i}(z,z)+\sum_{j=2}^{+\infty}K_{A,n,i}^{(j)}(z,z)$$

with, for any $z_1,z_2\in\mathbb{C}$,

$$K_{A,n,i}^{(2)}(z_1,z_2)=\int_A K_{n,i}(z_1,v)K_{n,i}(v,z_2)\ell(\mathrm{d}v)$$

and, for any $j\in\mathbb{M}$ such that $j\geq 3$,

$$K_{A,n,i}^{(j)}(z_1,z_2)=\int_A K_{n,i}(z_1,v)K_{A,n,i}^{(j-1)}(v,z_2)\ell(\mathrm{d}v).$$

Since, by hypothesis, there exists $M\in(0,+\infty)$ such that $\gamma_k\leq M$ for all $k\in\mathbb{N}$, and since $\|K_{n,i}\|_\infty=\frac{\gamma_i}{n\pi}$, we obtain recursively that for all $p\geq 1$,

$$0\leq \det[K_{n,i}](v_1,\dots,v_p)\leq\Big(\frac{\gamma_i}{n\pi}\Big)^p\leq\Big(\frac{M}{n\pi}\Big)^p$$

and then there exists a bounded sequence $(\epsilon_n)_{n\in\mathbb{N}}\subset\mathbb{R}$, independent of i, such that

$$\sum_{p=2}^{+\infty}\frac{(-1)^p}{p}\int_{A^p}\det[K_{n,i}](v_1,\dots,v_p)\ell(\mathrm{d}v_1)\dots\ell(\mathrm{d}v_p)=\frac{\epsilon_n}{n^2}.$$

In this way,

$$\mathbb{P}(\Phi_{n,i}(A)=0)=1-\int_A K_{n,i}(v,v)\ell(\mathrm{d}v)+\frac{\epsilon_n}{n^2}=1-\frac{\gamma_i|A|}{n\pi}+\frac{\epsilon_n}{n^2}.$$

Moreover, since, for any $z\in\mathbb{C}$,

$$\sum_{j=2}^{+\infty}K_{n,i}^{(j)}(z,z)\leq\sum_{j=2}^{+\infty}\|K_{n,i}\|_\infty^j=\frac{\|K_{n,i}\|_\infty^2}{1-\|K_{n,i}\|_\infty}$$

and since for all $x\in\mathbb{C}$, $K_{n,i}(x,x)=\frac{\gamma_i}{n\pi}$ and $\|K_{n,i}\|_\infty=\frac{\gamma_i}{n\pi}$, it follows that, for any $z\in\mathbb{C}$,

$$\frac{\gamma_i}{n\pi}\leq R_{A,n,i}(z)\leq\frac{\gamma_i}{n\pi}+\frac{\gamma_i^2}{n^2\pi^2-\gamma_i n\pi},$$

and then there exists a bounded sequence $(\eta_n)_{n\in\mathbb{N}}\subset\mathbb{R}$, independent of i, such that

$$\int_A R_{A,n,i}(z)\ell(\mathrm{d}z)=\frac{\gamma_i|A|}{n\pi}+\frac{\eta_n}{n^2}.$$

We can finally deduce that there exists a bounded sequence $(\epsilon'_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, independent of i, such that

$$\mathbb{P}(\Phi_{n,i}(A) = 1) = \frac{\gamma_i |A|}{n\pi} + \frac{\epsilon'_n}{n^2}.$$

Hence, on one hand, since $\mathbb{P}(\Phi_n(A) = 0) = \prod_{i=1}^n \mathbb{P}(\Phi_{n,i}(A) = 0)$, we get:

$$\begin{aligned}
\mathbb{P}(\Phi_n(A) = 0) &= \prod_{i=1}^{n} (1 - \frac{\gamma_i |A|}{n\pi} + \frac{1}{n^2}\epsilon_n) \\
&= \prod_{i=1}^{n} e^{\ln(1 - \frac{\gamma_i |A|}{n\pi} + \frac{1}{n^2}\epsilon_n)},
\end{aligned}$$

and then, since $\lim\limits_{n\to+\infty} \frac{1}{n}\sum_{i=1}^n \gamma_i = \gamma$, it follows that

$$\lim_{n\to+\infty} \mathbb{P}(\Phi_n(A) = 0) = e^{-\frac{\gamma}{\pi}|A|},$$

so (i) is verified.

On the other hand, since $\mathbb{P}(\Phi_n(A) = 1) = \sum_{i=1}^n \mathbb{P}(\Phi_{n,i}(A) = 1) \prod_{\substack{j=1 \\ j\neq i}}^n \mathbb{P}(\Phi_{n,j}(A) = 0)$, one has

$$\mathbb{P}(\Phi_n(A) = 1) = \sum_{i=1}^{n} \Big(\frac{\gamma_i |A|}{n\pi} + \frac{1}{n^2}\epsilon'_n\Big) \prod_{\substack{j=1 \\ j\neq i}}^{n} \Big(1 - \frac{\gamma_j |A|}{n\pi} + \frac{1}{n^2}\epsilon_n\Big),$$

and then, using previous arguments and since $(\gamma_k)_{k\in\mathbb{N}}$ is bounded,

$$\limsup_{n\to+\infty} \mathbb{P}(\Phi_n(A) = 1) \geq e^{-\frac{\gamma|A|}{\pi}} \frac{\gamma|A|}{\pi},$$

so (ii) holds, as expected.

□

Hypotheses of Theorem A.2.2 are quite restrictive because the intensities of each β-GPP are dependent of n. However, in practice, we mainly work with finite families of β-GPPs. Therefore, we can choose the value of the $(\gamma_i)_{i\in\{1,\ldots,n\}}$ such that they match the real values of the intensity of each β-GPP.

A.3 Statistical analysis

In this section we introduce the fitting method that is used to obtain the parameter β. We also present the results from the fitting of each deployment and operator in Paris, France.

Summary statistic

In order to fit the real deployment to the β-GPP model, we introduce the J-function that characterizes any point process. This function is a summary statistic based on inter-point distances. General information about summary statistics can be found in [53].

Definition A.3.1 (J-function).
*The **J-function** of a stationary point process Φ on $\mathbb{R}^d$ is defined for any $r \in \mathbb{R}_+$ by:*

$$J(r) = \frac{1 - G(r)}{1 - F(r)},$$

where F is the empty space function of Φ and G its nearest-neighbor distance distribution function, defined for some $u \in \mathbb{R}^d$ and any $r \in \mathbb{R}_+$ by:

$$F(r) = \mathbb{P}(\|u - \Phi\| \leq r)$$

and

$$G(r) = \mathbb{P}(\|u - \Phi \setminus \{u\}\| \leq r).$$

The J-function provides both a characterization of the point process and a direct information about its attractiveness or repulsiveness. More precisely, when $J < 1$, X is attractive, otherwise X is repulsive. The equality $J \equiv 1$ characterizes the PPP, where there is no interaction between the particles. For the case of the β-GPP, we get from [30] the following theorem.

Theorem A.3.2 (J-function of a β-GPP).
The J-function of the β-GPP with intensity $\frac{\gamma}{\pi}$ is given for any $r \in \mathbb{R}_+$ by:

$$J(r) = \frac{1}{1 - \beta + \beta e^{-\frac{\gamma}{\beta} r^2}}.$$

Note that for any β this J-function is bigger than one, which confirms that the β-GPP is a repulsive point process. When β tends to 0, this expression tends to 1, which corresponds to the J-function of a PPP.

This J-function allows to validate the β-GPP as a distribution model of the repartition of the base stations for each operator and each technology.

Fitting method

Thanks to R language and the spatstat package [6], the estimate of the J-function is derived from the raw data. Since we consider only a finite set of antennas, edge-effect might appear on the J-function estimate. We then have to keep a subset of the data to perform the estimation. Figure A.1 gives the window we considered for extracting data in Paris, France. It covers about 60% of the city and its shape is chosen to match the geographical borders.

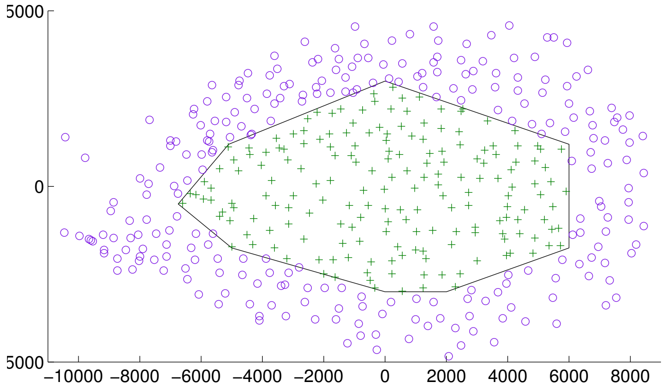


Figure A.1: Example of data sample for one GSM operator. The J-function is fitted on the points within the polygonal window.

The values of the J-function estimate are computed for $r \leq 600$m. Above 600 m, the estimation is not relevant due to the edge-effect. J is then directly fitted on the estimate and the parameter β is deduced. An example of fitting is given in Figure A.2. It is clear that the point process formed by the base stations locations is repulsive and fits well the theoretical model. Therefore, it outfits the PPP model, because the J-function a PPP is equal to one for all r. In the next paragraph we present the results we obtained on raw data.

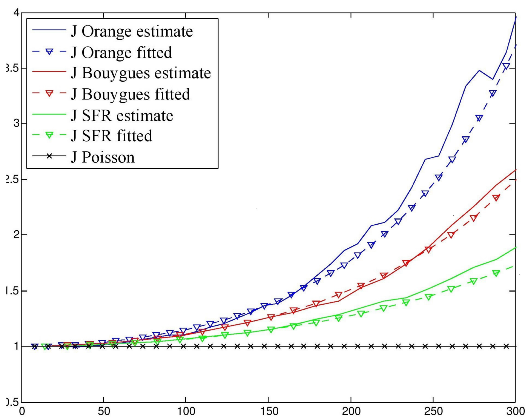


Figure A.2: Example of J-function fitting for Orange, SFR and Bouygues on the 3G 2100 MHz band. As a comparison, $J(r) = 1$ for all $r \in \mathbb{R}_+$ in the PPP case.

Fitting results and interpretation

Locations of the base stations are publicly available for the whole French territory and can be found online [2]. There are four operators in France and most of them provide 2G to 4G coverage. For each operator and each technology, numerical values of β and ρ from the fitting are given in Table A.1. Each intensity ρ is simply computed using the number of corresponding base stations in the window. The parameter β is then computed by the method of least squares applied to the J-function of the β-GPP and its estimation.

Table A.1: Numerical values of β and ρ per technology and operator.

		Orange	SFR	Bouygues	Free
GSM 900	β	0.81	0.76	0.65	NA
	ρ	2.39	2.65	2.63	NA
GSM 1800	β	0.84	0.85	0.71	NA
	ρ	3.00	2.39	3.59	NA
UMTS 900	β	NA	0.97	0.53	0.89
	ρ	NA	1.92	2.44	1.05
UMTS 2100	β	1.04	0.65	0.82	0.89
	ρ	3.27	3.48	4.04	1.05
LTE 800	β	1.02	0.93	0.67	NA
	ρ	0.67	1.65	1.87	NA
LTE 1800	β	NA	NA	0.75	NA
	ρ	NA	NA	3.46	NA
LTE 2600	β	0.93	0.67	0.63	0.89
	ρ	2.80	2.76	2.46	1.05

Values of β and ρ give some insights about the deployment strategy of each cellular network operators, especially about the coverage-capacity trade-off. Orange's high values of β and ρ suggest that this operator deployed (as the historic, previously state-owned operator) a network that fulfilled an optimal coverage and an optimal traffic capacity (densely deployed network). However, SFR and Bouygues first deployed a network with a minimum of antennas (in order to abide by the coverage requirement of the regulator) and then gradually increased traffic capacity on hot-spots (by increasing locally the number of antennas). This involves adding more antennas on sites that are already covered, thus creating clusters and decreasing the value of β and increasing the value of ρ. The French telecommunication regulator (ARCEP) published yearly reports [3] that suggest such evolution. We deduce that French operators used two different deployment strategies. The first strategy consists in fulfilling both coverage and optimal traffic capacity at once. While the second strategy is to deploy a network that abides to the coverage requirements in a first stage, then in a second stage to increase the number of antennas on hot-spots in order to improve the traffic capacity.

When deploying their 3G or 4G networks, operators reused and shared some existing 2G sites. Therefore, we consider that classifying the base station sites per operator is more relevant than classifying them by technologies. Table A.2 summaries these results. As expected, previous conclusions still hold as values of β are stable between the two tables. We also notice that Free, as a newcomer (2012), has a small amount of traffic to deal with, and

Table A.2: Numerical values of β and ρ per operator and for the superposition of all the sites.

	Orange	SFR	Bouygues	Free	**Overall**
β	0,94	0,70	0,81	0,89	**0,17**
ρ	3,48	3,70	4,23	1,05	**10,28**
Number of sites	185	197	225	56	**547**

therefore has deployed less antennas than its competitors. Data analysis also shows that the superposition of all sites is tending to a PPP as β is equal to 0.17. Therefore the PPP model still holds as an indicator of electromagnetic exposure of cellular networks.

A.4 Conclusion

In this work, we successfully show that β-GPP is a realistic model for base station distribution. The β parameter is inferred by using statistical tools on real data. Qualitative results on network deployment are then derived. We also prove theoretically that the independent superposition of multiple β-GPPs converges in distribution to a PPP justifying observations made on real deployments. This will have greater implications in modeling multi-tiers networks. We show that the values of ρ and β are characteristics of the coverage-capacity trade-off. Future works will investigate the impact ρ and β on the design of optimal deployment strategies.

Appendix B

French summary - Résumé en français

B.1 Chapitre 1

Chapitre 1 : Introduction

Le premier chapitre est une introduction générale qui présente rapidement les différents aspects de la théorie des processus ponctuels qui sont abordés au cours de la thèse, ainsi que le plan de la thèse et les nouvelles contributions qu'elle apporte.

1.1 Cadre

Cette section donne quelques définitions et propriétés élémentaires sur les processus ponctuels, formellement vus comme des sous-ensembles localement finis de points. Ces objets d'étude fournissent des outils mathématiques puissants avec des applications dans de nombreux domaines, en particulier lorsque la répartition de particules a besoin d'être analysée de façon mathématique. Le choix d'une catégorie de processus ponctuels pour modéliser cette répartition dépend fortement si les interactions entre les particules induit de l'attractivité, avec des amas de points, ou de la répulsivité, avec de l'espace entre les particules.

Dans la sous-section 1.1.1, nous donnons les moyens usuels pour caractériser un processus ponctuel : la fonctionnelle de Laplace, la fonction de Janossy, la fonction de corrélation et l'intensité de Papangelou, et caractérisons le processus ponctuel de Poisson comme le seul processus ponctuel sans interactions entre ses particules.

La sous-section 1.1.2 aborde succintement les deux opérations élémentaires qui feront l'objet de nos futures investigations : la superposition indépendante et l'amincissement indépendant.

Dans la sous-section 1.1.3, les familles de processus ponctuels usuelles sont présentées. Les processus ponctuels à base Poisson incluent les processus ponctuels de Cox, les processus ponctuels purement aléatoires et les processus ponctuels de Poisson conditionnels. Les deux principaux types de processus ponctuels répulsifs sont les processus de Gibbs et les processus déterminantaux, lesquels peuvent être inclus dans une classe plus large de processus ponctuels, les processus ponctuels α-déterminantaux et permanantaux. Un exemple clé de processus déterminantal est le processus ponctuel de Ginibre, à partir duquel sont construits les processus ponctuels de β-Ginibre. Nous présentons enfin une famille de processus ponctuels attractifs récemment introduite par Davydov et al. [20] qui étend la notion de stabilité aux processus ponctuels, les processus ponctuels discrets α-stables (DαS).

Dans la section 1.1.4, nous présentons les différents modes de convergence et distances sur les processus ponctuels, en particulier la distance de Kantorovich-Rubinstein, amenée à être abondamment utilisée dans la suite.

La méthode de Stein est présentée dans ses grands principes dans la section 1.1.5.

1.2 Contributions

Cette section présente les motivations et contributions de la thèse, et donne son plan.

Les principaux éléments de la théorie des processus ponctuels décrits précédemment sont présentés plus formellement dans le chapitre 2. Dans le chapitre 3, nous appliquons la méthode de Stein aux processus ponctuels de Poisson finis et déduisons des résultats de convergence en utilisant les intensités de Papangelou. Dans le chapitre 4, tenant compte de la structure en Poisson agrégatif des processus ponctuels DαS, nous adaptons pour eux le schéma adopté pour les processus ponctuels de Poisson. En appendice A, nous exposons les résultats donnés dans [38], où les processus ponctuels de β-Ginibre sont étudiés comme modèle pour la répartition des emplacements des stations de base dans un réseau sans fil.

Les contributions de cette thèse peuvent se résumer ainsi :

- Une nouvelle borne sur la distance entre des processus ponctuels basée sur l'intensité de Papangelou.
- De nombreuses applications de ce résultat pour des processus ponctuels avec des corrélations, certaines d'entre elles grandement utiles pour la compréhension des systèmes de télécommunication sans fil.
- Le développement du calcul de Malliavin pour les processus ponctuels discrets α-stables et l'obtention de taux de convergence pour la superposition et l'amincissement de tels processus ponctuels.

B.2 Chapitre 2

Chapitre 2 : Préliminaires

Dans ce chapitre, nous rappelons quelques définitions et propriétés basiques de la théorie des processus ponctuels. Dans la section 2.1, nous fixons les notations mathématiques et rappelons les fonctions caractérisant les processus ponctuels. Dans la section 2.2, nous rappelons quelques notions utiles sur la convergence de processus ponctuels et proposons une preuve pour établir que la distance de Kantorovich-Rubinstein associée à la distance discrète est égale à la distance en variation totale. Les principales transformations de processus ponctuels et leurs propriétés sont rappelées en section 2.3. Dans la section 2.4, nous présentons les processus ponctuels à base Poisson qui sont utilisés dans les chapitres suivants et la section 2.5 est centrée sur la classe des processus ponctuels α-déterminantaux/permanantaux.

2.1 Généralités sur les processus ponctuels

Cette section commence par fixer les notations utilisées tout au long du manuscrit, puis donne de façon formelle les éléments basiques de la théorie des processus ponctuels. Nous donnons les définitions et propriétés des mesures considérées : les mesures de comptage, les configurations, les processus ponctuels et leurs mesures d'intensité ; puis rappelons les fonctions capables de caractériser un processus ponctuel : la fonctionnelle de Laplace, la fonction génératrice, les mesure et fonction de Janossy, la fonction de corrélation, la mesure de Campbell réduite, la mesure de Palm et l'intensité de Papangelou. Les notions de couplage et couplage maximal sont également rappelées.

2.2 Convergence

Les différents modes de convergence considérés par la suite sont présentés dans cette section : la convergence vague, la convergence en loi, ainsi que les convergences associées à des distances sur les mesures aléatoires. Ces distances auxquelles nous nous intéressons sont la distance discrète, la distance en variation totale, la distance polonaise et la **distance de Kantorovich-Rubinstein** Δ^*, laquelle associée à une distance Δ est définie entre deux mesures de probabilité $\mathbb{P}_1$ et $\mathbb{P}_2$ sur $\mathbb{M}$ par

$$\Delta^*(\mathbb{P}_1,\mathbb{P}_2) := \inf_{\mathbf{C}\in\Sigma(\mathbb{P}_1,\mathbb{P}_2)} \int_{N_{\mathbb{X}}\times N_{\mathbb{X}}} \Delta(\omega_1,\omega_2)\mathbf{C}(\mathrm{d}(\omega_1,\omega_2))$$

où $\Sigma(\mathbb{P}_1,\mathbb{P}_2)$ désigne l'ensemble des mesures de probabilité sur $\mathbb{M}\times\mathbb{M}$ avec première loi marginale $\mathbb{P}_1$ et seconde loi marginale $\mathbb{P}_2$.

Nous démontrons ensuite, via une technique de couplage maximal, que la distance de Kantorovich-Rubinstein associée à la distance discrète entre deux processus ponctuels est la distance en variation totale. Le lemme 2.2.9 donne la construction d'un couplage maximal entre deux processus ponctuels finis, et on démontre alors (théorème 2.2.10) que si Φ_1,Φ_2 sont deux processus ponctuels finis et $(\widehat{\Phi}_1,\widehat{\Phi}_2)$ est un couplage maximal de (Φ_1,Φ_2), alors

$$\Delta_{TV}(\Phi_1,\Phi_2) = \mathbb{P}(\widehat{\Phi}_1 \neq \widehat{\Phi}_2) = \Delta^*_D(\Phi_1,\Phi_2).$$

Nous comparons ensuite les différentes topologies induites par les distances précédemment définies (théorème 2.2.11).

2.3 Transformations de processus ponctuels

Cette section fait le point sur les transformations usuelles de processus ponctuels : réduction à un sous-ensemble compact, superposition, amincissement, dilatation, randomisation, et rappelle les expressions des fonctions de corrélation, fonctions de Janossy, fonctionnelles de Laplace et fonctions génératrices correspondantes.

2.4 Processus ponctuels à base Poisson

Nous rappelons dans cette section les définitions et quelques propriétés basiques de processus ponctuels basés sur le processus de Poisson, en particulier leurs fonctionnelles de

Laplace, fonctions de Janossy et fonctions de corrélation.

Après avoir défini le processus ponctuel binomial, nous abordons le processus ponctuel de Poisson, défini comme l'unique processus ponctuel de mesure d'intensité M tel que, pour tous sous-ensembles relativement compacts disjoints Λ_1, Λ_2, les variables aléatoires $\Phi(\Lambda_1)$ et $\Phi(\Lambda_2)$ sont indépendantes. Réduit à un sous-ensemble compact, il peut aussi être caractérisé (théorème 2.4.3) comme un processus binomial, conditionnellement à un nombre poissonnien de points. Après avoir précisé ses fonctionnelle de Laplace, fonction de Janossy et fonction de corrélation, nous rappelons la formule de Mecke (théorème 2.4.6), donnée pour toute fonction mesurable $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$ par

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x, \Phi \setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[u(x, \Phi)]M(\mathrm{d}x).$$

Le processus de Poisson jouit également de propriétés de stabilité vis-à-vis de la superposition et l'amincissement, et vérifie (théorème 2.4.7) la propriété d'invariance suivante : pour tout $t \in [0, 1]$,

$$t \circ \Phi^{(1)} + (1-t) \circ \Phi^{(2)} \stackrel{\mathcal{D}}{=} \Phi,$$

où $\Phi^{(1)}$ et $\Phi^{(2)}$ sont des copies indépendantes du processus de Poisson Φ.

Les processus de Poisson finis peuvent être inclus dans une classe de processus ponctuels plus large, les processus ponctuels purement aléatoires. Un processus ponctuel purement aléatoire, conditionnellement à son nombre de points aléatoire N, est un processus ponctuel binomial avec exactement N points. Nous donnons sa fonctionnelle de Laplace, ses fonctions de Janossy et de corrélation et rappelons que, pour toute fonction mesurable $F : N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}[F(\Phi)] = \sum_{n=0}^{+\infty} p_n \int_{\mathbb{X}^n} F(x_1, \ldots, x_n)\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_n),$$

où Φ est le processus ponctuel purement aléatoire supporté par la mesure de probabilité μ et la distribution (p_n).

Les processus de Poisson sont également des cas particuliers de processus ponctuels de Cox et de processus ponctuels de Poisson conditionnels. Un processus ponctuel de Cox d'intensité aléatoire M est, conditionnellement à M, un processus ponctuel de Poisson de mesure d'intensité M, alors qu'un processus ponctuel de Poisson conditionnel est, conditionnellement à la réalisation d'un certain événement C, un processus ponctuel de Poisson. La dernière catégorie citée inclut les processus ponctuels de Poisson hard-core et bornés. Nous rappelons également pour tous ces processus ponctuels leurs fonctionnelles caractéristiques.

Enfin, les processus de Gibbs constituent un modèle de référence pour des répartitions de particules avec répulsion, nous rappelons leur définition.

2.5 Processus ponctuels α-déterminantaux/permanantaux

Dans cette section sont décrits les processus ponctuels déterminantaux et permanantaux (α-DPPP). Les définitions et propriétés énoncés proviennent essentiellement de [11].

B.3 Chapitre 3

Chapitre 3 : Méthode de Stein, intensité de Papangelou et applications

Dans ce chapitre, nous appliquons la méthode de Stein pour les processus ponctuels de Poisson finis et déduisons des résultats de convergence en utilisant les intensités de Papangelou. Dans la section 3.1, nous décrivons grossièrement la méthode de Stein appliquée à un processus ponctuel de Poisson fini. Dans la section 3.2, nous associons à un processus de Glauber ses semi-groupe, générateur infinitésimal et gradient, et établissons leurs propriétés. Nous déduisons dans la section 3.3 une formule de représentation, puis une majoration de la distance entre un processus ponctuel de Poisson fini et un autre processus ponctuel fini (théorème 3.3.2). Dans la section 3.4, les intensités de Papangelou sont étudiées. Après avoir proposé une définition de répulsivité, nous donnons quelques propriétés relatives aux processus ponctuels répulsifs, processus ponctuels finis, transformations et processus ponctuels classiques. Dans la section 3.5, nous appliquons la majoration donnée par le théorème 3.3.2 afin d'établir des résultats de convergence sur les processus ponctuels de Poisson, de Cox, purement aléatoires et de Poisson conditionnels. A partir de la majoration, nous donnons des résultats similaires sur les processus ponctuels répulsifs dans la section 3.6 : cela concerne la distance entre un processus ponctuel de Poisson et, respectivement, une superposition, un processus ponctuel déterminantal aminci et dilaté et un processus ponctuel de Gibbs. Dans la section 3.7, toujours à partir du théorème 3.3.2, nous fournissons une vitesse de convergence selon la distance polonaise pour un résultat établi par Kallenberg sur des processus ponctuels amincis.

3.1 Principe général de Stein

Cette section a pour but de présenter la méthode de Stein appliquée à un processus ponctuel de Poisson. Nous utilisons le principe de Stein et la construction donnée dans [25], mais nos preuves sont parfois différentes, mettant en lumière les propriétés de l'opération d'amincissement et l'invariance de la distribution du processus de Poisson (théorème 2.4.7) : pour tout processus ponctuel de Poisson Φ et tout $t \in [0,1]$,

$$t \circ \Phi^{(1)} + (1-t) \circ \Phi^{(2)} \stackrel{\mathcal{D}}{=} \Phi,$$

où $\Phi^{(1)}$ et $\Phi^{(2)}$ sont des copies indépendantes de Φ.

La première étape de la méthode de Stein consiste à caractériser l'objet cible, ici un processus ponctuel de Poisson fini. Le moyen utilisé est de considérer un opérateur fonctionnel L qui, à un niveau formel, satisfait pour un processus ponctuel fini Φ l'identité

$$\mathbb{E}[LF(\Phi)] = 0 \text{ pour une large classe de fonctions } F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$$

si et seulement si, Φ est un processus ponctuel de Poisson de mesure d'intensité finie M.

La seconde étape est de résoudre l'équation de Stein, c'est-à-dire de trouver, pour toute fonction test $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$, une fonction $H_F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ telle que, pour tout $\phi \in \widehat{N}_{\mathbb{X}}$,

$$LH_F(\phi) = \mathbb{E}[F(\zeta)] - F(\phi),$$

où ζ est un processus ponctuel de Poisson de mesure d'intensité finie M.

Nous utilisons l'approche par générateur basée sur la théorie des processus spatiaux de naissance et de mort. Dans notre cas, L est construit comme le générateur infinitésimal d'un processus de Markov, aussi appelé processus de Glauber, avec la distribution de ζ pour distribution invariante. Si $(P_t)_{t\geq 0}$ est le semi-groupe associé au processus de Glauber, on peut montrer que, pour tout $\phi \in \widehat{N}_{\mathbb{X}}$,

$$LH_F(\phi) = \int_0^{+\infty} LP_sF(\phi)\mathrm{d}s,$$

ce qui mène à la formule de représentation de Stein-Dirichlet :

$$\mathbb{E}[F(\zeta)] - F(\phi) = \int_0^{+\infty} LP_sF(\phi)\mathrm{d}s,$$

de laquelle nous pouvons déduire une majoration de $\Delta^*_{TV}(\zeta, \Phi)$, où Φ est un processus ponctuel fini. Tous ces éléments sont exposés avec plus de détails dans ce qui suit.

3.2 Semi-groupe, gradient, générateur infinitésimal

Une famille $(P_t)_{t\geq 0}$ d'opérateurs est un **semi-groupe** sur $\widehat{N}_{\mathbb{X}}$ si, pour tous $t, s \in \mathbb{R}_+$,

$$P_{t+s} = P_t \circ P_s.$$

Son **générateur infinitésimal** L est alors défini pour toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$ tel que $t \mapsto P_tF(\phi)$ est dérivable en 0 par :

$$LF(\phi) = \left.\frac{\mathrm{d}P_tF(\phi)}{\mathrm{d}t}\right|_{t=0}.$$

Le **processus de Glauber** $(G_t)_{t\geq 0}$ associé à un processus ponctuel de Poisson ζ de mesure d'intensité finie M est défini pour tous $t \in \mathbb{R}_+$ et $\phi \in \widehat{N}_{\mathbb{X}}$ par :

$$G_t(\phi) = e^{-t} \circ \phi + (1 - e^{-t}) \circ \zeta.$$

Pour tout $t \in \mathbb{R}_+$, l'opérateur P_t est défini pour toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$ par :

$$P_tF(\phi) = \mathbb{E}[F(G_t(\phi))] = \mathbb{E}[F(e^{-t} \circ \phi + (1 - e^{-t}) \circ \zeta)].$$

Ses dynamiques peuvent être décrites comme suit : on imagine un processus de Poisson homogène ζ_b sur $\mathbb{R}_+$ d'intensité $M(\mathbb{X})$. Les instants de saut de ζ_b déterminent les instants de naissance des particules de ζ, placées dans $\mathbb{X}$ selon la distribution $\frac{M(\cdot)}{M(\mathbb{X})}$. La durée de vie de chaque particule est exponentiellement distribuée avec paramètre 1.

On démontre alors que la famille $(P_t)_{t\geq 0}$ donnée par la définition précédente est bien un semi-groupe (théorème 3.2.4).

Pour tout $x \in \mathbb{X}$, le **gradient** D_x de direction x est défini, pour toute fonction mesurable $F : N_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in N_{\mathbb{X}}$, par

$$D_x F(\phi) = F(\phi + x) - F(\phi).$$

On vérifie alors la propriété de fermabilité suivante (théorème 3.2.6) : si $F, G : N_{\mathbb{X}} \to \mathbb{R}$ sont deux fonctions mesurables et bornées telles que $F(\phi) = G(\phi)$ $\mathbb{P}_\zeta(\mathrm{d}\phi)$-presque sûrement (p.s.), alors

$$D_x F(\phi) = D_x G(\phi)\ (M \otimes \mathbb{P}_\zeta)(\mathrm{d}x, \mathrm{d}\phi)\text{-p.s..}$$

On établit ensuite (théorème 3.2.7) que le générateur infinitésimal L associé au semi-groupe précédent est donné pour toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$ par

$$LF(\phi) = \int_{\mathbb{X}} D_x F(\phi) M(\mathrm{d}x) + \sum_{y \in \phi} (F(\phi \setminus y) - F(\phi)),$$

puis qu'un processus ponctuel Φ est un processus ponctuel de Poisson de mesure d'intensité M si et seulement si, pour toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$,

$$\mathbb{E}[LF(\Phi)] = 0.$$

Les gradient et semi-groupe considérés sont de plus liés par la propriété de commutation suivante (lemme 3.2.8) : pour tout $t \in \mathbb{R}_+$, tout $x \in \mathbb{X}$, toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$,

$$D_x P_t F(\phi) = e^{-t} P_t D_x F(\phi).$$

Une propriété d'ergodicité est également vérifiée (lemme 3.2.9) : pour tout $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\lim_{t \to +\infty} P_t F(\phi) = \mathbb{E}[F(\zeta)].$$

3.3 Formule de représentation et conséquences

On commence cette section en établissant la formule de représentation de Stein-Dirichlet (théorème 3.3.1) : pour tout $F \in \mathrm{Lip}_1(\widehat{N}_{\mathbb{X}}, \Delta_{TV})$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$,

$$\mathbb{E}[F(\zeta)] - F(\phi) = \int_0^{+\infty} LP_s F(\phi) \mathrm{d}s.$$

Il en découle le résultat fondamental suivant (théorème 3.3.2) : si ζ est un processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité finie $M(\mathrm{d}x) = m(x)\ell(\mathrm{d}x)$ et Φ un second processus ponctuel fini sur $\mathbb{X}$ d'intensité de Papangelou c, alors

$$\Delta_{TV}^*(\Phi, \zeta) \le \int_{\mathbb{X}} \mathbb{E}[|m(x) - c(x, \Phi)|] \ell(\mathrm{d}x).$$

3.4 Intensité de Papangelou et répulsivité

L'intensité de Papangelou permet en particulier de donner une définition précise de répulsivité pour un processus ponctuel : selon [35] un processus ponctuel Φ sur $\mathbb{X}$ d'intensité de Papangelou c est dit **répulsif** (selon c) si, pour tous $\omega, \phi \in N_{\mathbb{X}}$ tels que $\omega \subset \phi$ et tout $x \in \mathbb{X}$,

$$c(x, \phi) \le c(x, \omega).$$

Nous prolongeons cette définition en qualifiant ce processus ponctuel de **faiblement répulsif** (selon c) si, pour tout $\phi \in N_{\mathbb{X}}$ et tout $x \in \mathbb{X}$,

$$c(x, \phi) \le c(x, \varnothing).$$

Nous donnons ensuite les définitions de fonction et sous-ensemble croissants et décroissants : une fonction $f : N_{\mathbb{X}} \to \mathbb{R}$ est dite **croissante** (resp. **décroissante**) si, pour tous $\phi_1, \phi_2 \in N_{\mathbb{X}}$,

$$(\phi_1 \subset \phi_2) \implies (f(\phi_1) \le f(\phi_2)) \text{ (resp. } (\phi_1 \subset \phi_2) \implies (f(\phi_1) \ge f(\phi_2))),$$

et un sous-ensemble A de $N_{\mathbb{X}}$ est dit **croissant** (resp. **décroissant**) si $\mathbf{1}_A$ est croissant (resp. décroissant), c'est-à-dire si, pour tous $\phi_1 \in A$ et $\phi_2 \in N_{\mathbb{X}}$,

$$(\phi_1 \subset \phi_2) \implies (\phi_2 \in A) \text{ (resp. } (\phi_2 \subset \phi_1) \implies (\phi_2 \in A)).$$

Nous établissons quelques propriétés sur les processus faiblement répulsifs : si Φ est un processus ponctuel fini et faiblement répulsif sur $\mathbb{X}$ d'intensité de Papangelou c et probabilité de vide p_0, alors d'une part (lemme 3.4.3), pour tout $x \in \mathbb{X}$,

$$|c(x, \varnothing) - \rho(x)| \le (1 - p_0) c(x, \varnothing),$$

d'autre part (lemme 3.4.4),

$$\mathbb{E}[|c(x, \Phi) - \rho(x)|] \le 2(c(x, \varnothing) - \rho(x)).$$

Lorsque Φ est un processus ponctuel fini sur $\mathbb{X}$ d'intensité de Papangelou c, on démontre également (lemme 3.4.5) que

$$\mathbb{P}(|\Phi| = 1) = \mathbb{P}(|\Phi| = 0) \int_{\mathbb{X}} c(x, \varnothing) \mathrm{d}x.$$

Nous abordons dans la suite le calcul de l'intensité de Papangelou pour différentes transformations de processus ponctuels : la réduction à un sous-ensemble compact, la superposition, l'amincissement et la dilatation.

Tout d'abord, si Φ est un processus ponctuel sur $\mathbb{X}$ d'intensité de Papangelou c, Λ un sous-ensemble compact de $\mathbb{X}$ et $\Phi_{|\Lambda}$ la réduction de Φ à Λ, alors (théorème 3.4.6) son intensité de Papangelou c_Λ vérifie pour tout $x \in \mathbb{X}$

$$c_\Lambda(x, \Phi_{|\Lambda}) = c(x, \Phi) \mathbf{1}_{\{x \in \Lambda\}} \text{ p.s..}$$

Ensuite, si $\Phi_1, \dots, \Phi_n$ ($n \in \mathbb{N}$) sont des processus ponctuels indépendants sur $\mathbb{X}$ avec intensités de Papangelou respectives $c_1, \dots, c_n$ et Φ leur superposition, alors (théorème 3.4.7) son intensité de Papangelou c vérifie, pour tout $x \in \mathbb{X}$,

$$c\Big(x, \sum_{i=1}^{n} \Phi_i\Big) = \sum_{i=1}^{n} c_i(x, \Phi_i) \text{ p.s..}$$

Si $\Phi_1, \dots, \Phi_n$ ($n \in \mathbb{N}$) sont de plus faiblement répulsifs, alors leur superposition indépendante est aussi faiblement répulsive (corollaire 3.4.8).

Si Φ est un processus ponctuel sur $\mathbb{X}$, β une fonction de $\mathbb{X}$ dans $[0,1]$ et $\beta \circ \Phi$ le β-amincissement de Φ, alors (théorème 3.4.9) son intensité de Papangelou c_β vérifie, pour tout $x \in \mathbb{X}$,

$$c_\beta(x, \beta \circ \Phi) = \beta(x)\mathbb{E}[c(x, \Phi) \mid \beta \circ \Phi] \text{ p.s..}$$

Enfin, si Φ est un processus ponctuel sur $\mathbb{R}^d$ d'intensité de Papangelou c, ϵ un nombre réel positif et $\Phi^{(\epsilon)}$ la ϵ-dilatation de Φ, alors (théorème 3.4.10) son intensité de Papangelou $c^{(\epsilon)}$ est donnée pour tout $x \in \mathbb{R}^d$ et tout $\phi \in \widehat{N}_{\mathbb{R}^d}$ par

$$c^{(\epsilon)}(x, \phi) = \frac{1}{\epsilon} c(\epsilon^{-\frac{1}{d}} x, \epsilon^{-\frac{1}{d}} \phi).$$

On donne ensuite une expression de l'intensité de Papangelou pour chacun des processus ponctuels classiques.

Si Φ est un processus ponctuel de Poisson de mesure d'intensité $M(\mathrm{d}x) = m(x)\mathrm{d}x$, alors on sait (théorème 3.4.11) que son intensité de Papangelou c est donnée pour tout $x \in \mathbb{X}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$ par

$$c(x, \phi) = m(x).$$

Si Φ est un processus ponctuel purement aléatoire sur $\mathbb{X}$ supporté par une distribution $(p_n)_{n \in \mathbb{N}_0}$ telle que $p_n \neq 0$ pour tout $n \in \mathbb{N}_0$, et si μ est une mesure de probabilité telle que $\mu(\mathrm{d}x) = q(x)\ell(\mathrm{d}x)$, alors (théorème 3.4.12) l'intensité de Papangelou c de Φ est donnée pour tout $n \in \mathbb{N}_0$ et tous $x, x_1, \dots, x_n \in \mathbb{X}$ par

$$c(x, \{x_1, \dots, x_n\}) = (n+1)\frac{p_{n+1}}{p_n} q(x).$$

De plus, Φ est répulsif si et seulement si, pour tout $n \in \mathbb{N}_0$,

$$(n+1)p_{n+1}^2 \geq (n+2)p_n p_{n+2}$$

et faiblement répulsif si et seulement si, pour tout $n \in \mathbb{N}_0$,

$$p_0(n+1)p_{n+1} \leq p_n p_1.$$

Si Φ est un processus ponctuel de Poisson conditionnel de mesure d'intensité $M(\mathrm{d}x) = m(x)\mathrm{d}x$ et d'ensemble conditionnel C, alors son intensité de Papangelou c est donnée pour tout $n \in \mathbb{N}_0$ et tous $x, x_1, \dots, x_n \in \mathbb{X}$ par

$$c(x,\{x_1,\dots,x_n\}) = m(x)\mathbf{1}_{\{x_1,\dots,x_n,x\}\in C}\mathbf{1}_{\{x_1,\dots,x_n\}\in C}.$$

De plus, si C est décroissant, alors Φ est répulsif (théorème 3.4.13).

Si Φ est un processus ponctuel de Gibbs de paramètre de température $\theta > 0$ et énergie potentielle totale

$$U(x_1,\dots,x_n) = \sum_{r=1}^{n} \sum_{1\le i_1<\dots<i_r\le n} \Psi_r(x_{i_1},\dots,x_{i_r}),$$

où Ψ_r est le potentiel d'interaction de r^{e} ordre, alors son intensité de Papangelou c est donnée pour tout $x\in\mathbb{X}$ et tout $\phi\in\widehat{N}_{\mathbb{X}}$ par

$$c(x,\phi) = e^{-\theta(U(x\phi)-U(\phi))},$$

et Φ est répulsif (théorème 3.4.14).

Si Φ est un α-DPPP de noyau K et noyau associé J, alors son intensité de Papangelou c est donnée pour tout $x\in\mathbb{X}$ et tout $\phi\in\widehat{N}_{\mathbb{X}}$ par

$$c(x,\phi) = \frac{\det_\alpha J(x\phi,x\phi)}{\det_\alpha J(\phi,\phi)}.$$

Si $\alpha=-1$, alors Φ est répulsif (théorème 3.4.15).

3.5 Application à des processus ponctuels à base Poisson

Nous appliquons maintenant la majoration donnée par le théorème 3.3.2 pour les processus ponctuels à base Poisson. En particulier, nous rappelons les deux résultats suivants pour des processus ponctuels de Poisson et de Cox finis : si ζ_1,ζ_2 sont deux processus ponctuels de Poisson sur $\mathbb{X}$ de mesures d'intensité respectives M_1 et M_2, alors (théorème 3.5.1)

$$\Delta^*_{TV}(\zeta_1,\zeta_2) \le \Delta_{TV}(M_1,M_2),$$

et si Γ_1,Γ_2 sont deux processus ponctuels de Cox sur $\mathbb{X}$ dirigés par des mesures aléatoires presque sûrement finies respectives M_1 et M_2, alors (théorème 3.5.2)

$$\Delta^*_{TV}(\Gamma_1,\Gamma_2) \le \Delta^*_{TV}(M_1,M_2).$$

Considérant à présent une mesure finie M sur $\mathbb{X}$ telle que $M(\mathrm{d}x) = m(x)\mathrm{d}x$ et $\mu\in\mathbb{M}_1$ tel que $\mu(\mathrm{d}x) = \frac{m(x)}{M(\mathbb{X})}\mathrm{d}x$, si Φ est un processus ponctuel purement aléatoire sur $\mathbb{X}$ supporté par μ et $(p_n)_{n\in\mathbb{N}_0}$ est une distribution telle que $p_n\neq 0$ pour tout $n\in\mathbb{N}_0$, alors

$$\Delta^*_{TV}(\Phi,\zeta_M) \le \sum_{n=0}^{+\infty} \left|(n+1)p_{n+1} - M(\mathbb{X})p_n\right|,$$

où ζ_M est le processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité M (théorème 3.5.4).

Si Φ est un processus ponctuel de Poisson de mesure d'intensité finie $M(\mathrm{d}x) = m(x)\mathrm{d}x$ et si Φ_C est le processus ponctuel de Poisson conditionnel associé à Φ de mesure d'intensité M et condition $C\in\mathcal{N}_{\mathbb{X}}$, alors (théorème 3.5.5)

$$\Delta^*_{TV}(\Phi_C, \Phi) \leq \int_{\mathbb{X}} m(x)\mathbb{P}(\Phi_C x \notin C)\mathrm{d}x.$$

Si Φ est un processus ponctuel de Poisson d'intensité finie λ restreinte à un sous-ensemble relativement compact Λ de $\mathbb{X} = \mathbb{R}^d$ et si Φ_R est le processus ponctuel de Poisson hard-core associé à Φ de mesure paramètre M et paramètre $R > 0$, alors (corollaire 3.5.6)

$$\Delta^*_{TV}(\Phi_R, \Phi) \leq \frac{\lambda^2|\Lambda|}{p_R} V_d(R)$$

où

$$p_R = \mathbb{P}(\forall x, y \in \Phi, x \neq y \implies \Delta_{\mathbb{X}}(x, y) \geq R)$$

et

$$V_d(R) = \frac{\pi^{\frac{d}{2}} R^d}{\Gamma(\frac{d}{2})}.$$

Si Φ est un processus ponctuel de Poisson de mesure d'intensité finie $M(\mathrm{d}x) = m(x)\mathrm{d}x$ et si Φ_N est le processus ponctuel de Poisson borné associé à Φ de mesure paramètre M et paramètre $N \in \mathbb{N}_0$, alors (corollaire 3.5.7)

$$\Delta^*_{TV}(\Phi_N, \Phi) \leq \frac{e^{-M(\mathbb{X})}}{p_N} \frac{(M(\mathbb{X}))^{N+1}}{N!}$$

où $p_N = \mathbb{P}(\Phi(\mathbb{X}) \leq N)$.

3.6 Application à des processus faiblement répulsifs

Dans cette section, nous appliquons le théorème 3.3.2 aux processus ponctuels faiblement répulsifs.

Le premier résultat porte sur les superpositions (théorème 3.6.1) : pour tout $n \in \mathbb{N}$, Φ_n est la superposition de n processus ponctuels indépendants, finis et faiblement répulsifs $\Phi_{n,1}, \ldots, \Phi_{n,n}$, de fonctions de corrélation respectives $\rho_{n,1}, \ldots, \rho_{n,n}$ et ζ_M est un processus ponctuel de Poisson de mesure d'intensité $M(\mathrm{d}x) = m(x)\ell(\mathrm{d}x)$. Alors,

$$\Delta^*_{TV}(\Phi_n, \zeta_M) \leq R_n + 2n\Big(\max_{i \in \{1,\ldots,n\}} \int_{\mathbb{X}} \rho_{n,i}(x)\ell(\mathrm{d}x)\Big)^2,$$

où

$$R_n := \int_{\mathbb{X}} \Big| \sum_{i=1}^{n} \rho_{n,i}(x) - m(x) \Big| \ell(\mathrm{d}x).$$

Supposant de plus (remarque 3.6.2) qu'il existe une constante réelle C telle que, pour tout $n \in \mathbb{N}$,

$$\max_{i\in\{1,\dots,n\}} \int_{\mathbb{X}} \rho_{n,i}(x)\ell(\mathrm{d}x) \le \frac{C}{n},$$

on a, pour tout $n \in \mathbb{N}$,

$$\Delta^*_{TV}(\Phi_n, \zeta_M) \le R_n + \frac{2C^2}{n}.$$

En particulier (corollaire 3.6.3), si Φ_n est un processus ponctuel $(-1/n)$-déterminantal fini de noyau K et ζ un processus ponctuel de Poisson de mesure d'intensité $K(x,x)\mathrm{d}x$, alors

$$\Delta^*_{TV}(\Phi_n, \zeta) \le \frac{2}{n}\Big(\int_{\mathbb{X}} K(x,x)\mathrm{d}x\Big)^2.$$

Autre conséquence (corollaire 3.6.4) : si h est une fonction de densité sur $[0,1]$ telle que $h(0_+) := \lim_{x\to 0_+} h(x) \in \mathbb{R}$ et si Λ est un sous-ensemble compact de $\mathbb{R}_+$, alors, supposant que $X_{n,1},\dots,X_{n,n}$ sont n variables aléatoires i.i.d. de densité $h_n = \frac{1}{n}h(\frac{1}{n}\cdot)$, le processus ponctuel Φ_n défini par $\Phi_n = \{X_{n,1},\dots,X_{n,n}\} \cap \Lambda$ vérifie l'inégalité suivante :

$$\Delta^*_{TV}(\Phi_n, \zeta) \le \int_\Lambda \Big|h\Big(\frac{1}{n}x\Big) - h(0_+)\Big|\mathrm{d}x + \frac{2}{n}\Big(\int_\Lambda h\Big(\frac{1}{n}x\Big)\mathrm{d}x\Big)^2$$

où ζ est le processus ponctuel de Poisson homogène d'intensité $h(0_+)$ réduite à Λ.

Lorsque l'on superpose des processus ponctuels amincis, on obtient le résultat suivant (théorème 3.6.5) : si Φ est un processus ponctuel sur un sous-ensemble compact Λ de $\mathbb{X}$ d'intensité de Papangelou c et mesure d'intensité $M(\mathrm{d}x) = m(x)\mathrm{d}x$, si ζ est un processus ponctuel de Poisson de mesure d'intensité M, si, pour tout $n \in \mathbb{N}$, le processus ponctuel Φ_n est défini par

$$\Phi_n = \sum_{k=1}^n \frac{1}{n} \circ \Phi^{(k)},$$

où $\Phi^{(1)},\dots,\Phi^{(n)}$ sont n indépendantes copies de Φ et s'il existe une fonction intégrable $K:\ \Lambda \to \mathbb{R}_+$ telle que, pour tout $x \in \Lambda$, $\mathbb{V}[c(x,\Phi)] \le K(x)$, alors

$$\Delta^*_{TV}(\Phi_n, \zeta) \le \frac{1}{\sqrt{n}} \int_\Lambda \sqrt{K(x)}\mathrm{d}x.$$

On combine ensuite sur un processus déterminantal stationnaire un amincissement et une dilatation (théorème 3.6.6) : si $\zeta_{\Lambda,\lambda}$ désigne le processus ponctuel de Poisson homogène d'intensité λ réduit à un sous-ensemble compact Λ, si $\Phi_{\Lambda,\beta}$ est le processus ponctuel sur $\mathbb{R}^d$ obtenu par un β-amincissement et une β-dilatation sur le processus ponctuel déterminantal stationnaire Φ sur $\mathbb{R}^d$ d'intensité $\lambda \in \mathbb{R}$ et de noyau K, que l'on réduit à Λ, alors

$$\Delta^*_{TV}(\Phi_{\Lambda,\beta}, \zeta_{\Lambda,\lambda}) \le \frac{2\beta}{1-\beta}\lambda|\Lambda|.$$

L'application à des processus ponctuels de Gibbs donnée dans la suite (théorème 3.6.7) se concentre uniquement sur les processus ponctuels de Gibbs par paire, c'est-à-dire telle que, pour tout $r \in \mathbb{N} \setminus \{1,2\}$, $\Psi_r \equiv 0$: si $\epsilon \in \mathbb{R}_+$ et si Φ est un processus ponctuel de Gibbs sur $\mathbb{X}$ de paramètre de température $\theta > 0$, fonction de partition $C(\theta)$ et énergie potentielle totale

$$U(x_1, \dots, x_k) = \sum_{i=1}^{k} \Psi_1(x_i) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \Psi_2(x_i, x_j),$$

tel que $\int_{\mathbb{X}} e^{-\theta \Psi_1(x)} \mathrm{d}x < +\infty$, $\Psi_2 \geq 0$ et $\|\Psi_2\|_\infty \leq \epsilon$, alors

$$\Delta^*_{TV}(\Phi, \zeta_M) \leq (M(\mathbb{X}))^2 \theta \epsilon,$$

où ζ_M est le processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité $M(\mathrm{d}x) = e^{-\theta \Psi_1(x)} \mathrm{d}x$.

3.7 Extension d'un théorème de Kallenberg

Le but de cette section est de fournir une vitesse de convergence pour le théorème suivant (théorème 3.7.1), dû à Kallenberg : si $(\Phi_n)_{n \in \mathbb{N}}$ est une suite de processus ponctuels sur $\mathbb{X}$, $(p_n)_{n \in \mathbb{N}}$ une suite de fonctions de $\mathbb{X}$ dans $[0,1)$ telle que $(p_n)_{n \in \mathbb{N}}$ tend vers 0 uniformément, M une mesure aléatoire sur $\mathbb{X}$ et Γ_M un processus ponctuel de Cox dirigé par M, alors

$$p_n \Phi_n \xrightarrow[n \to +\infty]{loi} M \iff p_n \circ \Phi_n \xrightarrow[n \to +\infty]{loi} \Gamma_M.$$

On commence par démontrer (lemme 3.7.2) que si M_1, M_2 sont des mesures aléatoires sur $\mathbb{X}$ et $\Gamma_{M_1}, \Gamma_{M_2}$ des processus ponctuels de Cox dirigés par M_1, M_2 respectivement, alors

$$\Delta_P(\Gamma_{M_1}, \Gamma_{M_2}) = \overline{\Delta}_P(M_1, M_2),$$

avec $\overline{\Delta}_P$ désignant la distance polonaise sur $\mathbb{M}'_1$ associée à $g = (g_k)_{k \in \mathbb{N}}$ défini, pour tout $k \in \mathbb{N}$ et tout $\varphi \in \mathbb{M}$, par $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, où ζ_φ est un processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité φ.

On montre ensuite (lemme 3.7.3) que si $\varphi \in N_{\mathbb{X}}$ et si $p : \mathbb{X} \to [0,1)$ est une fonction mesurable, alors une version c de l'intensité de Papangelou de $p \circ \varphi$ par rapport à la mesure $p(x)\varphi(\mathrm{d}x)$ est fournie pour tout $x \in \mathbb{X}$ et tout $\eta \in N_{\mathbb{X}}$ par

$$c(x, \eta) = \mathbf{1}_{\{x \in \varphi \setminus \eta\}} \frac{1}{1 - p(x)},$$

d'où l'on déduit (lemme 3.7.4) que si Φ est un processus ponctuel sur $\mathbb{X}$, p une fonction de $\mathbb{X}$ dans [0,1) et $\Gamma_{p\Phi}$ un processus ponctuel de Cox dirigé par $p\Phi$, alors

$$\Delta^*_{TV}(\mathbb{P}_{p \circ \Phi}, \mathbb{P}_{\Gamma_{p\Phi}}) \leq 2\mathbb{E}\Big[\sum_{x \in \Phi} p^2(x)\Big].$$

Le résultat souhaité est finalement la suivant (théorème 3.7.5) : si Φ est un processus ponctuel sur $\mathbb{X}$, p une fonction mesurable de $\mathbb{X}$ dans [0,1), M une mesure aléatoire sur $\mathbb{X}$ et Γ_M un processus ponctuel de Cox dirigé par M, alors

$$\Delta_P(p \circ \Phi, \Gamma_M) \leq 2\mathbb{E}\Big[\sum_{x \in \Phi} p^2(x)\Big] + \overline{\Delta}_P(p\Phi, M),$$

avec $\overline{\Delta}_P$ désignant la distance polonaise sur $\mathbb{M}_1'$ associée à $g = (g_k)_{k \in \mathbb{N}}$ défini, pour tout $n \in \mathbb{N}$ et tout $\varphi \in \mathbb{M}$, par $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, où ζ_φ est un processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité φ.

B.4 Chapitre 4

Chapitre 4 : Processus ponctuels discrets α-stables

Dans ce chapitre, nous étudions les processus ponctuels discrets α-stables et adaptons sur eux la méthode de Stein utilisée pour les processus ponctuels de Poisson dans le chapitre 3.

Après avoir rappelé en section 4.1 les principaux résultats existant sur la théorie des processus ponctuels discrets α-stables, nous donnons en section 4.2 une expression pour l'intensité de Papangelou d'un processus ponctuel discret α-stable, et établissons trois généralisations de la formule de Mecke. Nous fournissons en section 4.3 un moyen de relier entre eux les processus ponctuels discrets α-stables - respectivement les mesures aléatoires α-stables - pour différentes valeurs de α. Dans la section 4.4, la méthode de Stein est appliquée aux processus ponctuels discrets α-stables. Nous construisons le processus de Glauber, établissons quelques propriétés de ses semi-groupe, générateur infinitésimal et gradients correspondants, et donnons une formule de représentation similaire à celle donnée pour le processus ponctuel de Poisson. Dans la section 4.5, nous introduisons les α-intensités de Papangelou et établissons quelques propriétés : nous donnons l'expression de la α-intensité de Papangelou d'un processus ponctuel discret α-stable, précisons le lien avec les intensités de Papangelou et fournissons quelques résultats concernant les transformations usuelles. Dans la section 4.6, quelques résultats de convergence sont démontrés lorsque α varie. Nous donnons quelques majorations pour la distance en variation totale entre un processus ponctuel discret α-stable et un processus ponctuel de Poisson, et plus généralement entre des processus ponctuels discrets α-stables dont les exposants α sont différents. Dans la section 4.7, nous donnons des résultats de convergence pour un α fixé. Deux adaptations du théorème de Kallenberg (section 3.7) sont établies, et nous déduisons un moyen d'approximation d'une mesure aléatoire strictement α-stable avec des processus ponctuels. Finalement, nous bornons la distance entre des processus ponctuels discrets α-stables avec différentes mesures spectrales et entre une superposition et un processus ponctuel discret α-stable.

4.1 Généralités sur les processus ponctuels discrets α-stables

On rappelle dans cette section quelques définitions et propriétés basiques de la théorie des mesures aléatoires et processus ponctuels stables, qui proviennent principalement de [20].

4.2 Intensité de Papangelou et formule de Mecke

Cette section présente quelques nouveaux résultats. Les premiers d'entre eux sont relatifs aux processus ponctuels de Sibuya.

Si $t \in [0,1)$ et Υ_t est le processus ponctuel de Sibuya t-dilaté sur $\mathbb{X}$ d'exposant α et mesure paramètre $\mu \in \mathbb{M}_1$ telle que $\mu(\mathrm{d}x) = q_\mu(x)\ell(\mathrm{d}x)$, alors (théorème 4.2.1) une version c_t de son intensité de Papangelou par rapport à ℓ est donnée pour tous $x \in \mathbb{X}$ et $\phi \in N_{\mathbb{X}} \setminus \{\varnothing\}$ par

$$c_t(x,\phi) = (|\phi| - \alpha)q_\mu(x) \text{ et } c_t(x,\varnothing) = \frac{\alpha t}{1-t} q_\mu(x).$$

On précise que l'intensité de Papangelou c_t d'un processus ponctuel de Sibuya t-dilaté sur $\mathbb{X}$ d'exposant α vérifie, pour tous $\omega, \phi \in \widehat{N_{\mathbb{X}}}$ tels que $\omega \subset \phi$ et tout $x \in \mathbb{X}$,

$$c_t(x,\omega) \le c_t(x,\phi),$$

si et seulement si $\alpha + t \le 1$. Notons que, dans ce cas, en adaptant notre définition de répulsivité à l'attractivité, Υ_t peut être vu comme un processus ponctuel attractif.

On montre également (théorème 4.2.2) qu'un processus ponctuel de Sibuya Υ sur $\mathbb{X}$ d'exposant α et mesure paramètre $\mu \in \mathbb{M}_1$ vérifie pour toute fonction $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$ mesurable l'égalité

$$\mathbb{E}\Big[\sum_{x\in\Upsilon} u(x, \Upsilon \setminus x)\Big] = \int_{\mathbb{X}} \mathbb{E}[(|\Upsilon| - \alpha)u(x,\Upsilon)]\mu(\mathrm{d}x) + \alpha \int_{\mathbb{X}} u(x,\varnothing)\mu(\mathrm{d}x).$$

Nous déduisons le résultat suivant (théorème 4.2.3) : si $\Phi = \sum_{\mu\in\zeta} \Upsilon_\mu$ est un processus ponctuel DαS tel que sa mesure spectrale σ est supportée par $\{\mu \in \mathbb{M}_1 : \mu \ll \ell\}$, alors son intensité de Papangelou c par rapport à ℓ vérifie, pour tout $x \in \mathbb{X}$,

$$c(x,\Phi) = \sum_{\mu\in\zeta} (|\Upsilon_\mu| - \alpha) q_\mu(x) + \alpha \int_{\mathbb{M}_1} q_\mu(x)\sigma(\mathrm{d}\mu) \text{ p.s.}$$

où, pour tout $\mu \in \mathbb{M}_1$ tel que $\mu \ll \ell$, $q_\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\ell}$.

Nous proposons alors d'adapter la formule de Mecke, connue pour les processus ponctuels de Poisson, aux processus ponctuels DαS.

Considérant $\Phi = \sum_{\mu\in\zeta} \Upsilon_\mu$ un processus ponctuel DαS tel que sa mesure spectrale σ est supportée par $\mathbb{M}_1$, il vient, pour toute fonction mesurable $u : \mathbb{X} \times N_{\mathbb{X}} \to \mathbb{R}_+$, les trois formules de Mecke suivantes : d'une part (théorème 4.2.4),

$$\mathbb{E}\Big[\sum_{x\in\Phi} u(x,\Phi\setminus x)\Big] = \int_{\mathbb{M}_1}\int_{\mathbb{X}} \mathbb{E}[(|\Upsilon_\mu| - \alpha)u(x, \Phi + \Upsilon_\mu)]\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu)$$
$$+ \alpha \int_{\mathbb{M}_1}\int_{\mathbb{X}} \mathbb{E}[u(x,\Phi)]\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu),$$

d'autre part (théorème 4.2.5),

$$\mathbb{E}\Big[\sum_{x\in\Phi}u(x,\Phi\setminus x)\Big]=\mathbb{E}\Big[\sum_{\mu\in\zeta}(|\Upsilon_\mu|-\alpha)\int_{\mathbb{X}}u(x,\Phi)\mu(\mathrm{d}x)\Big]+\alpha\int_{\mathbb{M}_1}\int_{\mathbb{X}}\mathbb{E}[u(x,\Phi)]\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu),$$

et enfin si, pour tous $\phi\in N_{\mathbb{X}}$ et $\mu\in\mathbb{M}_1^*$, $D_\mu u(\phi)<+\infty$, alors (théorème 4.2.6)

$$\mathbb{E}\Big[\sum_{y\in\Phi}(u(\Phi)-u(\Phi\setminus y))\Big]=\alpha\int_{\mathbb{M}_1}\mathbb{E}[D_\mu u(\Phi)]\sigma(\mathrm{d}\mu).$$

4.3 Lien entre α-stabilité et β-stabilité

Dans cette section, nous explorons le lien entre mesures aléatoires stables (StαS) et processus ponctuels stables de même mesure spectrale, mais exposants différents. Ces résultats fournissent d'intéressants outils pour comprendre, au moins de façon intuitive, les structures respectives des mesures aléatoires StαS et processus ponctuels DαS.

On considère pour les deux théorèmes suivants $\alpha,\beta,\gamma\in(0,1]$ tels que $\alpha=\beta\gamma$, une mesure σ localement finie sur $\mathbb{S}$ telle que, pour tout $B\in\mathcal{X}_0$,

$$\int_{\mathbb{S}}\mu(B)^\alpha\sigma(\mathrm{d}\mu)<+\infty,$$

l'application $\iota:\mu\in\mathbb{S}\mapsto\delta_\mu\in\mathbb{S}'$ et une mesure aléatoire StγS ξ' sur $\mathbb{M}_1$ de mesure spectrale $\sigma'=\sigma\circ\iota^{-1}$.

Le théorème 4.3.1 établit alors qu'une mesure aléatoire ξ telle que, conditionnellement à ξ', ξ est une mesure aléatoire StβS de mesure spectrale ξ', est une mesure aléatoire StαS sur $\mathbb{X}$ de mesure spectrale σ.

Le théorème 4.3.2 établit quant à lui que le processus ponctuel $\Phi=\sum_{\mu\in\Phi'}\Upsilon_\mu^\beta$ est un processus ponctuel DαS sur $\mathbb{X}$ de mesure spectrale σ.

4.4 Méthode de Stein pour des processus ponctuels DαS

Dans cette section, la méthode de Stein est appliquée pour des processus ponctuels finis DαS. Puisque un processus ponctuel DαS Φ a une représentation de type Poisson agrégatif $\sum_{\mu\in\zeta}\Upsilon_\mu$, il peut être identifié comme l'image d'une projection P sur $N_{\mathbb{X}}$ d'un processus ponctuel de Poisson marqué $\widetilde{\Phi}$ sur $\mathbb{M}_1\times\widehat{N}_{\mathbb{X}}$, définie de telle façon que la marque de chaque $\mu\in\zeta$ est un processus ponctuel de Sibuya Υ_μ, c'est-à-dire,

$$\widetilde{\Phi}=\sum_{\mu\in\zeta}\delta_{(\mu,\Upsilon_\mu)}.$$

Cette application P est définie par $P=S\circ Q$, où Q est la projection de $N_{\mathbb{M}_1\times\widehat{N}_{\mathbb{X}}}$ dans $N_{\widehat{N}_{\mathbb{X}}}$ et l'application $S:N_{\widehat{N}_{\mathbb{X}}}\to N_{\mathbb{X}}$ est définie, pour tout $\{\phi_1,\dots,\phi_n\}\in N_{\widehat{N}_{\mathbb{X}}}$, par

$$S(\{\phi_1,\dots,\phi_n\})=\phi_1+\dots+\phi_n.$$

Ceci peut être résumé par le schéma suivant :

$$\begin{array}{ccc} N_{\mathbb{M}_1 \times \widehat{N}_{\mathbb{X}}} & \to & N_{\mathbb{X}} \\ \sum_{\mu \in \zeta} \delta_{(\mu, \Upsilon_\mu)} & \stackrel{P}{\to} & \sum_{\mu \in \zeta} \Upsilon_\mu. \end{array}$$

De cette approche, nous pouvons déduire une version de la méthode de Stein qui adapte facilement celle utilisée pour un processus ponctuel de Poisson fini. Cependant, le processus ponctuel de Poisson ζ n'est pas observé et le gradient correspondant, qui serait défini, pour toute fonction mesurable $F : N_{\mathbb{M}_1 \times N_{\mathbb{X}}} \to \mathbb{R}$, tous $\phi \in N_{\mathbb{M}_1 \times N_{\mathbb{X}}}$, $\mu \in \mathbb{M}_1$ et $\eta \in N_{\mathbb{X}}$, par

$$D_{(\mu,\eta)} F(\phi) = F(\phi + (\mu, \eta)) - F(\phi)$$

ne peut pas être calculé dans les cas pratiques.

Intuitivement, si nous souhaitons conserver le même principe pour la définition du gradient discret, nous devons trouver quelque chose à ajouter à $\sum_{\mu \in \zeta} \Upsilon_\mu$. Le terme ainsi ajouté doit être similaire (d'une certaine façon) à un processus ponctuel de Sibuya Υ_μ. Notre idée est donc de considérer la quantité aléatoire $F(\phi + \Upsilon_\mu) - F(\phi)$, où $\phi \in N_{\mathbb{X}}$.

Nous débutons ces investigations en énonçant le lemme suivant (lemme 4.4.1) : si Ψ est un processus ponctuel sur $\mathbb{M}_1$, alors, pour tout $t \in [0, 1]$,

$$t \circ \sum_{\mu \in \Psi} \Upsilon_\mu \stackrel{\mathcal{D}}{=} \sum_{\mu \in \Psi} t \circ \Upsilon_\mu \stackrel{\mathcal{D}}{=} \sum_{\mu \in t^\alpha \circ \Psi} \Upsilon_\mu,$$

où, pour tout $\mu \in \mathbb{M}_1^*$, Υ_μ est un processus ponctuel de Sibuya d'exposant α et mesure paramètre μ.

Le **processus de Glauber** $(G_t)_{t \geq 0}$ associé à un processus ponctuel DαS Φ de mesure spectrale finie σ supportée par $\mathbb{M}_1$ est défini pour tous $t \in \mathbb{R}_+$ et $\phi \in \widehat{N}_{\mathbb{X}}$ par

$$G_t(\phi) = e^{-\frac{t}{\alpha}} \circ \phi + (1 - e^{-t})^{\frac{1}{\alpha}} \circ \Phi.$$

Pour tout $t \in \mathbb{R}_+$, l'operateur P_t est défini pour toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$ par :

$$P_t F(\phi) = \mathbb{E}[F(G_t(\phi))] = \mathbb{E}[F(e^{-\frac{t}{\alpha}} \circ \phi + (1 - e^{-t})^{\frac{1}{\alpha}} \circ \Phi)].$$

On obtient alors facilement que, pour tout $t \in \mathbb{R}_+$, toute fonction mesurable et bornée $F : \widehat{N}_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in \widehat{N}_{\mathbb{X}}$,

$$P_t F(\phi) = \mathbb{E}[F(e^{-\frac{t}{\alpha}} \circ \phi + \sum_{\mu \in (1 - e^{-t}) \circ \zeta} \Upsilon_\mu)].$$

Ses dynamiques peuvent être décrites comme suit : imaginons un processus de Poisson homogène ζ_b sur $\mathbb{R}_+$ d'intensité $\sigma(\mathbb{M}_1)$. Les instants de saut de ζ_b déterminent les instants de naissance des mesures de probabilité dans ζ, placées dans $\mathbb{M}_1$ selon la distribution $\frac{\sigma(\cdot)}{\sigma(\mathbb{M}_1)}$. Pour chaque nouvelle mesure $\mu \in \mathbb{M}_1$, un processus ponctuel de Sibuya Υ_μ est placé dans $\mathbb{X}$. La durée de vie de chaque processus ponctuel de Sibuya est distribuée exponentiellement de paramètre 1 (et ses particules meurent simultanément). La durée de vie de chaque particule de ϕ est distribuée exponentiellement de paramètre $\frac{1}{\alpha}$ et ces durées de vie sont indépendantes.

Si $\alpha = 1$, le processus de Glauber associé au processus ponctuel DαS de mesure spectrale σ défini dans cette section correspond au processus de Glauber associé au processus ponctuel de Poisson de mesure d'intensité $\mu(\mathrm{d}x)\sigma(d\mu)$, construit dans la section 3.1.

On montre alors (théorème 4.4.3) que la famille $(P_t)_{t\geq 0}$ donnée par la définition précédente est bien un semi-groupe.

La création d'un nouveau processus ponctuel de Sibuya dans le processus de Glauber est induite par la création de sa mesure de probabilité correspondante. Cela nous mène à considérer les deux définitions de gradient suivantes.

Pour tout $\omega \in N_{\mathbb{X}}$, le **gradient** D_ω dans la direction ω est défini, pour toute fonction mesurable $F : N_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in N_{\mathbb{X}}$, par

$$D_\omega F(\phi) = F(\phi + \omega) - F(\phi).$$

Pour tout $\mu \in \mathbb{M}_1^*$, le **gradient** D_μ dans la direction μ d'exposant α est défini, pour toute fonction mesurable et bornée $F : N_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in N_{\mathbb{X}}$, par

$$D_\mu^\alpha F(\phi) = \mathbb{E}_{\Upsilon_\mu}[D_{\Upsilon_\mu} F(\phi)] = \mathbb{E}_{\Upsilon_\mu}[F(\phi + \Upsilon_\mu) - F(\phi)],$$

où pour tout $\mu \in \mathbb{M}_1^*$, Υ_μ est un processus ponctuel de Sibuya d'exposant α et mesure paramètre μ.

En identifiant le point $x \in \mathbb{X}$ avec la configuration $\{x\} \in N_{\mathbb{X}}$, on observe que l'application $x \in \mathbb{X} \mapsto D_x$ définie dans la section 3.1 est la restriction à $\{\omega \in N_{\mathbb{X}} : |\omega| = 1\}$ de l'application $\omega \in N_{\mathbb{X}} \mapsto D_\omega$. De plus, considérant le cas $\alpha = 1$, on a, pour tout $\mu \in \mathbb{M}_1^*$, toute fonction mesurable et bornée $F : N_{\mathbb{X}} \to \mathbb{R}$ et tout $\phi \in N_{\mathbb{X}}$,

$$D_\mu^1 F(\phi) = \int_{\mathbb{X}} D_x F(\phi)\mu(\mathrm{d}x) = \int_{\mathbb{X}} \big(F(\phi + x) - F(\phi)\big)\mu(\mathrm{d}x).$$

Le processus ponctuel Φ vérifie comme pour le cas Poisson la propriété de fermabilité suivante (théorème 4.4.6) : si $F, G : N_{\mathbb{X}} \to \mathbb{R}$ sont deux fonctions mesurables et bornées telles que $F(\phi) = G(\phi)$ $\mathbb{P}_\Phi(\mathrm{d}\phi)$-p.s., alors

$$D_\mu F(\phi) = D_\mu G(\phi) \ (\sigma \otimes \mathbb{P}_\Phi)(\mathrm{d}\mu, \mathrm{d}\phi)\text{-p.s..}$$

On démontre alors pour le gradient la propriété suivante (théorème 4.4.7) : si $t \in [0,1]$ et $\mu \in \mathbb{M}_1^*$, alors, pour toute fonction mesurable et bornée $F : \widehat{N_{\mathbb{X}}} \to \mathbb{R}$,

$$D_{t\mu} F = t^\alpha D_\mu F.$$

On établit ensuite l'expression du générateur infinitésimal L associé au semi-groupe (théorème 4.4.8), donnée pour toute fonction mesurable et bornée $F : \widehat{N_{\mathbb{X}}} \to \mathbb{R}$ et tout $\phi \in \widehat{N_{\mathbb{X}}}$ par :

$$LF(\phi) = \int_{\mathbb{M}_1} D_\mu F(\phi)\sigma(\mathrm{d}\mu) + \frac{1}{\alpha}\sum_{y\in\phi}(F(\phi \setminus y) - F(\phi)).$$

puis les relations de commutation suivantes (lemme 4.4.9) : pour tout $t \in \mathbb{R}_+$, tout $\mu \in \mathbb{M}_1^*$, toute fonction mesurable et bornée $F : \widehat{N_{\mathbb{X}}} \to \mathbb{R}$ et tout $\phi \in \widehat{N_{\mathbb{X}}}$,

$$D_\mu P_t F(\phi) = P_t D_{e^{-\frac{t}{\alpha}}\mu} F(\phi) = e^{-t} P_t D_\mu F(\phi),$$

et, pour tout $x \in \mathbb{X}$,

$$D_x P_t F(\phi) = e^{-\frac{t}{\alpha}} P_t D_x F(\phi),$$

et enfin la propriété d'ergodicité suivante (lemme 4.4.10) : pour tout $F \in \mathrm{Lip}_1(\widehat{N_{\mathbb{X}}}, \Delta_D)$ et tout $\phi \in \widehat{N_{\mathbb{X}}}$,

$$\lim_{t \to +\infty} P_t F(\phi) = \mathbb{E}[F(\Phi)].$$

Il est alors possible de déduire une nouvelle formule de représentation de Stein-Dirichlet (théorème 4.4.11) : pour tout $F \in \mathrm{Lip}_1(\widehat{N_{\mathbb{X}}}, \Delta_D)$ et tout $\phi \in \widehat{N_{\mathbb{X}}}$,

$$\mathbb{E}[F(\Phi)] - F(\phi) = \int_0^{+\infty} L P_s F(\phi) \mathrm{d}s.$$

4.5 α-intensité de Papangelou

La α-intensité de Papangelou adapte la définition de l'intensité de Papangelou aux processus ponctuels DαS.

Pour Φ un processus ponctuel sur $\mathbb{X}$ et λ une mesure de Radon sur $\mathbb{M}_1$, l'application $c_\alpha : \mathbb{M}_1 \times N_{\mathbb{X}} \to \mathbb{R}_+$ est appelée (une version de) la **α-intensité de Papangelou** de Φ par rapport à λ si, pour toute fonction mesurable $u : N_{\mathbb{X}} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{y \in \Phi} (u(\Phi) - u(\Phi \setminus y))\Big] = \alpha \mathbb{E}\Big[\int_{\mathbb{M}_1} c_\alpha(\mu, \Phi) D_\mu u(\Phi) \lambda(\mathrm{d}\mu)\Big].$$

Par exemple, si Φ est un processus ponctuel DαS tel que sa mesure spectrale finie σ est absolument continue par rapport à une mesure de Radon λ sur $\mathbb{M}_1$, alors (théorème 4.5.2) l'application $c_\alpha : \mathbb{M}_1 \times N_{\mathbb{X}} \to \mathbb{R}_+$ définie pour tout $(\mu, \phi) \in \mathbb{M}_1 \times N_{\mathbb{X}}$ par

$$c_\alpha(\mu, \phi) = \frac{\mathrm{d}\sigma(\mu)}{\mathrm{d}\lambda}$$

est la α-intensité de Papangelou de Φ par rapport à λ.

Plus généralement, si $\Phi = \sum_{\mu \in \zeta} \Upsilon_\mu$ est un processus ponctuel sur $\mathbb{X}$ où ζ est un processus ponctuel sur $\mathbb{M}_1$ d'intensité de Papangelou c par rapport à une mesure de Radon λ et tel que les processus ponctuels de Sibuya de la somme sont indépendants, alors (théorème 4.5.3) la α-intensité de Papangelou c_α de Φ par rapport à λ vérifie, pour tout $\mu \in \mathbb{M}_1$,

$$c_\alpha(\mu, \Phi) = c(\mu, \zeta) \text{ p.s..}$$

Il est également possible de considérer une superposition indépendante (théorème 4.5.4) : si $\Phi_1, \dots, \Phi_n$ ($n \in \mathbb{N}$) sont des processus ponctuels indépendants sur $\mathbb{X}$ de α-intensités de Papangelou respectives $c_{\alpha,1}, \dots, c_{\alpha,n}$ et Φ leur superposition indépendante, alors la α-intensité de Papangelou c_α de Φ vérifie, pour tout $\mu \in \mathbb{M}_1$,

$$c_\alpha(\mu, \Phi) = \sum_{k=1}^{n} c_{\alpha,k}(\mu, \Phi_k) \text{ p.s..}$$

De même, pour un amincissement (théorème 4.5.5) : si Φ est un processus ponctuel sur $\mathbb{X}$, β une fonction de $\mathbb{X}$ dans $[0,1]$ et $\beta \circ \Phi$ le β-amincissement de Φ, alors son α-intensité de Papangelou $c_{\alpha,\beta}$ vérifie, pour tout $\mu \in \mathbb{M}_1$,

$$c_{\alpha,\beta}(\mu, \beta \circ \Phi) = \beta(x)\mathbb{E}[c_\alpha(\mu, \Phi) \mid \beta \circ \Phi] \text{ p.s..}$$

4.6 Résultats de convergence α-dépendants

Dans cette section, les résultats de convergence sont donnés pour Δ_{TV} au lieu de Δ^*_{TV}. En effet, cette dernière distance est trop forte pour être utilisée entre des processus ponctuels discrets stables, puisque le nombre de points de tels processus ponctuels a une espérance infinie.

Premier résultat établi (théorème 4.6.1) : si, pour tout $i \in \{1,2\}$, Φ_i est un processus ponctuel Dα_iS d'intensité aléatoire presque sûrement finie Stα_iS ξ_i, alors

$$\Delta_{TV}(\Phi_1, \Phi_2) \le \Delta^*_D(\xi_1, \xi_2).$$

On démontre ensuite (lemme 4.6.2) que la distance en variation totale entre un processus ponctuel DαS Φ_α de mesure spectrale finie σ et un processus ponctuel de Poisson Φ sur $\mathbb{X}$ de mesure d'intensité $\mu(\mathrm{d}x)\sigma(\mathrm{d}\mu)$ vérifie l'inégalité

$$\Delta_{TV}(\Phi_\alpha, \Phi) \le (1-\alpha)C(\alpha)\sigma(\mathbb{M}_1),$$

où, pour tout $a \in (0,1]$,

$$C(a) = 2\min\Big(\Big\{1+a, 1+2\frac{e^{-2}}{a}\Big\}\Big) \le 2\Big(1+\frac{\sqrt{2}}{e}\Big).$$

Plus généralement (théorème 4.6.3), si $\alpha, \beta \in (0,1]$ sont tels que $\alpha < \beta$, si Φ_α et Φ_β sont des processus ponctuels respectivement DαS et DβS et de même mesure spectrale finie σ, alors

$$\Delta_{TV}(\Phi_\alpha, \Phi_\beta) \le \Big(1-\frac{\alpha}{\beta}\Big)C\Big(\frac{\alpha}{\beta}\Big)\sigma(\mathbb{M}_1).$$

Dans le cas où Φ_α est un processus ponctuel DαS de mesure spectrale $\sigma = c\delta_\mu$, où $c \in \mathbb{R}_+$ et $\mu \in \mathbb{M}_1$ et où Φ est un processus ponctuel de Poisson de mesure d'intensité $c\mu$, on a (théorème 4.6.4), pour tout $\epsilon \in \mathbb{R}_+$,

$$\Delta_{TV}(\Phi_\alpha, \Phi) \le c^{\frac{1}{\alpha}}\epsilon + |c^{\frac{1}{\alpha}} - c| + \mathbb{P}(|X_\alpha - 1| \ge \epsilon),$$

où X_α est une variable aléatoire positive strictement α-stable dont la transformée de Laplace est donnée, pour tout $t \in \mathbb{R}_+$, par

$$\mathbb{E}[e^{-tX_\alpha}] = e^{-t^\alpha}.$$

On compare alors les majorations obtenues dans le lemme 4.6.2 et le théorème 4.6.4 pour le processus ponctuel DαS Φ_α de mesure spectrale $\sigma = \delta_\mu$ et le processus ponctuel de Poisson Φ de mesure d'intensité μ. D'après le lemme 4.6.2,

$$\Delta_{TV}(\Phi_\alpha, \Phi) \leq (1-\alpha)C(\alpha) \leq 4(1-\alpha),$$

donc il semble pertinent de considérer la quantité $p_\alpha(\epsilon) := \mathbb{P}(|X_\alpha - 1| \geq \epsilon)$ pour $\epsilon = 4(1-\alpha)$ et lorsque α est proche de 1. Puisque nous n'avons pas d'expression analytique pour la fonction de répartition de X_α, nous donnons seulement une approximation de la quantité $p_\alpha(4(1-\alpha))$ par la méthode de Monte-Carlo. Nous simulons un échantillon de $N = 10^6$ réalisations de X_α pour $\alpha \in [0.990, 1)$ et observons dans la figure 4.3 que $p_\alpha(4(1-\alpha))$ tend vers 1 - au lieu de 0 - lorsque α tend vers 1. Il semble donc que la majoration est meilleure dans le lemme 4.6.2 que dans le théorème 4.6.4.

Cependant, la même méthode utilisée pour $\epsilon = \sqrt{1-\alpha}$ permet d'observer (figure 4.3) que $p_\alpha(\sqrt{1-\alpha})$ semble être borné par $\sqrt{1-\alpha}$ et donc de conjecturer que

$$\Delta_{TV}(\Phi_\alpha, \Phi) \leq 2\sqrt{1-\alpha}.$$

Le dernier résultat proposé dans cette section est le suivant (théorème 4.6.5) : si, pour tout $i \in \{1,2\}$, Φ_i est un processus ponctuel Dα_iS d'intensité aléatoire St(α_i)S ξ_i, alors

$$\Delta_P(\Phi_1, \Phi_2) = \overline{\Delta}_P(\xi_1, \xi_2),$$

avec $\overline{\Delta}_P$ désignant la distance polonaise sur $\mathbb{M}_1'$ associée à $g = (g_k)_{k\in\mathbb{N}}$ définie, pour tout $k \in \mathbb{N}$ et tout $\varphi \in \mathbb{M}$, par $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, où ζ_φ est un processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité φ.

4.7 Résultats de convergence pour transformations

Dans cette section, des résultats de convergence sont établis pour des processus ponctuels construits par amincissements et superpositions, et sont aussi donnés pour Δ_{TV}.

Une version du théorème de Kallenberg cité dans le chapitre 3 est énoncée ainsi (théorème 4.7.1) : si $(\Phi_n)_{n\in\mathbb{N}}$ est une suite de processus ponctuels sur $\mathbb{X}$ et $(p_n)_{n\in\mathbb{N}}$ une suite de fonctions de $\mathbb{X}$ dans $[0,1)$ telle que $(p_n)_{n\in\mathbb{N}}$ tend vers 0 uniformément ; si ξ est une mesure aléatoire StαS et Ψ un processus ponctuel DαS, de même mesure spectrale finie σ, alors

$$p_n\Phi_n \xrightarrow[n\to+\infty]{loi} \xi \iff p_n \circ \Phi_n \xrightarrow[n\to+\infty]{loi} \Psi.$$

De plus, pour tout $n \in \mathbb{N}$,

$$\Delta_P(p_n \circ \Phi_n, \Psi) \leq 2\mathbb{E}\Big[\sum_{x\in\Phi_n} p_n^2(x)\Big] + \overline{\Delta}_P(p_n\Phi_n, \xi),$$

avec $\overline{\Delta}_P$ désignant la distance polonaise sur $\mathbb{M}_1'$ associée à $g = (g_k)_{k\in\mathbb{N}}$ définie, pour tout $k \in \mathbb{N}$ et tout $\varphi \in \mathbb{M}$, par $g_k(\varphi) = \mathbb{E}[f_k(\zeta_\varphi)]$, où ζ_φ est un processus ponctuel de Poisson sur $\mathbb{X}$ de mesure d'intensité φ.

Il est également possible d'approcher une mesure aléatoire StαS à l'aide d'un processus ponctuel DαS (corollaire 4.7.2) : si Φ est un processus ponctuel DαS de mesure d'intensité StαS ξ, alors la suite (ξ_n) de mesures aléatoires définie pour tout $n \in \mathbb{N}$ par

$$\xi_n = \frac{1}{n^{\frac{1}{\alpha}}} \sum_{k=1}^{n} \Phi^{(k)},$$

où $\Phi^{(1)}, \dots, \Phi^{(n)}$ sont n copies indépendantes de Φ, converge en loi vers ξ.

Un théorème de Kallenberg pour des sommes de processus ponctuels de Sibuya est ensuite énoncé (théorème 4.7.3) : si $(\Psi_n)_{n\in\mathbb{N}}$ est une suite de processus ponctuels sur $\mathbb{M}_1$, $(p_n)_{n\in\mathbb{N}}$ une suite de fonctions de $\mathbb{X}$ dans $[0,1)$ tendant uniformément vers 0, si de plus $\Phi_n = \sum_{\mu\in\Psi_n} \Upsilon_\mu$ où, conditionnellement à Ψ_n, $(\Upsilon_\mu)_{\mu\in\mathbb{M}_1}$ est une famille de processus ponctuels de Sibuya indépendants d'exposant α, si σ est un élément aléatoire presque sûrement fini de $\mathbb{M}$ et Γ un processus ponctuel de Cox dirigé par σ, si $p_n^\alpha \Psi_n \xrightarrow[n\to+\infty]{loi} \sigma$, alors

$$p_n \circ \Phi_n \xrightarrow[n\to+\infty]{loi} \sum_{\mu\in\Gamma} \Upsilon_\mu.$$

L'équivalent du théorème 3.3.2 pour les processus ponctuels stables est alors établi (théorème 4.7.4) : si Ψ est un processus ponctuel DαS sur $\mathbb{X}$ de mesure spectrale finie $\sigma(\mathrm{d}\mu) = m(\mu)\lambda(\mathrm{d}\mu)$ (avec $\lambda \in \mathbb{M}_R'$) et Φ est un second processus ponctuel fini sur $\mathbb{X}$ de α-intensité de Papangelou c_α par rapport à λ, alors

$$\Delta_{TV}(\Phi, \Psi) \leq \int_{\mathbb{M}_1} \mathbb{E}[|m(\mu) - c_\alpha(\mu, \Phi)|]\lambda(\mathrm{d}\mu).$$

En particulier (corollaire 4.7.5), si Φ_1 et Φ_2 sont deux processus ponctuels DαS de mesures spectrales finies respectives σ_1 et σ_2, alors

$$\Delta_{TV}(\Phi_1, \Phi_2) \leq \Delta_{TV}(\sigma_1, \sigma_2).$$

Le chapitre se termine en étudiant la convergence d'une superposition de processus ponctuels amincis (corollaire 4.7.6) : si Φ est un processus ponctuel fini sur $\mathbb{X}$ de α-intensité de Papangelou c_α par rapport à une mesure de Radon λ sur $\mathbb{M}_1$ telle que, pour tout $\mu \in \mathbb{M}_1$, $m(\mu) := \mathbb{E}[c_\alpha(\mu, \Phi)] \in \mathbb{R}_+$ et si Ψ est un processus ponctuel DαS sur $\mathbb{X}$ de mesure spectrale finie $\sigma(\mathrm{d}\mu) = m(\mu)\lambda(\mathrm{d}\mu)$, si, pour tout $n \in \mathbb{N}$, le processus ponctuel Φ_n est défini par

$$\Phi_n = \sum_{k=1}^{n} \frac{1}{n^{\frac{1}{\alpha}}} \circ \Phi^{(k)},$$

où $\Phi^{(1)}, \dots, \Phi^{(n)}$ sont n copies indépendantes de Φ et si il existe une fonction intégrable (par rapport à λ) $K : \mathbb{M}_1 \to \mathbb{R}_+$ telle que, pour tout $\mu \in \mathbb{M}_1$, $\mathbb{V}[c_\alpha(\mu, \Phi)] \leq K(\mu)$, alors

$$\Delta_{TV}(\Phi_n, \Psi) \leq \frac{1}{n^{\frac{1}{\alpha}-\frac{1}{2}}} \int_{\mathbb{M}_1} \sqrt{K(\mu)}\lambda(\mathrm{d}\mu).$$

B.5 Appendice A

Appendice A : Cas d'étude sur la régularité du déploiement d'un réseau cellulaire

A.1 Introduction

Les modèles d'emplacements d'émetteurs ont pour objectif de fournir des outils pour comprendre le déploiement d'un réseau effectif. Pour les entreprises de télécommunications, la connaissance a priori de la distribution des emplacements d'antennes aide à prédire et gérer les coûts de déploiement du réseau. De tels modèles mathématiques fournissent également des méthodes pratiques pour estimer la probabilité de couverture d'un réseau donné. Ces résultats intéressent potentiellement les régulateurs de communication ainsi que les autorités de santé publique, puisque l'exposition électromagnétique est désormais une problématique à l'échelle mondiale.

Le premier modèle introduit dans les réseaux radio est le réseau déterministe hexagonal. Bien que le maillage régulier des cellules donne une approximation d'une cellule type, il ne permet pas de rendre compte de la réalité propre du déploiement du réseau. Ce modèle s'avère aussi être optimiste quant à l'estimation des interférences [1]. La nature aléatoire des paramètres impliqués dans la stratégie de couverture rend difficile l'utilisation d'un modèle déterministe et régulier. Les idées provenant de la géométrie stochastique, spécialement sur les processus ponctuels - i.e. les processus ponctuels de Poisson, les processus ponctuels de type Matérn hard-core, les processus ponctuels de Ginibre et β-Ginibre - furent ensuite largement explorés dans la littérature concernant la communication sans fil. Les travaux précurseurs dans ce domaine ont été réalisés par Baccelli et al. sur les processus ponctuels de Poisson [5]. Il en découla de nombreux résultats, tels que le probabilité de couverture relative au rapport signal sur interférence plus bruit (SINR). Les derniers développements sur les modèles poissonniens incluent également la modélisation de réseaux hétérogènes (k-niveau) [33] et de signaux sur les réseaux sans fil [44]. Cependant, les positions des stations de base dans un réseau déployé de façon poissonnienne sont décorrélées entre elles, si bien que des agrégats de points peuvent apparaître. La distance moyenne inter-sites dans de telles configurations est donc plus petite que dans la réalité. Par conséquent, les modèles poissonniens génèrent plus d'interférences que pour un réseau réel. Les articles de Andrews et al. [1] et Nakata et al. [54] montrent que le processus ponctuel de Poisson fournit une prédiction plus pessimiste de probabilité d'interruption que dans les modèles répulsifs.

Les corrélations spatiales entre les locations des stations de base existent, puisque ces stations de base doivent être séparées les unes des autres pour optimiser la couverture et minimiser les interférences inter-sites. Pour prendre en compte ces effets, des modèles répulsifs (ou réguliers) furent introduits dans la littérature. Une approche simple est de transformer un processus ponctuel de Poisson en un processus ponctuel répulsif via un amincissement. De tels processus ponctuels sont appelés les processus ponctuels Matérn hard-core. Les interférences pour de tels réseaux ont été étudiées [39] mais les modèles hard-core restent difficiles à manipuler car la probabilité d'interruption ne peut pas être déduite analytiquement. Les processus ponctuels soft-core ont alors éveillé les intérêts de la communauté. Parmi ces processus, les processus ponctuels de Ginibre et β-Ginibre (tous deux déterminantaux) furent examinés dans le domaine de la communication sans fil. Ils furent tout d'abord introduits par Shirai et al. [66] en physique quantique pour modéliser les interactions entre fermions. Les travaux de Miyoshi et al. [52] et Deng et al. [30] ont permis l'obtention de la probabilité de couverture par rapport au SINR pour les modèles de Ginibre et β-Ginibre.

Dans ce travail, nous démontrons que la distribution des stations de base pour un opérateur et une technologie donnés peut être "fittée" avec un processus ponctuel de β-Ginibre dans la région de Paris. La distribution de toutes les stations de base, pour tous les opérateurs peut, quant à elle, être fittée avec un processus ponctuel de Poisson. Notre principale contribution est la justification théorique de ce phénomène. Nous prouvons que la superposition indépendante de différents processus ponctuels de β-Ginibre converge en distribution vers un processus ponctuel de Poisson. Enfin, nous tirons des conclusions sur la capacité de couverture pour les différents opérateurs. Des résultats qualitatifs sont déduits à partir des valeurs inférées de β et de l'intensité ρ. La fonction ρ peut donner des informations sur la stratégie de dimensionnement adoptée par l'opérateur, tandis que β fournit des indications sur la couverture réseau.

Les autres papiers existant sur les modèles de déploiement d'antennes considèrent essentiellement le calcul du SINR et de la probabilité de couverture pour une large catégorie de processus ponctuels. Nous nous intéressons plutôt à la validation du modèle de β-Ginibre et du modèle de superposition poissonnien sur des données réelles en région urbaine dense. Un tel cas d'étude a été rendu possible grâce à l'agence nationale des fréquences (ANFR) qui fournit les éléments nécessaires dans une base de données en accès ouvert [2].

Dans la section A.2, nous introduisons le théorème de convergence en distribution pour une superposition indépendante de processus ponctuels de β-Ginibre. Dans la section A.3, nous exposons la méthode utilisée pour fitter le modèle de β-Ginibre avec les données actuelles. Une interprétation qualitative des stratégies de déploiement est ensuite réalisée à partir des valeurs de β et ρ inférées.

A.2 Modèle théorique

Un des principaux apports de ce travail est l'étude de la superposition indépendante de plusieurs processus ponctuels de β-Ginibre. Le théorème de convergence clé pour les processus ponctuels de β-Ginibre que nous établissons est le suivant (théorème A.2.2) : si $n \in \mathbb{N}$ et Φ_n est la superposition indépendante de n processus ponctuels $\Phi_{n,1}, \ldots, \Phi_{n,n}$, tels que, pour tout $i \in \{1, ..., n\}$, $\Phi_{n,i}$ est un processus ponctuel de $\beta_{n,i}$-Ginibre d'intensité $\frac{\gamma_i}{n\pi}$, où $\beta_{n,i} \in (0,1]$ et $\gamma_i \in (0,+\infty)$, et si la suite $(\gamma_k)_{k\in\mathbb{N}^*}$ est bornée et

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} \gamma_i = \gamma, \text{ avec } \gamma \in [0,+\infty),$$

alors $(\Phi_n)_{n\in\mathbb{N}}$ converge en loi vers un processus ponctuel de Poisson homogène Φ d'intensité $\frac{\gamma}{\pi}$.

Une preuve détaillée de ce théorème est alors exposée et s'appuie sur le théorème A.2.1, qui permet de caractériser la convergence en loi.

A.3 Analyse statistique

Dans cette section, nous introduisons la méthode de fitting utilisée pour obtenir le paramètre β et présentons les résultats du fitting pour le déploiement de chacun des opérateurs mobiles à Paris.

Statistique sommaire

Afin de fitter le déploiement réel avec le modèle de β-Ginibre, nous introduisons la J-fonction associée au processus ponctuel qu'elle caractérise. Cette fonction est une statistique sommaire basée sur les distances inter-points. Des informations générales sur les statistiques sommaires peuvent être trouvées dans [10].

La **J-fonction** d'un processus ponctuel stationnaire Φ sur $\mathbb{R}^d$ est définie pour tout $r \in \mathbb{R}_+$ par

$$J(r) = \frac{1 - G(r)}{1 - F(r)},$$

où F est la fonction ensemble-vide de Φ et G la fonction de distribution de la distance au plus proche voisin, définie pour un $u \in \mathbb{R}^d$ fixé et tout $r \in \mathbb{R}_+$ par

$$F(r) = \mathbb{P}(\|u - \Phi\| \leq r)$$

et

$$G(r) = \mathbb{P}(\|u - \Phi \setminus \{u\}\| \leq r).$$

La J-fonction fournit à la fois une caractérisation du processus ponctuel et une information directe sur son attractivité ou sa répulsivité. Plus précisément, quand $J < 1$, Φ est attractif, sinon Φ est répulsif. L'égalité $J \equiv 1$ caractérise le processus ponctuel de Poisson, où il n'y a pas d'interactions entre les particules. Pour le cas du processus ponctuel de β-Ginibre, nous avons par [8] le théorème A.3.2, qui établit que la J-fonction d'un processus ponctuel de β-Ginibre d'intensité $\frac{\gamma}{\pi}$ est donnée pour tout $r \in \mathbb{R}_+$ par

$$J(r) = \frac{1}{1 - \beta + \beta e^{-\frac{\gamma}{\beta} r^2}}.$$

Notons que pour tout β cette J-fonction est plus grande que 1, ce qui confirme que le processus ponctuel de β-Ginibre est un processus ponctuel répulsif. Lorsque β tend vers 0, cette expression tend vers 1, ce qui correspond à la J-fonction d'un processus ponctuel de Poisson.

Cette J-function permet de valider le processus ponctuel de β-Ginibre comme un modèle de distribution de la répartition des stations de base pour chaque opérateur et chaque technologie.

Méthode de fitting

Grâce au langage R et au package spatstat [11], l'estimation de la J-fonction est directement déduite des données brutes. Puisque nous ne considérons qu'un ensemble fini d'antennes, des effets de bord risquent d'apparaître sur l'estimation de la J-fonction. Nous devons donc nous restreindre à un sous-ensemble des données pour améliorer notre estimation. La figure A.1 montre la fenêtre que nous considérons pour l'extraction des données. Cette fenêtre couvre environ 60% de la ville et sa forme est choisie en adéquation avec les limites géographiques. Les valeurs de l'estimation de la J-fonction sont calculées pour $r \leq 600$m. Au-dessus de 600 m, l'estimation n'est plus pertinente à cause des effets de bord. La fonction J est ensuite directement fittée sur l'estimation et la paramètre β est déduit. Un exemple de fitting est donné à la figure A.2. Il est clair que le processus ponctuel formé par les emplacements des stations de base est répulsif et fitte correctement le modèle théorique. En conséquence, cela permet de rejeter la modèle poissonnien puisque la J-fonction d'un processus ponctuel de Poisson est égale à 1 pour tout r. Dans le paragraphe suivant, nous présentons les résultats obtenus sur les données brutes.

Résultats du fitting et interprétation

Les emplacements des stations de base sont disponibles publiquement pour tout le territoire français et peuvent être trouvés en ligne [9]. Il y a quatre opérateurs en France ; ils fournissent une couverture 2G à 4G pour la plupart d'entre eux. Pour chaque opérateur et chaque technologie, les valeurs numériques de β et ρ obtenues par le fitting sont données à la table A.1. Chaque intensité ρ est simplement calculée en utilisant le nombre de stations de base correspondantes dans la fenêtre. Le paramètre β est ensuite calculé par la méthode des moindres carrés appliquée à la J-fonction du processus ponctuel de β-Ginibre et son estimation.

Les valeurs obtenues pour β et ρ donnent quelques indications sur la stratégie de déploiement d'un réseau cellulaire pour chaque opérateur, spécialement à propos de la capacité de couverture trade-off. Les hautes valeurs de β and ρ pour Orange suggèrent que cet opérateur a déployé (en tant qu'opérateur historique, propriété de l'état) un réseau qui réalise une couverture optimale et une capacité de trafic optimale (réseau densément déployé). Cependant, SFR et Bouygues ont les premiers déployé un réseau avec un minimum d'antennes (afin de se conformer aux exigences de couverture du régulateur) et a donc graduellement augmenté la capacité de trafic sur les hot-spots (en augmentant localement le nombre d'antennes). Cela implique d'ajouter plus d'antennes sur les sites qui sont déjà couverts, donc de créer des agrégats, de diminuer la valeur de β et d'augmenter la valeur de ρ. L'autorité de régulation des communications électroniques et des postes (ARCEP) a publié des rapports annuels [12] qui suggèrent une telle évolution. Nous déduisons que les opérateurs français utilisent deux stratégies de déploiement différentes. La première stratégie consiste à réaliser à la fois la couverture et une capacité de trafic optimale, tandis que la deuxième stratégie est de déployer un réseau qui se conforme aux exigences de couverture dans un premier temps, puis dans un second temps d'augmenter le nombre d'antennes en hot-spots pour améliorer la capacité de trafic.

Lorsqu'ils déploient leurs réseaux 3G ou 4G, les opérateurs réutilisent et partagent des sites 2G déjà existants. C'est pourquoi, nous considérons que classifier les sites de stations de base par opérateur est plus pertinent que de les classifier par technologie. La table A.2 donne un résumé de ces résultats. Comme espéré, les précédentes conclusions sont toujours valables car les valeurs de β sont stables entre les deux tables. Nous remarquons également que Free, en tant que nouveau venu (2012), a une petite quantité de trafic à traiter, et donc a déployé moins d'antennes que ses concurrents. L'analyse des données montre aussi que la superposition de tous les sites tend vers un processus ponctuel de Poisson car β est alors égal à 0.17. Le modèle poissonnien est donc toujours valable comme indicateur de l'exposition électromagnétique des réseaux cellulaires.

A.4 Conclusion

Dans ce travail, nous démontrons avec succès que les processus ponctuels de β-Ginibre sont un modèle réaliste pour la distribution des stations de base. Le paramètre β est inféré en utilisant des outils statistiques sur données réelles. Des résultats qualitatifs sur les déploiements de réseaux sont alors déduits. Nous prouvons aussi de façon théorique que la superposition indépendante de plusieurs processus ponctuels de β-Ginibre converge en distribution vers un processus ponctuel de Poisson, justifiant les observations faites sur les déploiements réels. Ces résultats auront certainement un impact très positif dans la modélisation des réseaux multi-niveaux. Nous montrons que les valeurs de ρ et β sont des caractéristiques de capacité de couverture trade-off. Les futurs travaux étudieront l'impact de ρ et β sur les stratégies de déploiement optimales.

Bibliography

[1] J. G. Andrews, F. Baccelli, and R. K. Ganti. A tractable approach to coverage and rate in cellular networks. *IEEE Transactions on Communications*, 59(11):3122 – 3134, Nov. 2011.

[2] ANFR. Cartoradio, 2014.

[3] ARCEP. La qualité des services mobiles, 2014.

[4] G. J. Babu and E. D. Feigelson. Spatial point processes in astronomy. *Journal of Statistical Planning and Inference*, 50(3):311–326, Mar. 1996.

[5] F. Baccelli and B. Blaszczyszyn. Stochastic Geometry and Wireless Networks, Volume I - Theory. *Foundations and Trends in Networking*, 3(3-4):249–449, Jan. 2010.

[6] A. Baddeley and R. Turner. Spatstat: an R package for analyzing spatial point patterns. *Journal of Statistical Software*, 12(6), Jan. 2005.

[7] A. D. Barbour. Stein's method and Poisson process convergence. *Journal of Applied Probability*, 25:175–184, 1988.

[8] A. D. Barbour and T. C. Brown. Stein's method and point process approximation. *Stochastic Processes and their Applications*, 43(1):9–31, Nov. 1992.

[9] A. D. Barbour and A. Xia. On Stein's factors for Poisson approximation in Wasserstein distance. *Bernoulli*, 12(6):943–954, 2006.

[10] R. Bardenet, F. Lavancier, X. Mary, and A. Vasseur. On a few statistical applications of determinantal point processes. In *ESAIM: Proceedings and Surveys*, volume 60, pages 180–202, Grenoble, 2017. Coeurjolly J.F., Leclercq-Samson A. (eds).

[11] I. Camilier and L. Deucresefond. Quasi-invariance and integration by parts for determinantal and permanental processes. *Journal of Functional Analysis*, 259(1):268–300, July 2010.

[12] F. Chazal, B. T. Fasy, F. Lecci, B. Michel, A. Rinaldo, and L. Wasserman. Subsampling Methods for Persistent Homology. *Proceedings of the 32nd International Conference on Machine Learning*, 37:2143–2151, 2015.

[13] L. H.-Y. Chen. Poisson approximation for dependent trials. *The Annals of Probability*, 3(3):534–545, 1975.

[14] G. Christoph and K. Schreiber. Discrete Stable Random Variables. *Statistics & Probability Letters*, 37(3):243–247, Mar. 1998.

[15] G. Christoph and K. Schreiber. Scaled Sibuya distribution and discrete self-decomposability. *Statistics & Probability Letters*, 48(2):181–187, June 2000.

[16] L. Coutin and L. Decreusefond. Stein's method for Brownian approximations. *Communications on Stochastic Analysis*, 7(3):349–372, 2013.

[17] D. R. Cox and V. Isham. *Point processes*, volume 12 of *Chapman & Hall/CRC Monographs on Statistics & Applied Probability*. CRC Press, 1980.

[18] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods.* Springer, New York, 2nd edition, 2003.

[19] Y. Davydov, I. Molchanov, and S. Zuyev. Strictly stable distributions on convex cones. *Electronic Journal of Probability*, 13(11):259–321, 2008.

[20] Y. Davydov, I. Molchanov, and S. Zuyev. Stability for random measures, point processes and discrete semigroups. *Bernoulli*, 17(3):1015–1043, 2011.

[21] L. Decreusefond. Wasserstein distance on configuration space. *Potential Analysis*, 28(3):283–300, May 2008.

[22] L. Decreusefond, I. Flint, N. Privault, and G. L. Torrisi. Determinantal Point Processes. In *Stochastic Analysis for Poisson Point Processes*, volume 7 of *Bocconi & Springer Series*, pages 311–342. Peccati G., Reitzner M., springer, cham edition, 2016.

[23] L. Decreusefond, I. Flint, and A. Vergne. A note on the simulation of the Ginibre point process. *Journal of Applied Probability*, 52(4):1003–1012, 2015.

[24] L. Decreusefond, A. Joulin, and N. Savy. Upper bounds on Rubinstein distances on configuration spaces and applications. *Communication on Stochastic Analysis and Applications*, 3(4):377–399, 2010.

[25] L. Decreusefond, M. Schulte, and C. Thäle. Functional Poisson approximation in Kantorovich-Rubinstein distance with applications to U-statistics and stochastic geometry. *The Annals of Probability*, 44(3):2147–2197, 2016.

[26] L. Decreusefond and A. Vasseur. Asymptotics of Superposition of Point Processes. In *Geometric Science of Information*, volume 9389 of *Lecture Notes in Computer Science*, pages 187–194. Nielsen F., Barbaresco F. (Eds), Paris, springer edition, 2015.

[27] L. Decreusefond and A. Vasseur. Asymptotics of some point processes transformations. *Full paper in preparation*, 2017.

[28] L. Decreusefond, A. Vasseur, and S. Zuyev. Asymptotics of discrete stable point processes. *Conference SPA 2017, Moscou*, 2017.

[29] P. Del Moral and J. Tugaut. On the stability and the uniform propagation of chaos properties of Ensemble Kalman-Bucy filters. *The Annals of Applied Probability*, to appear, 2017.

[30] N. Deng, W. Zhou, and M. Haengii. The Ginibre Point Process as a Model for Wireless Networks with Repulsion. *IEEE Transactions on Wireless Communications*, 14(1):107 – 121, Jan. 2014.

[31] L. Devroye. Random variate generation for the digamma and trigamma distributions. *Journal of Statistical Computation and Simulation*, 43:197–216, Nov. 1992.

[32] L. Devroye. A triptych of discrete distributions related to the stable law. *Statistics & Probability Letters*, 18(5):349–351, Dec. 1993.

[33] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews. Modeling and analysis of k- tier downlink heterogeneous cellular networks. *IEEE Journal on Selected Areas in Communications*, 30(3):550–560, Apr. 2012.

[34] A. C. Gatrell, T. C. Bailey, P. J. Diggle, and B. S. Rowlingson. Spatial Point Pattern Analysis and Its Application in Geographical Epidemiology. *Transactions of the Institute of British Geographers*, 21(1):256–274, 1996.

[35] H.-O. Georgii and H. J. Yoo. Conditional intensity and gibbsianness of determinantal point processes. *Journal of Statistical Physics*, 118(1-2):55–84, Jan. 2005.

[36] J. Ginibre. Statistical Ensembles of Complex, Quaternion, and Real Matrices. *Journal of Mathematical Physics*, 6:440–449, 1965.

[37] A. Goldman. The Palm measure and the Voronoi tessellation for the Ginibre process. *The Annals of Applied Probability*, 20(1):90–128, Jan. 2010.

[38] J.-S. Gomez, A. Vasseur, A. Vergne, L. Decreusefond, P. Martins, and W. Chen. A Case Study on Regularity in Cellular Network Deployment. *IEEE Wireless Communications Letters*, 4(4):421–424, Aug. 2015.

[39] M. Haenggi. Mean interference in hard-core wireless networks. *IEEE Communications Letters*, 15(8):792–794, Aug. 2011.

[40] M. Haenggi. *Stochastic Geometry for Wireless Networks*. Cambridge University Press, New York, 2012.

[41] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virag. Determinantal Processes and Independence. *Probability Surveys*, 3:206–229, May 2006.

[42] O. Kallenberg. *Random Measures*. Akademie-Verlag, New York, 3 edition, 1983.

[43] O. Kallenberg. *Foundations of moder probability. Probability and its applications.* Springer, 1997.

[44] H. P. Keeler, N. Ross, and A. Xia. When do wireless network signals appear Poisson? *CoRR*, abs/1411.3757, 2014.

[45] G. Last, G. Peccati, and M. Schulte. Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. *Probability Theory and Related Fields*, 165(3-4):667–723, Aug. 2016.

[46] F. Lavancier, J. Møller, and E. Rubak. Determinantal point process models and statistical inference : Extended version. *Journal of the Royal Statistical Society, series B*, 77(4):853–877, 2015. DOI: 10.1111/rssb.12096.

[47] D.-Z. Liu and Y. Wang. Universality for Products of Random Matrices I: Ginibre and Truncated Unitary Cases. *International Mathematics Research Notices*, 2016(11):3473–3524, Jan. 2016. DOI: 10.1093/imrn/rnv247.

[48] X. Lu, A. Vasseur, I. Flint, and D. Niyato. Application of discrete stable point processes to a jamming model. *Full paper in preparation*, 2017.

[49] P. Lévy. Stochastic Models with Infinite Variance. *Bulletin de la Société Mathématique de France*, 52:49–85, 1924.

[50] O. Macchi. The coincidence approach to stochastic point processes. *Advances in Applied Probability*, 7(1):83–122, 1975.

[51] K. Matthes, J. Kerstan, and J. Mecke. *Infinitely Divisible Point Processes*. John Wiley & Sons, Chichester-New York-Brisbane-Toronto, 1978. 10.1002/bimj.4710210410.

[52] N. Miyoshi and T. Shirai. A cellular network model with Ginibre configured base stations. *Advances in Applied Probability*, 46(3):832–845, 2014.

[53] J. Møller and R. P. Waagepetersen. *Statistical inference and simulation for spatial point processes*. CRC Press, 2003.

[54] I. Nakata and N. Miyoshi. Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations. *Performance Evaluation*, 78:7–17, Aug. 2014.

[55] J. P. Nolan. *Stable Distributions: Models for Heavy Tailed Data*. Springer, New York, 2016.

[56] I. Nourdin and G. Peccati. *Normal approximations with Malliavin Calculus: From Stein's method to universality*. Cambridge University Press, June 2012.

[57] F. Papangelou. The conditional intensity of general point processes and an application to line processes. *Z. Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 28(3):207–226, Sept. 1974.

[58] C. Preston. Spatial birth-and-death processes. *Proceedings of the 40th Session of the International Statistical Institute*, 2:371–391, 1975.

[59] G. Reinert. Three general approaches to Stein's method. *An introduction to Stein's method*, 4:183–221, Apr. 2005.

[60] D. Ruelle. *Statistical mechanics: Rigorous results*. W. A. Benjamin, Amsterdam, 1969.

[61] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. CRC Press, 1994.

[62] D. Schuhmacher and K. Stucki. Gibbs point process approximation: totoal variation bounds using Stein's method. *The Annals of Probability*, 42(5):1911–1951, 2014.

[63] D. Schuhmacher and A. Xia. A new metric between distributions of point processes. *Advances in Applied Probability*, 40(3):651–672, 2008.

[64] M. Schulte and C. Thäle. Distances Between Poisson k-Flats. *Methodology and Computing in Applied Probability*, 16(2):311–329, June 2014.

[65] H.-H. Shih. On Stein's method for infinite-dimensional Gaussian approximation in abstract Wiener spaces. *Journal of Functional Analysis*, 261(5):1236–1283, Sept. 2011.

[66] T. Shirai and Y. Takahashi. Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes. *Journal of Functional Analysis*, 205(2):414–463, Dec. 2003.

[67] M. Sibuya. Generalized hypergeometric, digamma and trigamma distributions. *Annals of the Institute of Statistical Mathematics*, 31(1):373–390, Dec. 1979.

[68] A. Soshnikov. Determinantal random point fields. *Russian Mathematical Surveys*, 55(5):923–975, 2000.

[69] C. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, 2:583–602, 1972.

[70] F. W. S. Steutel and K. Van Harn. Discrete Analogues of Self-Decomposability and Stability. *The Annals of Probability*, 7(5):893–899, 1979.

[71] D. Stoyan and A. Penttinen. Recent Applications of Point Process Methods in Forestry Statistics. *Statistical Science*, 15(1):61–78, 2000.

[72] H. Thorisson. *Coupling, Stationarity, and Regeneration*. Probability and its Applications. Springer, 2000.

[73] C. Villani. *Optimal Transport: Old and New*, volume 338 of *Springer*. Springer Science & Business Media, 2008.

Analyse asymptotique de processus ponctuels

Aurélien VASSEUR

RESUMÉ : La méthode de Stein constitue une des principales techniques pour la résolution de certains problèmes d'approximation en théorie des probabilités. Dans ce manuscrit, nous l'appliquons au contexte des processus ponctuels.

La première partie de ces investigations se concentre sur le processus ponctuel de Poisson. Sa propriété caractéristique d'indépendance fournit le moyen d'expliquer intuitivement pourquoi une suite de processus ponctuels de moins en moins répulsive peut converger vers un tel processus ponctuel. Ceci nous amène plus généralement à démontrer des résultats de convergence pour des suites de processus ponctuels construites à partir d'opérations telles que la superposition, l'amincissement ou l'homothétie. L'utilisation d'une distance sur les processus ponctuels, appelée distance de Kantorovich-Rubinstein, permet en outre l'obtention de taux de convergence.

La seconde partie est centrée sur une classe de processus ponctuels avec beaucoup d'attractivité, appelés processus ponctuels α-stables. Leur structure basée sur un processus ponctuel de Poisson nous permet d'élargir à ces processus la méthode utilisée précédemment et de proposer de nouveaux résultats, via certaines propriétés que nous établissons sur ces processus ponctuels.

MOTS-CLÉS : Processus ponctuel, géométrie stochastique, méthode de Stein, intensité de Papangelou, processus ponctuel de Poisson, stabilité, convergence.