Problèmes d'homogénéisation elliptique en présence de défauts

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Titre : Problèmes d'homogénéisation elliptique en présence de défauts

Résumé : Le travail de cette thèse a porté sur plusieurs problèmes d'homogénéisation d'équations elliptiques linéaires dans un cadre de coefficients oscillants non périodiques. Les classes de coefficients considérées sont supposées modéliser des géométries périodiques perturbées par des défauts de différentes natures. L'objectif de ces problèmes est d'expliciter les limites homogénéisées de suites de solutions quand un paramètre d'échelle tend vers 0 tout en précisant le taux de convergence. La principale difficulté de notre cadre est la résolution d'équations dites du correcteur associées à ces problèmes, équations linéaires elliptiques posées sur des ouverts non bornés.

Dans une première partie, on traite l'homogénéisation du problème de diffusion sous forme divergence quand le coefficient est une perturbation non locale d'un coefficient périodique. La perturbation ne disparaît jamais mais devient rare à l'infini. On prouve l'existence d'un correcteur adapté, on identifie la limite homogénéisée et on étudie la taux de convergence de la suite de solutions vers sa limite homogénéisée.

La deuxième partie traite également de l'homogénéisation du problème de diffusion mais dans le cas de coefficients presque invariants par translation à l'infini. La géométrie est alors caractérisée par l'intégrabilité à l'infini d'un certain gradient discret du coefficient, supposé appartenir à un espace de Lebesgue pour un exposant donné. Quand l'exposant de Lebesgue est strictement inférieur à la valeur de la dimension ambiante, on établit une adaptation discrète de l'inégalité de Gagliardo-Nirenberg-Sobolev afin de montrer que le coefficient appartient à une classe de coefficients périodiques perturbés par un défaut local. On prouve alors l'existence d'un correcteur et on identifie la limite homogénéisée de la suite de solutions. Dans le cas où l'exposant de Lebesgue est supérieur à la dimension, on exhibe des coefficients pour lesquels la suite de solutions possède plusieurs valeurs d'adhérence.

Finalement, dans la troisième partie, on s'intéresse à l'homogénéisation d'un problème de type Schrödinger stationnaire avec un potentiel oscillant appartenant à une classe particulière de potentiels périodiques perturbés par des défauts potentiellement non localisés dans l'espace. On montre à nouveau l'existence d'un correcteur adapté et on prouve la convergence de la suite de solutions vers une limite homogénéisée.

Mots clefs : Homogénéisation, Problèmes multi-échelles, Défauts, Équations aux dérivées partielles elliptiques, Équation de diffusion, Équation de Schrödinger stationnaire, Correcteurs, Taux de convergence.

Title : Elliptic homogenization problems with defects

Abstract : The purpose of this work is the homogenization of several second order linear elliptic equations with non-periodic oscillating coefficients. The classes of coefficients we consider are assumed to model periodic geometries perturbed by various types of defects. The aims of these problems are to identify the homogenized limit of sequences of solutions as a small scale parameter vanishes and to make precise the convergence rate, for several topologies. The main difficulty that arises in our settings is the study of the so-called corrector equations associated with these problems, which are posed on unbounded domains.

First, we consider an homogenization problem for the diffusion equation in divergence form when the coefficient is a non-local perturbation of a periodic coefficient. The perturbation does not vanish but becomes rare at infinity. We prove the existence of a corrector, identify the homogenized limit and study the convergence rates of the sequence of solutions to its homogenized limit.

The second part also regards the homogenization of the diffusion problem but, in this part, the coefficient is assumed to be almost translation-invariant at infinity. The geometry is then characterized by the integrability at infinity of a particular discrete gradient of the coefficient. In particular, this discrete gradient is assumed to belong to a Lebesgue space for a given exponent. When the Lebesgue exponent is less than the value of the ambient dimension, we establish a discrete adaptation of the Gagliardo-Nirenberg-Sobolev inequality in order to show that the coefficient actually belongs to a certain class of periodic coefficients perturbed by a local defect. We next prove the existence of a corrector and we identify the homogenized limit of the sequence of solutions. When the Lebesgue exponent is equal to or greater than the value of the ambient dimension, we exhibit admissible coefficients such that the sequence of solutions possesses different subsequences that converge to different limits.

Finally, in the third part, we consider the homogenization of the stationary Schrödinger equation with an highly oscillatory potential that belongs to a particular class of periodic potentials perturbed by possibly non-local defects. We show the existence of an adapted corrector and prove the convergence of the sequence of solutions to an homogenized limit.

Keywords : Homogenization, Multiscale problems, Defects, Elliptic partial differential equations, Diffusion equation, Stationary Schrödinger equation, Correctors, Convergence rates.

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Liste des publications

[Gou22a] R. Goudey, *A periodic homogenization problem with defects rare at infinity*, Networks and Heterogeneous Media 17, no.4, pp. 547–592, 2022.

[Gou22b] R. Goudey, *Elliptic homogenization with almost translation-invariant coefficients*, Asymptotic Analysis, 2022.

[GLB22] R. Goudey and C. Le Bris, *Linear elliptic homogenization for a class of highly oscillating non-periodic potentials*, Preprint, arXiv :2205.15600, 2022.

Les références [Gou22a], [Gou22b], [GLB22] sont reproduites (avec des adaptations mineures) dans les **Chapitres** 2, 3 et 4 respectivement.

Chapitre 1

Introduction générale

Sommaire

1.1 Introduction

L'homogénéisation d'un problème multi-échelles consiste à aborder des phénomènes issus d'un milieu hétérogène complexe, présentant des variations microscopiques dans sa structure, avec un point de vue macroscopique. Le milieu étudié possède au moins deux échelles : une échelle fine, ou plusieurs telles échelles, où on voit les hétérogénéités et une échelle macroscopique plus grossière. L'enjeu de cette approche est de déterminer si, à l'échelle macroscopique, les propriétés du milieu sont celles d'un milieu homogène, défini de manière unique, et qui doit alors être caractérisé. Si c'est le cas, on dit alors qu'il y a *homogénéisation* du milieu et on s'attend à ce que le milieu homogène obtenu soit associé à un certain procédé de "moyennisation" issu de la structure microscopique.

De nombreux modèles physiques traitent d'environnements hétérogènes : conduction thermique dans des matériaux composites ou possédant des micro-fractures, traversée d'un fluide à travers un milieu poreux, simulation de la densité du trafic routier etc... Une des principales motivations de la théorie de l'homogénéisation est l'approximation numérique de solutions u^ε d'équations aux dérivées partielles issues de ces modèles physiques et qui dépendent d'un paramètre d'échelle $\varepsilon > 0$, paramètre représentatif de la disproportion entre la plus petite des échelles et la plus grande. L'approximation numérique de ces solutions requiert souvent des discrétisations extrêmement coûteuses lorsque ce paramètre ε est petit et on cherche donc à approcher cette solution par une solution dite *homogénéisée* associée au problème homogène obtenu à l'échelle macroscopique. On identifie donc deux objectifs distincts : comprendre qu'il y a un milieu équivalent homogène physique dont on pourra déterminer les paramètres caractéristiques, par exemple la valeur du module de Young et du coefficient de Poisson du matériau équivalent, ou bien un objectif computationnel qui consiste à approcher numériquement la solution u^ε. En particulier, on attend de cette description homogénéisée d'être une bonne approximation du problème qui prendrait précisément en compte les informations à petite échelle. En étudiant les propriétés microscopiques d'un milieu, on espère alors pouvoir déterminer, physiquement ou numériquement, son comportement à l'échelle macroscopique par une approche mathématique. Les questions mathématiques liées aux problèmes d'homogénéisation sont alors à la fois d'ordre théorique : "Est-il possible de construire et de décrire une solution homogénéisée ?" mais aussi d'ordre pratique : "Peut-on préciser l'erreur liée à cette approximation en spécifiant le taux de convergence de la solution u^ε vers la solution homogénéisée dans les bonnes topologies et quand le paramètre d'échelle tend vers zéro ?".

En général, il n'est pas aisé de justifier mathématiquement ce processus d'homogénéisation puisque les différents paramètres qui interviennent dans ces problèmes sont souvent liés de manière très complexe et, lorsqu'ils existent, les milieux homogènes associés sont caractérisés à l'aide de procédés sophistiqués de moyennisation des propriétés du milieu hétérogène d'origine. Une hypothèse simplificatrice de microstructure périodique a ainsi été privilégiée dès le début de la théorie de l'homogénéisation, voir par exemple la référence classique de Bensoussan, Lions et Papanicolaou [18] sur le sujet. Plusieurs théorèmes de convergence à la fois *constructifs* (l'équation homogénéisée associée au problème peut être déterminée explicitement) et *précisés* (les taux de convergences vers la solution homogénéisée sont connus) ont alors été démontrés dans le cadre d'équations à coefficients périodiques.

Dans le but de développer une théorie de l'homogénéisation complète pour des modèles plus complexes que celui d'une géométrie parfaitement périodique, et donc plus proches des systèmes réels observables, plusieurs entreprises complémentaires se sont développées. Dans le contexte des équations aux dérivées partielles elliptiques et linéaires, on pense tout d'abord aux théories générales et abstraites de De Giorgi et de Tartar (voir par exemple [77, 90, 38]). Les résultats établis dans ce cadre sont cependant purement qualitatifs : l'existence d'une limite homogénéisée est établie à l'aide de théorème de compacité, on ne sait généralement pas expliciter cette limite et les taux de convergence sont inconnus.

Plusieurs extensions naturelles du cadre périodique ont également été développées afin d'obtenir des résultats à la fois constructifs et précisés : les cadres quasi et presque périodiques d'abord traités par Kozlov dans [64] ou Oleinik et Zhikov dans [82] et étoffés à travers plusieurs travaux d'Armstrong et ses collaborateurs comme [7, 10] ; le cadre ergodique et stationnaire, généralisation naturelle de la structure périodique dans un contexte aléatoire, initialement introduit dans les années 80, avec les travaux de Kozlov [65], Papanicolaou et Varadhan [84] et Yurinskii [93] par exemple, puis récemment développé par les résultats quantitatifs de Gloria, Neukamm et Otto [49, 50] et ceux d'Armstrong, Mourrat et Kuusi [8].

Durant la dernière décennie, Blanc, Le Bris et Lions ont parallèlement introduit des modèles

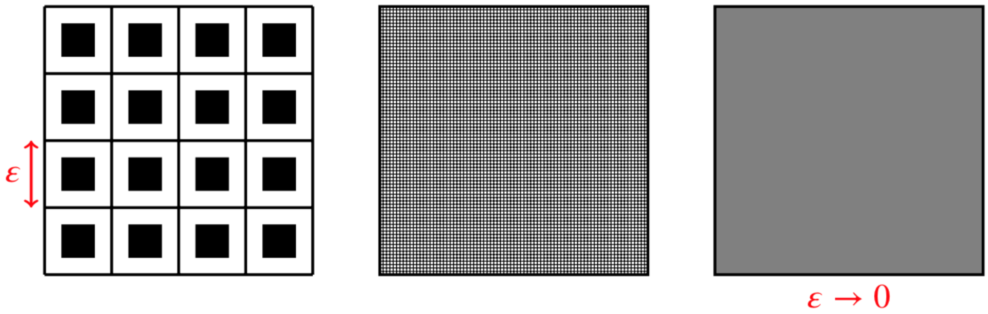

Figure 1.1 – Exemple du procédé d'homogénéisation d'un matériau possédant une microstructure périodique. À gauche, on représente un zoom du milieu hétérogène du milieu et, à droite, on illustre le matériau homogène équivalent après passage à la limite.

déterministes non-périodiques, donnant en particulier naissance à une classe de problèmes s'appuyant sur un principe de superposition : les coefficients sont la somme d'un coefficient périodique b_{per} et d'une perturbation $\tilde{b}$. Ces modèles sont donc caractérisés par la présence d'une géométrie périodique sous-jacente perturbée par des défauts de différentes natures. Ce nouveau cadre a d'abord été traité pour des équations linéaires sous forme divergence pour des perturbations *locales* de géométrie périodique, puis a progressivement été étendu à d'autres types de problèmes ou d'équations, voir par exemple [25, 26, 27, 28, 20] pour le cadre elliptique. On pourra voir aussi [35] et [1] pour des premiers travaux dans cette direction pour un cadre Hamilton-Jacobi. Dans le contexte elliptique et linéaire, ce cadre déterministe non-périodique permet notamment d'établir une théorie complète de l'homogénéisation, c'est-à-dire où il est possible d'expliciter la limite homogénéisée et d'obtenir des taux de convergence précis vers cette limite.

C'est dans la continuité de ce dernier cadre que s'inscrit le travail présenté dans cette thèse, notamment avec la volonté d'étendre ces résultats d'homogénéisation dans un contexte plus général de perturbations de géométries périodiques, pour des perturbations plus "méchantes" et pour différentes classes d'équations aux dérivées partielles elliptiques et linéaires. L'objectif est évidemment de proposer des cadres géométriques vérifiant des hypothèses ni trop spécifiques afin d'essayer de se rapprocher au maximum d'une réalité "pratique", ni trop abstraites qui, même en établissant des résultats théoriques élaborés, ne permettraient pas d'établir des résultats de convergence précis ou applicables d'un point de vue numérique.

1.2 Homogénéisation périodique

1.2.1 Équation de diffusion

Un des problèmes d'homogénéisation des équations aux dérivées partielles elliptiques d'ordre deux que nous étudierons dans les **Chapitres** 2 et 3 de cette thèse est le problème de diffusion avec un coefficient à oscillations rapides :

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon\right) = f & \text{sur } \Omega, \\ u^\varepsilon = 0 & \text{sur } \partial\Omega. \end{cases} \tag{1.1}$$

Ici Ω est un ouvert borné de $\mathbb{R}^d$ ($d \geq 1$) pris suffisamment régulier, $\varepsilon > 0$ est le petit paramètre d'échelle et $f \in L^2(\Omega)$ est donnée et est éventuellement régulière. Le champ de matrices a est supposé uniformément borné et elliptique, c'est-à-dire,

$$\exists M > 0, \quad |a(x)| \leq M, \quad \text{pour presque tout } x \in \mathbb{R}^d, \tag{1.2}$$

$$\exists \lambda > 0, \quad \lambda |x|^2 \leq \langle x, ax \rangle, \quad \text{pour presque tout } x \in \mathbb{R}^d. \tag{1.3}$$

Le cas où le coefficient $a = a_{per}$ est $\mathbb{Z}^d$-périodique est bien connu et a été largement étudié dans la littérature, voir par exemple [18, 59]. Sous cette hypothèse de périodicité, le comportement de u^ε ainsi que celui de son gradient sont d'abord intuités par une approche heuristique : l'*Ansatz du développement à deux échelles*. Cette approche, détaillée par exemple dans [3], consiste à postuler que u^ε admet un développement de la forme

$$u^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots,$$

où chaque fonction $u_k(x, y)$ est caractérisée par deux variables, une variable lente macroscopique x et une variable rapide microscopique $y = \frac{x}{\varepsilon}$, et est supposée périodique par rapport à la variable y. En supposant que x et y sont indépendantes lorsque $\varepsilon \to 0$ (on parle de séparation d'échelle), l'idée est alors d'introduire cet Anstaz dans l'équation (1.1) et d'identifier les u_k en égalisant les termes de même ordre en puissance de ε. Bien que cette approche ne soit pas rigoureusement justifiée, elle permet d'identifier un candidat pour être la limite de la suite u^ε et d'intuiter le comportement du gradient de cette suite. Tous calculs effectués, on s'attend ainsi à ce que u^ε converge vers une limite u^*, solution dans $H_0^1(\Omega)$ de l'équation dite homogénéisée :

$$\begin{cases} -\operatorname{div}((a_{per})^* \nabla u^*) = f & \text{sur } \Omega, \\ u^*(x) = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1.4}$$

où le coefficient $(a_{per})^*$ est une matrice constante de taille $d \times d$. Par ailleurs, le comportement de u^ε peut être décrit à l'aide d'une famille de fonctions $(w_{per,p})_{p \in \mathbb{R}^d}$, où la fonction $w_{per,p}$ est l'unique solution périodique à moyenne nulle de l'équation du correcteur :

$$-\operatorname{div}(a_{per}(\nabla w_{per,p} + p)) = 0. \tag{1.5}$$

Ces correcteurs $w_{per,p}$ sont essentiels pour établir une théorie de l'homogénéisation puisqu'ils permettent d'abord d'expliciter le coefficient homogénéisé $(a_{per})^*$:

$$((a_{per})^*)_{i,j} = \left\langle e_i^T a_{per} \left(e_j + \nabla w_{per,e_j}\right) \right\rangle := \int_Q e_i^T a_{per} \left(e_j + \nabla w_{per,e_j}\right) dy, \tag{1.6}$$

où $Q =]0, 1[^d$ désigne la cellule unité de $\mathbb{R}^d$ et $(e_i)_{i \in \{1,\dots,d\}}$ la base canonique. Les correcteurs sont également nécessaires pour décrire le comportement asymptotique de u^ε et notamment de son gradient puisqu'on s'attend à avoir :

$$u^\varepsilon \underset{\varepsilon \to 0}{\sim} u^* + \varepsilon \sum_{i=1}^d \partial_i u^* w_{per,e_i}(./\varepsilon).$$

Lorsque ε devient petit, on s'attend donc à ce que la solution u^ε soit similaire à la somme de la solution homogénéisée et d'un terme oscillant d'ordre ε. Ces oscillations d'amplitude ε ont

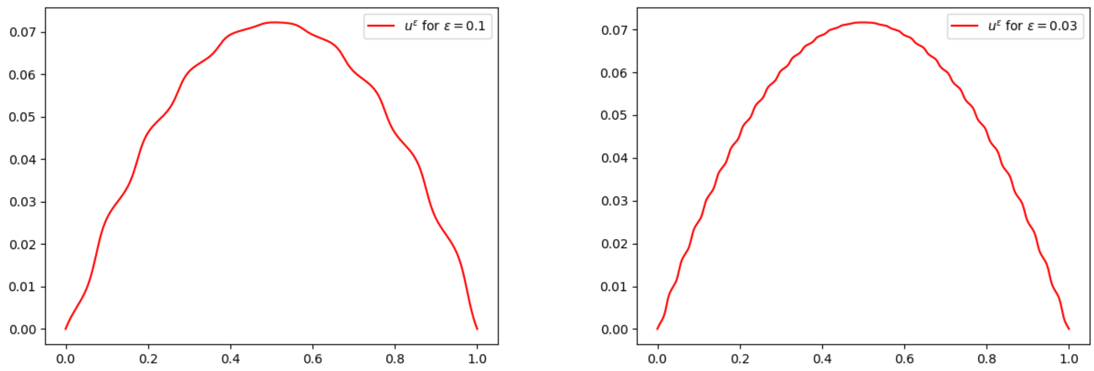

FIGURE 1.2 – Exemple de solution u^ε oscillant autour d'une solution homogénéisée en dimension $d = 1$. Ici u^ε est la parabole solution de (1.1) pour $a = 2 + \sin(2\pi x)$, $f = 1$ et $\Omega =]0, 1[$.

également lieu à l'échelle ε ce qui, après dérivation, empêche d'obtenir une convergence forte de ∇u^ε vers ∇u^*. On peut voir dans la Figure 1.2 un exemple monodimensionnel d'une telle solution oscillante.

En prenant en compte ces considérations formelles, il est alors possible de les justifier rigoureusement et on peut effectivement montrer la convergence, *faible* dans $H^1(\Omega)$ et *forte* dans $L^2(\Omega)$, de u^ε vers u^*, unique solution dans $H_0^1(\Omega)$ de (1.4). Ici, l'existence d'un correcteur périodique $w_{per,p}$ de (1.5) pour tout $p \in \mathbb{R}^d$ est une simple conséquence de l'inégalité de Poincaré-Wirtinger et du Lemme de Lax-Milgram. En introduisant alors la suite de reste

$$R^\varepsilon = u^\varepsilon - u^* - \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{per,e_i}(./\varepsilon),$$

on montre que R^ε converge fortement vers 0 dans $H^1(\Omega)$. Par ailleurs, des taux de convergence précis vers la solution homogénéisée ont été établis et pour tout $\Omega_1 \subset\subset \Omega$, on a

$$\left\{ \begin{array}{c} \|u^\varepsilon - u^*\|_{L^2(\Omega_1)} \leq C_1 \varepsilon \|f\|_{L^2(\Omega)}, \\ \|R^\varepsilon\|_{H^1(\Omega_1)} \leq C_2 \varepsilon \|f\|_{L^2(\Omega)}. \end{array} \right.$$

Le taux de convergence jusqu'au bord est également connu (voir par exemple [59, (1.51) p.28]) :

$$\|R^\varepsilon\|_{H^1(\Omega)} \leq C_3 \sqrt{\varepsilon} \|f\|_{L^2(\Omega)}.$$

Sous des hypothèses de régularité Höldérienne du coefficient a_{per} et de régularité de f et Ω, le contrôle de ∇R^ε peut aussi être raffiné. En effet, pour $\Omega_1 \subset\subset \Omega$, $r \geq 2$ et $\beta \in]0, 1[$, il est possible de montrer que

$$\begin{aligned} &\|\nabla R^\varepsilon\|_{L^r(\Omega_1)} \leq C_4 \varepsilon \|f\|_{L^r(\Omega)}, \\ &\|\nabla R^\varepsilon\|_{L^\infty(\Omega_1)} \leq C_5 \varepsilon \ln\left(1 + \varepsilon^{-1}\right) \|f\|_{C^{0,\beta}(\Omega)}. \end{aligned}$$

Ces deux dernières estimées, beaucoup plus techniques à établir que les précédentes, sont par exemple prouvées dans [62] en utilisant les résultats de [11].

Insistons sur le fait que les preuves des résultats d'homogénéisation reposent fortement sur des propriétés de convergence faible vérifiées par les fonctions périodiques, notamment la convergence faible de $a_{per}(./\varepsilon)\nabla w_{per,p}(./\varepsilon)$ vers $\langle a_{per}\nabla w_{per,p}\rangle$ qui permet d'identifier le coefficient homogénéisé (1.6) et une propriété essentielle du correcteur : sa sous-linéarité stricte à l'infini dans le sens où $\varepsilon w_{per,p}(./\varepsilon)$ converge uniformément vers 0 sur Ω.

1.2.2 Équation de Schrödinger stationnaire

Un autre problème d'homogénéisation elliptique linéaire auquel nous nous intéressons dans le **Chapitre** 4 de cette thèse est également bien connu dans le cadre périodique (voir [18, Chapitre 1, Section 12]) : le problème de type Schrödinger stationnaire avec potentiel rapidement oscillant donné par l'équation elliptique linéaire d'ordre deux

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{1}{\varepsilon}V(./\varepsilon)u^\varepsilon + \nu u^\varepsilon = f & \text{sur } \Omega, \\ u^\varepsilon = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1.7}$$

où $\Omega \subset \mathbb{R}^d$ est borné, $f \in L^2(\Omega)$, $\nu \in \mathbb{R}^d$ est fixé et $V = V_{per} \in L^\infty(\mathbb{R}^d)$ est un potentiel $\mathbb{Z}^d$-périodique. En raison du terme explosif $\frac{1}{\varepsilon}V_{per}(./\varepsilon)$ et afin d'assurer l'existence de limites non triviales pour u^ε, le potentiel V_{per} est supposé être à moyenne nulle, c'est-à-dire $\langle V_{per}\rangle = 0$. Pour ce problème, un développement à deux échelles totalement similaire à celui de la section précédente permet de montrer que le comportement de la suite de solutions $(u^\varepsilon)_{\varepsilon>0}$ est fortement lié à un correcteur périodique w_{per} solution de

$$\Delta w_{per} = V_{per}, \tag{1.8}$$

et on s'attend à ce que u^ε converge vers u^*, solution de l'équation homogénéisée

$$\begin{cases} -\Delta u^* + (V_{per})^* u^* + \nu u^* = f & \text{sur } \Omega, \\ u^* = 0 & \text{sur } \partial\Omega. \end{cases}$$

Le potentiel homogénéisé est constant et est donné par

$$(V_{per})^* = \langle w_{per}V_{per}\rangle = -\langle|\nabla w_{per}|^2\rangle, \tag{1.9}$$

la deuxième égalité étant une application simple de la formule de Green après multiplication de (1.8) par w_{per}. On note que le caractère bien posé de (1.7) n'est pas évident, l'homogénéisation de ce problème est en effet établie dans [18] en montrant tout d'abord la convergence de la première valeur propre λ_1^ε de l'opérateur $-\Delta+\frac{1}{\varepsilon}V(./\varepsilon)+\nu$ avec conditions au bord homogènes de Dirichlet vers la première valeur propre de l'opérateur homogénéisé. Précisément,

$$\lim_{\varepsilon\to 0}\lambda_1^\varepsilon = \mu_1 + (V_{per})^* + \nu,$$

où μ_1 est la première valeur propre de $-\Delta$ avec conditions au bord homogènes de Dirichlet. Sous l'hypothèse suffisante

$$\mu_1 + (V_{per})^* + \nu > 0,$$

il est alors prouvé que la solution u^ε est bien définie pour ε petit et qu'elle converge, *faiblement* dans $H^1(\Omega)$ et *fortement* dans $L^2(\Omega)$, vers u^*. De plus, la suite de restes

$$R^\varepsilon = u^\varepsilon - u^* - \varepsilon u^* w_{per}(./\varepsilon)$$

converge fortement vers 0 dans $H^1(\Omega)$. Il est également établi dans [94] que la convergence a lieu à vitesse ε.

Ces résultats sont de nouveau établis en utilisant principalement deux propriétés du correcteur périodique w_{per} : la sous-linéarité stricte à l'infini, toujours dans le sens où $\varepsilon w_{per}(./\varepsilon)$ converge uniformément vers 0, et la convergence faible de $|\nabla w_{per}(./\varepsilon)|^2$ vers sa moyenne afin d'identifier le potentiel homogénéisé (1.9).

1.3 Introduction des problèmes avec défauts

La contribution principale de cette thèse consiste à étendre les résultats d'homogénéisation périodique pour les deux problèmes (1.1) et (1.7) que nous venons de présenter à des modèles plus complexes de géométries non périodiques, caractérisés par des cadres périodiques perturbés par des défauts. Une des difficultés principales induites par ces problèmes d'homogénéisation, d'apparence simple puisque décrits par des équations elliptiques linéaires bien connues, est dictée par la nature *non linéaire* et *non locale* de l'application qui, à des coefficients donnés, ici respectivement a pour (1.1) et V pour (1.7), associe la solution de l'équation associée. Une des motivations est de comprendre comment une variation des coefficients va affecter la solution u^ε dans le régime où ε devient asymptotiquement petit.

Dans ce but, il est d'abord essentiel de comprendre les propriétés requises par ces modèles pour espérer établir une théorie de l'homogénéisation complète, d'introduire les modèles de perturbations de la périodicité que nous considérerons et d'identifier précisément les difficultés principales induites par le manque de périodicité dans l'étude des problèmes d'homogénéisation elliptique considérés.

1.3.1 Premiers pas vers le non-périodique : un exemple élémentaire illustratif

Une première étape élémentaire dans la compréhension des problèmes d'homogénéisation elliptique est d'établir quelles sont les conditions fondamentales requises sur les coefficients des équations pour espérer établir la convergence des problèmes vers un problème homogénéisé *qui peut* être précisément identifié. On se place tout d'abord dans un cadre monodimensionnel le plus simple possible et dans lequel on fait disparaître tous les opérateurs différentiels :

$$a\left(\frac{\cdot}{\varepsilon}\right) u^\varepsilon = f \quad \text{sur }]0,1[.$$

On suppose ici que f est une fonction régulière sur $]0,1[$ et que le coefficient a est borné et coercif dans le sens de (1.2) et (1.3). Nous allons voir que ce problème, bien qu'extrêmement simple, reflète en réalité des difficultés algébriques présentes dans le cadre des équations aux dérivées partielles plus complexes que nous considérons dans la suite. Quand ε tend vers 0, il est alors légitime de se demander si u^ε converge, au moins dans $\mathcal{D}'(]0,1[)$, vers une fonction u^* solution d'un problème de la forme

$$a^* u^* = f. \tag{1.10}$$

Bien entendu, ici on ne peut pas espérer mieux qu'une convergence faible de u^ε puisque $a(./\varepsilon)$ ne converge fortement dans aucun espace lorsque ε tend vers 0. L'objectif, qui sera le même tout au long de ce travail de thèse, est d'aller au delà de résultats abstraits de compacité et de répondre à la question suivante : La suite u^ε converge-t-elle vers une limite u^* qui peut être *explicitement identifiée*, de manière unique (et non à extraction près) ?

Étant donnée la positivité stricte du coefficient a, la solution u^ε est trivialement donnée par

$$u^\varepsilon = \frac{f}{a\left(\frac{\cdot}{\varepsilon}\right)}.$$

On retrouve déjà une des principales difficultés qui seront présentes dans l'étude des problèmes (1.1) et (1.7) : le coefficient a et la solution u^ε sont liés de manière *non-linéaire*. Ici, on remarque que la question de la convergence de u^ε quand le paramètre ε tend vers 0 est équivalente à la convergence de $\displaystyle\int_0^1 \frac{f}{a\left(\frac{\cdot}{\varepsilon}\right)}\psi$ pour tout $\psi \in \mathcal{D}(]0,1[)$, question naturellement reliée à la convergence faible de l'inverse du coefficient $a(./\varepsilon)$.

Une condition nécessaire pour que u^ε admette une limite est donc la convergence pour la topologie faible-$\star$ de L^∞ de $\dfrac{1}{a(./\varepsilon)}$. D'un certain point de vue, puisque

$$\int_0^1 \frac{1}{a(y/\varepsilon)}dy = \varepsilon \int_0^{1/\varepsilon} \frac{1}{a(y)}dy,$$

l'existence d'une limite faible de $\dfrac{1}{a(./\varepsilon)}$ quand ε tend vers 0 peut se traduire par le fait que $\dfrac{1}{a}$ admette une moyenne au sens de la limite sur des grands volumes, précisément au sens de la limite sur des boules de rayon $\dfrac{1}{\varepsilon}$. On peut donc aisément se convaincre qu'il est nécessaire que le coefficient a possède une structure bien particulière afin que certaines transformations non-linéaires de a (ici l'inverse de a) admettent une moyenne. Si c'est le cas, alors u^ε converge effectivement vers une solution u^* d'un problème homogénéisé de la forme (1.10) et, en notant $\left\langle \dfrac{1}{a} \right\rangle$ la limite faible de $\dfrac{1}{a(./\varepsilon)}$, le coefficient homogénéisé peut être explicitement identifié et on a $a^* = \left\langle \dfrac{1}{a} \right\rangle^{-1}$.

On peut donc déjà donner plusieurs exemples connus de structures admissibles dans lesquels on sait montrer l'existence d'une telle limite faible et également établir des résultats d'homogénéisation similaires pour les problèmes plus complexes aux dérivées partielles :

— Le cadre périodique que l'on a déjà évoqué. En effet, si $a = a_{per}$ est périodique, il est bien connu que $\dfrac{1}{a_{per}(./\varepsilon)} \overset{\varepsilon\to 0}{\rightharpoonup} \left\langle \dfrac{1}{a_{per}} \right\rangle$ faiblement dans $L^\infty(\mathbb{R}) - \star$.

- Le cadre presque-périodique, extension naturelle du cadre périodique pour lequel il est également bien connu que la propriété d'existence de moyenne est conservée par passage à l'inverse.
- Le cadre de coefficients aléatoires stationnaires $a(x, \omega)$ pour lesquels les convergences faibles sont assurées, en les bons sens par rapport à la variable ω, par la théorie ergodique.

Chacune de ces classes possède en réalité une propriété fondamentale : elle est stable par opérations algébriques et l'existence de moyennes est conservée par produit ou par inverse. On souhaite donc que les classes de coefficients que l'on considérera pour modéliser des géométries périodiques perturbées vérifient des propriétés similaires.

Dans notre contexte de perturbation de la géométrie périodique, un premier exemple fondamental admissible que nous étofferons dans la suite est donné par des coefficients perturbés de la forme

$$a = a_{per} + \tilde{a}, \tag{1.11}$$

où a_{per} est périodique et $\tilde{a} \in L^\infty(\mathbb{R}^d)$ est une perturbation dite localisée qui disparaît à l'infini dans le sens $\lim_{|x|\to\infty} \tilde{a}(x) = 0$. Pour de tels coefficients, on a

$$\frac{1}{a} = \frac{1}{a_{per}} - \frac{\tilde{a}}{a_{per}(a_{per} + \tilde{a})}.$$

On peut montrer que $|\tilde{a}(./\varepsilon)|$ converge faiblement vers 0 et, puisque $a_{per} + \tilde{a}$ vérifie (1.3), cela implique alors

$$\frac{1}{a(./\varepsilon)} \xrightarrow{\varepsilon\to 0} \left\langle \frac{1}{a_{per}} \right\rangle.$$

Le problème homogénéisé (1.10) est donc identique au problème homogénéisé associé au cadre périodique sans défaut. Dans les problèmes d'équations aux dérivées partielles elliptiques que nous étudierons, nous verrons que ce type de propriétés reste vrai. Notons cependant que ce phénomène est assez spécifique, c'est une conséquence du caractère elliptique et diffusif des équations considérées et il pourrait possiblement ne pas être vérifié pour d'autres classes d'équations.

1.3.2 Classes de défauts

Nous présentons maintenant les différentes classes de défauts de la géométrie périodique que nous considérerons tout au long de notre travail. Premièrement, un des fils conducteurs des problèmes avec défauts étudiés dans cette thèse est donné par des milieux dont la géométrie est modélisée par des coefficients de la forme

$$b = \sum_{k\in\mathbb{Z}^d} \varphi(. - X_k), \tag{1.12}$$

où $\varphi \in \mathcal{D}(\mathbb{R}^d)$ et $(X_k)_{k\in\mathbb{Z}^d}$ est une suite de $\mathbb{R}^d$. Cet ensemble particulier de fonctions est directement inspiré d'un cadre général introduit par Blanc, Le Bris et Lions dans [22] concernant le calcul d'énergies moyennes de systèmes infinis et non-périodiques de particules. Dans ce cadre, dont nous ne détaillerons ici que brièvement les caractéristiques, les particules sont décrites par l'ensemble de leurs positions $(X_k)_{k\in\mathbb{Z}^d}$ et l'existence de moyennes d'énergies est intimement reliée à l'existence de moyennes de fonctions appartenant à des ensembles nommés $\mathcal{A}(\{X_k\})$, fermeture pour une norme uniforme de l'algèbre engendrée par les fonctions de la forme (1.12). L'ensemble des positions est supposé vérifier un ensemble d'hypothèses de répartition permettant d'assurer l'existence de moyennes et donné par

$$\sup_{x\in\mathbb{R}^d} \#\left\{k \in \mathbb{Z}^d \,\middle|\, |x - X_k| \leq 1\right\} < +\infty, \tag{$\mathcal{H}1$}$$

$$\exists R > 0 \ \inf_{x\in\mathbb{R}^d} \#\left\{k \in \mathbb{Z}^d \,\middle|\, |x - X_k| \leq R\right\} > 0, \tag{$\mathcal{H}2$}$$

$$\exists \lim_{R\to\infty} \frac{1}{|B_R|} \#\left\{k \in \mathbb{Z}^d \,\middle|\, X_k \in B_R\right\}, \tag{$\mathcal{H}3.1$}$$

et pour tout, $n \in \mathbb{N}$, tout $h = (h_1, h_2, ..., h_n) \in \mathbb{R}^{nd}$, tout $\delta > 0$ suffisamment petit, la limite suivante existe :

$$l(h, \delta) = \lim_{R\to\infty} \frac{1}{|B_R||B_\delta|^n} \left\{X_{i_0}, ..., X_{i_n} \mid X_{i_0} - X_{i_k} \in B_\delta(h_k), 1 \leq k \leq n\right\}. \tag{$\mathcal{H}3.n$}$$

De manière plus explicite, ces hypothèses assurent que

($\mathcal{H}1$) : il n'y a pas d'accumulation d'un nombre arbitrairement grand de particules,

($\mathcal{H}2$) : il n'existe pas de domaine arbitrairement grand dont les particules sont absentes,

($\mathcal{H}3.1$) : il y a asymptotiquement en les grands volumes, un nombre linéaire de particules par unité de volume,

($\mathcal{H}3.n$) : les particules sont localisées de sorte que les n-uplets de plus proches voisins soient en nombre linéaire et elles sont approximativement espacées de "périodes" h.

L'hypothèse ($\mathcal{H}3.n$) peut alors être reformulée de manière analytique (voir [22, Proposition 2.4]) puisque, pour tout $n \in \mathbb{N}$, elle assure l'existence d'une mesure $l^n(h_1, ..., h_n)$, positive et uniformément localement bornée, donnée par

$$l^n(h_1, ..., h_n) = \lim_{R\to\infty} \frac{1}{|B_R|} \sum_{X_{k_0}\in B_R} ... \sum_{X_{k_n}\in B_R} \delta_{(X_{k_0}-X_{k_1}, ..., X_{k_0}-X_{k_n})}(h_1, ..., h_n).$$

Sous ces hypothèses et pour toute fonction f de la forme

$$f(x) = \sum_{(k_1,...,k_n)\in\mathbb{Z}^n} \varphi_1(x - X_{k_1})\varphi_2(x - X_{k_2})...\varphi_n(x - X_{k_n}),$$

où φ_1,...,φ_n appartiennent à $\mathcal{D}(\mathbb{R}^d)$, on a alors l'existence d'une moyenne $\langle f\rangle \in \mathbb{R}$ au sens suivant

$$\begin{aligned} \langle f\rangle &= \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f \\ &= \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{n-1}} \varphi_1(x)\varphi_2(x - h_1)...\varphi(x - h_{n-1})\, dl^{n-1}(h_1, ...h_{n-1})\, dx. \end{aligned} \tag{1.13}$$

Bien que le problème de minimisation d'énergie d'un système de particules soit a priori très distant des problèmes d'homogénéisation que nous étudions au cours de cette thèse, nous avons vu que les questions d'homogénéisation sont intrinsèquement liées à des propriétés de convergence faible et donc, dans un certain sens, à des questions d'existence de moyennes de fonctions. De plus, la structure d'algèbre et plus précisément des propriétés du type (1.13) introduites ci-dessus paraissent adaptées pour traiter les problèmes d'homogénéisation. Si un coefficient oscillant est de la forme (1.12), les études de cas monodimensionnels comme celle présentée en Section 1.3.1 nous montrent effectivement qu'il est nécessaire d'imposer que certaines transformations non-linéaires de ce coefficient admettent une moyenne (typiquement les puissances ou l'inverse du coefficient). Il est donc suffisant d'imposer que toutes les puissances du coefficient considéré possèdent une moyenne. Ce cadre très général a alors originellement été utilisé par Blanc, Le Bris et Lions comme *inspiration* et *motivation* afin de proposer des hypothèses géométriques admissibles pour espérer obtenir des résultats asymptotiques dans le contexte des problèmes d'homogénéisation. On peut par exemple évoquer les problèmes elliptiques linéaires de [25, 26, 27] pour des perturbations *locales* de la géométrie périodique, générés par des fonctions de la forme (1.12) pour $X_k = k + Z_k$ où Z_k possède un nombre fini de termes non nuls. Nous reviendrons plus en détails sur ce cas dans la suite de cette section. Un cas d'homogénéisation stochastique pour des coefficients stationnaires a également été considéré dans [23] en utilisant un ensemble de points aléatoires $X_k(\omega)$ vérifiant un équivalent aléatoire des hypothèses ($\mathcal{H}1$)-($\mathcal{H}2$)-($\mathcal{H}3.1$)-($\mathcal{H}3.n$).

Pour les différents problèmes elliptiques que nous étudierons, bien que la possibilité d'établir une théorie de l'homogénéisation dans le contexte très général de la classe de fonctions (1.12) sous les hypothèses ($\mathcal{H}1$)-($\mathcal{H}2$)-($\mathcal{H}3.1$)-($\mathcal{H}3.n$) (ou dans un contexte général similaire) reste encore un problème ouvert, ce cadre permet de générer un certain nombre de modèles de géométrie périodique perturbée par des défauts. Le choix $X_k = k$ pour tout $k \in \mathbb{Z}^d$, qui vérifie trivialement les hypothèses de répartition ci-dessus, permet effectivement de modéliser une géométrie périodique. Une façon naturelle de perturber cette géométrie consiste alors à considérer

$$X_k = k + Z_k,$$

où la suite Z_k va permettre de décrire explicitement des défauts de périodicité. En considérant des coefficients de la forme

$$b = \sum_{k \in \mathbb{Z}^d} \varphi(. - k - Z_k), \tag{1.14}$$

ou appartenant à l'algèbre engendrée par les fonctions de cette forme, et en faisant varier les caractéristiques du paramètre Z_k, il est alors possible d'identifier plusieurs classes de défauts qui vont plus ou moins impacter le milieu périodique sous-jacent modélisé par des fonctions périodiques de la forme $b_{per} = \sum_{k \in \mathbb{Z}^d} \varphi(. - k)$.

Notons par ailleurs que les coefficients de la forme (1.14) (ou appartenant à l'algèbre engendrée par les fonctions de la forme (1.14)) servent essentiellement de point de départ pour comprendre les problèmes d'homogénéisation dans des cadres non-périodiques et peuvent mener à l'élaboration de modèles bien plus généraux de perturbations de la géométrie périodique. Un procédé d'extension possible pour un ensemble de points Z_k donné consiste, par exemple, à considérer l'adhérence de l'algèbre engendrée par la classe de coefficients (1.14) pour différentes normes. Le choix de la norme utilisée doit bien évidemment conserver l'existence de moyennes à la limite, c'est-à-dire conserver les propriétés de convergence faible de ces coefficients. Ceci permet à la fois d'obtenir des classes de perturbations plus grandes, directement issues de la classe de fonctions (1.14) et d'obtenir des ensembles de coefficients possédant une structure de Banach, plus adaptée à l'étude des problèmes aux dérivées partielles.

Nous identifions alors différentes catégories de défauts que nous considérerons durant ce travail de thèse. Ces défauts sont décrits par les propriétés de la suite Z_k et vont plus ou moins impacter la géométrie sous-jacente. Nous présentons ces classes de défauts par ordre croissant d'impact sur la structure périodique, elles sont illustrées en Figure 1.3.

Défauts localisés. Une première façon de perturber le cadre périodique parfait est d'insérer un défaut très localisé en considérant

$$Z_k = \begin{cases} Z_0 \neq 0 & \text{si } k = 0, \\ 0 & \text{si } k \neq 0. \end{cases} \tag{1.15}$$

Dans le contexte de coefficients de la forme (1.14), on voit alors que b peut s'écrire comme $b = b_{per} + \tilde{b}$ où $b_{per} = \sum_{k\in\mathbb{Z}^d} \varphi(. - k)$ est périodique et $\tilde{b} = \varphi(. - Z_0) - \varphi$ est une perturbation à support compact. Puisqu'une telle perturbation de la géométrie périodique n'a aucun impact à grande distance, on s'attend à ce que les propriétés "moyennes" du milieu étudié soient identiques à celles du milieu périodique sans défaut sous-jacent. On peut donc intuiter que le processus d'homogénéisation, dans le cadre des problèmes elliptiques considérés, mène à l'"effacement" des défauts dans notre contexte, comme on peut le vérifier dans les travaux [20, 25, 26, 27]. Autrement dit, s'il existe, le milieu homogénéisé obtenu est identique au milieu homogénéisé associé à la structure hétérogène périodique sous-jacente et sans défaut décrite par b_{per}. Cela se traduit mathématiquement par la convergence faible de $|\tilde{b}(./\varepsilon)|$ vers 0 pour la topologie de $L^\infty - \star$. Notons tout de même que cette perturbation localisée va, certes, disparaître à l'échelle macroscopique, mais si l'objectif est d'approcher numériquement les suites de solutions u^ε à des échelles microscopiques (et donc pour des normes suffisamment fines comme L^∞), ces défauts vont nécessairement se voir puisque c'est précisément à cette échelle que les défauts ont un impact non négligeable sur le milieu ambiant.
L'exemple de la suite Z_k définie par (1.15) n'est bien sûr qu'un choix possible parmi d'autres pour modéliser des défauts localisés de la structure périodique. On peut par exemple évoquer ses extensions immédiates données par des suites où Z_k est non nul pour un nombre fini de $k \in \mathbb{Z}^d$ ou encore par une décroissance très rapide de Z_k à l'infini. La considération de défauts Z_k localisés permet également de générer une classe un peu plus générale de perturbations qui disparaissent à l'infini. Ainsi, si Z_k est définie par (1.15), la classe de défauts introduite dans [25, 26, 27] et décrite par les coefficients de la forme $b = b_{per} + \tilde{b}$ où $\tilde{b} \in L^p(\mathbb{R}^d)$ pour un certain $p \in]1, +\infty[$, est obtenue par fermeture de l'algèbre engendrée par les fonctions (1.14) pour la norme $\|b\| := \|b_{per}\|_{L^2_{per}} + \|\tilde{b}\|_{L^p(\mathbb{R}^d)}$.

Défauts non localisés mais rares. Un autre exemple de perturbation de géométrie périodique qui, en moyenne, n'impacte pas la structure périodique et auquel nous nous intéresserons est le cas de défauts rares à l'infini. Contrairement aux défauts localisés, ce cadre assure une contribution non nulle à l'infini de la suite Z_k, qui est donc non localisée dans l'espace. Un exemple typique en dimension $d = 1$ est

$$Z_k = \begin{cases} Z_p \neq 0 & \text{si } \exists p \in \mathbb{N} \text{ tel que } k \in \{-2^p, 2^p\}, \\ 0 & \text{sinon.} \end{cases}$$

La rareté de ces défauts est alors représentée dans le fait que les Z_k s'éloignent exponentiellement les uns des autres à l'infini. Comme dans le cas précédent, on verra qu'en cas d'existence, le milieu homogénéisé obtenu est identique au milieu homogénéisé associé à la structure hétérogène périodique sous-jacente. En effet, pour un coefficient $b = b_{per} + \tilde{b}$ où

$$\tilde{b} = \sum_{p\in\mathbb{Z}} \varphi\left(. - \operatorname{sign}(p)2^{|p|} - Z_p\right) - \varphi\left(. - \operatorname{sign}(p)2^{|p|}\right),$$

on a

$$\begin{aligned}\int_0^1 |\tilde{b}(x/\varepsilon)|dx &= \varepsilon \int_0^{1/\varepsilon} |\tilde{b}(x)|dx \\ &= \varepsilon \sum_{p\in\mathbb{Z}} \int_0^{1/\varepsilon} \left|\varphi\left(y - \operatorname{sign}(p)2^{|p|} - Z_p\right) - \varphi\left(y - \operatorname{sign}(p)2^{|p|}\right)\right| dy.\end{aligned}$$

Seul un nombre fini de termes est non nul dans la somme ci-dessus. Si φ est à support dans $[-A, A]$ et Z_p est bornée, alors l'intégrale associée à un $p \in \mathbb{Z}$ donné est effectivement nulle dès que $2^{|p|} > \dfrac{1}{\varepsilon} + A + \sup\limits_{p\in\mathbb{Z}} |Z_p|$, soit

$$|p| > \frac{\ln\left(\dfrac{1}{\varepsilon} + A + \sup\limits_{p\in\mathbb{Z}} |Z_p|\right)}{\ln(2)}.$$

On en déduit donc

$$\int_0^1 |\tilde{b}(x/\varepsilon)|dx \leq C\varepsilon|\ln(\varepsilon)|\|\varphi\|_{L^1(\mathbb{R}^d)} \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{1.16}$$

où $C > 0$ est indépendante de ε. Ceci implique, qu'en moyenne, les défauts rares à l'infini n'affectent pas le milieu périodique et cette rareté est mathématiquement encodée dans la borne logarithmique de (1.16). En revanche, comme nous l'avons souligné dans le cas précédent, ces défauts ont un impact à l'échelle microscopique et devront donc être pris en compte dans l'approximation numérique des solutions u^ε.
Dans le **Chapitre** 2, nous introduirons un espace fonctionnel de défauts rares à l'infini se comportant comme une somme de fonctions $L^2(\mathbb{R}^d)$ tronquées au voisinage de certains points. Ce cadre est en réalité directement inspiré de l'algèbre engendrée par la classe de fonctions (1.14) pour un choix particulier de Z_k rare à l'infini et après fermeture de cette algèbre pour une norme adaptée.

Défauts complètement non localisés. Il est également possible de considérer des défauts de périodicité plus violents, non localisés dans l'espace et qui affectent fortement la géométrie périodique sous-jacente.
Dans ce cas, les suites Z_k sont non-périodiques, ni presque-périodiques, ne disparaissent d'aucune manière à l'infini et possèdent potentiellement des valeurs isolées de 0 pour un nombre non négligeable de points $k \in \mathbb{Z}^d$. Nous présenterons plusieurs exemples de telles suites dans le **Chapitre** 4, des simulations déterministes de suites de variables aléatoires identiquement distribuées par exemple. Dans ce chapitre, nous considérerons effectivement de tels défauts dans le contexte de l'homogénéisation du problème de Schrödinger (1.7) et nous verrons que l'existence de moyennes des coefficients de la forme (1.14) ou de leurs puissances est conditionnée à des hypothèses sur les corrélations de la suite Z_k. Le choix de s'intéresser au problème

de Schrödinger (1.7) plutôt qu'au problème de diffusion (1.1) pour de tels défauts est en réalité une considération simplificatrice dans notre étude. Le problème de diffusion soulève en effet des difficultés bien supérieures, évoquées en Section 1.5, que nous ne savons pas traiter pour de telles perturbations et avec nos connaissances actuelles.

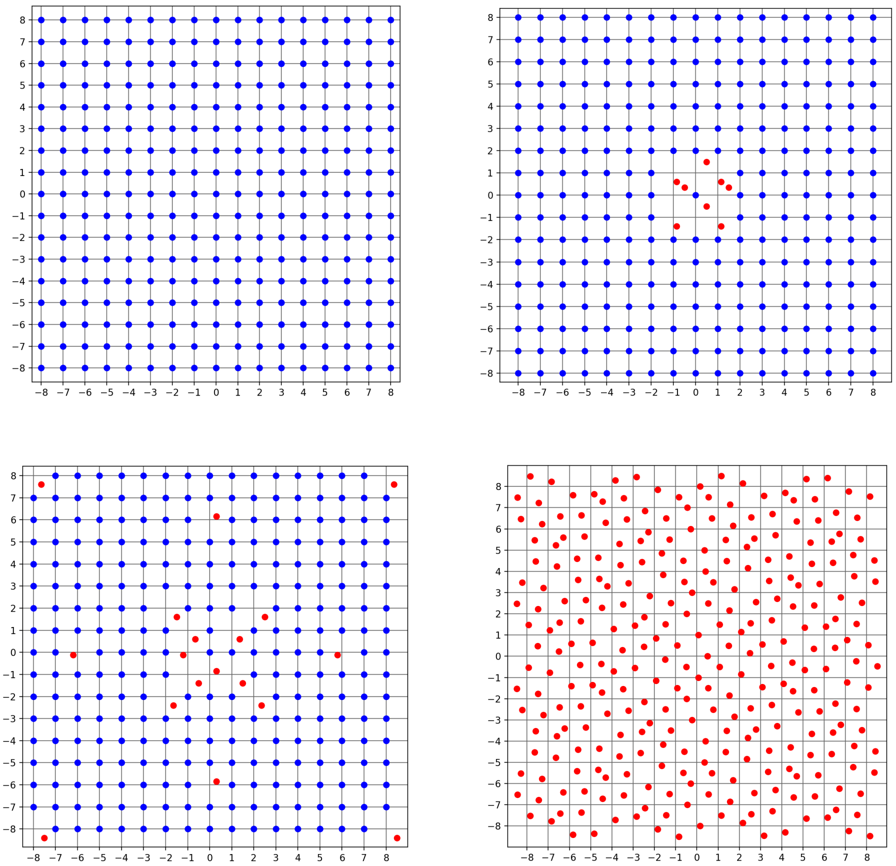

FIGURE 1.3 – Illustration des points $X_k = k + Z_k$ pour différentes perturbations Z_k. En bleu, les points non-perturbés ($Z_k = 0$). En rouge, les points perturbés ($Z_k \neq 0$).
Haut à gauche : Structure périodique de référence. **Haut à droite** : Défauts localisés. **Bas à gauche** : Défauts non localisés mais rares. **Bas à droite** : Défauts complètement non localisés.

Pour terminer cette présentation des différentes catégories de perturbations de la géométrie périodique que nous considérons dans cette thèse, nous introduisons également une classe de coefficients b, dont l'étude sera centrale dans le **Chapitre** 3 et qui nous permettra de modéliser un milieu non-périodique ayant tout de même une structure proche d'un milieu périodique à l'infini. Cette classe est caractérisée par une certaine invariance par translation des coefficients à l'infini. Nous introduisons ainsi, pour toute fonction $g \in L^1_{loc}(\mathbb{R}^d)$, le gradient discret de g noté δg et défini par

$$\delta g := (\delta_i g)_{i\in\{1,\dots,d\}} := (g(.+e_i) - g)_{i\in\{1,\dots d\}}\,. \tag{1.17}$$

On voit que ce gradient discret décrit l'écart de la fonction g à une fonction $\mathbb{Z}^d$-périodique en tout point de l'espace. Les coefficients b permettant de simuler une géométrie proche d'une géométrie périodique sont alors caractérisés par une propriété de décroissance à l'infini de leur gradient discret, qui sera typiquement donnée dans le **Chapitre** 3 par

$$\delta b \in \left(L^p(\mathbb{R}^d)\right)^d. \tag{1.18}$$

D'un point de vue appliqué, ce cadre considère que l'on s'intéresse à un milieu dont on ignore les propriétés structurelles (contrairement au modèle de coefficient (1.14) où on suppose regarder un milieu périodique connu fabriqué de manière imparfaite) et l'étude des propriétés d'invariance par translation consiste à le tester afin de déterminer s'il est issu d'un milieu périodique ou non. A priori, on peut penser que ce modèle ne rentre pas dans une des catégories de perturbations localisées ou non-localisées que nous venons juste de présenter ci-dessus. En réalité, nous montrerons dans le **Chapitre** 3 qu'un coefficient b vérifiant l'hypothèse (1.18) avec $p < d$ peut s'écrire comme la somme d'un coefficient périodique b_{per} et d'une perturbation $\tilde{b}$ possédant certaines propriétés d'intégrabilité à l'infini et modélise donc une perturbation localisée d'un milieu périodique.

Pour conclure cette section, notons que du point de vue de la modélisation des matériaux, notre cadre de perturbations de la périodicité a pour but d'étoffer les résultats d'homogénéisation pour des milieux non-périodiques en essayant de se rapprocher au mieux d'une réalité pratique. Nous sommes conscients que ces modèles requièrent également des hypothèses de structure relativement fortes puisque nous présupposons par exemple la superposition d'une géométrie périodique, de période connue, et de défauts. Notre approche nécessite notamment de déterminer en amont la période ambiante par une analyse fréquentielle (une hypothèse de $\mathbb{Z}^d$-périodicité est faite dans tout ce travail) ainsi que la nature des défauts. Cependant, ces modèles peuvent être considérés en complémentarité de l'approche stochastique stationnaire qui demande également des hypothèses assez rigides (stationnarité des milieux, connaissance de la loi de probabilité ambiante...), principalement lorsque les données issues des observations ne sont pas suffisantes pour pouvoir vérifier ces hypothèses. Bien que cette question n'ait pas été résolue durant ce travail de thèse, on espère également pouvoir enrichir ces modèles en les adaptant à des cadres stochastiques de perturbations de milieux stationnaires (voir Section 1.5).

1.3.3 Problématiques majeures

Pour les deux problèmes (1.1) et (1.7) que nous étudierons dans cette thèse dans des cadres de perturbation de la géométrie périodique, nous rappelons que les enjeux principaux sont d'établir une théorie de l'homogénéisation pour laquelle on peut à la fois identifier explicitement la limite de u^ε et approcher précisément le comportement de cette suite à petites échelles, son comportement local en particulier.

A la manière du cas périodique introduit en Section 1.2, l'approche formelle du développement à deux échelles permet d'obtenir une intuition quant au comportement asymptotique des solutions u^ε pour les problèmes (1.1) et (1.7). On s'attend alors à ce que ce comportement asymptotique soit fortement lié à un correcteur solution d'une équation elliptique linéaire d'ordre deux donnée par

$$-\operatorname{div}(a(\nabla w + p)) = 0, \quad \forall p \in \mathbb{R}^d, \tag{1.19}$$

dans le cadre du problème de diffusion (1.1) et par

$$\Delta w = V, \tag{1.20}$$

pour l'homogénéisation de l'équation de Schrödinger stationnaire (1.7). Après identification du candidat potentiel u^* pour être la limite homogénéisée, on s'attend à ce que u^ε vérifie un développement du type $u^\varepsilon \sim u^* + \varepsilon w(./\varepsilon)F(x) + ...$ où F dépend de u^* ou de ses dérivées.

L'étude de u^ε nécessite donc de connaître et d'évaluer ces correcteurs en tout point de Ω/ε, domaine défini par

$$\Omega/\varepsilon = \{x/\varepsilon, \ x \in \Omega\},$$

et d'obtenir des bornes uniformes sur cet ensemble afin d'espérer montrer la convergence de u^ε et d'identifier précisément son comportement asymptotique. Puisque Ω/ε devient asymptotiquement grand, cela implique de résoudre les équations du correcteur sur des domaines non bornés et dans la plupart des cas, il est impossible d'utiliser les outils classiques de résolution sur des ouverts bornés, outils fortement liés à des propriétés de compacité (on pense par exemple à des propriétés de type inégalité de Poincaré et au lemme de Lax-Milgram).

Une première possibilité pour pallier cette difficulté est de considérer une catégorie de problèmes possédant une structure "compacte" sous-jacente induite par certaines propriétés d'invariance par translation ou, plus généralement, d'invariance par une action de groupe. La connaissance de propriétés moyennes du correcteur permet alors d'obtenir un grand nombre d'informations concernant ses propriétés locales et d'obtenir des bornes uniformes sur tout l'espace. C'est par exemple le cas dans le contexte périodique pour lequel la compacité du tore périodique permet de conclure, dans le contexte presque-périodique qui s'appuie sur les propriétés du compactifié de Bohr ou dans le contexte ergodique stationnaire reposant sur de bonnes propriétés de l'espace de probabilité abstrait. Les problèmes de perturbation de la géométrie périodique que nous étudions dans cette thèse ne rentrent cependant pas dans cette catégorie puisque, même dans le cas de perturbations localisées autour d'un point, les propriétés moyennes du problème ne "voient" pas ces défauts et ne permettent donc pas d'identifier exhaustivement ses propriétés locales, sur tout l'espace.

Une approche naturelle consiste donc à adopter des méthodes "constructives" faisant intervenir les fonctions de Green associées aux opérateurs différentiels étudiés sur des ouverts non bornés. Les problèmes induits sont alors très proches de questions d'analyse harmonique et de la théorie des opérateurs de Calderòn-Zygmund (détaillée par exemple dans [73]) pour lesquels il est connu que l'obtention de borne uniforme globale est impossible sans hypothèse de structure assez forte. Typiquement, nous verrons que pour le problème (1.19), la considération de défauts localisés encodés dans des coefficients de la forme $a = a_{per} + \tilde{a}$ dont la seule information serait la convergence vers 0 à l'infini de $\tilde{a}$ mène à la résolution d'équations de la forme

$$-\operatorname{div}(a\nabla u) = \operatorname{div}(f), \tag{1.21}$$

pour des classes de fonctions f très grandes. Par exemple si f appartient à l'ensemble $\mathcal{C}^0_0(\mathbb{R}^d)$ des fonctions continues qui tendent vers 0 à l'infini, on ne peut a priori pas obtenir des bornes uniformes sur tout l'espace vérifiées par u, ou au minimum par son gradient (un contre-exemple est proposé en Annexe A). Ces bornes étant requises pour établir une théorie de l'homogénéisation et établir des estimées de convergence, cette barrière mathématique impose donc de considérer des cadres fonctionnels un peu plus spécifiques. Une grande partie du travail ce cette thèse va donc consister à introduire des hypothèses de structure à la fois fonctionnelles et géométriques qui, couplées à une compréhension des propriétés des opérateurs différentiels Δ et $-\operatorname{div}(a\nabla.)$ et de leur noyaux de Green respectifs, permettront de construire des solutions des équations de correcteur adaptées à nos problèmes d'homogénéisation.

Comme nous l'avons également évoqué en Section 1.2, la question de l'existence d'un correcteur w n'est cependant pas l'unique condition requise pour établir une généralisation des résultats du cadre périodique et pour décrire précisément le comportement asymptotique de u^ε dans nos cadres de perturbations. Le correcteur doit vérifier certaines propriétés asymptotiques supplémentaires :

— Des propriétés de convergence faible de $\nabla w(./\varepsilon)$ quand ε tend vers 0 afin d'identifier explicitement la limite homogénéisée u^*. Dans le cadre périodique, on peut effectivement voir que l'existence des coefficients homogénéisés donnés par (1.6) pour l'équation de diffusion et par (1.9) pour l'équation de Schrödinger reposent respectivement sur la convergence faible vers une constante de $a_{per}(./\varepsilon)\nabla w_{per}(./\varepsilon)$ et de $|\nabla w_{per}(./\varepsilon)|^2$. L'existence de telles limites est bien connue dans le cadre périodique mais des propriétés similaires doivent être établies pour les cadres non-périodiques et les limites en question explicitement identifiées.

— D'autre part, la description du comportement de u^ε quand ε tend vers 0 repose sur une propriété de sous-linéarité stricte à l'infini du correcteur. On doit montrer que le correcteur construit vérifie $\lim\limits_{\varepsilon \to 0} \varepsilon w(./\varepsilon) = 0$ sur tout le domaine Ω pour espérer prouver une convergence forte des restes $R^\varepsilon(x) = u^\varepsilon(x) - u^*(x) - \varepsilon w(x/\varepsilon)F(x)$ vers 0. Nous verrons qu'une condition suffisante pour assurer cette propriété est une certaine condition d'annulation du gradient de w donnée par la convergence faible vers 0 de $\nabla w(./\varepsilon)$ quand ε tend vers 0.

— En lien avec le point précédent, on peut également voir que l'obtention de taux de convergence pour ∇R^ε est conditionnée à la compréhension du comportement asymptotique de $w(./\varepsilon)$, précisément à la donnée de taux de convergence uniformes de $\varepsilon w(./\varepsilon)$ sur l'ouvert Ω.

L'ensemble de ces contraintes est relié aux questions de convergence faible de la Section 1.3.1 et nécessite d'imposer des hypothèses géométriques supplémentaires sur les classes de défauts que l'on considère. Comme nous l'avons déjà précisé, nous verrons que dans le contexte des perturbations (1.14) présentées en Section 1.3.2, ces contraintes se traduisent par des conditions nécessaires d'existence de moyennes ou de corrélations de la suite Z_k couplées à des vitesses de convergence de ses moyennes de Cesàro.

1.3.4 L'exemple des défauts localisés de $L^q(\mathbb{R}^d)$

C'est dans l'esprit de tout ce que nous venons d'évoquer dans la section précédente que les premiers problèmes d'homogénéisation elliptique pour des perturbations localisées intégrables de la géométrie périodique ont été introduits puis résolus dans [20, 25, 26, 27]. On rappelle que ces travaux concernent le problème de diffusion (1.1) et les défauts de périodicités sont encodés dans des coefficients de la forme $a = a_{per} + \tilde{a}$ où $\tilde{a} \in L^q(\mathbb{R}^d)$, pour $q \in]1, +\infty[$.

1.3.4.1 Le cas simple monodimensionnel

Considérons tout d'abord le cas le plus simple de la dimension $d = 1$. Puisque les équations considérées peuvent être résolues explicitement pour cette dimension, ce cadre va servir d'illustration pour motiver les approches utilisées dans le cas de dimensions supérieures et, plus généralement, pour les différents cas de défauts que nous considérerons dans cette thèse.

Ici, on considère $\Omega =]0,1[$, $f \in L^2(\Omega)$ et, pour tout $\varepsilon > 0$, on note u^ε l'unique solution dans $H^1_0(0,1)$ du problème suivant :

$$\begin{cases} -\dfrac{d}{dx}\left((a_{per}+\tilde{a})\,(./\varepsilon)\dfrac{d}{dx}u^\varepsilon\right) = f \quad \text{sur }]0,1[, \\ \qquad u^\varepsilon(0) = u^\varepsilon(1) = 0. \end{cases} \tag{1.22}$$

On suppose également que $a = a_{per} + \tilde{a}$ et a_{per} vérifient les propriétés (1.2) et (1.3), il est donc clair que cette équation est bien posée pour tout $\varepsilon > 0$.

La résolution directe du problème (1.22) nous fournit les expressions explicites suivantes :

$$\begin{aligned} (u^\varepsilon)'(x) &= (a_{per}+\tilde{a})^{-1}\,(x/\varepsilon)\,(-F(x)+C^\varepsilon)\,, \\ u^\varepsilon(x) &= \int_0^x (a_{per}+\tilde{a})^{-1}\,(y/\varepsilon)\,(-F(y)+C^\varepsilon)\,dy. \end{aligned}$$

où on a noté $F(x) = \displaystyle\int_0^x f(y)dy$ et où C^ε est une constante d'intégration donnée par les conditions au bord :

$$C^\varepsilon = \left(\int_0^1 (a_{per}+\tilde{a})^{-1}(y/\varepsilon)dy\right)^{-1}\int_0^1 (a_{per}+\tilde{a})^{-1}(y/\varepsilon)F(y)dy.$$

Exactement comme en Section 1.3.1, on peut montrer que lorsque $\tilde{a} \in L^q(\mathbb{R})$, pour $q \in]0,1[$, on a la convergence faible de $(a_{per}+\tilde{a})^{-1}(./\varepsilon)$ vers $\langle a_{per}^{-1}\rangle$ pour la topologie $L^\infty(\mathbb{R}) - \star$. Ceci implique en particulier que u^ε converge, fortement dans $L^2(\Omega)$ et faiblement dans $H^1(\Omega)$, vers u^*, unique solution dans $H^1_0(\Omega)$ de

$$\begin{cases} \dfrac{d}{dx}\left(a^*\dfrac{d}{dx}u^*\right) = f \quad \text{sur }]0,1[, \\ u^*(0) = u^*(1) = 0, \end{cases}$$

où $a^* = \langle a_{per}^{-1}\rangle^{-1}$. Le coefficient homogénéisé est donc identique à celui obtenu dans un cadre périodique sans défaut, c'est-à-dire pour $\tilde{a} = 0$.

A la manière de l'étude du cas périodique, nous pouvons également introduire un correcteur w solution du problème suivant :

$$-\frac{d}{dx}\left((a_{per}+\tilde{a})\left(1+\frac{d}{dx}w\right)\right) = 0 \quad \text{sur } \mathbb{R}.$$

La résolution directe de ce problème nous amène donc à définir w par :

$$w(x) = -x + C_1\int_0^x (a_{per}+\tilde{a})^{-1}\,(y)dy,$$

où C_1 est une constante d'intégration. La condition de sous-linéarité à l'infini du correcteur, c'est-à-dire la convergence de $\varepsilon w(./\varepsilon)$ vers 0, nécessaire pour espérer approcher u^ε dans $H^1(\Omega)$ impose alors $C_1 = a^*$. Finalement, en dérivant w, on obtient :

$$w'(x) = -1 + a^*\,(a_{per}+\tilde{a})^{-1}\,(x).$$

Par un calcul simple, on a l'égalité suivante sur la dérivée du correcteur :

$$w'(x) = -1 + \frac{a^*}{a_{per}} - \frac{a^*\tilde{a}}{a_{per}\left(a_{per}+\tilde{a}\right)} = w'_{per} + \tilde{w}'$$

où $w'_{per} = -1 + \dfrac{a^*}{a_{per}}$ est la dérivée de l'unique (à constante additive près) solution périodique du problème du correcteur lorsque $a = a_{per}$. De plus, puisque a_{per} et $a_{per} + \tilde{a}$ vérifient (1.2) et (1.3) et $\tilde{a}$ appartient à $L^q(\mathbb{R})$, on a que $\tilde{w} = -\dfrac{a^*\tilde{a}}{a_{per}\left(a_{per}+\tilde{a}\right)}$ appartient aussi à $L^q(\mathbb{R})$. La dérivée du correcteur admet donc la même structure "périodique + défaut L^q" que le coefficient. Cette propriété motivera particulièrement nos approches dans l'étude du problème (1.1) en dimension supérieure puisque c'est exactement sous cette forme que nous chercherons le gradient d'une solution du problème du correcteur. Comme pour le cadre périodique, on peut alors considérer une approximation u^ε en norme H^1 définie par

$$u^{\varepsilon,1}(x) = u^*(x) + \varepsilon\left(u^*\right)'(x)\left(1 + w(x/\varepsilon)\right),$$

et on peut calculer explicitement :

$$\begin{aligned}(R^\varepsilon)'(x) = (u^\varepsilon)'(x) - (u^{\varepsilon,1})'(x) &= \frac{1}{\left(a_{per}+\tilde{a}\right)(x/\varepsilon)}\left(C^\varepsilon - \int_0^1 F(y)dy\right) \\ &+ \varepsilon(u^*)''(x)\left(1 - \frac{x}{\varepsilon} + a^*\int_0^{x/\varepsilon}\left(\frac{1}{a_{per}} - \frac{\tilde{a}}{a_{per}\left(a_{per}+\tilde{a}\right)}\right)(y)dy\right).\end{aligned}$$

En utilisant la périodicité de a_{per} et le fait que $\tilde{a} \in L^q(\mathbb{R})$ couplé à l'inégalité de Hölder, on peut alors montrer l'existence d'une constante $C > 0$ indépendante de ε telle que

$$\|\left(R^\varepsilon\right)'\|_{L^2(0,1)} \leq C\varepsilon^{\frac{1}{q}}.$$

Deux remarques peuvent alors être tirées de cette inégalité. Premièrement, le correcteur adapté permet d'obtenir une approximation de la dérivée de u^ε puisque $(R^\varepsilon)'$ converge vers 0 lorsque ε tend vers 0. Deuxièmement, l'impact des défauts est visible dans le taux de convergence puisque, dans le cas périodique sans défaut où $a = a_{per}$, il est connu que le taux de convergence est d'ordre ε.

1.3.4.2 Principaux résultats en dimension $d \geq 2$

En dimension $d \geq 2$, l'existence d'un correcteur adapté à ce cadre périodique perturbé est beaucoup plus délicate puisque l'équation (1.19), posée sur un domaine non borné, ne peut plus être résolue explicitement. Ce problème est précisément le point central des travaux menés dans [25, 26, 27]. Motivés par les considérations monodimensionnelles que nous venons juste de présenter, les auteurs établissent ainsi, pour tout $p \in \mathbb{R}^d$, l'existence d'un correcteur solution de (1.19) de la forme $w_p = w_{per,p} + \tilde{w}_p$ dont le gradient possède la même structure que le coefficient a : c'est la somme d'un gradient périodique et d'une perturbation dans $L^q(\mathbb{R}^d)$. Dans ce but, pour tout $f \in \left(L^q(\mathbb{R}^d)\right)^d$, ils montrent qu'il est suffisant d'établir l'existence d'une solution u avec un gradient dans $\left(L^q(\mathbb{R}^d)\right)^d$ de l'équation donnée par (1.21). Ces questions d'existence mènent alors à considérer des propriétés de continuité dans $L^q(\mathbb{R}^d)$ de l'opérateur

$-\operatorname{div}(a\nabla .)$, similaires à celles soulevées par la théorie des opérateurs de Calderòn-Zygmund. L'existence est ainsi établie en utilisant les travaux fondateurs [11, 14] de Avellaneda et Lin relatifs aux propriétés du noyau de Green pour l'opérateur $-\operatorname{div}(a_{per}\nabla .)$ à coefficients périodiques ainsi que des méthodes de type concentration-compacité inspirées de [70].

Après avoir montré l'existence d'un correcteur, la limite homogénéisée u^* est précisément identifiée, elle est solution de (1.4) où le coefficient homogénéisé est, de nouveau, identique à celui obtenu dans le cadre périodique pour lequel $a = a_{per}$ n'est pas perturbé. De plus, l'obtention des propriétés asymptotiques du correcteur s'appuie sur des propriétés spécifiques des espaces $L^q(\mathbb{R}^d)$ (inégalités de type Morrey par exemple) et des résultats quantitatifs de convergence sont démontrés dans [20]. En considérant

$$R^\varepsilon = u^\varepsilon - u^* - \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon),$$

les auteurs ont par exemple montré que, sous certaines hypothèses de régularité sur f et Ω, pour tout $r \geq 2$ et $\Omega_1 \subset\subset \Omega$,

$$\|\nabla R^\varepsilon\|_{L^r(\Omega_1)} \leq C\varepsilon^{\nu_q} \|f\|_{L^r(\Omega)},$$
$$\|\nabla R^\varepsilon\|_{L^\infty(\Omega_1)} \leq C\varepsilon^{\nu_q} \ln\left(1+\varepsilon^{-1}\right) \|f\|_{\mathcal{C}^{0,\beta}(\Omega)}$$

où $\nu_q > 0$ est un exposant connu qui ne dépend que de q et de la dimension d.

Insistons également sur le fait que, dans ce cadre, l'existence d'un correcteur adapté aux défauts est essentiel d'un point de vue pratique. Les estimées ci-dessus montrent que le correcteur permet d'approcher u^ε précisément à l'échelle microscopique, c'est-à-dire pour des topologies suffisamment fines (typiquement L^∞ ou L^r pour r grand). C'est à cette échelle, qui est primordiale pour des applications numériques en pratique, que les défauts de périodicité ont un impact sur la solution u^ε et que la nécessité de considérer un correcteur qui prend précisément en compte ces perturbations est la plus importante. Un raisonnement effectué dans [25, Section 4] dans un cadre de défauts $L^2(\mathbb{R}^d)$, montre effectivement l'impossibilité de se "tromper" de correcteur pour obtenir des résultats de convergence à l'échelle microscopique. En choisissant uniquement la partie périodique w_{per} du correcteur w dans l'approximation de u^ε (c'est-à-dire en considérant $R^\varepsilon_{per} = u^\varepsilon - u^* - \sum_{i=1}^{d} \partial_i u^* w_{per,e_i}(./\varepsilon)$), les auteurs montrent alors que les résultats d'approximation sont nettement dégradés. Cela illustre donc l'importance de construire un correcteur adapté w solution de (1.19).

L'objectif de notre travail est en particulier d'étendre ces résultats à un panel de perturbations le plus large possible, pour des défauts localisé ou non et pour des classes de coefficients plus générales comme celles introduites en Section 1.3.2.

1.4 Contributions de la thèse

Cette thèse est composée de trois parties principales traitant des problèmes (1.1) ou (1.7) dans différents cadres de perturbation de la géométrie périodique. Quelques résultats indépendants reliés au problème de diffusion (1.1) et établis durant cette thèse sont également présentés en Annexe A. Nous résumons ici les trois principales contributions de cette thèse. Chacune de ces contributions est détaillée dans le chapitre correspondant.

1.4.1 Chapitre 2 : Problème d'homogénéisation périodique avec défauts rares à l'infini

Le **Chapitre** 2 reproduit l'article [Gou22a]. L'étude menée dans ce chapitre est une extension des travaux de [20, 25, 26, 27] et concerne le problème d'homogénéisation pour l'équation de diffusion (1.1) dans le cadre d'un milieu à structure périodique perturbée par des défauts non localisés. Dans cette étude, nous considérons le problème d'homogénéisation (1.1) pour une classe de coefficients composée de fonctions matricielles de la forme (1.11). Le coefficient a_{per} appartient à $\left(L^2_{per}(\mathbb{R}^d)\right)^{d\times d}$ et décrit une géométrie périodique sous-jacente qui, à la différence du cadre présenté en Section 1.3.4, est perturbée par un défaut $\tilde{a}$ qui ne disparaît jamais à l'infini mais devient *rare* lorsqu'on s'éloigne de l'origine. A la manière des exemples de perturbations donnés en Section 1.3.2, nous considérons précisément des coefficients $\tilde{a}$ qui se comportent formellement comme une somme infinie de perturbations localisées qui s'éloignent exponentiellement les unes des autres lorsque qu'on s'écarte de l'origine. L'objectif est alors d'étendre les résultats d'homogénéisation bien connus du contexte périodique sans défaut présenté en Section 1.2.1 à notre problème (1.1)-(1.11). Un développement multi-échelles formel dans notre cas montre, de nouveau, que le comportement de u^ε est lié à un correcteur w_p solution de (1.19) pour tout $p \in \mathbb{R}^d$, équation posée sur tout l'espace $\mathbb{R}^d$.

Cadre mathématique. Dans cette étude nous proposons un cadre fonctionnel permettant de formaliser la notion de défauts rares à l'infini. L'idée est d'abord d'introduire un ensemble de points $\mathcal{G} = \{x_p\}_{p\in\mathbb{Z}^d}$ où chaque point x_p va modéliser la présence d'un défaut de le géométrie périodique modélisée par le coefficient a_{per}. On introduit également le diagramme de Voronoï associé à l'ensemble de points $\mathcal{G}$ et pour $x_p \in \mathcal{G}$, on note alors V_{x_p} la cellule de Voronoï contenant le point x_p et définie par

$$V_{x_p} = \bigcap_{x_q\in\mathcal{G}\setminus\{x_p\}} \left\{x \in \mathbb{R}^d \middle| |x - x_p| \le |x - x_q|\right\}.$$

Notre but étant d'assurer que les défauts sont suffisamment rares à l'infini, il est nécessaire d'imposer une distribution appropriée des points x_p dans l'espace. On suppose donc que l'ensemble $\mathcal{G}$ vérifie les trois hypothèses géométriques suivantes :

$$\forall x_p \in \mathcal{G}, \quad \left|V_{x_p}\right| < \infty, \tag{H1}$$

$$\exists C_1 > 0,\ C_2 > 0,\ \forall x_p \in \mathcal{G}, \quad C_1 \le \frac{1 + |x_p|}{D\left(x_p, \mathcal{G}\setminus\{x_p\}\right)} \le C_2, \tag{H2}$$

$$\exists C_3 > 0, \forall x_p \in \mathcal{G}, \quad \frac{Diam\left(V_{x_p}\right)}{D\left(x_p, \mathcal{G}\setminus\{x_p\}\right)} \le C_3, \tag{H3}$$

où $D(.,.)$ est la distance euclidienne, $|V_{x_p}|$ désigne le volume de V_{x_p} et $Diam\left(V_{x_p}\right)$ son diamètre. L'hypothèse (H2) est l'hypothèse principale de notre étude puisqu'elle implique que la distance relative entre chaque point augmente lorsqu'on s'éloigne de l'origine. Elle assure que la distance entre un point x_p et tous les autres croît exactement comme $|x_p|$ et contraint notamment les points x_p à être de plus en plus rares à l'infini. Dans le **Chapitre** 2, nous montrerons effectivement qu'une conséquence essentielle de cette hypothèse est que le nombre de points x_p contenu dans une boule B_R de rayon $R > 0$ est borné par le logarithme de R. A la différence de (H2), les hypothèses (H1) et (H3) sont seulement des hypothèses techniques non restrictives puisqu'elles consistent à limiter la taille des cellules de Voronoï. Dans un certain sens, elles assurent que nous considérons le pire scénario possible, c'est-à-dire un scénario où l'ensemble $\mathcal{G}$ contient autant de points que possibles tout en satisfaisant (H2).

Bien que nous établissions toutes les propriétés géométriques vérifiées par les cellules de Voronoï V_{x_p} et nécessaires à l'élaboration d'une théorie complète de l'homogénéisation de (1.1) dans la généralité des hypothèses (H1),(H2) et (H3), nous nous intéressons particulièrement à un exemple illustratif vérifiant ces hypothèses et pour lequel les coordonnées des points x_p sont des puissances de 2. Nous détaillons cet ensemble dans le **Chapitre** 2 (voir (2.8)-(2.9)).

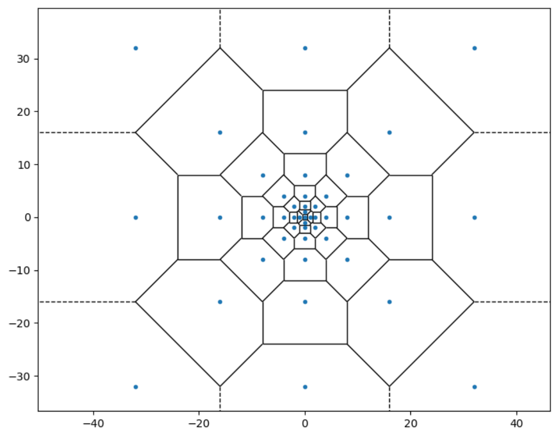

FIGURE 1.4 – Exemple d'un ensemble de points en dimension $d = 2$ vérifiant les hypothèses (H1)-(H2)-(H3) et son diagramme de Voronoï.

L'espace fonctionnel permettant de décrire nos défauts rares à l'infini est alors défini par :

$$\mathcal{B}^2(\mathbb{R}^d) = \left\{ f \in L^2_{unif}(\mathbb{R}^d) \;\middle|\; \exists f_\infty \in L^2(\mathbb{R}^d),\ \lim_{|x_p|\to\infty} \int_{V_{x_p}} |f(x) - \tau_{-p} f_\infty(x)|^2 dx = 0 \right\}, \tag{1.23}$$

où $\tau_{-p} f := f(. - x_p)$ pour toute fonction f et

$$L^2_{unif}(\mathbb{R}^d) = \left\{ f \in L^2_{loc}(\mathbb{R}^d),\ \sup_{x\in\mathbb{R}^d} \|f\|_{L^2(B_1(x))} < \infty \right\}.$$

Intuitivement, une fonction de $\mathcal{B}^2(\mathbb{R}^d)$ se comporte, localement au voisinage de chaque point x_p, comme une fonction fixe de $L^2(\mathbb{R}^d)$ tronquée sur le domaine V_{x_p}.

On considère alors un coefficient matriciel de la forme (1.11) avec $\tilde{a} \in \mathcal{B}^2(\mathbb{R}^d)^{d\times d}$ et on note $\tilde{a}_\infty$ la limite L^2 associée à $\tilde{a}$ et définie par (1.23). Pour $\alpha \in]0, 1[$, on note également $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ l'espace des fonctions uniformément bornées sur $\mathbb{R}^d$ et uniformément α-Hölderiennes, et on fait les hypothèses de coercivité et de régularité suivantes :

$$\exists \lambda > 0 \text{ tel que pour tous } x, \xi \in \mathbb{R}^d \quad \lambda|\xi|^2 \le \langle a(x)\xi, \xi\rangle, \quad \lambda|\xi|^2 \le \langle a_{per}(x)\xi, \xi\rangle, \tag{1.24}$$

et

$$a_{per},\ \tilde{a},\ \tilde{a}_\infty \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}, \qquad \alpha \in]0,1[. \tag{1.25}$$

Résultats principaux. Notre étude se concentre sur le cas où la dimension d est supérieure ou égale à 3. Les cas en dimensions $d = 1$ et $d = 2$ sont en effet spécifiques et les preuves requièrent des adaptations que nous évoquerons tout au long du **Chapitre** 2. Cela est en particulier lié au comportement asymptotique de la fonction de Green de l'opérateur Laplacien pour ces deux dimensions. On établit les résultats suivants :

Théorème 1.1. *On suppose que $d \geq 3$, que a est de la forme (1.11) avec $\tilde{a} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d}$ et vérifie (1.24)-(1.25). Pour tout $p \in \mathbb{R}^d$, il existe une unique (à constante additive près) fonction $w_p \in H^1_{loc}(\mathbb{R}^d)$ telle que $\nabla w_p \in \left(L^2_{per}(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d)\right)^d \cap \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, solution de :*

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(p + \nabla w_p)) = 0 \quad \textit{dans } \mathbb{R}^d, \\ \displaystyle\lim_{|x|\to\infty} \frac{|w_p(x)|}{1+|x|} = 0. \end{cases} \tag{1.26}$$

De plus, la suite u^ε de solutions de (1.1) converge, fortement dans $L^2(\Omega)$ et faiblement dans $H^1(\Omega)$, vers u^* solution dans $H^1_0(\Omega)$ de (1.4).

Théorème 1.2. *On suppose que $\Omega \subset \mathbb{R}^d$ est un domaine borné de régularité $C^{2,1}$. Soit $\Omega_1 \subset\subset \Omega$. Sous les hypothèses du Théorème 1.1, on définit $u^{\varepsilon,1} = u^* + \varepsilon \displaystyle\sum_{i=1}^d \partial_i u^* w_{e_i}(./\varepsilon)$ où w_{e_i} est défini par le Théorème 1.1 pour $p = e_i$ et u^* est l'unique solution dans $H^1_0(\Omega)$ de (1.4). Alors $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ vérifie les estimées suivantes :*

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \varepsilon \|f\|_{L^2(\Omega)},$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \varepsilon \|f\|_{L^2(\Omega)},$$

où C_1 et C_2 sont deux constantes positives indépendantes de f et ε.

Comme c'était déjà le cas pour le cas de perturbations de $L^q(\mathbb{R}^d)$ présentées en Section 1.3.4, le gradient du correcteur associé à notre problème possède alors la même structure "périodique + perturbation rare à l'infini" que le coefficient a. Le problème homogénéisé est également donné par (1.4) et est identique à celui associé au cadre périodique sans défaut. La démonstration du résultat d'existence d'un correcteur du Théorème 1.1 repose sur une adaptation des méthodes employées dans [27]. Précisément, en cherchant à construire un correcteur comme la somme du correcteur périodique $w_{per,p}$ solution de (1.5) associé à a_{per} et d'une perturbation $\tilde{w}_p$ telle que $\nabla \tilde{w}_p \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, on montre que la résolution de

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})\nabla u) = \operatorname{div}(f), \\ \nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d, \end{cases} \tag{1.27}$$

pour tout $f \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, est suffisante pour établir l'existence d'une solution de (1.26). L'étude de (1.27) est alors effectuée en deux étapes : une étude préliminaire de l'opérateur à coefficient périodique $\nabla\left(-\operatorname{div}(a_{per}\nabla .)\right)^{-1}\operatorname{div}$ de $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ dans $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, en utilisant à la fois les propriétés du noyau de Green de $-\operatorname{div}(a_{per}\nabla .)$ données par les travaux de Avellaneda et Lin [14] et les propriétés géométriques de l'espace $\mathcal{B}^2(\mathbb{R}^d)$, puis une extension de ces résultats à l'opérateur général $\nabla\left(-\operatorname{div}((a_{per}+\tilde{a})\nabla.)\right)^{-1}\operatorname{div}$ par une méthode dite de concentration-compacité introduite dans la preuve de [27, Proposition 2.1]. Une fois l'existence d'un correcteur établie, le Théorème 1.2 d'homogénéisation est obtenu en adaptant les résultats montrés dans [20] pour le cas de défauts localisés et en utilisant les propriétés structurelles et géométriques de l'espace $\mathcal{B}^2(\mathbb{R}^d)$ afin d'obtenir des taux de convergence précis.

Nous proposons également une extension naturelle des Théorèmes 1.1 et 1.2 en introduisant, pour tout $r \in]1,+\infty[$, les espaces de défauts

$$\mathcal{B}^r(\mathbb{R}^d) = \left\{ f \in L^r_{unif}(\mathbb{R}^d) \;\middle|\; \exists f_\infty \in L^r(\mathbb{R}^d),\ \lim_{|x_p|\to\infty} \int_{V_{x_p}} |f(x) - \tau_{-p} f_\infty(x)|^r dx = 0 \right\},$$

définis comme $\mathcal{B}^2(\mathbb{R}^d)$ mais en utilisant la topologie de L^r.

Théorème 1.3. *On suppose que $d \geq 3$, $r \in]1,+\infty[$ et a est de la forme (1.12) avec $\tilde{a} \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^{d\times d}$ et vérifie (1.24)-(1.25). Pour tout $p \in \mathbb{R}^d$, il existe une unique (à l'addition d'une constante près) fonction $\tilde{w}_p \in W^{1,r}_{loc}(\mathbb{R}^d)$ telle que $w_p = w_{per,p} + \tilde{w}_p$ est solution de l'équation du correcteur (1.26), où $w_{per,p}$ est l'unique solution périodique (à constante additive près) de (1.5) et $\nabla\tilde{w}_p \in \left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.*

Théorème 1.4. *On suppose que $\Omega \subset \mathbb{R}^d$ est un domaine borné de régularité $C^{2,1}$ et que $r \neq d$. Soit $\Omega_1 \subset\subset \Omega$. Sous les hypothèses du Théorème 1.3, on définit $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon)$ où w_{e_i} est le correcteur donné par le Théorème 1.3 pour $p = e_i$ et $u^* \in H^1(\Omega)$ est solution de (1.4). On définit également*

$$\nu_r = \min\left(1, \frac{d}{r}\right) \in]0,1],$$

et

$$\mu_r = \begin{cases} 0 & si\ r < d, \\ \dfrac{1}{r} & sinon. \end{cases}$$

Alors $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ vérifie les estimées suivantes :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \left(\log|\varepsilon|\right)^{\mu_r} \varepsilon^{\nu_r} \|f\|_{L^2(\Omega)},$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \left(\log|\varepsilon|\right)^{\mu_r} \varepsilon^{\nu_r} \|f\|_{L^2(\Omega)},$$

où C_1 et C_2 sont deux constantes positives indépendantes de f et ε.

Les preuves de ces deux résultats suivent essentiellement le même schéma que le cas de défauts $\mathcal{B}^2$. La principale difficulté additionnelle est que, lorsque $r \neq 2$, il est impossible d'utiliser certaines propriétés "Hilbertiennes" induites par la structure de L^2 et les preuves doivent donc être adaptées.

1.4.2 Chapitre 3 : Homogénéisation elliptique avec coefficients presque invariants par translation à l'infini

Le **Chapitre** 3 reproduit l'article [Gou22b]. Nous considérons toujours l'homogénéisation du problème de diffusion (1.1) dans un cadre de géométrie périodique perturbée mais, à la différence des travaux de [20, 25, 26, 27] et du cadre de défauts rares à l'infini introduit dans la section précédente, nous ne supposons pas, a priori, que le coefficient a est de la forme spécifique (1.11), c'est-à-dire la superposition d'une géométrie périodique connue et d'un défaut identifié. Ici, comme dans le cadre que nous avons introduit en fin de Section 1.3.2, le coefficient a est seulement caractérisé par une certaine invariance par $\mathbb{Z}^d$-translation à l'infini, donnée typiquement en dimension $d = 1$ par l'intégrabilité de la fonction $\delta a := a(. + 1) - a$ sur $\mathbb{R}$.

Nous supposons toujours que a vérifie les hypothèses de coercivité et de borne uniforme données respectivement par (1.2) et (1.3) afin d'assurer que le problème (1.1) est bien posé. Nous supposons également

$$a \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}, \qquad \text{for } \alpha \in]0, 1[, \tag{1.28}$$

hypothèse de régularité additionnelle requise dans notre approche afin d'appliquer certains résultats de régularité elliptique et d'utiliser des estimées ponctuelles vérifiée par la fonction de Green des équations sous forme divergence à coefficients périodiques.

Afin de formaliser notre cadre de coefficients non-périodiques et presque invariants par translation à l'infini en dimension d quelconque, pour tout $p \in [1, +\infty[$, nous définissons l'espace

$$\mathbf{A}^p = \left\{ g \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \delta g \in \left(L^p(\mathbb{R}^d)\right)^d \right\}.$$

Ici le gradient discret δ est défini par (1.17).

Pour notre étude, les coefficients a considérés pour modéliser une géométrie asymptotiquement $\mathbb{Z}^d$-périodique sont supposés vérifier :

$$\exists p \in [1, +\infty[, \ \forall i, j \in \{1, ..., d\}, \quad a_{i,j} \in \mathbf{A}^p. \tag{1.29}$$

Une telle hypothèse assure que δa converge vers 0 à l'infini dans un certain sens et, par conséquent, que le comportement de a est proche d'un coefficient $\mathbb{Z}^d$-périodique lorsqu'on s'éloigne de l'origine (voir figure 1.5 pour des exemples en dimensions $d = 1$ et $d = 2$).

En supposant que le coefficient a vérifie (1.2), (1.3), (1.28) et (1.29), les objectifs sont donc, à nouveau, de déterminer si toute la suite u^ε converge vers une solution homogénéisée u^* (et pas seulement le long d'une sous-suite), s'il est possible d'expliciter le coefficient homogénéisé a^* et s'il existe un correcteur strictement sous-linéaire à l'infini, solution de (1.19).

Le cas $p < d$. Lorsque $\delta a \in \left(L^p(\mathbb{R}^d)\right)^d$ pour $p < d$, notre approche est une adaptation d'un cas continu détaillé en Section 3.1.1 pour lequel le coefficient a vérifie $\nabla a \in L^p(\mathbb{R}^d)$ et admet une décomposition de la forme $a = c + \tilde{a}$. Précisément, une conséquence de l'inégalité de Gagliardo-Nirenberg-Sobolev montre que ce coefficient est la somme d'une constante c et d'une perturbation localisée intégrable $\tilde{a} \in \left(L^{p^*}(\mathbb{R}^d)\right)^{d\times d}$, pour $p^* = \frac{pd}{d-p}$ l'exposant de Sobolev associé à p. Dans notre cas discret, on montre que le coefficient a décrit une géométrie périodique perturbée par un défaut *localisé* qui, à moyennisation locale près, appartient à

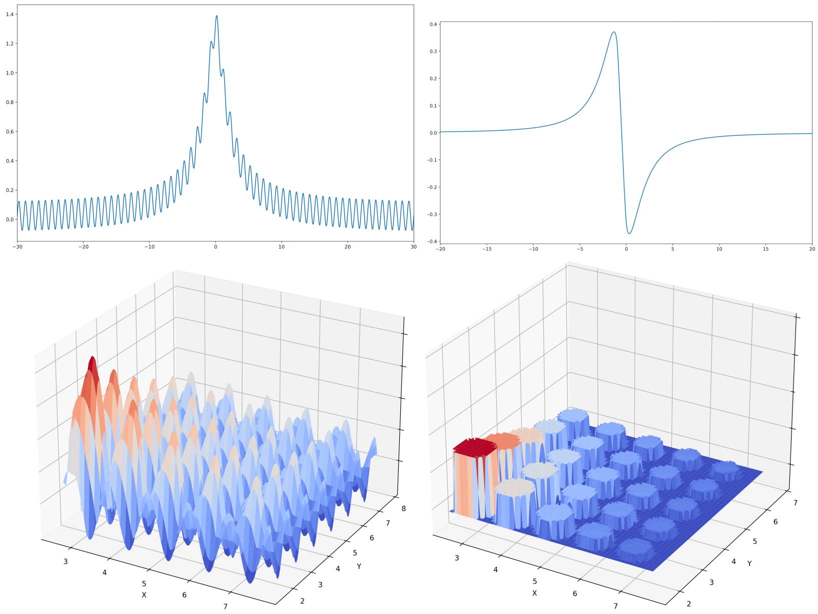

FIGURE 1.5 – Exemples de coefficients a vérifiant l'hypothèse (1.29) en dimensions $d = 1$ (gauche : $a(x)$; droite : $\delta a(x)$) et $d = 2$ (gauche : $a(x, y)$; droite : $|\delta_2 a(x, y)|$).

$L^{p^*}(\mathbb{R}^d)$. Dans ce but, on introduit un opérateur $\mathcal{M}$ qui permet de décrire les moyennes locales d'une fonction $f \in L^1_{loc}(\mathbb{R}^d)$ et défini par :

$$\mathcal{M}(f)(z) = \int_{Q+z} f(x)dx.$$

On introduit aussi deux espaces fonctionnels :

$$\mathcal{E}^p = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \mathcal{M}\left(|f|\right) \in L^{p^*}(\mathbb{R}^d) \right\},$$

$$\mathcal{A}^p = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \mathcal{M}\left(|f|\right) \in L^{p^*}(\mathbb{R}^d) \text{ et } \delta f \in (L^p(\mathbb{R}^d))^d \right\},$$

munis des normes suivantes :

$$\|f\|_{\mathcal{E}^p} = \|\mathcal{M}\left(|f|\right)\|_{L^{p^*}(\mathbb{R}^d)},$$

$$\|f\|_{\mathcal{A}^p} = \|\mathcal{M}\left(|f|\right)\|_{L^{p^*}(\mathbb{R}^d)} + \|\delta f\|_{(L^p(\mathbb{R}^d))^d}.$$

Le résultat principal concernant les fonctions de $\mathbf{A}^p$ quand $p < d$ est donné dans la proposition suivante :

Proposition 1.1. *On suppose que $p < d$. Soit $f \in \mathbf{A}^p$, alors il existe une unique fonction périodique f_{per} telle que $f - f_{per} \in \mathcal{E}^p$. De plus, il existe une constante $C > 0$ indépendante de f telle que :*

$$\|f - f_{per}\|_{\mathcal{E}^p} \leq C\|\delta f\|_{L^p(\mathbb{R}^d)}.$$

Cette proposition est une adaptation discrète de l'inégalité de Gagliardo-Nirenberg-Sobolev et assure que toute fonction $f \in \mathbf{A}^p$ est la somme d'une fonction périodique et d'une "perturbation" de $\mathcal{A}^p$. Dans le cas particulier où $p < d$, on a alors les résultats suivants :

Théorème 1.5. *On suppose $d \geq 2$ et que a vérifie (1.2), (1.3) (1.28) et (1.29) pour $1 < p < d$. On note a_{per} l'unique coefficient périodique donné par la Proposition 1.1 tel que $\tilde{a} := a - a_{per} \in (\mathcal{A}^p)^{d\times d}$. Soient $q \in \mathbb{R}^d$ et $w_{per,q}$ la solution périodique, unique à l'addition d'une constante près, de $-\operatorname{div}\left(a_{per}\left(\nabla w_{per,q} + q\right)\right) = 0$ sur $\mathbb{R}^d$. Alors il existe $\tilde{w}_q \in L^1_{loc}(\mathbb{R}^d)$ solution de*

$$\begin{cases} -\operatorname{div}(a(\nabla w_{per,q} + \nabla \tilde{w}_q + q)) = 0 & \text{sur } \mathbb{R}^d, \\ \displaystyle\lim_{|x|\to\infty} \frac{|\tilde{w}_q(x)|}{1+|x|} = 0, \end{cases}$$

telle que $\nabla \tilde{w}_q \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Une telle solution $\tilde{w}_q$ est unique à l'addition d'un constante près.

De plus, la suite u^ε de solutions de (1.1) converge, fortement dans $L^2(\Omega)$ et faiblement dans $H^1(\Omega)$, vers u^* solution de (1.4).

Théorème 1.6. *On suppose que $\Omega \subset \mathbb{R}^d$ est un domaine borné de régularité $C^{2,1}$ et que l'exposant $1 < p < d$ vérifie $p \neq \dfrac{d}{2}$. Soit $\Omega_1 \subset\subset \Omega$. Sous les hypothèses du Théorème 1.5, on définit $u^{\varepsilon,1} = u^* + \varepsilon \displaystyle\sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon)$ où $w_{e_i} = w_{per,e_i} + \tilde{w}_{e_i}$ est le correcteur donné par le Théorème 1.5 pour $q = e_i$ et $u^* \in H^1(\Omega)$ est solution de (1.4). On définit également*

$$\mu_p := \begin{cases} \dfrac{d}{p^*} & \text{si } p > \dfrac{d}{2}, \\ 1 & \text{si } p < \dfrac{d}{2}. \end{cases}$$

Alors, pour tout $r \geq 2$, $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ vérifie l'estimée suivante :

$$\|\nabla R^\varepsilon\|_{L^r(\Omega_1)} \leq C\varepsilon^{\mu_p}\|f\|_{L^r(\Omega)},$$

où C est une constante positive indépendante de f et ε.

Le Théorème 1.5 donne l'existence d'un correcteur adapté au problème (1.1) et montre la convergence de u^ε vers une limite homogénéisée u^*. On montre à nouveau que le gradient du correcteur partage la même structure que le coefficient a et que la perturbation de $\mathcal{A}^p$ n'affecte pas la solution homogénéisée u^* puisque le coefficient homogénéisé est le même que celui obtenu après homogénéisation de (1.1) quand $a = a_{per}$. La preuve s'appuie également sur les techniques introduites dans [27] et repose essentiellement sur l'étude de l'équation de diffusion générale

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u) = \operatorname{div}(f),$$

lorsque f appartient à l'espace $(\mathcal{A}^p)^d$. Pour cela, on montre la continuité de l'opérateur à coefficients périodiques $\nabla\left(-\operatorname{div}(a_{per}\nabla .)\right)^{-1}\operatorname{div}$ de $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ dans $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ à l'aide des les propriétés du noyau de Green de $-\operatorname{div}(a_{per}\nabla .)$, puis on étend cette propriété à l'opérateur général $\nabla\left(-\operatorname{div}((a_{per}+\tilde{a})\nabla .)\right)^{-1}\operatorname{div}$ par concentration-compacité.

La convergence de ∇R^ε pour la topologie de $W^{1,r}$ quand $r \geq 2$ et les taux de convergence donnés par le Théorème 1.6 sont une conséquence directe des résultats établis dans [20] et des propriétés des fonctions à gradient dans $\mathcal{A}^p$, vérifiées en particulier par la partie pertubative $\tilde{w}_{e_i}$ de nos correcteurs.

En réalité, nous montrons aussi que l'hypothèse de régularité Hölderienne (1.28), combinée à l'hypothèse (1.29), implique que la perturbation $\tilde{a}$ appartient à $\left(L^q(\mathbb{R}^d)\right)^{d\times d}$ pour un exposant $p^* < q$ identifié (voir la Proposition 3.7). Ceci implique que les résultats de [25, 26, 27] couvrent notre cadre et montrent l'existence d'un correcteur $w = w_{per} + \tilde{w}$ où $\tilde{w}$ est solution de (1.19) telle que $\nabla\tilde{w} \in \left(L^q(\mathbb{R}^d)\right)^d$ pour cet exposant q particulier. Les résultats des Théorèmes 1.5 et 1.6 sont cependant plus forts dans notre approche : ils montrent que la perturbation $\tilde{w}$ de notre correcteur possède un gradient dans $(\mathcal{A}^p)^d$ et, puisque nous verrons que $p^* < q$, ils fournissent de meilleures propriétés d'intégrabilité à l'infini. Nous montrons effectivement que les taux de convergence théoriques de ∇R^ε sont améliorés si nous supposons $\tilde{a} \in (\mathcal{A}^p)^{d\times d}$ plutôt qu'uniquement $\tilde{a} \in \left(L^q(\mathbb{R}^d)\right)^{d\times d}$. Au delà de montrer de meilleurs taux de convergence, cette étude permet également de proposer une méthodologie concernant l'étude du problème d'homogénéisation (1.1) pour des coefficients à gradient discret intégrable. Notre approche est en effet similaire mais aussi *indépendante* des preuves et arguments utilisés dans le contexte des perturbations de L^q. En particulier, l'objectif est d'illustrer le fait que les méthodes employées dans [27, 26, 25] nécessitent seulement de connaître le comportement global des moyennes de a à l'infini.

Le cas $p \geq d$. Lorsque $p \geq d$, on montre que l'homogénéisation du problème (1.1) n'est pas toujours possible. On établit effectivement l'existence de coefficients a tels que les suites u^ε associées possèdent plusieurs valeurs d'adhérence. Les contre-exemples de coefficients considérés sont caractérisés par une oscillation lente à l'infini et les différentes limites obtenues sont également solutions de problèmes de la forme (1.4) mais pour lesquels les coefficients a^* ne sont pas forcément constants.

1.4.3 Chapitre 4 : Homogénéisation elliptique pour une classe de potentiels oscillants non-périodiques

Le **Chapitre** 4 reproduit l'article [GLB22]. Dans ce chapitre, on s'intéresse à l'homogénéisation de l'équation de Schrödinger stationnaire avec un potentiel rapidement oscillant donnée par (1.7), où $V \in L^\infty(\mathbb{R}^d)$ est un potentiel non-périodique qui modélise une géométrie périodique perturbée. Comme dans le cas périodique abordé en Section 1.2.2, le potentiel V est supposé être à "moyenne" nulle dans le sens suivant :

$$\lim_{\varepsilon\to 0} V(./\varepsilon) = 0 \quad \text{dans } L^\infty(\mathbb{R}^d) - \star. \tag{1.30}$$

Les potentiels non-périodiques que nous considérons sont inspirés du cadre introduit dans [22] que nous avons présenté en Section 1.3.2 et sont donnés par

$$V(x) = g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(x - k - Z_k), \tag{1.31}$$

où g_{per} est une fonction Q-périodique, φ appartient à $\mathcal{D}(\mathbb{R}^d)$ et $Z := (Z_k)_{k\in\mathbb{Z}^d}$ est une suite vectorielle qui satisfait $Z \in \left(l^\infty(\mathbb{Z}^d)\right)^d$. Dans ce contexte, le potentiel V est une perturbation du potentiel périodique

$$V_{per} = g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(x - k) \tag{1.32}$$

par le défaut

$$\tilde{V} := \sum_{k\in\mathbb{Z}^d} \left(\varphi(x-k-Z_k) - \varphi(x-k)\right). \tag{1.33}$$

En particulier, l'objectif est d'étendre les résultats d'homogénéisation du problème (1.7) du contexte périodique de la Section 1.2.2 à des cas de perturbations où la suite Z, qui modélise évidemment les défauts de périodicité, n'est pas localisée dans l'espace et ne disparaît d'aucune manière à l'infini.

La principale difficulté du cadre non-périodique défini par (1.31) est toujours l'étude de l'équation du correcteur, donnée ici par le problème du Laplacien (1.20). Comme nous l'avons précisé en Section 1.3.3, cette équation doit être résolue au moins sur Ω/ε, domaine asymptotiquement non borné lorsque ε tend vers 0.

Dans un premier temps, montrer l'existence d'un correcteur w solution de (1.20) sur tout l'espace $\mathbb{R}^d$ n'est a priori pas évident puisque le potentiel V n'est pas périodique. Nous verrons que, contrairement aux résultats établis dans les **Chapitres** 2 et 3, les difficultés propres liées au cadre (1.31) nous amèneront en réalité à considérer une suite de correcteurs w_ε dépendants de ε et solutions de (1.20) seulement sur Ω/ε. Nous reviendrons sur ce point dans la suite. Formellement, notre approche pour montrer l'existence d'un correcteur w solution de (1.20) consiste à nouveau à utiliser la structure particulière du potentiel V, c'est-à-dire une perturbation du potentiel périodique (1.32) par la perturbation (1.33) afin de construire un correcteur de la forme

$$w = w_{per} + \tilde{w},$$

où $\Delta w_{per} = V_{per}$ et $\nabla \tilde{w}$ est défini par

$$\nabla \tilde{w} = \sum_{k\in\mathbb{Z}^d} \nabla G * \left(\varphi(.-k-Z_k) - \varphi(.-k)\right). \tag{1.34}$$

Ici, on a noté G la fonction de Green associée à l'opérateur Δ sur $\mathbb{R}^d$. L'égalité formelle (1.34) nous permet alors d'identifier la difficulté principale relative à la bonne définition de $\nabla\tilde{w}$. En effet, $\nabla G(x-k)$ se comporte comme $\dfrac{1}{|x-k|^{d-1}}$ à l'infini et la convergence de la série apparaissant dans (1.34) n'est pas évidente. En particulier, il faudrait montrer que la convolution apparaissant dans la somme (1.34) permet d'obtenir un taux de décroissance par rapport à k (pour k grand) augmenté de plus d'un exposant, le taux de décroissance $\dfrac{1}{|k|^d}$ étant précisément le taux critique pour la convergence de la somme en dimension ambiante d. Un développement de Taylor de la fonction φ, sur lequel nous nous appuierons largement dans le travail du **Chapitre** 4, permet d'obtenir (toujours de manière formelle) l'égalité suivante :

$$\nabla \tilde{w} = \sum_{k\in\mathbb{Z}^d} \nabla G * \left(-Z_k.\nabla\varphi(.-k) + \int_0^1 (1-t) Z_k^T D^2\varphi(.-k-tZ_k) Z_k dt\right).$$

Par linéarité du problème, on distingue alors deux termes qui ont des contributions différentes dans la convergence de la somme. Le premier est lié à la série

$$\sum_{k\in\mathbb{Z}^d} \nabla G * \left(\int_0^1 (1-t) Z_k^T D^2\varphi(.-k-tZ_k) Z_k dt\right), \tag{1.35}$$

qui est normalement convergente dans L^∞ puisqu'elle ne contient que des dérivées secondes de φ et, par intégration par parties, donne des dérivées d'ordre trois de la fonction de Green D^3G qui décroissent comme $\frac{1}{|x|^{d+1}}$. Le second terme, lié à la série

$$\sum_{k\in\mathbb{Z}^d} \nabla G * \left(-Z_k.\nabla\varphi(.-k)\right), \tag{1.36}$$

est beaucoup plus délicat puisque la contribution de dérivées secondes de G ne permet d'obtenir qu'une décroissance critique en $\frac{1}{|k|^d}$. Dans le **Chapitre** 4, nous verrons que plusieurs conditions spécifiques sur la fonction φ ou la suite Z permettent d'obtenir une convergence de la somme (1.36) dans L^∞ mais dans un contexte le plus général possible, nous ne pourrons montrer qu'une convergence de la somme dans l'espace des fonctions à oscillations moyennes bornées $BMO(\mathbb{R}^d)$ en utilisant les propriétés de l'opérateur de Calderòn-Zygmund $T : f \mapsto \nabla^2 G * f$. Ceci va alors impliquer quelques technicités dans l'utilisation du correcteur pour montrer l'homogénéisation de (1.7). Notamment, afin d'obtenir des bornes uniformes en ε vérifiées par le gradient du correcteur, cette contrainte nous imposera de considérer une suite de correcteurs w_ε ayant une dépendance en le paramètre d'échelle ε comme nous l'avons évoqué plus haut. De plus ces bornes seront vérifiées uniquement sur le domaine Ω/ε et non sur tout l'espace $\mathbb{R}^d$.

Dans un second temps, en supposant l'existence d'une telle suite de correcteurs w_ε établie, une seconde difficulté induite par le problème (1.7) consiste à assurer que cette suite w_ε est adaptée au problème d'homogénéisation (1.7). Premièrement, il est effectivement nécessaire de montrer que $\varepsilon w_\varepsilon(./\varepsilon)$ converge uniformément vers 0 sur Ω. Deuxièmement, à la manière du cadre périodique présenté en Section 1.2.2, on s'attend également à ce que le potentiel homogénéisé soit lié à la limite faible de $|\nabla w_\varepsilon(./\varepsilon)|^2$. L'existence de cette limite doit donc être établie et sa valeur doit être identifiée. Pour montrer ces deux points, il est alors nécessaire d'imposer des propriétés supplémentaires vérifiées par la fonction φ ou par la suite Z. Nous choisissons d'imposer uniquement des contraintes sur la suite Z dans un souci de généralité.

Une première hypothèse nécessaire pour montrer la convergence de $\varepsilon w_\varepsilon(./\varepsilon)$ est liée à l'existence d'une moyenne pour Z et on suppose qu'il existe $\langle Z\rangle \in \mathbb{R}^d$ telle que

$$\forall R>0, x_0\in\mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^d}{|B_R|} \sum_{k\in B_R(x_0)/\varepsilon} Z_k = \langle Z\rangle. \tag{A1}$$

Nous montrons la nécessité d'une telle hypothèse à travers un exemple monodimensionnel dans le **Chapitre** 4. Par ailleurs, la convergence faible de $|\nabla w_\varepsilon(./\varepsilon)|^2$ est liée aux corrélations d'ordre deux de Z. Ainsi, un premier ensemble d'hypothèses lié aux propriétés de la série (1.36) concerne les auto-corrélations de Z. En notant $\overline{Z}_k := Z_k - \langle Z\rangle$, on suppose l'existence d'une famille de constantes $\mathcal{C}_{l,i,j}$ pour tout $l\in\mathbb{Z}^d$ et $i,j\in\{1,...,d\}$, telle que

$$\bullet\ \forall R>0, x_0\in\mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^d}{|B_R|} \sum_{k\in B_R(x_0)/\varepsilon} (\overline{Z}_k)_i(\overline{Z}_k)_j = \mathcal{C}_{l,i,j}. \tag{A2.a}$$

- $\exists \delta : \mathbb{R}^+ \to \mathbb{R}^+$, $\forall i, j \in \{1, ..., d\}$, $\forall R > 0, \forall x_0 \in \mathbb{R}^d, \exists\, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\sup_{|l| \leq \frac{1}{\delta(\varepsilon)}} \left| \frac{\varepsilon^d}{|B_R|} \sum_{k \in B_R(x_0)/\varepsilon} (\overline{Z}_k)_i (\overline{Z}_{k+l})_j - \mathcal{C}_{l,i,j} \right| \leq \gamma(\varepsilon) \text{ et } \left\{ \begin{array}{l} \lim\limits_{\varepsilon \to 0} \gamma(\varepsilon) |\ln(\varepsilon)| = 0, \\ \lim\limits_{\varepsilon \to 0} \varepsilon^{-1} \delta(\varepsilon) = 0. \end{array} \right. \tag{A2.b}$$

- $\forall i, j \in \{1, ...d\}$,

$$x \mapsto \sum_{|l| \leq L} \mathcal{C}_{l,i,j} \left(\partial_i \partial_j G * \varphi \right) (x - l) \text{ converge dans } L^1_{loc}(\mathbb{R}^d) \text{ quand } L \to +\infty. \tag{A2.c}$$

Ici le taux de convergence $|\ln(\varepsilon)|$ qui apparaît dans l'hypothèse (A2.b) est relié au taux de décroissance critique $|k|^{-d}$ induit par les dérivées secondes de la fonction de Green G.

Ces hypothèses (A2.a) (A2.b) et (A2.c), toutes exactement quadratiques, sont cependant seulement suffisantes pour étudier la partie du correcteur issue de la somme (1.36) qui est linéaire par rapport à Z. Les non-linéarités par rapport à Z qui apparaissent dans (1.31) et qui sont retranscrites dans le reste du développement de Taylor de φ dans (1.35) nécessitent d'imposer une hypothèse plus forte sur les corrélations d'ordre deux de Z :

$$\begin{gathered} \forall F \in \mathcal{C}^0(\mathbb{R}^d \times \mathbb{R}^d),\ \forall l \in \mathbb{Z}^d,\ \exists C_{F,l} \in \mathbb{R},\ \forall R > 0, \forall x_0 \in \mathbb{R}^d, \\ \lim_{\varepsilon \to 0} \frac{\varepsilon^d}{|B_R|} \sum_{k \in B_R(x_0)/\varepsilon} F(Z_k, Z_{k+l}) = C_{F,l}. \end{gathered} \tag{A3}$$

Dans le **Chapitre** 4, nous donnons plusieurs exemples de suites Z qui vérifient les hypothèses (A1) à (A3). Ces suites appartiennent aussi bien au cadre de perturbations locales $\left(\lim\limits_{|k| \to \infty} Z_k = 0 \right)$, qu'au cadre de perturbations non-locales $\left(\lim\limits_{|k| \to \infty} Z_k \neq 0 \right)$. Notons que, à la différence des études présentées dans les chapitres précédents dans le contexte du problème de diffusion (1.1), notre cadre permet de couvrir des cas de perturbations localisées pour des suites Z_k à décroissance lente (typiquement où Z_k n'appartient à aucun $\left(l^p(\mathbb{Z}^d)\right)^d$), ou encore des perturbations complètement non localisées qui ne disparaissent d'aucune manière à l'infini.

Résultats principaux. Le résultat principal concernant l'existence d'un correcteur dans ce contexte est alors donné par le théorème suivant :

Théorème 1.7. *On suppose que $d \geq 2$ et que V est un potentiel de la forme (1.31). On suppose également (1.30) et que $Z \in \left(l^\infty(\mathbb{Z}^d)\right)^d$ satisfait les hypothèses (A1), (A2.a), (A2.b), (A2.c) et (A3). Alors, pour tout $R > 0$ et tout $\varepsilon > 0$, il existe $W_{\varepsilon,R} \in L^1_{loc}(\mathbb{R}^d)$ solution de*

$$\Delta W_{\varepsilon,R} = V \quad \textit{sur } B_{R/\varepsilon}, \tag{1.38}$$

telle que $\left(\nabla W_{\varepsilon,R}(./\varepsilon)\right)_{\varepsilon>0}$ est bornée dans $\left(L^p(B_R)\right)^d$ pour tout $p \in [1, +\infty[$ et

$$\left\{ \begin{array}{l} \nabla W_{\varepsilon,R}(./\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} 0 \quad \textit{faiblement dans } L^p(B_R),\ \forall p \in [1, +\infty[, \\ \varepsilon W_{\varepsilon,R}(./\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} 0 \quad \textit{fortement dans } L^\infty(B_R), \\ \exists \mathcal{M} \in \mathbb{R}, \quad |\nabla W_{\varepsilon,R}|^2(./\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} \mathcal{M} \quad \textit{faiblement dans } L^p(B_R),\ \forall p \in [1, +\infty[. \end{array} \right.$$

Ici, on voit alors la différence principale avec le cas périodique : le correcteur obtenu est "ε-dépendant". Dans le **Chapitre** 4, nous montrons effectivement qu'il est de la forme

$$W_{\varepsilon,R} = w - x. \fint_{B_{4R}} \nabla w(./\varepsilon),$$

où w est une solution particulière de (1.38) avec un gradient dans $\left(BMO(\mathbb{R}^d)\right)^d$.

L'étude de la convergence de u^ε est alors réalisée en deux étapes. Premièrement, le correcteur donné par le Théorème 1.7 permet de montrer la convergence de la première valeur propre λ_1^ε de l'opérateur $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon) + \nu$ avec conditions au bord homogènes de Dirichlet $\mu_1 - \mathcal{M} + \nu$, où μ_1 est la première valeur propre de l'opérateur -Δ avec les mêmes conditions au bord. On peut ainsi établir que le problème (1.7) est bien posé lorsque

$$\mu_1 - \mathcal{M} + \nu > 0, \tag{1.39}$$

et lorsque ε est suffisamment petit. Sous l'hypothèse suffisante (1.39), on peut alors montrer la convergence forte de u^ε vers u^* dans $L^2(\Omega)$, solution de

$$\begin{cases} -\Delta u^* - \mathcal{M}u^* + \nu u^* = f & \text{sur } \Omega, \\ u^* = 0 & \text{sur } \partial\Omega, \end{cases}$$

et établir le résultat suivant :

Théorème 1.8. *Sous les hypothèses du Théorème 1.7, on définit $W_{\varepsilon,\Omega} := W_{\varepsilon,R}$ où $R = Diam(\Omega)$. On suppose également que l'hypothèse (1.39) est vérifiée. Alors, pour ε suffisamment petit, il existe une unique solution $u^\varepsilon \in H_0^1(\Omega)$ de (1.7). De plus la suite*

$$R^\varepsilon := u^\varepsilon - u^* - \varepsilon u^* W_{\varepsilon,\Omega}(./\varepsilon)$$

converge fortement vers 0 dans $H^1(\Omega)$ quand ε tend vers 0.

On retrouve alors un résultat similaire à celui présenté en Section 1.2.2 et, aux technicalités liées aux propriétés du correcteur donné par le Théorème 1.7 près, la preuve du Théorème 1.8 s'appuie sur les méthodes introduites dans [18, Chapitre 1, Section 12] pour le cadre périodique. Dans un second temps, en suivant une méthode de [94] également introduite pour le cadre périodique, on utilise le résultat de convergence de l'opérateur $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon) + \nu$ sous l'hypothèse (1.39) afin d'établir la convergence de toutes ses valeurs propres vers celles de l'opérateur homogénéisé. Cela nous permet d'établir la généralisation suivante du Théorème 1.8 :

Théorème 1.9. *Sous les hypothèses du Théorème 1.7, on suppose que $\mu_l - \mathcal{M} + \nu \neq 0$ pour tout $l \in \mathbb{N}^*$, où μ_l est la l-ième valeur propre (en comptant les multiplicités) de -Δ sur Ω avec conditions au bord homogènes de Dirichlet. Alors les conclusions du Théorème 1.8 restent vraies.*

Dans ce travail nous n'obtenons pas de taux de convergence de la suite R^ε en norme H^1. Les preuves que nous proposons pour démontrer les Théorèmes 1.8 et 1.9 nous permettent d'intuiter que la vitesse de convergence de ∇R^ε est reliée à la vitesse de convergence de $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ vers 0 dans $L^\infty(\Omega)$ ainsi que, dans un certain sens, à la vitesse de convergence de $|\nabla W_{\varepsilon,\Omega}|^2(./\varepsilon)$ vers $\mathcal{M}$. Cependant, ces vitesses de convergence sont inconnues dans notre cadre relativement général donné par (1.31) et par les hypothèses (A1)-(A2.a)-(A2.b)-(A2.c)-(A3). Des cas monodimensionnels simples montrent notamment qu'il est nécessaire d'imposer des contraintes supplémentaires sur Z_k (taux de convergence des moyennes de Cesàro notamment) ou sur φ pour estimer la vitesse de décroissance de $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ vers 0.

1.5 Perspectives

Durant cette thèse, différents problèmes d'homogénéisation elliptique linéaire pour des géométries périodiques perturbées par des défauts ont été abordés. Plusieurs approches intéressantes pourraient alors être considérées dans la continuation de ce travail.

Perturbation de la géométrie aléatoire stationnaire. Une première extension naturelle des modèles de perturbation de la géométrie périodique pourrait consister à regarder des modèles de perturbation de géométrie aléatoire stationnaire. En notant $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé et $(\tau_k)_{k\in\mathbb{Z}^d}$ une action du groupe $(\mathbb{Z}^d, +)$ sur Ω préservant la mesure $\mathbb{P}$ et ergodique, on pourrait alors considérer le problème de diffusion (1.1) pour des coefficients aléatoires stationnaires perturbés. Dans l'esprit des modèles présentés dans cette thèse, on pense dans un premier temps à des coefficients de la forme

$$a(.,\omega) := a_s(.,\omega) + \tilde{a}(.,\omega),$$

où a_s est stationnaire dans le sens $a_s(x+k,\omega) = a_s(x,\tau_k\omega)$, presque partout en x et presque sûrement, et $\tilde{a}$ est une perturbation à définir. Cette perturbation pourrait appartenir à un espace du type $L^p(\mathbb{R}^d, L^p(\Omega))$ par exemple. Plus généralement, en s'inspirant du cadre déterministe des coefficients presque invariant par translation à l'infini introduit dans le **Chapitre** 3, on pourrait également considérer des coefficients tels que

$$\delta a \in L^p(\mathbb{R}^d, L^p(\Omega)), \quad p < d,$$

où

$$\delta a(x,\omega) := (\delta_i a(x,\omega))_{i\in\{1,\dots,d\}} := (f(x+e_i, \tau_{-e_i}\omega) - a(x,\omega))_{i\in\{1,\dots d\}}\,.$$

Dans de tels cadres, les méthodes utilisées pour les problèmes déterministes (les méthodes de type concentration-compacité introduites dans [27] en particulier), ne permettent plus de conclure en raison d'un certain manque de propriétés de compacité de l'espace probabilisé. Il serait alors nécessaire de développer des techniques adaptées pour résoudre les équations du correcteur.

Approche numérique. Bien que cette thèse n'ait pas abordé les aspects computationnels, on pourrait aussi s'intéresser à l'étude numérique de nos modèles. Deux difficultés apparaissent dans nos cadres non-périodiques :

a) Les équations du correcteur (1.19) et (1.20) sont posées sur des ouverts non bornés.
b) Les coefficients homogénéisés a^* et V^* s'expriment à partir des correcteurs comme des moyennes au sens de la limite sur des grands volumes :

$$(a^*)_{i,j} = \lim_{R\to\infty} \frac{1}{R^d} \int_{[-R/2,R/2]^d} e_i^T a(y) \left(\nabla w_{e_j}(y) + e_j\right) dy,$$

ou

$$V^* = -\lim_{R\to\infty} \frac{1}{R^d} \int_{[-R/2,R/2]^d} |\nabla w(y)|^2 dy.$$

Il convient donc d'introduire des troncatures adaptées pour approcher numériquement les correcteurs ainsi que les solutions homogénéisées. Il s'agirait par exemple d'utiliser la structure particulière des coefficients pour optimiser ces approches numériques.

Homogénéisation du problème de diffusion dans un cadre plus général. Pour conclure nous pouvons rapidement considérer un dernier problème d'une difficulté a priori bien supérieure aux problèmes étudiés dans cette thèse et qui n'a pas pu être résolu ici. Ce problème, déjà évoqué par Blanc, Le Bris et Lions durant l'étude de leurs premiers problèmes d'homogénéisation pour des géométries périodiques perturbées, consisterait à établir l'homogénéisation de l'équation de diffusion (1.1) dans un contexte non-périodique général similaire à celui introduit dans le **Chapitre** 4 pour l'équation de Schrödinger stationnaire (1.7). On pourrait par exemple considérer des coefficients de la forme

$$a = \sum_{k \in \mathbb{Z}^d} \varphi(. - k - Z_k),$$

ou d'une forme similaire, pour une suite Z_k qui ne disparaît d'aucune manière à l'infini. L'étude du problème de diffusion (1.1) et la résolution de l'équation du correcteur (1.19) dans ce cadre sont cependant nettement plus ardues en comparaison avec l'étude du problème de Schrödinger stationnaire. Ici, la suite de solutions u^ε et le coefficient a sont notamment liés de manière *complètement non-linéaire*. D'un certain point de vue, le problème de Schrödinger (1.7) est une version *bilinéaire* (V multiplie l'inconnue u^ε) très simplifiée de (1.1). Une des difficultés serait notamment de déterminer s'il existe des conditions sur la suite Z_k permettant d'établir l'existence d'un correcteur solution de (1.19) adapté au problème d'homogénéisation.

Chapitre 2

Problème d'homogénéisation périodique avec défauts rares à l'infini

Ce chapitre reproduit un article publié dans *Networks and Heterogeneous Media* [Gou22a].

On s'intéresse ici à l'homogénéisation de l'équation de diffusion $-\operatorname{div}(a(./\varepsilon)\nabla u^\varepsilon) = f$ quand le coefficient a est une perturbation non-locale d'un coefficient périodique. La perturbation ne disparaît jamais à l'infini mais devient rare dans un sens précisé dans le texte. On prouve l'existence d'un correcteur, on identifie la limite homogénéisée et on étudie le taux de convergence de u^ε vers sa limite homogénéisée.

Sommaire

A periodic homogenization problem with defects rare at infinity

2.1 Introduction

2.1.1 Motivation

The purpose of this paper is to address the homogenization problem for a second order elliptic equation in divergence form with a certain class of oscillating coefficients :

$$\begin{cases} -\operatorname{div}(a(x/\varepsilon)\nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon(x) = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) sufficiently regular (the regularity will be made precise later on) and f is a function in $L^2(\Omega)$. The class of (matrix-valued) coefficients a considered is that of the form

$$a_{per} + \tilde{a}, \tag{2.2}$$

which describes a periodic geometry encoded in the coefficient a_{per} and perturbed by a coefficient $\tilde{a}$ that represents a non-local perturbation (a "defect") that, although it does not vanish at infinity, becomes rare at infinity. More specifically, we consider coefficients $\tilde{a}$ that locally behave like $L^2(\mathbb{R}^d)$ functions in the neighborhood of a set of points localized at an exponentially increasing distance from the origin. Formally, the coefficient $\tilde{a}$ is an infinite sum of localized perturbations, increasingly distant from one another. A prototypical one-dimensional example of such a defect reads as $\displaystyle\sum_{k\in\mathbb{Z}} \phi(x - \operatorname{sign}(k)2^{|k|})$ for some fixed $\phi \in \mathcal{D}(\mathbb{R})$, where $|k|$ denotes the absolute value of k and $\operatorname{sign}(k)$ denotes its sign. It is depicted in Figure 2.1

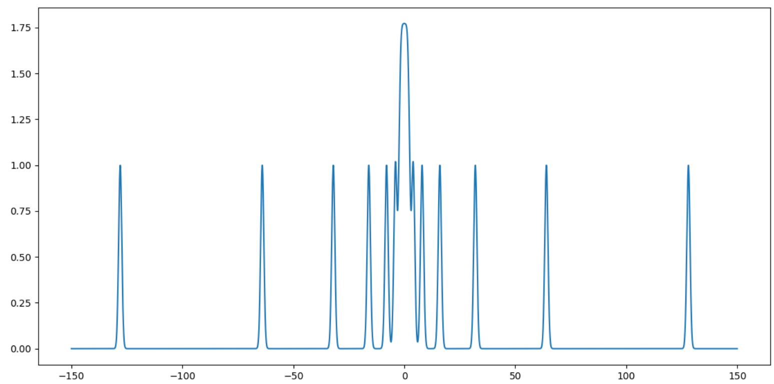

FIGURE 2.1 – Prototype perturbation in dimension $d = 1$.

Homogenization theory for the unperturbed periodic problem (2.1)-(2.2) when $\tilde{a} = 0$ is well-known (see for instance [18, 59]). The solution u^ε converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to u^*, solution to the homogenized problem :

$$\begin{cases} -\operatorname{div}(a^*\nabla u^*) = f & \text{in } \Omega, \\ u^*(x) = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.3}$$

where a^* is a constant matrix. The convergence in the $H^1(\Omega)$ norm is obtained upon introducing a corrector $w_{per,p}$ defined for all p in $\mathbb{R}^d$ as the periodic solution (unique up to the addition of a constant) to :

$$-\operatorname{div}(a_{per}(\nabla w_{per,p} + p)) = 0 \quad \text{in } \mathbb{R}^d. \tag{2.4}$$

This corrector allows to both make explicit the homogenized coefficient

$$(a^*)_{i,j} = \int_Q e_i^T a_{per}(y) \left(e_j + \nabla w_{per,e_j}\right) dy, \tag{2.5}$$

(where Q denotes the d-dimensional unit cube, (e_i) the canonical basis of $\mathbb{R}^d$) and define the approximation

$$u^{\varepsilon,1} = u^*(.) + \varepsilon \sum_{i=1}^{d} \partial_i u^*(.) w_{per,e_i}(./\varepsilon), \tag{2.6}$$

such that $u^{\varepsilon,1} - u^\varepsilon$ strongly converges to 0 in $H^1(\Omega)$ (see [3] for more details). In addition, convergence rates can be made precise, with in particular :

$$\begin{aligned} &\|\nabla u^\varepsilon - \nabla u^{\varepsilon,1}\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\|f\|_{L^2(\Omega)}, \\ &\|\nabla u^\varepsilon - \nabla u^{\varepsilon,1}\|_{L^2(\Omega_1)} \leq C\varepsilon\|f\|_{L^2(\Omega)} \quad \text{for every } \Omega_1 \subset\subset \Omega, \end{aligned}$$

for some constants independent of f.

Our purpose here is to extend the above results to the setting of the *perturbed* problem (2.1)-(2.2). The main difficulty is that the corrector equation

$$-\operatorname{div}\left((a_{per} + \tilde{a})\left(\nabla w_p + p\right)\right) = 0,$$

(formally obtained by a two-scale expansion (see again [3] for the details) and analogous to (2.4) in the periodic case) is defined on the whole space $\mathbb{R}^d$ and cannot be reduced to an equation posed on a bounded domain, as is the case in periodic context in particular. This prevents us from using classical techniques. The present work follows up on some previous works [20, 27, 26, 25] where the authors have developed an homogenization theory in the case where $\tilde{a} \in L^p(\mathbb{R}^d)$ for $p \in]1, \infty[$. The existence and uniqueness (again up to an additive constant) of a corrector, the gradient of which shares the same structure "periodic + L^p" as the coefficient a, is established. Convergence rates are also made precise. Similarly to [20, 27, 26, 25], we aim to show here, in a context of a perturbation rare at infinity, there also exists a corrector (unique up to the addition of a constant), and such that its gradient has the structure (2.2) of the diffusion coefficient : it can be decomposed as a sum of the gradient of a periodic corrector and a gradient that becomes rare at infinity (in a sense similar to that for $\tilde{a}$, and made precise below).

2.1.2 Functional setting

We introduce here a suitable functional setting to describe the class of defects we consider.

In order to formalize our mathematical setting, we first define a generic infinite discrete set of points denoted by $\mathcal{G} = \{x_p\}_{p\in\mathbb{Z}^d}$. In the sequel, each point x_p actually models the presence of a defect in the periodic background modeled by a_{per} and our aim is to ensure these defects are sufficiently rare at infinity.

We next introduce the Voronoi diagram associated with our set of points. For $x_p \in \mathcal{G}$, we denote by V_{x_p} the Voronoi cell containing the point x_p and defined by

$$V_{x_p} = \bigcap_{x_q\in\mathcal{G}\setminus\{x_p\}} \left\{x \in \mathbb{R}^d \middle| |x - x_p| \leq |x - x_q|\right\}. \tag{2.7}$$

We now consider three geometric assumptions that ensure an appropriate distribution of the points in the space. The set $\mathcal{G}$ is required to satisfy the following three conditions :

$$\forall x_p \in \mathcal{G}, \quad \left|V_{x_p}\right| < \infty, \tag{H1}$$

$$\exists C_1 > 0,\ C_2 > 0,\ \forall x_p \in \mathcal{G}, \quad C_1 \leq \frac{1 + |x_p|}{D\left(x_p, \mathcal{G} \setminus \{x_p\}\right)} \leq C_2, \tag{H2}$$

$$\exists C_3 > 0, \forall x_p \in \mathcal{G}, \quad \frac{Diam\left(V_{x_p}\right)}{D\left(x_p, \mathcal{G} \setminus \{x_p\}\right)} \leq C_3, \tag{H3}$$

where $|A|$ denotes the volume of a subset $A \subset \mathbb{R}^d$, $Diam(A)$ the diameter of A and $D(.,.)$ the euclidean distance.

Assumption (H2) is the most significant assumption in our case since it implies that the points are increasingly distant from one another far from the origin. It in particular implies

$$\lim_{x_p\in\mathcal{G},\ |x_p|\to\infty} D\left(x_p, \mathcal{G} \setminus \{x_p\}\right) = +\infty.$$

More precisely, it ensures the distance between a point x_p and the others has the same growth as the norm $|x_p|$ and, therefore, requires the Voronoi cell V_{x_p} (which contains a ball of radius $\dfrac{D\left(x_p, \mathcal{G} \setminus \{x_p\}\right)}{2}$ as a consequence of its definition) to be sufficiently large. In particular, this assumption ensures that the defects modeled by the points x_p are sufficiently rare at infinity. In particular, we show in Section 2.2 that Assumption (H2) ensures that the number of points x_p contained in a ball B_R of radius $R > 0$ is bounded by the logarithm of R. This property is an essential element for the methods used in the proof of this article.

In contrast to (H2), Assumptions (H1) and (H3) are only technical and not very restrictive. They limit the size of the Voronoi cells. In the case where these assumptions are not satisfied, our main results of Theorems 2.1 and 2.2 stated below still hold. Their proofs have to be adapted, upon splitting the Voronoi cells in several subsets such that each subset satisfies geometric constraints similar to (H1), (H2) and (H3). To some extent, our assumptions (H1) and (H3) ensure we consider the worst case scenario, where the set $\mathcal{G}$ contains as many points as possible while satisfying (H2).

In addition, although we establish in Section 2.2 all the geometric properties satisfied by the Voronoi cells V_{x_p} which are required in our approach to study the homogenization problem (2.1) with the whole generality of Assumptions (H1), (H2) and (H3), we choose, for the sake of illustration and for pedagogic purposes, to work with a particular set of points (for which the coordinates are powers of 2) and to establish our main results of homogenization in this specific setting. There are, of course, many alternative sets that satisfy (H1), (H2) and (H3) but our specific choice is convenient. To define our specific set of points, we first introduce a constant $C_0 > 1$ and a set of indices $\mathcal{P}_{C_0}$ defined by :

$$\mathcal{P}_{C_0} = \left\{ p \in \mathbb{Z}^d \;\middle|\; \max_{p_i \neq 0} \{|p_i|\} \leq C_0 + \min_{p_i \neq 0} \{|p_i|\} \right\}. \tag{2.8}$$

Our specific set of points (see Figure 2.2) is then defined by :

$$\mathcal{G}_{C_0} = \left\{ x_p = \left(\operatorname{sign}(p_i) 2^{|p_i|}\right)_{i \in \{1,...d\}} \;\middle|\; (p_1, ..., p_d) \in \mathcal{P}_{C_0} \right\}. \tag{2.9}$$

We use here the convention $\operatorname{sign}(0) = 0$. The set of indices (2.8) contains only the points with integer coordinates on the axes $\operatorname{Span}(e_i)$ and the points close to each diagonal of the form $\operatorname{Span}(e_{i_1} + ... + e_{i_k})$ for $k \in \{2, .., d\}$ and $(i_1, ..., i_k) \in \{1, ..., d\}^k$. In this way, the points of $\mathcal{G}_{C_0}$ are exponentially distant from each other with respect to the norm of p. In Section 2.2, we show that the set $\mathcal{G}_{C_0}$ defined by (2.9) indeed satisfies Assumptions (H1), (H2) and (H3).

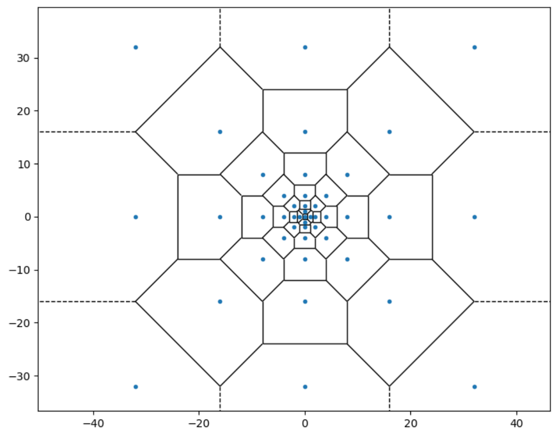

FIGURE 2.2 – Example of points in ambient dimension 2 that satisfy our assumptions along with their associated Voronoi diagram.

In the sequel, we use the following notation :

- B_R : the ball of radius $R > 0$ centered at the origin; $B_R(x)$: the ball of radius $R > 0$ and center $x \in \mathbb{R}^d$; $A_{R,R'}$: the set $B_R \setminus B_{R'}$ for $R > R' > 0$.
- $Q_R(x)$: the set $\left\{ y \in \mathbb{R}^d \;\middle|\; \max_i |y_i - x_i| \leq R \right\}$ for $R > 0$ and $x \in \mathbb{R}^d$; Q_R : the set $Q_R(0)$.
- $\sharp B$: the cardinality of a discrete set B.
- 2^p : the point $x_p \in \mathcal{G}_{C_0}$ for $p \in \mathcal{P}_{C_0}$; τ_p : the translation τ_{2^p} where $\tau_x f = f(. + x)$; V_p : the Voronoi cell V_{2^p}.

- $|p|$: the norm defined by $\max_{i\in\{1,\ldots,d\}} |p_i|$.

In addition, for a normed vector space $(X, \|.\|_X)$ and a matrix-valued function $f \in X^n$, $n \in \mathbb{N}$, we use the notation $\|f\|_X \equiv \|f\|_{X^n}$ when the context is clear.

We associate to (2.8)-(2.9) the following functional space :

$$\mathcal{B}^2(\mathbb{R}^d) = \left\{ f \in L^2_{unif}(\mathbb{R}^d) \;\middle|\; \exists f_\infty \in L^2(\mathbb{R}^d),\ \lim_{|p|\to\infty} \int_{V_p} |f(x) - \tau_{-p} f_\infty(x)|^2 dx = 0 \right\}, \quad (2.10)$$

equipped with the norm

$$\|f\|_{\mathcal{B}^2(\mathbb{R}^d)} = \|f_\infty\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2_{unif}(\mathbb{R}^d)} + \sup_{p\in\mathcal{P}_{C_0}} \|f - \tau_{-p} f_\infty\|_{L^2(V_p)}. \quad (2.11)$$

In (2.10), (2.11) we have denoted by :

$$L^2_{unif}(\mathbb{R}^d) = \left\{ f \in L^2_{loc}(\mathbb{R}^d),\ \sup_{x\in\mathbb{R}^d} \|f\|_{L^2(B_1(x))} < \infty \right\},$$

and

$$\|f\|_{L^2_{unif}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \|f\|_{L^2(B_1(x))}.$$

Intuitively, a function in $\mathcal{B}^2(\mathbb{R}^d)$ behaves, locally at the "vicinity" of each point x_p, as a fixed L^2 function truncated over the domain V_p. We show several properties of the functional space $\mathcal{B}^2(\mathbb{R}^d)$ in Section 2.3. As specified above, in the sequel we focus on homogenization problem (2.1) with non-local perturbations induced by the particular setting (2.8)-(2.9)-(2.10). We note, however, that the definition of $\mathcal{B}^2(\mathbb{R}^d)$ can be naturally adapted to the generality of Assumptions (H1)-(H2)-(H3) and the homogenization results established in the present study can of course be extended to this general setting. More precisely, most of our proofs only involve the general structure of the functional space $\mathcal{B}^2(\mathbb{R}^d)$ and several geometric properties related to the rarity of the points x_p that are established under our general assumptions in Section 2.2. The specific geometric properties of the set (2.9) are only explicitly used to study the equation $-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(f)$ when $f \in \mathcal{B}^2(\mathbb{R}^d)$, particularly to establish the convergence of several sums involving the asymptotic behavior of the Green function of the Laplacian operator (see Lemmas 2.3, 2.4 and 2.8). However, these results are not specific to the set (2.9). We explain how to adapt their proofs under our general assumptions in Remarks 2.4 and 2.7

2.1.3 Main results

We henceforth assume that the ambient dimension d is equal to or larger than 3. The one-dimensional and two-dimensional contexts are specific. Some results or proofs must be adapted in these particular cases but we will not proceed in that direction in all details. This is due to the asymptotic behavior of the Green function of the Laplacian operator in these two dimensions. In these two particular cases, we claim that it is still possible to show the existence of the corrector defined by Theorem 2.1 below. However, the method used in Lemmas 2.3 and 2.4, both useful for the proof of Theorem 2.1, need to be adapted. The one-dimensional context can be addressed easily because the solution to (2.14) is explicit. The two-dimensional case requires more work. We explain how to adapt our proof in Remark 2.5. In contrast, in dimensions $d = 1$ and $d = 2$, the convergence rates of Theorem 2.2 no longer hold. Indeed, the corrector w_p is then not necessarily bounded (see Lemma 2.8 for details). We are only able to prove weaker results in these cases. Additional details about these cases may be found in Remarks 2.5, 2.8, and 2.9.

For $\alpha \in]0,1[$, we denote by $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ the space of *uniformly* Hölder continuous and bounded functions with exponent α, that is :

$$\mathcal{C}^{0,\alpha}(\mathbb{R}^d) = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We consider a matrix-valued coefficient of the form (2.2) with $a_{per} \in L^2_{per}(\mathbb{R}^d)^{d\times d}$ and $\tilde{a} \in \mathcal{B}^2(\mathbb{R}^d)^{d\times d}$. We denote by $\tilde{a}_\infty$ the matrix-valued limit L^2-function associated with $\tilde{a}$, where each coefficient $(\tilde{a}_\infty)_{i,j}$ is the limit L^2-function associated with $(\tilde{a})_{i,j} \in \mathcal{B}^2(\mathbb{R}^d)$ and defined in (2.10). We assume that a_{per}, $\tilde{a}$ and $\tilde{a}_\infty$ satisfy :

$$\exists \lambda > 0 \text{ such that for all } x, \xi \in \mathbb{R}^d \quad \lambda|\xi|^2 \leq \langle a(x)\xi, \xi\rangle, \quad \lambda|\xi|^2 \leq \langle a_{per}(x)\xi,\xi\rangle, \tag{2.12}$$

and

$$a_{per},\ \tilde{a},\ \tilde{a}_\infty \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)^{d\times d}, \qquad \alpha \in]0,1[. \tag{2.13}$$

The coercivity (2.12) and the L^∞ bound on a ensure that the sequence of solutions $(u^\varepsilon)_{\varepsilon>0}$ to (2.1) converges in $weak - H^1(\Omega)$ and $strong - L^2(\Omega)$ up to an extraction when $\varepsilon \to 0$. Classical results of homogenization show the limit u^* is a solution to a diffusion equation of the form (2.3) for some matrix-valued coefficient a^* to be determined. The questions that we examine in this paper are : What is the diffusion coefficient a^* of the homogenized equation ? Is it possible to define an approximate sequence of solutions $u^{\varepsilon,1}$ as in (2.6) ? For which topologies does this approximation correctly describe the behavior of u^ε ? What is the convergence rate ?

In answer to our first question, we prove in Proposition 2.13 that the homogenized coefficient a^* is constant and is the same as in the periodic case. This result is a direct consequence of Proposition 2.10 which ensures that the perturbations of $\mathcal{B}^2(\mathbb{R}^d)$ have a zero average in a strong sens. Consequently, our perturbations are "small" at the macroscopic scale and do not affect the homogenization that occurs in the periodic case associated with the periodic coefficient a_{per}. In reply to the other questions, our main results are contained in the following two theorems :

Theorem 2.1. *For every $p \in \mathbb{R}^d$, there exists a unique (up to an additive constant) function $w_p \in H^1_{loc}(\mathbb{R}^d)$ such that $\nabla w_p \in \left(L^2_{per}(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d)\right)^d \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)^d$, solution to :*

$$\begin{cases} -\operatorname{div}((a_{per}+\tilde{a})(p+\nabla w_p)) = 0 \quad \text{in } \mathbb{R}^d, \\[2mm] \displaystyle\lim_{|x|\to\infty} \frac{|w_p(x)|}{1+|x|} = 0. \end{cases} \tag{2.14}$$

Theorem 2.2. *Assume Ω is a $C^{2,1}$-bounded domain. Let $\Omega_1 \subset\subset \Omega$. We define $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon)$ where w_{e_i} is defined by Theorem 2.1 for $p = e_i$ and u^* is the solution to (2.3). Then $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ satisfies the following estimates :*

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \varepsilon \|f\|_{L^2(\Omega)}, \tag{2.15}$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \varepsilon \|f\|_{L^2(\Omega)}, \tag{2.16}$$

where C_1 and C_2 are two positive constants independent of f and ε.

Our article is organized as follows. In Section 2.2 we prove some geometric properties satisfied by our set of points $\mathcal{G}_{C_0}$, in particular we show that it satisfies Assumptions (H1), (H2) and (H3). In section 2.3 we study the properties of $\mathcal{B}^2(\mathbb{R}^d)$ and its elements. In Section 2.4 we prove Theorem 2.1. Finally, in Section 2.5 we obtain the expected homogenization convergences stated in Theorem 2.2. We conclude this introduction section with some comments.

2.1.4 Extensions and perspectives

A first possible extension of the above results, which we study in Appendix 2.6, consists in considering the functional spaces $\mathcal{B}^r$ for $r \neq 2$, $1 < r < \infty$, defined similarly to $\mathcal{B}^2$, but using the L^r topology. As in the study of the $L^r(\mathbb{R}^d)$ defects in [27, 26], we show some modifications for the convergence rates of Theorem 2.2 depending upon the value of r and the ambient dimension d. Indeed, in this case, some results related to the strict sub-linearity of the corrector allow us to show that the convergence rate of R^ε is $\varepsilon^{\frac{d}{r}}|\log(\varepsilon)|^{\frac{1}{r}}$ if $r > d$ and ε else.

In addition, although we have not pursued in these directions, we believe it is possible to extend the above results in several other manners.

1) First, under additional assumptions satisfied by the function f, we expect the estimates of Theorem 2.2 to hold, with possibly different rates, in other norms than L^2 such as L^q, for $1 < q < \infty$ or $\mathcal{C}^{0,\alpha}$, for $\alpha \in]0,1[$. It seems that such questions could be addressed by adapting the proofs of Section 2.5 and consider the methods employed in [20] using the behavior of the Green function associated with problem (2.1).
2) We also believe that it is possible to show results analogous to that of Theorems 2.1 and 2.2 in the case of equations not in divergence form, instead of (2.1),

$$-a_{ij}\partial_{ij} u = f,$$

where a is a periodic coefficients perturbed by a defect in $\mathcal{B}^2(\mathbb{R}^d)$ of the form (2.2). One way to address this question could be to adapt the methods of [27, Section 3] in the case of local perturbations, that is, to show the existence of an invariant measure $m = m_{per} + \tilde{m}$ in $L^2_{per} + \mathcal{B}^2(\mathbb{R}^d)$ solution to :

$$-\partial_{i,j}\left(a_{i,j} m_{i,j}\right) = 0 \quad \text{in } \mathbb{R}^d,$$

such that $\inf m > 0$. Indeed, using the method presented in [12], this study could be then reduced to a problem of divergence form operator as soon as such a measure m exists and the results established in this article could allow to conclude.

3) In the same way, another possible generalization concerns advection-diffusion equation in the form :
$$-a_{ij}\partial_{ij}u + b_j\partial_j u = f \quad \text{in } \mathbb{R}^d,$$
where a and b are two periodic coefficients perturbed by a defect in $\mathcal{B}^2(\mathbb{R}^d)$. The method [28] is likely to be adapted to this case, showing the existence of an invariant measure m in $L^2_{per} + \mathcal{B}^2(\mathbb{R}^d)$ solution to
$$-\partial_i\left(\partial_j\left(a_{i,j}m_{i,j}\right) + b_i m_{i,j}\right) = 0 \quad \text{in } \mathbb{R}^d.$$

2.2 Geometric properties of the Voronoi cells

We start by studying the geometric properties of the Voronoi cells associated to every sets of points $\mathcal{G}$ satisfying the general Assumptions (H1), (H2) and (H3). In particular, we show these assumptions ensure the rarity of the points x_p in the space proving, in Proposition 2.3 and Corollary 2.1, that the number of points of $\mathcal{G}$ contained in a ball of radius $R > 0$ is bounded by the logarithm of R. In Propositions 2.2 and 2.4, we also show two technical properties regarding the size and the structure of the cells. All these properties are actually fundamental for the rest of our work since they allow us to prove several results regarding the existence and uniqueness of solutions to the class (2.31) of diffusion equations $-\operatorname{div}(a\nabla u) = \operatorname{div}(f)$ studied in Section 2.4. In particular, as we shall see in the proof of Lemma 2.3, we use these geometric properties to bound several integrals in order to define a solution to equation (2.35), that is (2.31) with $a = a_{per}$, using the associated Green function. To conclude this section, we also show that our specific set of points $\mathcal{G}_{C_0}$, defined by (2.9), satisfies (H1), (H2) and (H3).

2.2.1 General properties

In this subsection only, we proceed with the whole generality of Assumptions (H1), (H2) and (H3) and we introduce several useful geometric properties satisfied by every sets of points $\mathcal{G}$ satisfying these assumptions. These properties relate to the size of the Voronoi cells, their volume and their distribution in the space $\mathbb{R}^d$.

To start with, we show two properties regarding the volume of the Voronoi cells.

Proposition 2.1. *There exist $C_1 > 0$ and $C_2 > 0$ such that for every $x \in \mathcal{G}$, we have the following bounds :*
$$C_1|x|^d \leq |V_x| \leq C_2|x|^d.$$

Proof. For every $x \in \mathcal{G}$, using the definition of the Voronoï diagram, we have the following inclusion :
$$B_{D(x,\mathcal{G}\setminus\{x\})/2}(x) \subset V_x.$$
Therefore, there exists a constant $C(d) > 0$ such that :
$$C(d)D(x,\mathcal{G}\setminus\{x\})^d = \left|B_{D(x,\mathcal{G}\setminus\{x\})/2}(x)\right| \leq |V_x| \leq Diam(V_x)^d.$$
We conclude using (H2) and (H3). □

Proposition 2.2. *There exists a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ such that $(V_{x_n} - x_n)$ is an increasing sequence of sets and :*

$$\bigcup_{n\in\mathbb{N}} (V_{x_n} - x_n) = \mathbb{R}^d.$$

Proof. We consider a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ such that the sequence $|x_n|$ is increasing and $\lim\limits_{n\to\infty} |x_n| = \infty$ (such a choice is always possible according to Assumptions (H1) and (H2)). Since we have assumed that $\mathcal{G}$ satisfies (H2), there exists $C > 0$ such that for all $n \in \mathbb{N}$:

$$D(x_n, \mathcal{G} \setminus \{x_n\}) \geq C|x_n|.$$

Therefore, as a consequence of the definition of the Voronoi cells, the ball $B_{C|x_n|/2}(x_n)$ is included in V_{x_n} and, by translation, the ball $B_{C|x_n|/2}$ is included in $V_{x_n} - x_n$. Since $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence such that $\lim\limits_{n\to\infty} |x_n| = \infty$, we use (H1) and we obtain, up to an extraction, that V_{x_n} is included in $B_{C|x_{n+1}|/2}(x_n)$. Thus

$$\forall n \in \mathbb{N}, \quad V_{x_n} - x_n \subset B_{C|x_{n+1}|/2} \subset V_{x_{n+1}} - x_{n+1}.$$

The sequence $(V_{x_n} - x_n)$ is therefore an increasing sequence of sets and, in addition,

$$\mathbb{R}^d = \bigcup_{n\in\mathbb{N}} B_{C|x_n|/2} \subset \bigcup_{n\in\mathbb{N}} (V_{x_n} - x_n).$$

We directly deduce that $\mathbb{R}^d = \bigcup\limits_{n\in\mathbb{N}} (V_{x_n} - x_n)$. $\square$

The next results ensure a certain distribution of the Voronoi cells in the space. In particular, we prove that the number of cells contained in a ball of radius $R > 0$ increases at most as the logarithm of this radius. This property reflects the rarity of our points far from the origin and is essential in our approach.

Proposition 2.3. *There exists a constant $C(d) > 0$ that depends only of the ambient dimension d such that :*

$$\sharp \{x \in \mathcal{G} | x \in A_{2^n,2^{n+1}}\} \leq C(d).$$

Proof. Let $x \in \mathcal{G}$ such that $x \in A_{2^n,2^{n+1}}$. The definition of the Voronoi cells ensures that the distance $D(x, \partial V_x)$ is equal to $\frac{D(x,\mathcal{G}\setminus\{x\})}{2}$. Property (H2) gives the existence of a constant $C_1 > 0$ independent of x such that :

$$\frac{D(x, \mathcal{G} \setminus \{x\})}{2} \geq C_1 \frac{|x|}{2} \geq C_1 2^{n-1}.$$

Then, the ball $B_{C_1 2^{n-1}}(x)$ is contained in V_x, that is x is the only element of $\mathcal{G}$ in this ball. In addition, since $|x| \leq 2^{n+1}$, we obtain the following inclusion using a triangle inequality :

$$B_{C_1 2^{n-1}}(x) \subset B_{(C_1+4)2^{n-1}}.$$

Since this inclusion is valid for every $x \in \mathcal{G} \cap A_{2^n,2^{n+1}}$ we obtain :

$$\bigcup_{x\in\mathcal{G}\cap A_{2^n,2^{n+1}}} B_{C_1 2^{n-1}}(x) \subset B_{(C_1+4)2^{n-1}}.$$

Therefore, there exists $C_2(d) > 0$ such that :

$$\left| \bigcup_{x \in \mathcal{G} \cap A_{2^n, 2^{n+1}}} B_{C_1 2^{n-1}}(x) \right| \leq \left| B_{(C_1+4)2^{n-1}} \right| \leq C_2(d) 2^{d(n-1)}. \tag{2.17}$$

Next, we know that the Voronoi cells are disjoint and it follows that the collection of balls $(B_{C_1 2^{n-1}}(x))_{x \in \mathcal{G} \cap A_{2^n, 2^{n+1}}}$ is also disjoint. Thus, there exists $C_3(d) > 0$ such that :

$$\begin{aligned} \left| \bigcup_{x \in \mathcal{G} \cap A_{2^n, 2^{n+1}}} B_{C_1 2^{n-1}}(x) \right| &= \sharp \left\{ x \in \mathcal{G} | x \in A_{2^n, 2^{n+1}} \right\} \left| B_{C_1 2^{n-1}} \right| \\ &= \sharp \left\{ x \in \mathcal{G} | x \in A_{2^n, 2^{n+1}} \right\} C_3(d) 2^{d(n-1)}. \end{aligned} \tag{2.18}$$

With (2.17) and (2.18), we conclude that :

$$\sharp \left\{ x \in \mathcal{G} | x \in A_{2^n, 2^{n+1}} \right\} \leq \frac{C_2(d)}{C_3(d)}.$$

□

Corollary 2.1. *There exists $C > 0$ such that for every $R > 0$ and $x_0 \in \mathbb{R}^d$:*

$$\sharp \left\{ x \in \mathcal{G} | V_x \cap B_R(x_0) \neq \emptyset \right\} < C \log(R). \tag{2.19}$$

Proof. We start by proving the result if $R = 2^n$ for $n \in \mathbb{N}^*$. Without loss of generality, we can assume that n is sufficiently large to ensure there exists x in $\mathcal{G} \cap B_{2^n}(x_0)$. Using a triangle inequality, we remark that if $y \in B_{2^n}(x_0)$ we have :

$$|x - y| \leq |x - x_0| + |y - x_0| \leq 2^{n+1} \leq D\left(y, \mathbb{R}^d \setminus B_{2^{n+3}}(x_0)\right).$$

That is, if $\tilde{x} \in \mathcal{G}$ is such that $\tilde{x} \notin B_{2^{n+3}}(x_0)$, every point $y \in B_{2^n}(x_0)$ is closer to x than to $\tilde{x}$, that is $V_{\tilde{x}} \cap B_{2^n}(x_0) = \emptyset$. Therefore, we have

$$\sharp \left\{ x \in \mathcal{G} | V_x \cap B_{2^n}(x_0) \neq \emptyset \right\} \leq \sharp \left\{ x \in \mathcal{G} | x \in B_{2^{n+3}}(x_0) \right\}.$$

Next, if $|x_0| \leq 2^{n+4}$, we have $B_{2^{n+3}(x_0)} \subset B_{2^{2n+7}}$ and we use Proposition 2.3 to obtain the existence of a constant $C > 0$ independent of n such that :

$$\begin{aligned} \sharp \left\{ x \in \mathcal{G} | x \in B_{2^{n+3}}(x_0) \right\} &\leq \sharp \left\{ x \in \mathcal{G} | x \in B_{2^{2n+7}} \right\} \\ &= \sum_{k=0}^{2n+6} \sharp \left\{ x \in \mathcal{G} \middle| x \in A_{2^k, 2^{k+1}} \right\} + \sharp \left\{ x \in \mathcal{G} | x \in B_1 \right\} \\ &\leq Cn. \end{aligned}$$

If $|x_0| > 2^{n+4}$, we denote by $m \geq n + 4$ the unique integer such that $2^m < |x_0|$ and $|x_0| \leq 2^{m+1}$. In this case, we use a triangle inequality and we have

$$B_{2^{n+3}(x_0)} \subset A_{|x_0| + 2^{n+3}, |x_0| - 2^{n+3}} \subset A_{2^{m+2}, 2^{m-1}}.$$

Proposition 2.3 gives the existence of $C > 0$ independent of x_0 and n such that :

$$\begin{aligned}\sharp \left\{x \in \mathcal{G} | x \in B_{2^{n+3}}(x_0)\right\} &\leq \sharp \left\{x \in \mathcal{G} | x \in A_{2^{m+2},2^{m-1}}\right\} \\ &= \sum_{k=-1}^{1} \sharp \left\{x \in \mathcal{G} \middle| x \in A_{2^{m+k},2^{m+k+1}}\right\} \leq C.\end{aligned}$$

Finally, we have estimate (2.19) in the particular case $R = 2^n$.

Next, for any $R > 0$, we have :

$$R = 2^{\log_2(R)} \leq 2^{[\log_2(R)]+1},$$

where $[.]$ denotes the integer part. Thus, we obtain the following upper bound :

$$\sharp \left\{x \in \mathcal{G} | V_x \cap B_R(x_0) \neq \emptyset\right\} \leq \sharp \left\{x \in \mathcal{G} | V_x \cap B_{2^{[\log_2(R)]+1}}(x_0) \neq \emptyset\right\} \leq C \left([\log_2(R)] + 1\right),$$

and we can conclude. □

To conclude this section, we now introduce a particular set (denoted by W_x in the proposition below) containing a point $x \in \mathcal{G}$ which is both bigger than the cell V_x and far from all the others points of $\mathcal{G}$. As we shall see in Lemmas 2.3 and 2.4, this set is actually a technical tool that allows us to show the existence of the corrector stated in Theorem 2.1.

Proposition 2.4. *For every $x \in \mathcal{G}$, there exists a convex open set W_x of $\mathbb{R}^d$ and C_1, C_2, C_3, C_4 and C_5 five positive constants independent of x such that :*

(i) $V_x \subset W_x$,

(ii) $Diam(W_x) \leq C_1|x|$ *and* $D(V_x, \partial W_x) \geq C_2|x|$,

(iii) $\forall y \in \mathcal{G} \setminus \{x\}$, $D(y, W_x) \geq C_3|x|$,

(iv) $\sharp \left\{y \in \mathcal{G} | V_y \cap W_x \neq \emptyset\right\} \leq C_4$,

(v) $\forall y \in \mathcal{G} \setminus \{x\}$, $D(V_y \setminus W_x, V_x) \geq C_5|y|$.

Proof. Let x be in $\mathcal{G}$. In the sequel, we denote $I_{x,y} = \left\{z \in \mathbb{R}^d \,\middle|\, |z - x| \leq |z - y|\right\}$ and φ_x the homothety of center x and ratio $\frac{3}{2}$. For $y \in \mathcal{G} \setminus \{x\}$, we denote by $H_{x,y}$ the set defined by :

$$H_{x,y} = \varphi_x \left(I_{x,y}\right).$$

The set $H_{x,y}$ can be easily determined, it is the half-space defined by :

$$H_{x,y} = \left\{z \in \mathbb{R}^d \,\middle|\, |z - x| \leq |z - y|\right\} + \frac{1}{4}\overrightarrow{xy} = I_{x,y} + \frac{1}{4}\overrightarrow{xy}.$$

We finally consider :

$$W_x = \bigcap_{y \in \mathcal{G}\setminus\{x\}} H_{x,y},$$

which is actually the image of the cell V_x by the homothety φ_x (see figure 2.3).

We next prove that W_x satisfies (i), (ii), (iii), (iv) and (v).

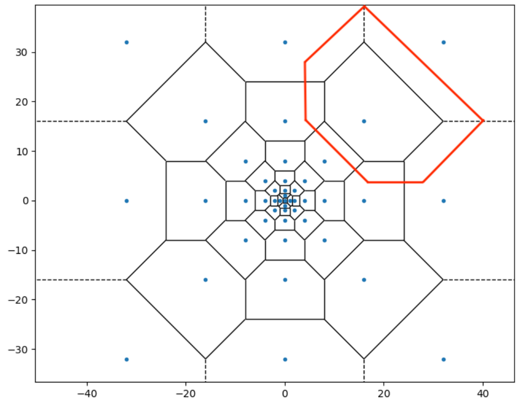

FIGURE 2.3 – Example for the choice of the open subset W_x (in red) when $d = 2$.

(i) : For every $y \in \mathcal{G} \setminus \{x\}$ we have $I_{x,y} \subset H_{x,y}$ and therefore, we obtain using definition (2.7) of V_x :

$$V_x = \bigcap_{y \in \mathcal{G} \setminus \{x\}} I_{x,y} \subset \bigcap_{y \in \mathcal{G} \setminus \{x\}} H_{x,y} = W_x,$$

and we have the first inclusion.

(ii) : W_x is a $\frac{3}{2}$-dilation of V_x, thus we have $Diam(W_x) = \frac{3}{2} Diam(V_x)$. We use (H2) and (H3) to obtain the first estimate. Next, the definitions of the sets $H_{x,y}$ and W_x give :

$$D(V_x, \partial W_x) = \frac{1}{4} \inf_{y \in \mathcal{G} \setminus \{x\}} |x - y| = \frac{1}{4} D\left(x, \mathcal{G} \setminus \{x\}\right).$$

We conclude using (H2).

(iii) : Let y be in $\mathcal{G} \setminus \{x\}$. By definition, for every $v \in W_x$, there exists $u \in I_{x,y}$ such that $v = u + \frac{1}{4}\overrightarrow{xy}$. Therefore, we use the triangle inequality and we have :

$$|v - y| \geq D\left(y, I_{x,y}\right) - \frac{1}{4}|x - y| = \frac{1}{2}|x - y| - \frac{1}{4}|x - y| = \frac{1}{4}|x - y|.$$

Taking the infimum over all $v \in W_x$ in the above inequality and using (H2), we finally obtain :

$$D(y, W_x) \geq \frac{1}{4}|x - y| \geq \frac{1}{4} D\left(x, \mathcal{G} \setminus \{x\}\right) \geq C\frac{1}{4}|x|,$$

where $C > 0$ is independent of x and y.

(iv) : First, we have proved there exists a constant $C_1 \geq 1$ independent of x such that we have $Diam\left(W_x\right) \leq C_1|x|$. Second, using Assumption (H2), we know there exists a constant $C_2 > 0$ such that for every $y \in \mathcal{G}$ we have $D\left(y, \mathcal{G} \setminus \{y\}\right) \geq C_2|y|$. Let $k > 2$ be an integer such that :

$$C_2 2^{k-2} - 1 > 4C_1. \tag{2.20}$$

We denote $n \in \mathbb{N}$, the unique integer such that $x \in A_{2^n, 2^{n+1}}$. Here, it is sufficient to establish a bound for x sufficiently large, thus without loss of generality, we can assume that $n > k$. We next show that if $y \in \mathcal{G}$ satisfies $|y| \leq 2^{-k-1}|x| \leq 2^{n-k}$ or $|y| \geq 2^k|x| \geq 2^{n+k}$, then $W_x \cap V_y = \emptyset$.

We start by assuming that $y \in \mathcal{G} \cap \left(\mathbb{R}^d \setminus B_{2^{n+k}}\right)$. Since

$$Diam\left(W_x\right) \leq C_1|x| \leq \; C_1 2^{n+1},$$

we have $W_x \subset B_{C_1 2^{n+1}}(x)$. Therefore, using a triangle inequality we obtain $W_x \subset B_{C_1 2^{n+2}}$. Our aim here is to prove that $I_{y,x} \cap B_{C_1 2^{n+2}} = \emptyset$ in order to deduce $I_{y,x} \cap W_x = \emptyset$. For every $z \in I_{y,x}$:

$$|z| \geq |z-x| - |x| \geq D\left(x, I_{x,y}\right) - |x|.$$

In addition, for every $y \in \mathcal{G} \setminus \{y\}$, we have $D\left(x, I_{x,y}\right) = \dfrac{1}{2}|x-y| \geq \dfrac{1}{2} D\left(y, \mathcal{G} \setminus \{y\}\right)$ and we deduce that :

$$\begin{aligned}
|z| &\geq D\left(y, \mathcal{G} \setminus \{y\}\right) - |x| \\
&\geq \frac{C_2}{2}|y| - |x| \\
&\geq C_2 2^{n+k-1} - 2^{n+1} \\
&\geq 2^{n+1}\left(C_2 2^{k-2} - 1\right) \geq C_1 2^{n+3}.
\end{aligned}$$

Therefore, $I_{x,y} \subset \left(\mathbb{R}^d \setminus B_{C_1 2^{n+3}}\right)$ and we obtain $W_x \cap I_{x,y} = \emptyset$. Since, $V_y = \displaystyle\bigcap_{z \in \mathcal{G} \setminus \{y\}} I_{z,y}$, we deduce that $V_y \cap W_x = \emptyset$.

Next we assume that $y \in B_{2^{n-k}}$ and we want to prove that $V_y \cap H_{x,y} = \emptyset$. As above, we can show that $V_y \subset B_{C_1 2^{n-k+1}}$ and for every $z \in H_{x,y}$:

$$\begin{aligned}
|z| &\geq \frac{1}{4}|x-y| - |y| \\
&\geq \frac{1}{4} C_2 |x| - |y| \\
&\geq 2^{n-k}\left(C_2 2^{k-2} - 1\right) \geq C_1 2^{n-k+2}.
\end{aligned}$$

Therefore $H_{x,y} \subset B_{C_1 2^{n-k+2}}$ and we have $V_y \cap H_{x,y} = \emptyset$. We deduce that $V_y \cap W_x = \emptyset$. To conclude, we use Proposition 2.3 and we obtain the existence of a constant $C_3 > 0$ independent of n such that :

$$\sharp\left\{x \in \mathcal{G} | x \in A_{2^n, 2^{n+1}}\right\} \leq C_3,$$

and therefore :

$$\begin{aligned}
\sharp\left\{y \in \mathcal{G} | V_y \cap W_x \neq \emptyset\right\} &\leq \sum_{m=-k}^{k} \left\{x \in \mathcal{G} | x \in A_{2^m, 2^{m+1}}\right\} \\
&\leq \sum_{m=n-k}^{n+k} C_3 = (2k+1)\, C_3.
\end{aligned}$$

We have finally proved (iv).

(v) : Let y be in $\mathcal{G} \setminus \{x\}$. We first assume that $2^{-k-1}|y| > |x|$, where k is defined as in (2.20) and is independent of x. In the proof of (iv) above, we have shown that $W_y \cap V_x = \emptyset$. Therefore, using Property (i) and (ii) of W_y we easily obtain that there exists a constant $M_1 > 0$ independent of x and y such that $D(V_x, V_y) > M_1|y|$ and we can conclude. Next, we assume that $2^{-k-1}|y| \leq |x|$. Using again Property (i) and (ii) of W_x, we obtain the existence of $M_2 > 0$ independent of x and y such that $D\left(V_x, V_y \setminus W_x\right) \geq M_2|x| \geq M_2 2^{-k-1}|y|$. Finally, we have proved (v) with $C_5 = \min(M_1, M_2 2^{-k-1})$.

□

2.2.2 The particular case of the "2^p"

We next prove that the set $\mathcal{G}_{C_0}$ defined by (2.9) satisfies Assumptions (H1), (H2) and (H3). In order to avoid many unnecessary technical details, we study here the Voronoi diagram only for $d = 3$ and, in the sequel, we admit that these properties still hold in higher dimension. We also consider the cell V_p only for $p = (p_1, p_2, p_3) \in (\mathbb{R}^{+*})^3$. Since the distribution of the points 2^p is symmetric with respect to the origin, the other cases are similar and we omit them.

Proof of (H1). Let $p = (p_1, p_2, p_3)$ be in $\mathcal{P}_{C_0} \cap (\mathbb{R}^{+*})^3$. We first prove the following inclusion :

$$V_p \subset \prod_{i=1}^{3} \left[2^{p_i - 1}, 2^{|p|+3}\right]. \tag{2.21}$$

To this aim, we want to show that if $(x, y, z) \notin \prod_{i=1}^{3} \left[2^{p_i-1}, 2^{|p|+3}\right]$, then there exists $x_q \in \mathcal{P}_{C_0} \setminus \{x_p\}$ such that the point (x, y, z) is closer to x_q than to x_p and therefore $(x, y, z) \notin V_p$. We consider $(x, y, z) \in (\mathbb{R}^+)^3$ and we start by assuming that $x < 2^{p_1-1}$. We have

$$D((x, y, z), x_p)^2 = |x - 2^{p_1}|^2 + |y - 2^{p_2}|^2 + |z - 2^{p_3}|^2,$$

and

$$D((x, y, z), (0, 2^{p_2}, 2^{p_3}))^2 = |x|^2 + |y - 2^{p_2}|^2 + |z - 2^{p_3}|^2.$$

Since $x < 2^{p_1-1}$, we use a triangle inequality and

$$|x - 2^{p_1}| > 2^{p_1} - 2^{p_1-1} = 2^{p_1-1} > |x|.$$

We obtain that $D((x, y, z), x_p)^2 > D((x, y, z), (0, 2^{p_2}, 2^{p_3}))^2$. That is, (x, y, z) is closer to $(0, 2^{p_2}, 2^{p_3}) \in \mathcal{G}_{C_0}$ than to x_p and we deduce that $(x, y, z) \notin V_p$. We can therefore conclude that V_p is included in $\{(x, y, z) \in \mathbb{R}^3 \mid 2^{p_1-1} \leq x\}$.
We next assume that $x > 2^{|p|+3}$. Since $|p| \geq p_1$, we have :

$$\begin{aligned} |x - 2^{p_1}|^2 &= \left(x - 2^{|p|+1} + 2^{|p|+1} - 2^{p_1}\right)^2 \\ &\geq \left(x - 2^{|p|+1} + 2^{|p|+1} - 2^{|p|}\right)^2 \\ &= |x - 2^{|p|+1}|^2 + 2^{|p|+1}(x - 2^{|p|+1}) + 2^{2|p|}. \end{aligned}$$

Using $x > 2^{|p|+3}$, it follows :

$$|x - 2^{p_1}|^2 > |x - 2^{|p|+1}|^2 + 13 \times 2^{2|p|}. \tag{2.22}$$

On the other hand, we have

$$|y-2^{p_2}|^2 = |y|^2 - 2^{p_2+1}y + 2^{2p_2},$$

$$|y-2^{p_2+1}|^2 = |y|^2 - 2^{p_2+2}y + 2^{2p_2+2}.$$

We obtain

$$|y-2^{p_2}|^2 \geq |y-2^{p_2+1}|^2 - 3\times 2^{p_2} \geq |y-2^{p_2+1}|^2 - 3\times 2^{2|p|}.$$

Similarly, we can show that $|z-2^{p_3}|^2 \geq |z-2^{p_3+1}|^2 - 3\times 2^{|p|}$ and, using (2.22), we have

$$\begin{aligned} D((x,y,z),x_p)^2 &> |x-2^{|p|+1}|^2 + |y-2^{p_2+1}|^2 + |z-2^{p_3+1}|^2 + 7\times 2^{|p|} \\ &> D((x,y,z),(2^{|p|+1},2^{p_2+1},2^{p_3+1}))^2. \end{aligned}$$

Now we claim that $(2^{|p|+1},2^{p_2+1},2^{p_3+1}) \in \mathcal{G}_{C_0}$. Indeed, since $(p_1,p_2,p_3) \in \mathcal{P}_{C_0}$, we have using (2.8) :

$$\begin{aligned} \max\{|p|+1,p_2+1,p_3+1\} &= \max\{p_1+1,p_2+1,p_3+1\} \\ &\leq \min\{p_1+1,p_2+1,p_3+1\} + C_0 \\ &\leq \min\{|p|+1,p_2+1,p_3+1\} + C_0. \end{aligned}$$

Since $D((x,y,z),x_p)^2 > D((x,y,z),(2^{|p|+1},2^{p_2+1},2^{p_3+1}))^2$, we therefore conclude that the point (x,y,z) is closer to $(2^{|p|+1},2^{p_2+1},2^{p_3+1})$ than to x_p and that $(x,y,z) \notin V_p$.

Using the symmetry of the distribution, we can use exactly the same argumentation to treat the cases $y < 2^{p_2-1}$, $y > 2^{|p|+3}$, $z < 2^{p_3-1}$ and $z > 2^{|p|+3}$. We have finally established inclusion (2.21). Since the volume of the cube $\prod_{i=1}^{3}\left[2^{p_i-1},2^{|p|+3}\right]$ is bounded by $8^3.2^{3|p|}$, we can deduce that :

$$|V_p| \leq 8^3.2^{3|p|}.$$

(H1) is proved. □

Proof of (H2). Let p be in $\mathcal{P}_{C_0}\cap(\mathbb{R}^{+*})^3$. We have :

$$D\left(x_p,\mathcal{G}_{C_0}\setminus\{x_p\}\right) \leq D(x_p,0) = |x_p|,$$

and therefore :

$$1 \leq \frac{1+|x_p|}{D\left(x_p,\mathcal{G}_{C_0}\setminus\{x_p\}\right)}.$$

To show the upper bound, we consider $x_q \in \mathcal{G}_{C_0}\setminus\{x_p\}$. Without loss of generality, we can assume $|p_1| = |p|$ and there are three cases :

– If $|q_1| \neq |p_1|$, then :

$$\begin{aligned} D(x_p,x_q) &\geq \left|\mathrm{sign}(p_1)2^{|p_1|} - \mathrm{sign}(q_1)2^{|q_1|}\right| \\ &\geq \left|2^{|p_1|} - 2^{|q_1|}\right| = 2^{|p_1|}\left|1-2^{|q_1|-|p_1|}\right| \\ &\geq 2^{|p_1|}\frac{1}{2} = 2^{|p|-1}. \end{aligned}$$

– If $p_1 = q_1$, since $p \in \mathcal{P}_{C_0}$, we have $\max(|p_2|, |p_3|) \geq |p| - C_0$. Since $x_q \neq x_p$, we obtain as above :

$$D(x_p, x_q) \geq \max\left(2^{|p_2|}\left|1 - 2^{|q_2|-|p_2|}\right|, 2^{|p_3|}\left|1 - 2^{|q_3|-|p_3|}\right|\right) \geq 2^{|p|-C_0-1}.$$

– If $p_1 = -q_1$, we have :

$$D(x_p, x_q) \geq \left|\operatorname{sign}(p_1)2^{|p_1|} - \operatorname{sign}(q_1)2^{|q_1|}\right| = 2^{|p|+1}.$$

In the three cases we conclude there exists $C > 0$ independent of q such that $D(x_p, x_q) \geq C2^{|p|}$. Finally, since $|x_p| = \left(2^{2p_1} + 2^{2p_2} + 2^{2p_3}\right)^{1/2} \leq \sqrt{3}.2^{|p|}$, we obtain the existence of a constant $C_1 > 0$ independent of p such that :

$$\frac{1 + |x_p|}{D\left(x_p, \mathcal{G}_{C_0} \setminus \{x_p\}\right)} \leq C_1.$$

□

Proof of (H3). Let $p = (p_1, p_2, p_3)$ be in $\mathcal{P}_{C_0} \cap (\mathbb{R}^{+*})^3$. We use (2.21) to bound the diameter of V_p by the diameter of the cube $[0, 2^{|p|+3}]^3$, that is :

$$Diam\left(V_p\right) \leq \sqrt{3}.2^{|p|+3}.$$

In addition, (H2) shows the existence of $C > 0$ such that for every $x_p \in \mathcal{G}$, we have :

$$D\left(x_p, \mathcal{G}_{C_0} \setminus \{x_p\}\right) \geq C|x_p| \geq C2^{|p|},$$

and we obtain (H3). □

We finally conclude this section establishing an estimate regarding the norm of each element x_p of $\mathcal{G}_{C_0}$. Using Proposition 2.1, the next property shall be useful to estimate the volume of the Voronoi cells in our particular case.

Proposition 2.5. *There exists $C_1 > 0$ and $C_2 > 0$ such that for every p in $\mathcal{P}_{C_0}$, we have :*

$$C_1 2^{|p|} \leq |x_p| \leq C_2 2^{|p|}. \tag{2.23}$$

Proof. For $p \in \mathcal{P}_{C_0}$, we have :

$$|x_p| = \left(\sum_{i \in 1,\dots,d} 2^{2|p_i|}\right)^{1/2}.$$

We first use the inequality $|p_i| \leq |p|$ to obtain the upper bound. That is :

$$|x_p| \leq \left(\sum_{i \in 1,\dots,d} 2^{2|p|}\right)^{1/2} \leq \sqrt{d}2^{|p|}.$$

For the lower bound, we denote $j = \operatorname{argmax}_{i \in \{1,\dots,d\}} |p_i|$ and we have :

$$|x_p| \geq 2^{|p_j|} = 2^{|p|}.$$

We have established the norm estimate (2.23). □

In the sequel of this work, we only consider the specific set $\mathcal{G}_{C_0}$, defined by (2.9), for a fixed arbitrary constant $C_0 > 1$. Therefore, for the sake of clarity and without loss of generality, we will denote $\mathcal{G}$ and $\mathcal{P}$ instead of $\mathcal{G}_{C_0}$ and $\mathcal{P}_{C_0}$.

2.3 Properties of the functional space $\mathcal{B}^2(\mathbb{R}^d)$

In this section we prove some properties satisfied by the functional space $\mathcal{B}^2(\mathbb{R}^d)$. The following results are heavily based upon the geometric distribution of the x_p. They are key for the understanding of the structure of $\mathcal{B}^2$ and to establish the homogenization of problem (2.1).

To start with, we show the uniqueness of a limit L^2-function f_∞ in $L^2(\mathbb{R}^d)$ defined in (2.10) and characterizing each element of $\mathcal{B}^2(\mathbb{R}^d)$. This result ensures that the definition of the function space $\mathcal{B}^2(\mathbb{R}^d)$ is consistent.

Proposition 2.6. *Let f be a function of $\mathcal{B}^2(\mathbb{R}^d)$. Then, the limit function $f_\infty \in L^2(\mathbb{R}^d)$ defined in (2.10) is unique.*

Proof. We assume there exist two functions f_∞ and g_∞ in $L^2(\mathbb{R}^d)$ such that

$$\lim_{|p|\to\infty} \|f - \tau_{-p} f_\infty\|_{L^2(V_p)} = \lim_{|p|\to\infty} \|f - \tau_{-p} g_\infty\|_{L^2(V_p)} = 0.$$

By a triangle inequality, we obtain for every $p \in \mathcal{P}$:

$$\|\tau_{-p} f_\infty - \tau_{-p} g_\infty\|_{L^2(V_p)} \leq \|f - \tau_{-p} f_\infty\|_{L^2(V_p)} + \|f - \tau_{-p} g_\infty\|_{L^2(V_p)} \underset{|p|\to+\infty}{\longrightarrow} 0.$$

In addition, we have $\|\tau_{-p} f_\infty - \tau_{-p} g_\infty\|_{L^2(V_p)} = \|f_\infty - g_\infty\|_{L^2(V_p - 2^p)}$. According to Proposition 2.2, we can find a sequence $(p_n)_{n\in\mathbb{N}} \in \mathcal{P}$ such that $\lim_{n\to\infty} |p_n| = \infty$ and :

$$\bigcup_{n\in\mathbb{N}} (V_{p_n} - 2^{p_n}) = \mathbb{R}^d.$$

We can finally conclude that $\|f_\infty - g_\infty\|_{L^2(\mathbb{R}^d)} = 0$, that is $f_\infty = g_\infty$ in $L^2(\mathbb{R}^d)$. □

We next study the structure of the space $\mathcal{B}^2(\mathbb{R}^d)$ showing two essential properties that shall allow us to establish the existence of the corrector in Section 2.4. In particular, we prove in Proposition 2.7 that $\mathcal{B}^2(\mathbb{R}^d)$ is a Banach space.

Proposition 2.7. *The space $\mathcal{B}^2(\mathbb{R}^d)$ equipped with the norm defined by (2.11), is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}^2(\mathbb{R}^d)$. Definitions (2.10) and (2.11) ensure the existence of a Cauchy sequence $f_{n,\infty}$ in $L^2(\mathbb{R}^d)$ such that for every $n \in \mathbb{N}$,

$$\lim_{|p|\to\infty} \|f_n - \tau_{-p} f_{n,\infty}\|_{L^2(V_p)} = 0.$$

Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $k > 0$:

$$\begin{aligned}
&\|f_{n+k} - f_n\|_{L^2_{unif}} \leq \varepsilon, \\
&\|f_{n+k,\infty} - f_{n,\infty}\|_{L^2(\mathbb{R}^d)} \leq \varepsilon, \\
&\sup_{p\in\mathcal{P}} \|\left(f_{n+k} - \tau_{-p} f_{n+k,\infty}\right) - \left(f_n - \tau_{-p} f_{n,\infty}\right)\|_{L^2(V_p)} \leq \frac{\varepsilon}{2}.
\end{aligned} \tag{2.24}$$

Since L^2 and L^2_{unif} are Banach spaces, there exist $f \in L^2_{unif}(\mathbb{R}^d)$ and $f_\infty \in L^2(\mathbb{R}^d)$ such that $f_n \underset{n\to+\infty}{\longrightarrow} f$ in $L^2_{unif}(\mathbb{R}^d)$ and $f_{n,\infty} \underset{n\to+\infty}{\longrightarrow} f_\infty$ in $L^2(\mathbb{R}^d)$. We consider the limit in (2.24) when $k \to \infty$ and we obtain :

$$\sup_{p\in\mathcal{P}} \| (f - \tau_{-p} f_\infty) - (f_n - \tau_{-p} f_{n,\infty}) \|_{L^2(V_p)} \leq \frac{\varepsilon}{2}.$$

Since ε can be chosen arbitrary small, we deduce :

$$\lim_{n\to\infty} \sup_{p\in\mathcal{P}} \| (f - \tau_{-p} f_\infty) - (f_n - \tau_{-p} f_{n,\infty}) \|_{L^2(V_p)} = 0.$$

The function f is therefore the limit of f_n for the norm (2.11). We just have to show that $f \in \mathcal{B}^2(\mathbb{R}^d)$ to conclude. Indeed, for a fixed $n > N$ and for p sufficiently large, we have :

$$\|f_n - \tau_{-p} f_{n,\infty}\|_{L^2(V_p)} \leq \frac{\varepsilon}{2}.$$

Using a triangle inequality, it follows :

$$\begin{aligned}\|f - \tau_{-p} f_\infty\|_{L^2(V_p)} &\leq \|f_n - \tau_{-p} f_{n,\infty}\|_{L^2(V_p)} + \sup_{p\in\mathcal{P}} \| (f - \tau_{-p} f_\infty) - (f_n - \tau_{-p} f_{n,\infty}) \|_{L^2(V_p)} \\ &\leq \varepsilon.\end{aligned}$$

Finally, we obtain $\displaystyle\lim_{|p|\to\infty} \|f - \tau_{-p} f_\infty\|_{L^2(V_p)} = 0.$ □

Proposition 2.8. *Let $\alpha \in]0,1[$, then $\mathcal{C}^{0,\alpha}(\mathbb{R}^d) \cap \mathcal{B}^2(\mathbb{R}^d)$ is dense in $\left(\mathcal{B}^2(\mathbb{R}^d), \|.\|_{\mathcal{B}^2(\mathbb{R}^d)}\right)$.*

Proof. We consider $f \in \mathcal{B}^2(\mathbb{R}^d)$ and $f_\infty \in L^2(\mathbb{R}^d)$ the associated limit function defined by (2.10). First, for any $\varepsilon > 0$, there exists $\phi \in \mathcal{D}(\mathbb{R}^d)$ such that $\|\phi - f_\infty\|_{L^2(\mathbb{R}^d)} < \frac{\varepsilon}{3}$, thus $\|\tau_{-p}\phi - \tau_{-p} f_\infty\|_{L^2(V_p)} \leq \frac{\varepsilon}{3}$ for all $p \in \mathcal{P}$. Second, since $f \in \mathcal{B}^2(\mathbb{R}^d)$:

$$\exists P^* \in \mathbb{N},\ \forall p \in \mathcal{P},\ |p| > P^* \ \Rightarrow \|f - \tau_{-p} f_\infty\|_{L^2(V_p)} < \frac{\varepsilon}{3}.$$

Since ϕ is compactly supported there also exists P, which we can always assume larger than P^*, such that for every $|p| > P$ and for all $q \neq p$, we have $(\tau_{-q}\phi)_{|V_p} = 0$.

The finite sum $\displaystyle\sum_{|q|\leq P} 1_{V_q} f$ (where 1_A denotes the indicator function of A) is compactly supported and then belongs to $L^2(\mathbb{R}^d)$. Again, we can find $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\left\| \psi - \sum_{|q|\leq P} 1_{V_q} f \right\|_{L^2(\mathbb{R}^d)} \leq \frac{\varepsilon}{3}.$$

We fix $g = \psi + \displaystyle\sum_{|p|>P} \tau_{-p}\phi$ and we want to show that g is a good approximation of f in $\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, that is g is close to f on each V_p, uniformly in p. First, we have :

$$g_{|V_p} = \begin{cases} \psi & \text{if } |p| \leq P, \\ \psi + \tau_{-p}\phi & \text{else.} \end{cases}$$

Therefore g is bounded and we can easily prove that $g \in \mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ where the associated limit function in $L^2(\mathbb{R}^d)$ is given by $g_\infty = \phi$. Furthermore, g is in $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ since it is a $\mathcal{C}^\infty$ function and all of its derivatives are bounded. Indeed, for every k in $\mathbb{N}^d$, we denote $\partial_k = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}...\partial_{x_d}^{k_d}$ and we have :

$$(\partial_k g)_{|V_p} = \begin{cases} \partial_k \psi & \text{if } |p| \leq P, \\ \partial_k\psi + \tau_{-p}\partial_k \phi & \text{else.} \end{cases}$$

and $\partial_k g$ is clearly bounded.

Let $p \in \mathcal{P}$, we consider two cases. If $|p| \leq P$, then :

$$\|g - f\|_{L^2(V_p)} = \left\| \psi - \sum_{|q| \leq P} 1_{V_q} f \right\|_{L^2(V_p)} \leq \varepsilon.$$

Else, if $|p| > P$, using that $\sum\limits_{|q|\leq P} 1_{V_q} f$ has support in $\bigcup\limits_{|q|\leq P} V_q$ we have :

$$\|\psi\|_{L^2(V_p)} = \left\| \psi - \sum_{|q| \leq P} 1_{V_q} f \right\|_{L^2(V_p)} \leq \left\| \psi - \sum_{|q| \leq P} 1_{V_q} f \right\|_{L^2(\mathbb{R}^d)} \leq \frac{\varepsilon}{3}.$$

$$\begin{aligned} \|g - f\|_{L^2(V_p)} &= \|\psi + \tau_{-p}\phi - f\|_{L^2(V_p)} \\ &\leq \|\psi\|_{L^2(V_p)} + \|\tau_{-p}\phi - \tau_{-p} f_\infty\|_{L^2(V_p)} + \|\tau_{-p} f_\infty - f\|_{L^2(V_p)} \\ &\leq \varepsilon. \end{aligned}$$

And we can conclude. □

We now establish a property regarding multiplication of elements of $\mathcal{B}^2(\mathbb{R}^d)$.

Proposition 2.9. *Let g and h be in $\mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We assume the associated L^2 function of g, denoted by g_∞, is in $L^\infty(\mathbb{R}^d)$, then $hg \in \mathcal{B}^2(\mathbb{R}^d)$.*

Proof. Since $g_\infty \in L^\infty(\mathbb{R}^d)$, we clearly have $g_\infty h_\infty \in L^2(\mathbb{R}^d)$. Using that for all $p \in \mathcal{P}$:

$$gh - \tau_{-p}(g_\infty h_\infty) = (h - \tau_{-p}h_\infty)\tau_{-p}g_\infty + (g - \tau_{-p}g_\infty)h.$$

We have by the triangle inequality :

$$\begin{aligned} \|gh - \tau_{-p}(g_\infty h_\infty)\|_{L^2(V_p)} \leq &\|h - \tau_{-p}h_\infty\|_{L^2(V_p)}\|g_\infty\|_{L^\infty(\mathbb{R}^d)} \\ &+ \|g - \tau_{-p}g_\infty\|_{L^2(V_p)}\|h\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

It follows, taking the limit for $|p| \to \infty$, that $gh \in \mathcal{B}^2(\mathbb{R}^d)$ and that $(gh)_\infty = g_\infty h_\infty$. □

Our next result is one of the most important properties for the sequel. As we shall see in section 2.5, it first implies that the homogenized coefficient in our setting is the same as the homogenized coefficient in the periodic case, that is, without perturbation. In addition, it gives some information about the growth of the corrector defined in Theorem 2.1 (in particular, we give a proof in proposition 2.11 of the strict sublinearity of the corrector). We will use all of these properties to prove the convergence stated in Theorem 2.2 in our case.

Proposition 2.10. *Let $u \in \mathcal{B}^2(\mathbb{R}^d)$. Then, for every $x_0 \in \mathbb{R}^d$:*

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)|dx = 0, \tag{2.25}$$

with the following convergence rate :

$$\frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)|dx \leq C \left(\frac{\log R}{R^d}\right)^{\frac{1}{2}}, \tag{2.26}$$

where $C > 0$ is independent of R and x_0.

Proof. We fix $R > 0$. Using the Cauchy-Schwarz inequality, we have :

$$\begin{aligned}\frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)|dx &\leq \frac{1}{\sqrt{|B_R|}} \left(\int_{B_R(x_0)} |u(x)|^2 dx\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{|B_R|}} \left(\sum_{p\in\mathcal{P}} \int_{V_p \cap B_R(x_0)} |u(x)|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

Since the number of V_p such that $B_R(x_0) \cap V_p \neq \emptyset$ is bounded by $\log(R)$ according to Corollary 2.1, we obtain :

$$\frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)|dx \leq \frac{(\log R)^{\frac{1}{2}}}{\sqrt{|B_R|}} \sup_p \|u\|_{L^2(V_p)} \leq C(d) \left(\frac{\log(R)}{R^d}\right)^{\frac{1}{2}} \sup_p \|u\|_{L^2(V_p)}.$$

Here, $C(d)$ depends only on the ambient dimension d. The last inequality yields (2.26) and conclude the proof. □

Corollary 2.2. *Let $u \in \mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $|u(./\varepsilon)|$ converges to 0 in the weak-$\star$ topology of $L^\infty(\mathbb{R}^d)$ when $\varepsilon \to 0$.*

Proof. We fix $R > 0$ and we first consider $\varphi = 1_{B_R}$. For any $\varepsilon > 0$, we have :

$$\begin{aligned}\left|\int_{\mathbb{R}^d} |u(x/\varepsilon)|\varphi(x)dx\right| &\leq \int_{B_R} |u(x/\varepsilon)|\, dx \\ &\overset{y=x/\varepsilon}{=} \varepsilon^d \int_{B_{R/\varepsilon}} |u(y)|\, dy \\ &= |B_R| \frac{\varepsilon^d}{|B_R|} \int_{B_{R/\varepsilon}} |u(y)|\, dy \\ &= \|\varphi\|_{L^1(\mathbb{R}^d)} \frac{\varepsilon^d}{|B_R|} \int_{B_{R/\varepsilon}} |u(y)|\, dy.\end{aligned}$$

We next use (2.26) in the right-hand term and we obtain the existence of $C > 0$ independent of ε and φ such that :

$$\left|\int_{\mathbb{R}^d} u(x/\varepsilon)\varphi(x)dx\right| \leq C\|\varphi\|_{L^1(\mathbb{R}^d)} \left(\varepsilon^d \log(1/\varepsilon)\right)^{\frac{1}{2}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

We conclude using the density of simple functions in $L^1(\mathbb{R}^d)$. □

We next introduce the notion of sub-linearity which is actually a fundamental property in homogenization. Indeed, in order to precise the convergence of the approximated sequence of solutions (2.6), we have to study the behavior of the sequences $\varepsilon w_{e_i}(./\varepsilon)$ when $\varepsilon \to 0$. The convergence to zero of these sequences and the understanding of the rate of convergence are key for establishing estimates (2.15) and (2.16) stated in Theorem 2.2. In the sequel, we therefore study this phenomenon for the functions with a gradient in $\mathcal{B}^2(\mathbb{R}^d)$.

Definition 2.1. *A function u is strictly sub-linear at infinity if :*

$$\lim_{|x|\to\infty} \frac{|u(x)|}{1+|x|} = 0.$$

In the next proposition we prove the sub-linearity of all the functions u such that $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^d$. We assume, for this general property only, that $d \geq 2$.

Proposition 2.11. *Assume $d \geq 2$. Let $u \in H^1_{loc}(\mathbb{R}^d)$ with $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^d$. Then u is strictly sub-linear at infinity and for all $s > d$, there exists $C > 0$ such that for every $x, y \in \mathbb{R}^d$ with $x \neq y$:*

$$|u(x) - u(y)| \leq C \left|\log(|x-y|)\right|^{\frac{1}{s}} |x-y|^{1-\frac{d}{s}}. \tag{2.27}$$

Proof. Let $x, y \in \mathbb{R}^d$ with $x \neq y$ and fix $r = |x-y|$. Since $\nabla u \in \left(L^\infty(\mathbb{R}^d)\right)^d$, we have $\nabla u \in \left(L^s_{loc}(\mathbb{R}^d)\right)^d$ for every $s \geq 1$. We next fix $s > d$. We know there exists a constant $C > 0$, depending only on d, such that :

$$|u(x) - u(y)| \leq C r \left(\frac{1}{r^d}\int_{B_r(x)} |\nabla u(z)|^s \, dz\right)^{\frac{1}{s}}.$$

This estimate is established for instance in [42, Remark p.268] as corollary of the Morrey's inequality ([42, Theorem 4 p.266]). Since $s > d \geq 2$, we use the boundedness of ∇u to obtain :

$$|u(x) - u(y)| \leq C \|\nabla u\|^{(s-2)/s}_{L^\infty(\mathbb{R}^d)} r \left(\frac{1}{r^d}\int_{B_r(x)} |\nabla u(z)|^2 \, dz\right)^{\frac{1}{s}}. \tag{2.28}$$

We next split the integral of (2.28) on each V_p such that $V_p \cap B_r(x) \neq \emptyset$ and we have :

$$|u(x) - u(y)| \leq C \|\nabla u\|^{(s-2)/s}_{L^\infty(\mathbb{R}^d)} r \left(\frac{1}{r^d}\sum_{p\in\mathcal{P}}\int_{B_r(x)\cap V_p} |\nabla u(z)|^2 \, dz\right)^{\frac{1}{s}}.$$

We finally use Corollary 2.1 and we obtain the existence of a constant $C_1 > 0$ such that :

$$|u(x) - u(y)| \leq C_1 \|\nabla u\|^{(s-2)/s}_{L^\infty(\mathbb{R}^d)} \|\nabla u\|^{2/s}_{\mathcal{B}^2(\mathbb{R}^d)} \left|\log(r)\right|^{\frac{1}{s}} r^{1-\frac{d}{s}}.$$

This inequality is true for all $s > d$, which allows us to conclude. In addition, the sub-linearity of u is obtained fixing $y = 0$ and letting $|x|$ go to the infinity in estimate (2.27). □

Remark 2.1. *In the case $d = 1$, since $s \geq 2$, the above proof gives :*

$$|u(x) - u(y)| \leq C \left|\log|x-y|\right|^{\frac{1}{2}} |x-y|^{\frac{1}{2}}.$$

The last proposition of this section gives an uniform estimate of the integral remainders of the functions of $\mathcal{B}^2(\mathbb{R}^d)$. The idea here is that the functions of $\mathcal{B}^2(\mathbb{R}^d)$ behave like a fixed L^2-functions at the vicinity of the points of $\mathcal{G}$ and therefore, have to be small in a L^2 sense far from these points. This property will be used in the proof of Lemma 2.3 in next section to establish an estimate in $\mathcal{B}^2(\mathbb{R}^d)$ satisfied by the solutions to diffusion equation (2.35).

Proposition 2.12. *Let f be in $\mathcal{B}^2(\mathbb{R}^d)$ and f_∞ the associated limit function in $L^2(\mathbb{R}^d)$. For any $\varepsilon > 0$, there exists $R^* > 0$ such that for every $R > R^*$ and every $p, q \in \mathcal{P}$:*

$$\left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-p} f_\infty|^2\right)^{1/2} < \varepsilon,$$

where $B_R(2^q)^c$ denotes the set $\mathbb{R}^d \setminus B_R(2^q)$. Therefore, we have the following limit :

$$\lim_{R\to\infty} \sup_{\substack{(p,q)\in\mathcal{P}^2 \\ p\neq q}} \left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-p} f_\infty|^2\right)^{1/2} = 0.$$

Proof. Let $\varepsilon > 0$. First, for every $R > 0$, $p, q \in \mathcal{P}$ we use a triangle inequality and we obtain the following upper bound :

$$\begin{aligned}\left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-p} f_\infty|^2\right)^{1/2} &\leq \left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-q} f_\infty|^2\right)^{1/2} + \left(\int_{V_q \cap B_R(2^q)^c} |\tau_{-q} f_\infty|^2\right)^{1/2} \\ &\quad + \left(\int_{V_q \cap B_R(2^q)^c} |\tau_{-p} f_\infty|^2\right)^{1/2} \\ &= I_1^{p,q}(R) + I_2^{p,q}(R) + I_3^{p,q}(R).\end{aligned}$$

We want to bound the three terms $I_1^{p,q}(R)$, $I_2^{p,q}(R)$ and $I_3^{p,q}(R)$ by ε uniformly in p, q.

We start by considering $I_1^{p,q}$. We have assumed that $f \in \mathcal{B}^2(\mathbb{R}^d)$, then, by definition, there exists $P > 0$ such that for every $q \in \mathcal{P}$ satisfying $|q| > P$, we have :

$$\left(\int_{V_q} |f - \tau_{-q} f_\infty|^2\right)^{1/2} < \frac{\varepsilon}{3}.$$

In addition, since the volume of each V_q is finite according to assumption (H1), there exists $R_1 > 0$ such that for every $|q| \leq P$, $B_{R_1}(2^q)^c \cap V_q = \emptyset$. Therefore, as soon as $|q| \leq P$ and $R \geq R_1$, we have $I_{p,q}(R) = 0$. Finally, considering successively the case $|q| \leq P$ and the case $|q| > P$, we obtain for every $q, p \in \mathcal{P}$ and $R \geq R_1$:

$$I_1^{p,q}(R) < \frac{\varepsilon}{3}.$$

We next study the second term $I_2^{p,q}$. Since f_∞ is in $L^2(\mathbb{R}^d)$, there exists $R_2 > 0$, which we can always assume larger than R_1, such that for every $q \in \mathcal{P}$:

$$\left(\int_{B_{R_2}(2^q)^c} |\tau_{-q} f_\infty(y)|^2 \, dy\right)^{1/2} \overset{x=y-2^q}{=} \left(\int_{B_{R_2}^c} |f_\infty(x)|^2 \, dx\right)^{1/2} < \frac{\varepsilon}{3}. \tag{2.29}$$

And we directly obtain, for every $R \geq R_2$:

$$I_2^{p,q}(R) < \frac{\varepsilon}{3}.$$

Finally, in order to bound the last term, we know that $\lim\limits_{|l| \to \infty} D\left(2^l, \mathcal{G} \setminus \{2^l\}\right) = +\infty$ as a consequence of Assumption (H2). Therefore, there exists a finite number of indices l such that :

$$D\left(V_l, \mathcal{G} \setminus \{2^l\}\right) \leq R_2. \tag{2.30}$$

Thus, we deduce the existence of a positive radius R_3 independent of q, p such that for every l satisfying (2.30) we have $B_{R_3}(2^l)^c \cap V_l = \emptyset$. Again we can always assume R_3 larger than R_2. There are two cases depending on the value of q :

1) If q satisfies (2.30), we have $B_{R_3}(2^q)^c \cap V_q = \emptyset$ and we obtain $I_3^{p,q}(R_3) = 0$.
2) Else, for every $y \in V_q$, we have $|y - 2^p| > R_2$. Therefore :

$$I_3^{p,q}(R_3) \leq \left(\int_{V_q} |\tau_{-p} f_\infty|^2\right)^{1/2} \overset{x=y-2^p}{=} \left(\int_{V_q - 2^p} |f_\infty|^2\right)^{1/2} \leq \left(\int_{B_{R_2}^c} |f_\infty|^2\right)^{1/2}.$$

Using (2.29), we have for every $R \geq R_3$, $I_3^{p,q}(R) \leq \frac{\varepsilon}{3}$.

In the two cases , we obtain for $R \geq R_3$:

$$I_3^{p,q}(R) \leq \frac{\varepsilon}{3}.$$

Since the values of R_3 is independent of p and q we can conclude the proof for $R^* = R_3$.

□

2.4 Existence result for the corrector equation

This section is devoted to the proof of Theorem 2.1. Equation (2.14) being posed on the whole space $\mathbb{R}^d$, we need to use here the geometric distribution of the 2^p and introduce some constructive techniques involving the fundamental solution of the operator $-\operatorname{div}(a\nabla .)$ to solve it. To start with, we establish some general results on equations

$$-\operatorname{div}(a\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d, \tag{2.31}$$

for coercive coefficients a of the form (2.2) and right hand side f in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ in order to deduce the existence of the corrector stated in Theorem 2.1. For this purpose, we consider the following strategy adapted from [27] : We first study diffusion problem (2.35) in the periodic context, that is, when the diffusion coefficient $a = a_{per}$ is periodic. Secondly we show in lemma 2.5 the continuity of the associated reciprocal linear operator $\nabla\left(-\operatorname{div} a\nabla\right)^{-1}\operatorname{div}$ from $\mathcal{B}^2(\mathbb{R}^d)$ to $\mathcal{B}^2(\mathbb{R}^d)$. Finally, we use this continuity in order to generalize the existence results of the periodic context to the general context when a is a perturbed coefficient of the form (2.2). To this end, we apply a method based on the connexity of the set $\mathcal{I} = [0, 1]$ as we shall see in the proof of Lemma 2.6.

2.4.1 Preliminary uniqueness results

We begin by establishing the uniqueness of a solution u to (2.31) such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)^d$. This result is actually essential in the proof of Theorem 2.1 since it both ensures the uniqueness of the corrector solution (2.14) and also allows us to establish the continuity estimate of Lemma 2.5 which is key in our approach to show the existence of a solution to (2.31).

Lemma 2.1. *Let a be an elliptic and bounded coefficient, and $u \in H^1_{loc}(\mathbb{R}^d)$, such that*

$$\sup_{p\in\mathcal{P}} \int_{V_p} |\nabla u|^2 < \infty,$$

be a solution to :

$$-\operatorname{div}(a\nabla u) = 0 \quad \textit{in } \mathbb{R}^d, \tag{2.32}$$

in the sense of distribution. Then $\nabla u = 0$.

Proof. we consider $u \in H^1_{loc}(\mathbb{R}^d)$ solution of (2.32). Since u is a solution to (2.32), there exists $C > 0$ such that for every $R > 0$, we have the following estimate (for details see for instance [46, Proposition 2.1 p.76] and [46, Remark 2.1 p.77]) :

$$\int_{B_R} |\nabla u|^2 \leq \frac{C}{R^2}\int_{A_{R,2R}} |u - \langle u\rangle_{A_{R,2R}}|^2,$$

where :

$$\langle u\rangle_{A_{R,2R}} = \frac{1}{|A_{R,2R}|}\int_{A_{R,2R}} u(x)dx.$$

We use the Poincaré-Wirtinger inequality on the right-hand side and we obtain :

$$\int_{B_R} |\nabla u|^2 \leq C\int_{A_{R,2R}} |\nabla u|^2.$$

Furthermore, we can write this inequality in the following form :

$$\int_{B_R} |\nabla u|^2 \leq \frac{C}{1+C}\int_{B_{2R}} |\nabla u|^2. \tag{2.33}$$

In addition, using Proposition (2.19), we know there exists a constant $C_1 > 0$ independent of R such that :

$$\int_{B_{2R}} |\nabla u|^2 = \sum_{V_p\cap B_{2R}\neq\emptyset}\int_{V_p\cap B_{2R}} |\nabla u|^2 \leq C_1\log(2R)\sup_p\int_{V_p} |\nabla u|^2. \tag{2.34}$$

Next, we define $F(R) = \displaystyle\int_{B_R} |\nabla u|^2$. The inequalities (2.33) and (2.34) yield for all $R > 0$ and for every $n \in \mathbb{N}^*$, we obtain

$$F(R) \leq \left(\frac{C}{1+C}\right)^n F(2^nR) \leq C_1\left(\frac{C}{1+C}\right)^n \log(2^nR)\sup_p\int_{V_p} |\nabla u|^2.$$

Since $\dfrac{C}{1+C} < 1$, we have :

$$\lim_{n\to\infty} \Big(\frac{C}{1+C}\Big)^n \log(2^n R) = 0,$$

and it therefore follows, letting n go to infinity, that $F(R) = 0$ for all $R > 0$, thus $\nabla u = 0$. $\square$

Corollary 2.3. *Let $f \in \mathcal{B}^2(\mathbb{R}^d)^d$, then a solution u of (2.31) with $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)^d$ is unique up to an additive constant.*

Remark 2.2. *Here the restriction made on the dimension is actually not necessary. The result and the proof of Lemma 2.1 of uniqueness still hold if we assume $d = 1$ or $d = 2$.*

Remark 2.3. *We remark that Assumptions (2.2) and (2.13) regarding the structure and the regularity of the coefficient a are not required to establish the uniqueness result of Lemma 2.1. In the proof, we only use the "Hilbert" structure of L^2, induced by the assumptions satisfied by u, and the fact that a is elliptic and bounded.*

2.4.2 Existence results in the periodic problem

Now that uniqueness has been dealt with, we turn to the existence of the solution to (2.31). We need to first establish it for a periodic coefficient considering the equation :

$$-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(f) \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{2.35}$$

We start by introducing the Green function $G_{per} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ associated with the operator $-\operatorname{div}(a_{per}\nabla .)$ on $\mathbb{R}^d$. That is, the unique solution to

$$\begin{cases} -\operatorname{div}_x\left(a_{per}(x)\nabla_x G_{per}(x,y)\right) = \delta_y(x) & \text{in } \mathcal{D}'(\mathbb{R}^d), \\ \displaystyle\lim_{|x-y|\to\infty} G_{per}(x,y) = 0. \end{cases} \tag{2.36}$$

According to the results established in [14, Section 2] about the asymptotic growth of the Green function (see also [11, theorem 13, proof of lemma 17] and [51] for bounded domain or [6, proposition 8] for additional details), there exists $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that for every $x, y \in \mathbb{R}^d$ with $x \neq y$:

$$|\nabla_y G_{per}(x,y)| \leq C_1 \frac{1}{|x-y|^{d-1}}, \tag{2.37}$$

$$|\nabla_x G_{per}(x,y)| \leq C_2 \frac{1}{|x-y|^{d-1}}, \tag{2.38}$$

$$|\nabla_x \nabla_y G_{per}(x,y)| \leq C_3 \frac{1}{|x-y|^{d}}. \tag{2.39}$$

We first introduce a result of existence in the $L^2(\mathbb{R}^d)$ case. The following lemma allows us to define a solution to (2.35) using the Green function when f belongs to $\left(L^2(\mathbb{R}^d)\right)^d$. The proof of this result is established in [14].

Lemma 2.2. *Let f be in $\left(L^2(\mathbb{R}^d)\right)^d$, then the function :*

$$u = \int_{\mathbb{R}^d} \nabla_y G_{per}(.,y).f(y)dy, \tag{2.40}$$

is a solution in $H^1_{loc}(\mathbb{R}^d)$ to (2.35) such that $\nabla u \in \left(L^2(\mathbb{R}^d)\right)^d$.

Our aim is now to generalize the above result to our case and, in particular, to give a sense to the function u define by (2.40) when $f \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. The idea here is to split the function f into a sum of L^2-functions f_p compactly supported in each V_p for $p \in \mathcal{P}$. Using Lemma 2.2, we shall obtain the existence of a collection u_p of solution to (2.35) when $f = f_p$. The main difficulty here is to show that the function u defined as the sum of the u_p is bounded.

Lemma 2.3. *Let $f \in \left(L^2_{loc}(\mathbb{R}^d)\right)^d$ such that $\sup\limits_{p\in\mathcal{P}} \|f\|_{L^2(V_p)} < \infty$, then the function u defined by*

$$u = \int_{\mathbb{R}^d} \nabla_y G_{per}(.,y)f(y)dy \tag{2.41}$$

is a solution in $H^1_{loc}(\mathbb{R}^d)$ to (2.35). In addition, u is the unique solution to (2.35) which satisfies $\sup\limits_{p\in\mathcal{P}} \|\nabla u\|_{L^2(V_p)} < \infty$ and there exists $C > 0$ independent of f and u such that we have the following estimate :

$$\sup_{p\in\mathcal{P}} \|\nabla u\|_{L^2(V_p)} \leq C \sup_{p\in\mathcal{P}} \|f\|_{L^2(V_p)}. \tag{2.42}$$

Proof. Step 1 : u is well defined

We start by proving that definition (2.41) makes sense and, in particular, that the above integral defines a function u solution to (2.35) in $H^1_{loc}(\mathbb{R}^d)$. In the sequel the letter C denotes a generic constant that may change from one line to another. For every $q \in \mathcal{P}$, we first introduce a set W_q and five constants C_1, C_2, C_3, C_4 and C_5 independent of q and defined by Proposition 2.4 such that :

(i) $V_q \subset W_q$,

(ii) $Diam(W_q) \leq C_1 2^{|q|}$, and $D(V_q, \partial W_q) \geq C_2 2^{|q|}$,

(iii) $\forall r \in \mathcal{P} \setminus \{q\}$, $Dist(2^r, W_q) \geq C_3 2^{|q|}$,

(iv) $\sharp\{r \in \mathcal{P} | V_r \cap W_q \neq \emptyset\} \leq C_4$,

(v) $\forall r \in \mathcal{P} \setminus \{q\}$, $D\left(V_q, V_r \setminus W_q\right) \geq C_5 2^{|r|}$.

To start with, we define for each $q \in \mathcal{P}$ the function :

$$u_q = \int_{\mathbb{R}^d} \nabla_y G_{per}(.,y)f(y)1_{V_q}(y)dy. \tag{2.43}$$

Lemma 2.2 ensures this function is a solution in $H^1_{loc}(\mathbb{R}^d)$ to :

$$\begin{cases} -\operatorname{div}(a_{per}\nabla u_q) = \operatorname{div}(f 1_{V_p}) & \text{in } \mathbb{R}^d, \\ \nabla u_q \in \left(L^2(\mathbb{R}^d)\right)^d. \end{cases}$$

Considering the gradient of (2.43), we have for every $x \in \mathbb{R}^d \setminus V_q$:

$$\nabla u_q(x) = \int_{\mathbb{R}^d} \nabla_x \nabla_y G_{per}(x,y) f(y) 1_{V_q}(y) dy.$$

Next, for every $N \in \mathbb{N}^*$, we define :

$$U_N = \sum_{q\in\mathcal{P},\ |q|\leq N} u_q,$$

and

$$S_N = \nabla U_N = \sum_{q\in\mathcal{P},\ |q|\leq N} \nabla u_q. \tag{2.44}$$

We next show that the two series U_N ans S_N are convergent in $L^2_{loc}(\mathbb{R}^d)$. To this aim, since the collection $(V_p)_{p\in\mathcal{P}}$ is a partition of $\mathbb{R}^d$, it is sufficient to prove that they normally converge in $L^2(V_p)$ for every $p \in \mathcal{P}$. We fix $p \in \mathcal{P}$ and for every $q \in \mathcal{P}$ such that $V_q \cap W_p = \emptyset$, we use the Cauchy-Schwarz inequality to obtain :

$$\begin{aligned}
\|u_q\|_{L^2(V_p)} &= \left(\int_{V_p} \left|\int_{V_q} \nabla_y G_{per}(x,y) f(y) dy\right|^2 dx\right)^{1/2} \\
&\leq \left(\int_{V_p}\int_{V_q} |\nabla_y G_{per}(x,y)|^2 dy \int_{V_q} |f(y)|^2 dy\, dx\right)^{1/2}.
\end{aligned}$$

Next, estimate (2.37) gives :

$$\|u_q\|_{L^2(V_p)} \leq C \sup_{r\in\mathcal{P}} \|f\|_{L^2(V_r)} \left(\int_{V_p}\int_{V_q} \frac{1}{|x-y|^{2d-2}} dy\, dx\right)^{1/2}. \tag{2.45}$$

Since $V_q \cap W_p = \emptyset$, Property (v) gives the existence of $C > 0$ such that for every $x \in V_p$ and $y \in V_q$, we have $|x-y| \geq C 2^{|q|}$. We next use Propositions 2.1 and 2.5 to obtain the existence of a constant $C > 0$ independent of p and q such that $|V_q| \leq C 2^{d|q|}$. Finally :

$$\begin{aligned}
\|u_q\|_{L^2(V_p)} &\leq C \sup_{r\in\mathcal{P}} \|f\|_{L^2(V_r)} \left(\int_{V_p}\int_{V_q} \frac{1}{2^{(2d-2)|q|}} dy\, dx\right)^{1/2} \\
&\leq C \sup_{r\in\mathcal{P}} \|f\|_{L^2(V_r)} \left(\int_{V_p} \frac{|V_q|}{2^{(2d-2)|q|}} dx\right)^{1/2} \\
&\leq C \sup_{r\in\mathcal{P}} \|f\|_{L^2(V_r)} |V_p|^{1/2} \frac{1}{2^{|q|(d/2-1)}}.
\end{aligned}$$

We thus obtain the following upper bound :

$$\begin{aligned}
\sum_{q\in\mathcal{P}} \|u_q\|_{L^2(V_p)} &= \sum_{\substack{q\in\mathcal{P},\\ V_q\cap W_p\neq\emptyset}} \|u_q\|_{L^2(V_p)} + \sum_{\substack{q\in\mathcal{P},\\ V_q\cap W_p=\emptyset}} \|u_q\|_{L^2(V_p)} \\
&\leq \sum_{\substack{q\in\mathcal{P},\\ V_q\cap W_p\neq\emptyset}} \|u_q\|_{L^2(V_p)} + C \sum_{q\in\mathcal{P}} \frac{1}{2^{|q|(d/2-1)}}.
\end{aligned}$$

The first sum is finite according to Property (iv) and we only have to prove the convergence of the second one. We have assumed $d > 2$ and consequently $d/2 - 1 > 0$. In addition, since the number of $q \in \mathcal{P}$ such that $|q| = n \in \mathbb{N}$ is bounded independently of n (as a consequence of Proposition 2.3), we have :

$$\sum_{q\in\mathcal{P}} \frac{1}{2^{|q|(d/2-1)}} \leq C \sum_{n\in\mathbb{N}} \frac{1}{2^{n(d/2-1)}} < \infty. \tag{2.46}$$

Therefore, for every $p \in \mathcal{P}$, the absolute convergence of U_N to u in $L^2(V_p)$ is proved. That is, since the sequence of the sets V_q defines a partition of $\mathbb{R}^d$, U_N converges to u in $L^2_{loc}(\mathbb{R}^d)$. Using asymptotic estimate (2.39) for $\nabla_x \nabla_y G_{per}$ we can conclude with the same arguments to prove the convergence of S_N in $L^2_{loc}(\mathbb{R}^d)$. In addition, the gradient operator being continuous in $\mathcal{D}'(\mathbb{R}^d)$, we have :

$$\sum_{q\in\mathcal{P}} \nabla u_q = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \nabla U_N = \nabla u.$$

To complete the proof, we have to show that u is a solution to (2.35). Let N be in $\mathbb{N}$. By linearity of the operator $\operatorname{div}(\mathrm{a}_{\mathrm{per}}\nabla.\,)$, U_N is a solution in $H^1_{loc}(\mathbb{R}^d)$ to :

$$-\operatorname{div}\left(a_{per}\nabla U_N\right) = \operatorname{div}\left(\sum_{q\in\mathcal{P},\ |q|\leq N} 1_{V_q} f\right) \quad \text{in } \mathbb{R}^d. \tag{2.47}$$

We take the L^2_{loc}-limit when $N \to \infty$ in (2.47) and we obtain :

$$-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d.$$

Therefore, u is a solution to (2.35) in $\mathcal{D}'(\mathbb{R}^d)$.

Step 2 : Proof of Estimate (2.42)

Let p be in $\mathcal{P}$, we want to split u in two parts. For every $x \in V_p$, we write :

$$\begin{aligned} u(x) &= \int_{W_p} \nabla_y G_{per}(x,y) f(y)dy + \int_{\mathbb{R}^d\setminus W_p} \nabla_y G_{per}(x,y) f(y)dy \\ &= I_{1,p}(x) + I_{2,p}(x). \end{aligned}$$

$I_{1,p}$ and $I_{2,p}$ are two distributions (they are in $L^2_{loc}(\mathbb{R}^d)$), so we can consider their gradients in a distribution sense. In addition, $I_{2,p}$ is a differentiable function on V_p and

$$\nabla I_{2,p}(x) = \int_{\mathbb{R}^d\setminus W_p} \nabla_x \nabla_y G_{per}(x,y) f(y)dy.$$

We start by establishing a bound for $\|\nabla I_{1,p}\|_{L^2(V_p)}$. First, we use estimate (2.37) for $\nabla_y G_{per}$ and we obtain :

$$\|I_{1,p}\|^2_{L^2(W_p)} \leq C \int_{W_p} \left(\int_{W_p} \frac{1}{|x-y|^{d-1}} |f(y)|dy \right)^2 dx.$$

We next apply the Cauchy-Schwarz inequality :

$$\|I_{1,p}\|^2_{L^2(W_p)} \leq C \int_{W_p} \left(\int_{W_p} \frac{1}{|x-y|^{d-1}} dy \right) \left(\int_{W_p} \frac{1}{|x-y|^{d-1}} |f(y)|^2 dy \right) dx.$$

Property (ii) implies that $W_p \subset Q_{C_1 2^{|p|}}(2^p)$. Therefore, for every $x \in W_p$ and $y \in W_p$, we have by a triangle inequality that $x - y \in Q_{C2^{|p|+1}}$ and then :

$$\int_{W_p} \frac{1}{|x-y|^{d-1}} dy \leq \int_{Q_{C2^{|p|+1}}} \frac{1}{|y|^{d-1}} dy \leq C2^{|p|}. \tag{2.48}$$

Using (2.48) and the Fubini theorem, we finally obtain :

$$\|I_{1,p}\|^2_{L^2(W_p)} \leq C2^{|p|} \int_{W_p} |f(y)|^2 \int_{Q_{C_1 2^{|p|+1}}(2^p)} \frac{1}{|x-y|^{d-1}} dx dy \leq C2^{2|p|} \|f\|^2_{L^2(W_p)}. \tag{2.49}$$

Lemma 2.2 ensures that $I_{1,p}$ is a solution in $\mathcal{D}'(\mathbb{R}^d)$ to :

$$-\operatorname{div}(a_{per} \nabla I_{1,p}) = \operatorname{div}(f 1_{W_p}). \tag{2.50}$$

Since Property (ii) ensures $D(\partial V_p, W_p) \geq C2^{|p|}$, we can apply a classical inequality of elliptic regularity (see for instance [47, Theorem 4.4 p.63]) to equation (2.50) in order to establish the following estimate :

$$\|\nabla I_{1,p}\|^2_{L^2(V_p)} \leq C \left(\frac{1}{2^{2|p|}} \|I_{1,p}\|^2_{L^2(W_p)} + \|f\|^2_{L^2(W_p)} \right), \tag{2.51}$$

and we deduce from previous inequalities (2.49) and (2.51) that :

$$\|\nabla I_{1,p}\|^2_{L^2(V_p)} \leq C \|f\|^2_{L^2(W_p)}. \tag{2.52}$$

In addition, we have :

$$\|f\|^2_{L^2(W_p)} \leq \sum_{\substack{q \in \mathcal{P}, \\ V_q \cap W_p \neq \emptyset}} \|f\|^2_{L^2(V_q)} \leq \sum_{\substack{q \in \mathcal{P}, \\ V_q \cap W_p \neq \emptyset}} \sup_{r \in \mathcal{P}} \|f\|^2_{L^2(V_r)}.$$

Next, we use a triangle inequality and Property (iv) of W_p to obtain :

$$\|f\|^2_{L^2(W_p)} \leq C \sup_{r \in \mathcal{P}} \|f\|^2_{L^2(V_r)}.$$

We apply this inequality in (2.52) and we finally obtain :

$$\|\nabla I_{1,p}\|_{L^2(V_p)} \leq C \sup_{r \in \mathcal{P}} \|f\|_{L^2(V_r)}, \tag{2.53}$$

where $C > 0$ is independent of p.

We next prove a similar bound for $\|\nabla I_{2,p}\|_{L^2(V_p)}$. To start with, we want to show there exists a constant $C > 0$ such that :

$$\|\nabla I_{2,p}\|_{L^\infty(V_p)} \leq C \frac{1}{2^{d|p|/2}} \sup_{r \in \mathcal{P}} \|f\|_{L^2(V_r)}. \tag{2.54}$$

To this aim, we fix $x \in V_p$ and we use estimate (2.39) for $\nabla_x \nabla_y G_{per}$ to obtain :

$$
\begin{aligned}
|\nabla I_{2,p}(x)| &\leq C \sum_{q \neq p} \int_{V_q \setminus W_p} \frac{1}{|x-y|^d} |f(y)| dy \\
&\leq C \sum_{q \neq p} \left(\int_{V_q \setminus W_p} \frac{1}{|x-y|^{2d}} dy \right)^{1/2} \left(\int_{V_q \setminus W_p} |f(y)|^2 dy \right)^{1/2} \\
&\leq C \sum_{q \neq p} \left(\int_{V_q \setminus W_p} \frac{1}{|x-y|^{2d}} dy \right)^{1/2} \sup_{r \in \mathcal{P}} \|f\|_{L^2(V_r)}
\end{aligned}
$$

Next, using Property (ii) of W_p, there exists C>0 such that for every $q \neq p$,

$$
D(V_q \setminus W_p, V_p) > C 2^{|p|},
$$

and it follows :

$$
\begin{aligned}
\sum_{|q|<|p|} \left(\int_{V_q \setminus W_p} \frac{1}{|x-y|^{2d}} dy \right)^{1/2} &\leq C \sum_{|q|<|p|} \left(\int_{V_q \setminus W_p} \frac{1}{2^{|p|2d}} dy \right)^{1/2} \\
&\leq C \sum_{|q|<|p|} \left(\frac{|V_q|}{2^{|p|2d}} \right)^{1/2} \\
&\leq C \sum_{|q|<|p|} \frac{2^{|q|d/2}}{2^{|p|d}}.
\end{aligned}
$$

The last inequality is actually a direct consequence of Propositions 2.1 and 2.5. In addition, we have proved in proposition 2.3 there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$, the number of $q \in \mathcal{P}$ such that $|q| = n$ is bounded by C. Therefore we have :

$$
\sum_{|q|<|p|} \frac{2^{|q|d/2}}{2^{|p|d}} = \sum_{n=0}^{|p|} \sum_{q \in \mathcal{P}, |q|=n} \frac{2^{|q|d/2}}{2^{|p|d}} \leq C \sum_{n=0}^{|p|} \frac{2^{nd/2}}{2^{|p|d}} = C \frac{2^{|p|d/2}}{2^{|p|d}} = C \frac{1}{2^{|p|d/2}}.
$$

And finally :

$$
\sum_{|q|<|p|} \left(\int_{V_q \setminus W_p} \frac{1}{|x-y|^{2d}} dy \right)^{1/2} \leq C \frac{1}{2^{|p|d/2}}. \tag{2.55}
$$

Furthermore, we have with similar arguments :

$$
\begin{aligned}
\sum_{|q|\geq|p|} \left(\int_{V_q \setminus W_p} \frac{1}{|x-y|^{2d}} dy \right)^{1/2} &\leq C \sum_{|q|\geq|p|} \left(\int_{V_q \setminus W_p} \frac{1}{2^{|q|2d}} dy \right)^{1/2} \\
&\leq C \sum_{|q|\geq|p|} \left(\frac{|V_q|}{2^{|q|2d}} \right)^{1/2} \\
&\leq C \sum_{|q|\geq|p|} \frac{1}{2^{|q|d/2}}.
\end{aligned}
$$

And we obtain again :

$$\sum_{|q|\geq|p|}\frac{1}{2^{|q|d/2}}\leq C\sum_{n\geq|p|}\frac{1}{2^{nd/2}}=C\frac{1}{2^{|p|d/2}}.$$

That is :

$$\sum_{|q|\geq|p|}\left(\int_{V_q\setminus W_p}\frac{1}{|x-y|^{2d}}dy\right)^{1/2}\leq C\frac{1}{2^{|p|d/2}}. \tag{2.56}$$

Using estimates (2.55) and (2.56), we have finally proved (2.54) and it follows :

$$\begin{aligned}\|\nabla I_{2,p}\|_{L^2(V_p)} &\leq |V_p|^{1/2}\,\|\nabla I_{2,p}\|_{L^\infty(V_p)}\\ &\leq C2^{|p|d/2}\frac{1}{2^{|p|d/2}}\sup_{r\in\mathcal{P}}\|f\|_{L^2(V_r)}.\end{aligned}$$

Therefore we have the existence of a constant $C>0$ independent of p such that :

$$\|\nabla I_{2,p}\|_{L^2(V_p)}\leq C\sup_{r\in\mathcal{P}}\|f\|_{L^2(V_r)}. \tag{2.57}$$

For every $p\in\mathcal{P}$, using estimates (2.53) and (2.57) and a triangle inequality, we conclude that :

$$\|\nabla u\|_{L^2(V_p)}\leq\|\nabla I_{1,p}\|_{L^2(V_p)}+\|\nabla I_{2,p}\|_{L^2(V_p)}\leq C\sup_{r\in\mathcal{P}}\|f\|_{L^2(V_r)}.$$

We finally obtain expected estimate (2.42) taking the supremum over all $p\in\mathcal{P}$ in the above inequality. □

To conclude the study of problem (2.35) with a periodic coefficient, we next show that the solution to (2.41) given in Lemma 2.3 has a gradient in $\mathcal{B}^2(\mathbb{R}^d)$.

Lemma 2.4. *Let $f\in\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, then the function u defined by (2.41) is the unique solution to (2.35) such that $\nabla u\in\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$.*

Proof. We want to prove there exists a function $g\in\left(L^2(\mathbb{R}^d)\right)^d$ such that

$$\lim_{|p|\to\infty}\|\nabla u-\tau_{-p}g\|_{L^2(V_p)}=0.$$

In this proof, the letter C also denotes a generic constant independent of p, u and f that may change from one line to another. Using the result of Lemma 2.2, we can define a function $u_\infty\in L^2_{loc}(\mathbb{R}^d)$ by :

$$u_\infty(x)=\int_{\mathbb{R}^d}\nabla_y G_{per}(x,y)f_\infty(y)dy$$

solution in $\mathcal{D}'(\mathbb{R}^d)$ to :

$$-\operatorname{div}(a_{per}\nabla u_\infty)=\operatorname{div}(f_\infty)\quad\text{in }\mathbb{R}^d, \tag{2.58}$$

such that $\nabla u_\infty\in\left(L^2(\mathbb{R}^d)\right)^d$. For every $p\in\mathcal{P}$, by subtracting a 2^p-translation of (2.58) from (2.35), the periodicity of a_{per} implies :

$$-\operatorname{div}\left(a_{per}\nabla\left(u-\tau_{-p}u_\infty\right)\right)=\operatorname{div}\left(f-\tau_{-p}f_\infty\right).$$

For every $p \in \mathcal{P}$, in the sequel we denote $u_p = u - \tau_{-p}u_\infty$ and $f_p = f - \tau_{-p}f_\infty$. In order to prove $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, the idea is to show that $\lim\limits_{|p|\to\infty} \displaystyle\int_{V_p} |\nabla u_p|^2 dx = 0$. We start by fixing $\varepsilon > 0$. Since $f \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, Proposition 2.12 gives the existence of a radius $R > 0$, such that for every $p, q \in \mathcal{P}$,

$$\left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-p}f_\infty|^2\right)^{1/2} < \varepsilon. \tag{2.59}$$

In the sequel, the idea is to repeat step by step the method used in the proof of Lemma 2.3. For $p \in \mathcal{P}$, we thus introduce the set W_p as in the previous proof and we split u_p in two parts. For every $x \in V_p$, we can write :

$$\begin{aligned} u_p(x) &= \int_{W_p} \nabla_y G_{per}(x,y) f_p(y) dy + \int_{\mathbb{R}^d \setminus W_p} \nabla_y G_{per}(x,y) f_p(y) dy \\ &= I_{1,p}(x) + I_{2,p}(x). \end{aligned}$$

In the sequel, we denote A_p the set $W_p \setminus V_p$. As in the previous proof (see the details of the proof of estimate (2.52)) we can show that :

$$\|\nabla I_{1,p}\|^2_{L^2(V_p)} \leq C\|f_p\|^2_{L^2(W_p)},$$

and we next prove that $\lim\limits_{|p|\to\infty} \|f_p\|^2_{L^2(W_p)} = 0$. First, since $f \in \left(\mathcal{B}(\mathbb{R}^d)\right)^d$, we already know that $\lim\limits_{|p|\to\infty} \displaystyle\int_{V_p} |f_p|^2 = 0$ and we only have to treat the integration term on A_p. Using Property (iii) of W_p, we know that the distance $D(2^q, W_p)$, for $q \neq p$, is bounded from below by $2^{|p|}$. Therefore, if $2^{|p|} > R$, we obtains :

$$A_p = \bigcup_{\substack{q \in \mathcal{P}\setminus\{p\} \\ V_q \cap W_p \neq \emptyset}} V_q \cap W_p \subset \bigcup_{\substack{q \in \mathcal{P}\setminus\{p\} \\ V_q \cap W_p \neq \emptyset}} V_q \cap B_R(2^q)^c.$$

In addition, Property (iv) of W_p gives the existence of a constant $C > 0$ such that the cardinality of the set of q satisfying $V_q \cap W_p \neq \emptyset$ is bounded by C. Estimate (2.59) therefore implies that

$$\int_{A_p} |f_p|^2 \leq \sum_{\substack{q \in \mathcal{P}\setminus\{p\} \\ V_q \cap W_p \neq \emptyset}} \int_{V_q \cap B_R(2^q)^c} |f_p|^2 \leq C\varepsilon.$$

Since ε can be chosen arbitrarily small, we finally obtain $\lim\limits_{|p|\to\infty} \displaystyle\int_{A_p} |f_p|^2 = 0$, that is

$$\lim_{|p|\to\infty} \|\nabla I_{1,p}\|^2_{L^2(V_p)} = 0.$$

We next prove that $\lim\limits_{|p|\to\infty} \|\nabla I_{2,p}\|^2_{L^2(V_p)} = 0$. We split $\nabla I_{2,p}$ in two parts such that for every

$x \in V_p$:

$$\begin{aligned}\nabla I_{2,p}(x) &= \sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \int_{(V_q\setminus W_p)\cap B_R(2^q)^c} \nabla_x\nabla_y G_{per}(x,y) f_p(y)dy \\ &+ \sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \int_{(V_q\setminus W_p)\cap B_R(2^q)} \nabla_x\nabla_y G_{per}(x,y) f_p(y)dy \\ &= J_{1,p}(x) + J_{2,p}(x).\end{aligned}$$

We want to estimate $\|J_{1,p}\|_{L^2(V_p)}$ and $\|J_{1,p}\|_{L^2(V_p)}$. We proceed exactly in the same way as in the previous proof (see the details of estimate (2.54)) and, using estimate (2.59), we obtain the following inequalities :

$$\|J_{1,p}\|_{L^\infty(V_p)} \leq C\frac{1}{2^{d|p|/2}} \sup_{q\in\mathcal{P}} \|f_p\|_{L^2(V_q\cap B_R(2^q)^c)} \leq C\frac{1}{2^{d|p|/2}}\varepsilon, \tag{2.60}$$

and

$$\|J_{2,p}\|_{L^\infty(V_p)} \leq CR^d\frac{|p|}{2^{|p|d}} \sup_{q\in\mathcal{P}} \|f_p\|_{L^2(V_q)}. \tag{2.61}$$

To conclude, we consider $P > 0$ such that for every $p \in \mathcal{P}$ satisfying $|p| > P$, we have :

$$R^d\frac{|p|}{2^{|p|d/2}} < \varepsilon.$$

Therefore, for every $|p| > P$, we use (2.60) and (2.61) and we obtain :

$$\begin{aligned}\|\nabla I_{2,p}\|_{L^2(V_p)} &\leq \|J_{1,p}\|_{L^2(V_p)} + \|J_{2,p}\|_{L^2(V_p)} \\ &\leq |V_p|^{1/2}\left(\|J_{1,p}\|_{L^\infty(V_p)} + \|J_{2,p}\|_{L^\infty(V_p)}\right) \\ &\leq C2^{|p|d/2}\left(\frac{1}{2^{|p|d/2}}\varepsilon + R^d\frac{|p|}{2^{|p|d}}\right) \leq C\varepsilon.\end{aligned}$$

Since we can choose ε arbitrarily small, we conclude that $\lim_{|p|\to\infty}\|\nabla I_{2,p}\|_{L^2(V_p)} = 0$. Finally, by a triangle inequality we have $\lim_{|p|\to\infty}\|\nabla u_p\|_{L^2(V_p)} = 0$, that is $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. □

Remark 2.4. *It is important to note that the essential point of the two above proofs is the convergence of the sums of the form* $\sum_{q\in\mathcal{P}}\int_{V_q}\frac{1}{|x-y|^d}f(y)dy$ *given in estimates* (2.45), (2.55) *and* (2.56). *Although we use here the particular distribution of the* 2^p, *these convergence results are not specific to the set* (2.9) *considered in this study. They are actually ensured by Assumptions* (H1), (H2) *and* (H3), *particularly by the logarithmic bound given in Proposition 2.3 and Corollary 2.1. Indeed, with the notations of Section 2.1.2 and given Assumptions* (H1)-(H2)-(H3), *we could similarly argue to obtain estimates such as in* (2.55)-(2.56) *by splitting the sums over each annulus* $A_n := A_{2^n,2^{n+1}}$ *and studying* $\sum_{n\in\mathbb{N}}\sum_{x_q\in\mathcal{G}\cap A_n}\int_{V_{x_q}}\frac{1}{|x-y|^d}f(y)dy$. *The results of existence stated in this section therefore still hold if we consider a generic set* $\mathcal{G}$ *satisfying our general assumptions.*

Remark 2.5. *In the two-dimensional context, the results of Lemmas 2.2, 2.3 and 2.4 remain true since estimates (2.37), (2.38) and (2.37) still hold. However the proof requires some additional technicalities, in particular to prove that the function u defined by (2.41) makes sense. In this case the series (2.46) does not actually converges but it is still possible to prove that the series of the gradients (2.44) converges. Here, the difficulty is to show that the limit of (2.44), denoted by T here, is the gradient in a distribution sense of a solution to (2.35). To this end, it is actually sufficient to show that $\partial_i T_j = \partial_j T_i$ for every $i, j \in \{1, ..., d\}$. This result is obtained considering the property of the limit of (2.44) in $\mathcal{D}(\mathbb{R}^d)$.*

2.4.3 Existence results in the general problem

Our aim is now to generalize the results established in the case of periodic coefficients to our original problem (2.31). Here, our approach is to prove in Lemma 2.5 the continuity of the linear operator $\nabla\left(-\operatorname{div} a\nabla\right)^{-1}\operatorname{div}$ from $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ in order to apply a method adapted from [27] and based on the connexity of the set $[0,1]$. This method is used in the proof of existence of Lemma 2.6. Finally, this result allows us to prove the existence of a corrector stated in Theorem 2.1.

Actually, we could have proved Lemmas 2.5 and 2.6 simultaneously but, in the interest of clarity, we first prove a priori estimate (2.62) and next, we establish the existence result in the general case.

Lemma 2.5 (A priori estimate)**.** *There exists a constant $C > 0$ such that for every f in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ and u solution in $\mathcal{D}'(\mathbb{R}^d)$ to (2.31) with ∇u in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, we have the following estimate :*

$$\|\nabla u\|_{\mathcal{B}^2(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{B}^2(\mathbb{R}^d)}. \tag{2.62}$$

Proof. We give here a proof by contradiction using a compactness-concentration method. We assume that there exists a sequence f_n in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ and an associated sequence of solutions u_n such that ∇u_n is in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ and :

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u_n) = \operatorname{div}(f_n), \tag{2.63}$$

$$\lim_{n\to\infty} \|f_n\|_{\mathcal{B}^2(\mathbb{R}^d)} = 0, \tag{2.64}$$

$$\forall n \in \mathbb{N} \quad \|\nabla u_n\|_{\mathcal{B}^2(\mathbb{R}^d)} = 1. \tag{2.65}$$

First of all, a property of the supremum bound ensures that for every $n \in \mathbb{N}$, there exists $x_n \in \mathbb{R}^d$ such that :

$$\|\nabla u_n\|_{L^2_{unif}} \geq \|\nabla u_n\|_{L^2(B_1(x_n))} \geq \|\nabla u_n\|_{L^2_{unif}} - \frac{1}{n}.$$

Next, in the spirit of the method of concentration-compactness [70], we denote $\bar{u}_n = \tau_{x_n} u_n$, $\bar{f}_n = \tau_{x_n} f_n$, $\bar{a}_n = \tau_{x_n} a$ and $\bar{\tilde{a}}_n = \tau_{x_n}\tilde{a}$ and we have for every $n \in \mathbb{N}$:

$$\|\nabla u_n\|_{L^2_{unif}} \geq \|\nabla \bar{u}_n\|_{L^2(B_1)} \geq \|\nabla u_n\|_{L^2_{unif}} - \frac{1}{n}. \tag{2.66}$$

Next, for every $n \in \mathbb{N}$, $\bar{u}_n$ is a solution to :

$$-\operatorname{div}(\bar{a}_n \nabla \bar{u}_n) = \operatorname{div}(\bar{f}_n) \quad \text{in } \mathbb{R}^d.$$

Since the norm of L^2_{unif} is invariant by translation, (2.64) and (2.65) ensure that $\bar{f}_n$ strongly converges to 0 in $L^2_{unif}(\mathbb{R}^d)$ and that the sequence $(\nabla \bar{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2_{unif}(\mathbb{R}^d)$. Therefore, up to an extraction, $\nabla \bar{u}_n$ weakly converges to a function $\nabla \bar{u}$ in $L^2_{loc}(\mathbb{R}^d)$.
The idea is now to study the limit of $\bar{a}_n$. To start with, we denote $\mathbf{x}_n = (x_{n,i} \bmod(1))_{i\in\{1,\dots,d\}}$. Since a_{per} is periodic, we have $\tau_{x_n} a_{per} = \tau_{\mathbf{x}_n} a_{per}$. In addition, the sequence $\mathbf{x}_n$ belongs to the unit cube of $\mathbb{R}^d$ and, therefore, it converges (up to an extraction) to $\mathbf{x} \in \mathbb{R}^d$. Since a_{per} is Holder continuous, $\tau_{\mathbf{x}_n} a_{per}$ converges uniformly to $\tau_{\mathbf{x}} a_{per}$, which is also in $\left(L^2_{per}(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$.
In order to study the convergence of $\bar{\tilde{a}}_n$, we consider several cases depending on x_n :

1. If x_n is bounded, it converges (up to an extraction) to $x_{lim} \in \mathbb{R}^d$. Then, since $\tilde{a}$ is Holder-continuous, $\bar{\tilde{a}}_n$ strongly converges in $L^2_{loc}(\mathbb{R}^d)$ to $\tau_{x_{lim}} \tilde{a} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d}$.
2. If x_n is not bounded, since $(V_p)_{p\in\mathcal{P}}$ is a partition of $\mathbb{R}^d$, there exists an unbounded sequence $(p_n)_n$ in $\mathcal{P}$ such that $x_n = 2^{p_n} + t_n$ with $t_n \in V_{p_n} - 2^{p_n}$.
   - If t_n is bounded, it converges (up to an extraction) to $t_{lim} \in \mathbb{R}^d$. In this case, for any compact subset K of $\mathbb{R}^d$, we have

$$\begin{aligned}
\|\bar{\tilde{a}}_n - \tilde{a}_\infty(. + t_{lim})\|_{L^2(K)} &\leq \|\tilde{a}(. + 2^{p_n} + t_n) - \tilde{a}_\infty(. + t_n)\|_{L^2(K)} \\
&\quad + \|\tilde{a}_\infty(. + t_n) - \tilde{a}_\infty(. + t_{lim})\|_{L^2(K)} \\
&= \|\tilde{a} - \tau_{-p_n}\tilde{a}_\infty\|_{L^2(K+2^{p_n}+t_n)} \\
&\quad + \|\tilde{a}_\infty(. + t_n) - \tilde{a}_\infty(. + t_{lim})\|_{L^2(K)}.
\end{aligned}$$

     First, since t_n is bounded and p_n is unbounded, we have $K + 2^{p_n} + t_n \subset V_{p_n}$ for n sufficiently large. Therefore, $\|\tilde{a} - \tau_{-p_n}\tilde{a}_\infty\|_{L^2(K+2^{p_n}+t_n)}$ converges to 0 when $n \to \infty$. Second, $\tilde{a}_\infty$ is Holder-continuous and t_n converges to t_{lim}. Thus, $\tilde{a}_\infty(.+t_n)$ converges uniformly to $\tilde{a}_\infty(.+t_{lim})$ and $\|\tilde{a}_\infty(.+t_n) - \tilde{a}_\infty(.+t_{lim})\|_{L^2(K)}$ converges to 0. Finally, $\bar{\tilde{a}}_n$ converges to $\tilde{a}_\infty(. + t_{lim})$ in $L^2(K)$ for every compact subset K.

   - If t_n is unbounded, we can always assume that $|t_n| \to \infty$ up to an extraction. We have for every K compact of $\mathbb{R}^d$,

$$\begin{aligned}
\|\bar{\tilde{a}}_n\|_{L^2(K)} &\leq \|\tilde{a}(. + 2^{p_n} + t_n) - \tilde{a}_\infty(. + t_n)\|_{L^2(K)} + \|\tilde{a}_\infty(. + t_n)\|_{L^2(K)} \\
&= \|\tilde{a} - \tau_{-p_n}\tilde{a}_\infty\|_{L^2(K+2^{p_n}+t_n)} + \|\tilde{a}_\infty\|_{L^2(K+t_n)}.
\end{aligned}$$

     First, since $\tilde{a}_\infty \in \left(L^2(\mathbb{R}^d)\right)^{d\times d}$ and t_n is unbounded we have that $\|\tilde{a}_\infty\|_{L^2(K+t_n)}$ converges to 0 when $n \to \infty$. Secondly, we introduce the set $W_{2^{p_n}}$ defined as in Proposition 2.4. For every $R > 0$, the properties of $W_{2^{p_n}}$ allow to show that there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $K + 2^{p_n} + t_n \subset W_{2^{p_n}}$ and :

$$K + 2^{p_n} + t_n \subset \bigcup_{\substack{q\in\mathcal{P} \\ V_q \cap W_{2^{p_n}} \neq \emptyset}} V_q \setminus B_R(2^q).$$

     Using Proposition 2.4, we know that the number of q such that $V_q \cap W_{2^{p_n}} \neq \emptyset$ is uniformly bounded with respect to n and Proposition 2.12 finally ensures that $\|\tilde{a} - \tau_{-p_n}\tilde{a}_\infty\|_{L^2(K+2^{p_n}+t_n)} \to 0$. Therefore, $\bar{\tilde{a}}_n$ strongly converges to 0 in $L^2_{loc}(\mathbb{R}^d)$.

In any case, the sequence $a_{per} + \tilde{\bar{a}}_n$ therefore converges to a coefficient $A = \tau_{\mathbf{x}} a_{per} + \tilde{A}$, where $\tilde{A}$ is of the form

$$\tilde{A} = \begin{cases} \tau_{x_{lim}} \tilde{a} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d} & \text{if } x_n \text{ is bounded,} \\ \tau_{t_{lim}} \tilde{a}_\infty \in \left(L^2(\mathbb{R}^d)\right)^{d\times d} & \text{if } x_n = 2^{p_n} + t_n \text{ for } p_n \text{ not bounded and } t_n \text{ bounded,} \\ 0 & \text{if } x_n = 2^{p_n} + t_n \text{ for } p_n \text{ and } t_n \text{ not bounded.} \end{cases}$$

In the three cases, as a consequence of Assumptions (2.12) and (2.13), the coefficient A is clearly bounded, elliptic and belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$. Moreover, as a consequence of the uniform Holder-continuity (with respect to n) of $\bar{a}_n - A$, the convergence of $\bar{a}_n$ to A is also valid in $L^\infty_{loc}(\mathbb{R}^d)$.

The next step of the proof is to study the limit $\nabla \bar{u}$ of $\nabla \bar{u}_n$ in these three cases. First, since $\bar{a}_n$ strongly converges to A in $L^2_{loc}(\mathbb{R}^d)$, considering the weak limit in (2.63) when $n \to \infty$, we obtain

$$-\operatorname{div}(A\nabla \bar{u}) = 0 \quad \text{in } \mathbb{R}^d. \tag{2.67}$$

We now state that $\nabla \bar{u} = 0$. Indeed,

1. if x_n is bounded, assumption (2.65) ensures that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$, we have :

$$\|\nabla \bar{u}_n\|_{L^2(V_p)} = \|\nabla u_n\|_{L^2(V_p + x_n)} \leq C.$$

Therefore, the property of lower semi-continuity satisfied by the norm $\|.\|_{L^2}$ implies

$$\forall p \in \mathcal{P}, \quad \|\nabla \bar{u}\|_{L^2(V_p)} \leq \liminf_{n\to\infty} \|\nabla \bar{u}_n\|_{L^2(V_p)} < C.$$

And we obtain $\sup\limits_p \|\nabla \bar{u}\|_{L^2(V_p)} < \infty$. Finally, since A is elliptic and bounded and $\bar{u}$ is solution to (2.67), the uniqueness results of Lemma 2.1 gives $\nabla \bar{u} = 0$ on $\mathbb{R}^d$.

2. if x_n is not bounded, we know that $x_n = 2^{p_n} + t_n$ where $|p_n| \to \infty$. For every $n \in \mathbb{N}$:

$$\|\nabla \bar{u}_n\|_{L^2(V_{p_n} - x_n)} = \|\nabla u_n\|_{L^2(V_{p_n})} \leq 1.$$

Up to an extraction, the sequence $V_{p_n} - x_n$ is an increasing sequence of sets, and we can show that $\bigcup\limits_{n\in\mathbb{N}} (V_{p_n} - x_n) = \mathbb{R}^d$ (see the proof of Proposition 2.2). Consequently, for every $R > 0$, there exists $N \in \mathbb{N}$ such that $B_R \subset (V_{p_N} - x_N)$ and

$$\forall n > N, \quad \|\nabla \bar{u}_n\|_{L^2(B_R)} \leq 1.$$

Using again lower semi-continuity, we have for every $R > 0$:

$$\|\nabla \bar{u}\|_{L^2(B_R)} \leq \liminf_{n\to\infty} \|\nabla \bar{u}_n\|_{L^2(B_R)} \leq 1.$$

We obtain that $\nabla \bar{u} \in L^2(\mathbb{R}^d)$. Since A is bounded and elliptic, a result of uniqueness established in [25, Lemma 1] finally ensures that $\nabla \bar{u} = 0$.

We are now able to show that $\nabla \bar{u}_n$ strongly converges to 0 in $L^2(B_1)$. To this aim, we note that, for every n, the addition of a constant to $\bar{u}_n$ does not affect $\nabla \bar{u}_n$. Then, without loss of generality, we can always assume that $\int_{B_2} \bar{u}_n = 0$ and the Poincaré-Wirtinger inequality gives the existence of a constant $C > 0$ independent of n such that :

$$\|\bar{u}_n\|_{L^2(B_2)} \leq C \|\nabla \bar{u}_n\|_{L^2(B_2)}.$$

$\bar{u}_n$ is therefore bounded in $H^1(B_2)$ according to Assumption (2.65). The Rellich theorem ensures that, up to an extraction, $\bar{u}_n$ strongly converges to $\bar{u}$, that is to 0, in $L^2(B_2)$. Since $\bar{u}_n$ is solution to (2.63), a classical inequality of elliptic regularity gives the following estimate (see for instance [47, Theorem 4.4 p.63]) :

$$\int_{B_1} |\nabla \bar{u}_n|^2 \leq C \left(\int_{B_2} |\bar{u}_n|^2 + \int_{B_2} |\bar{f}_n|^2 \right),$$

where C depends only of a and the ambient dimension d. We therefore consider the limit when $n \to \infty$ to conclude that $\nabla \bar{u}_n$ strongly converges to 0 in $L^2(B_1)$. We next use (2.66) and the strong convergence of $\nabla \bar{u}_n$ to 0 in $L^2(B_1)$ to conclude that

$$\lim_{n \to \infty} \|\nabla u_n\|_{L^2_{unif}(\mathbb{R}^d)} = 0.$$

That is, ∇u_n strongly converges to 0 in $L^2_{unif}(\mathbb{R}^d)$.

In order to conclude this proof, we will show that ∇u_n actually converges to 0 in $\mathcal{B}^2(\mathbb{R}^d)$ and obtain a contradiction.

First of all, we study the behavior of the sequence $\nabla u_{n,\infty}$. For $p \in \mathcal{P}$, we consider the 2^p-translation of (2.63) and we have

$$-\operatorname{div}((a_{per} + \tau_{-p}\tilde{a})\tau_{-p}\nabla u_n) = \operatorname{div}(\tau_{-p} f_n).$$

Letting $|p|$ go to the infinity, for every $n \in \mathbb{N}$, we obtain that $\nabla u_{n,\infty}$ is a solution to :

$$-\operatorname{div}((a_{per} + \tilde{a}_\infty)\nabla u_{n,\infty}) = \operatorname{div}(f_{n,\infty}) \quad \text{in } \mathbb{R}^d.$$

An estimate established in [27, Proposition 2.1], gives the existence of a constant $C > 0$ independent of n such that :

$$\|\nabla u_{n,\infty}\|_{L^2(\mathbb{R}^d)} \leq C \|f_{n,\infty}\|_{L^2(\mathbb{R}^d)}.$$

By assumption, we have $\lim_{n \to \infty} \|f_{n,\infty}\|_{L^2(\mathbb{R}^d)} = 0$ and we deduce that $\nabla u_{n,\infty}$ strongly converges to 0 in $L^2(\mathbb{R}^d)$, that is :

$$\lim_{n \to \infty} \|\nabla u_{n,\infty}\|_{L^2(\mathbb{R}^d)} = 0.$$

The last step is to establish that :

$$\lim_{n \to \infty} \sup_p \|\nabla u_n\|_{L^2(V_p)} = 0.$$

Let $\varepsilon > 0$. Since $\tilde{a}$ belongs to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d \times d}$ and is uniformly continuous, a direct consequence of Proposition 2.12 gives the existence of $R > 0$ such that :

$$\forall q \in \mathcal{P}, \quad \|\tilde{a}\|_{L^\infty(V_q \cap B_R(2^q)^c)} < \frac{\varepsilon}{2}.$$

In addition, since ∇u_n strongly converges to 0 in L^2_{unif}, there exists $N \in \mathbb{N}$ such that :

$$\forall n > N, \quad \|\nabla u_n\|_{L^2_{unif}(\mathbb{R}^d)} < \frac{\varepsilon}{2|B_R|\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)}}.$$

Using the last two inequalities, we obtain for every $q \in \mathcal{P}$:

$$\begin{aligned}\int_{V_q} |\tilde{a}(x)\nabla u_n(x)|^2 dx &\leq \int_{V_q \cap B_R(2^q)^c} |\tilde{a}(x)\nabla u_n(x)|^2 dx + \int_{V_q \cap B_R(2^q)} |\tilde{a}(x)\nabla u_n(x)|^2 dx \\ &\leq \|\tilde{a}\|_{L^\infty(V_q\cap B_R(2^q)^c)} \int_{V_q \cap B_R(2^q)^c} |\nabla u_n(x)|^2 dx \\ &+ \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \int_{V_p \cap B_R(2^q)} |\nabla u_n(x)|^2 dx \\ &\leq \|\tilde{a}\|_{L^\infty(V_q\cap B_R(2^q)^c)} \sup_p (\|\nabla u_n\|_{L^2(V_p)}) + \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} |B_R| \|\nabla u_n\|_{L^2_{unif}(\mathbb{R}^d)} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore :

$$\lim_{n\to\infty} \sup_p \int_{V_p} |\tilde{a}(x)\nabla u_n(x)|^2 dx = 0.$$

We next consider equation (2.63) and we use Lemma 2.1 to ensure that, up to the addition of a constant, u_n is the unique solution to :

$$-\operatorname{div}(a_{per}\nabla u_n) = \operatorname{div}(f_n + \tilde{a}\nabla u_n) \quad \text{in } \mathbb{R}^d.$$

such that $\sup_p \|\nabla u_n\|_{L^2(V_p)} < \infty$. Then, Estimate (2.42) established in Lemma 2.4 gives the existence of a constant $C > 0$ independent of n such that :

$$\sup_p \|\nabla u_n\|_{L^2(V_p)} \leq C \left(\sup_p \|f_n\|_{L^2(V_p)} + \sup_p \|\tilde{a}\nabla u_n\|_{L^2(V_p)} \right).$$

Letting n go to the infinity, we deduce that $\lim_{n\to\infty} \sup_p \|\nabla u_n\|_{L^2(V_p)} = 0$. We can finally conclude that

$$\lim_{n\to\infty} \|\nabla u_n\|_{\mathcal{B}^2(\mathbb{R}^d)} = 0,$$

and, since ∇u_n satisfies (2.65), we have a contradiction.

□

Lemma 2.6. *Let f be in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. Then, there exists $u \in H^1_{loc}(\mathbb{R}^d)$ solution to (2.31) such that $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$.*

Proof. First of all, we remark that it is sufficient to prove this existence result when $f \in \mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$. Indeed, if we denote $\Phi = \nabla\left(-\operatorname{div} a\nabla\right)^{-1} \operatorname{div}$ the reciprocal linear operator from $\left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ associated with equation (2.31) and we assume that Φ is well defined, Lemma 2.5 ensures it is continuous with respect to the norm of $\mathcal{B}^2(\mathbb{R}^d)$. Then, we are able to conclude in the general case using the density result stated in Property 2.8. In the sequel of this proof, we therefore assume that f belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.

To start with, we show a preliminary result of regularity satisfied by the solutions to (2.31). Assuming $f \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, we want to prove that a solution u to (2.31) such that $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ also satisfies $\nabla u \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Indeed, if u is such a solution to (2.31), a consequence of a regularity result established in [47, Theorem 5.19 p.87] (see also [46, Theorem 3.2 p.88]) gives the existence of $C > 0$ such that for all $x \in \mathbb{R}^d$:

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(B_1(x))} \leq C \left(\|\nabla u\|_{L^2_{unif}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \right). \tag{2.68}$$

Therefore, ∇u belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d) \cap \mathcal{B}^2(\mathbb{R}^d)\right)^d$.

In the sequel of the proof, we use an argument of connexity adapted from [27]. Let $\mathbf{P}(a)$ the following assertion : "There exists a solution $u \in \mathcal{D}'(\mathbb{R}^d)$ to :

$$-\operatorname{div}(a\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d$$

such that $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$."
For $t \in [0,1]$, we denote $a_t = a_{per} + t\tilde{a}$ and we define the following set :

$$\mathcal{I} = \left\{ t \in [0,1] \mid \forall s \in [0,t], \mathbf{P}(a_s) \text{ is true} \right\}.$$

Our aim is to show that $\mathbf{P}(a_1) = \mathbf{P}(a)$ is true. To this end, we will prove that $\mathcal{I}$ is non empty, closed and open for the topology of $[0,1]$ and conclude that $\mathcal{I} = [0,1]$.

$\mathcal{I}$ is not empty
For $t = 0$, the existence of a solution u such that $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ is a direct consequence of Lemma 2.4. We just have to use (2.68) to show the uniform Holder continuity of the gradient of the solution.

$\mathcal{I}$ is open
We assume there exists $t \in \mathcal{I}$ and we will find $\varepsilon > 0$ such that $[t, t+\varepsilon] \subset \mathcal{I}$. For $f \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, we want to solve :

$$-\operatorname{div}((a_t + \varepsilon\tilde{a})\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d, \tag{2.69}$$

where $\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. According to Proposition 2.9, for such a solution, we have $\varepsilon\tilde{a}\nabla u \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Next, we remark that equation (2.69) is equivalent to :

$$\nabla u = \Phi_t(\varepsilon\tilde{a}\nabla u + f), \tag{2.70}$$

where Φ_t is the reciprocal linear operator associated with the equation when $a = a_t$. Lemma 2.5 and estimate (2.68) imply the continuity of the operator Φ_t from $\left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ for the norm $\|.\|_{\mathcal{B}^2(\mathbb{R}^d)} + \|.\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}$. We fix ε such that :

$$\varepsilon \left(\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} + \|\tilde{a}_\infty\|_{L^\infty(\mathbb{R}^d)} \right) \|\Phi_t\|_{\mathcal{L}\left(\left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d\right)} < 1.$$

Therefore $g \to \Phi_t(\varepsilon\tilde{a}g + f) \in \mathcal{L}\left(\left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d\right)$ is a contraction in a Banach space. Finally, we can apply the Banach fixed-point theorem to obtain the existence and the uniqueness of a solution to (2.70) and we deduce that $[t, t+\varepsilon] \subset \mathcal{I}$.

$\mathcal{I}$ is closed

We assume there exist a sequence $(t_n) \in \mathcal{I}^{\mathbb{N}}$ and $t \in [0,1]$ such that $\lim_{n\to\infty} t_n = t$ and $t_n < t$. For every t_n, there exists u_n solution to :

$$-\operatorname{div}(a_{t_n}\nabla u_n) = f \quad \text{in } \mathbb{R}^d,$$

such that $\nabla u_n \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. For every $n \in \mathbb{N}$, Lemma 2.5 gives the existence of a constant C_n such that :

$$\|\nabla u_n\|_{B^2(\mathbb{R}^d)} \leq C_n \|f\|_{B^2(\mathbb{R}^d)}.$$

We first assume that C_n is bounded independently of n by a constant $C > 0$. Therefore, up to an extraction, ∇u_n weakly converges to a gradient ∇u in $L^2_{loc}(\mathbb{R}^d)$ and, using the lower semi-continuity of the L^2-norm, we have

$$\|\nabla u\|_{L^2_{unif}(\mathbb{R}^d)} + \sup_p \|\nabla u\|_{L^2(V_p)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^2_{unif}(\mathbb{R}^d)} + \sup_p \|\nabla u_n\|_{L^2(V_p)} \leq C\|f\|_{\mathcal{B}^2(\mathbb{R}^d)}.$$

In addition, for every $n \in \mathbb{N}$, u_n is a solution to the equivalent equation :

$$-\operatorname{div}(a_t\nabla u_n) = \operatorname{div}(f + (a_{t_n} - a_t)\nabla u_n) \tag{2.71}$$

Next, since t_n converges to t, we directly obtain that a_{t_n} converges to a_t in $\mathcal{B}^2(\mathbb{R}^d)$. In addition, since ∇u_n is bounded by a constant independent of n in $L^2_{unif}(\mathbb{R}^d)$, the sequence $(a_{t_n} - a_t)\nabla u_n$ strongly converges to 0 in $L^2_{loc}(\mathbb{R}^d)$. We can therefore consider the limit in (2.71) when $n \to \infty$ and deduce that u is a solution to :

$$-\operatorname{div}(a_t\nabla u) = \operatorname{div}(f).$$

We have to prove that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$. For every $m,\ n \in \mathbb{N}$, $u_n - u_m$ is a solution to :

$$-\operatorname{div}(a_t(\nabla u_n - \nabla u_m)) = \operatorname{div}((a_{t_n} - a_t)\nabla u_n - (a_{t_m} - a_t)\nabla u_m),$$

and we have the following estimate :

$$\|\nabla u_n - \nabla u_m\|_{\mathcal{B}^2(\mathbb{R}^d)} \leq C\|(a_{t_n} - a_t)\nabla u_n - (a_{t_m} - a_t)\nabla u_m\|_{\mathcal{B}^2(\mathbb{R}^d)}.$$

Therefore, u_n is a Cauchy-sequence in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ and since this space is a Banach space, we directly obtain that ∇u belongs to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ and we have the expected result.

Now, we want to prove that C_n is bounded independently of n using a proof by contradiction. We assume there exist two sequences f_n and ∇u_n in $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ such that :

$$-\operatorname{div}(a_{t_n}\nabla u_n) = \operatorname{div}(f_n) \quad \text{in } \mathbb{R}^d,$$

$$\lim_{n\to\infty} \|f_n\|_{\mathcal{B}^2(\mathbb{R}^d)} = 0,$$

$$\forall n \in \mathbb{N} \quad \|\nabla u_n\|_{\mathcal{B}^2(\mathbb{R}^d)} = 1.$$

For every $n \in \mathbb{N}$, the above equation is equivalent to :

$$-\operatorname{div}(a_t\nabla u_n) = \operatorname{div}(f_n + (a_{t_n} - a_t)\nabla u_n).$$

We can next remark that the boundedness of ∇u_n in $\mathcal{B}^2(\mathbb{R}^d)$ ensures that $(a_{t_n} - a_t)\nabla u_n$ strongly converges to 0 in $\mathcal{B}^2(\mathbb{R}^d)$ when $n \to \infty$. Finally, we can conclude exactly as in the proof of Lemma 2.5.

Since $[0, 1]$ is a connected space, we can conclude that $\mathcal{I} = [0, 1]$. In addition, if $u \in \mathcal{D}'(\mathbb{R}^d)$ is such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d) \subset L^2_{loc}(\mathbb{R}^d)$, the result of [39, corollary 2.1] finally ensures that $u \in L^2_{loc}(\mathbb{R}^d)$. $\square$

In the above proof, we have proved the following result :

Corollary 2.4. *Let $f \in \mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ and $u \in H^1_{loc}(\mathbb{R}^d)$ solution to (2.31) such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$. Then $\nabla u \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$.*

Remark 2.6. *Again, we do not need the restriction that we did on the dimension to prove the existence results stated in this section and we can easily generalize the existence of a solution to (2.31) in a two-dimensional context.*

2.4.4 Existence of the corrector

To conclude this section, we finally give a proof of Theorem 2.1 and, therefore, we obtain the existence of a unique corrector solution to (2.14) such its gradient belongs to $L^2_{per}(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d)$. To this end, we remark that corrector equation (2.14) is equivalent to a an equation in form (2.31) and we use the preliminary results of uniqueness and existence proved in this section.

Proof of theorem 2.1. Existence
Let p be in $\mathbb{R}^d$. We want to find a solution to (2.14) of the form $w_{per,p} + \tilde{w}_p$ where $w_{per,p}$ is the unique periodic corrector (that is the unique periodic solution to the corrector equation (2.14) when $\tilde{a} = 0$) and such that $\nabla \tilde{w}_p \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. First of all, we remark that equation (2.14) is equivalent to :

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla \tilde{w}_p) = \operatorname{div}(\tilde{a}(p + \nabla w_{per,p})) \quad \text{in } \mathbb{R}^d.$$

It is well known that $\nabla w_{per,p} \in \left(L^2_{per}(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and therefore, using the periodicity of $\nabla w_{per,p}$, we can easily show that $\tilde{a}(p + \nabla w_{per,p}) \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Then, the existence of $\tilde{w}_p$ such that $\nabla \tilde{w}_p \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ is given by Lemma 2.6 and Corollary 2.4. Since $\nabla \tilde{w}_p \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^d$, the sub-linearity at infinity of $\tilde{w}_p$ is a direct consequence of Proposition 2.11.

Uniqueness
We assume there exist two solutions u_1 and u_2 to (2.14) such that ∇u_1 and ∇u_2 belong to $\left(L^2_{per}(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d)\right)^d$. We denote $v = u_1 - u_2$ and we have $\nabla v = g_{per} + \tilde{g}$ where $g_{per} \in \left(L^2_{per}(\mathbb{R}^d)\right)^d$ and $\tilde{g} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. For every $q \in \mathcal{P}$, we have $\tau_q \nabla v = g_{per} + \tau_q \tilde{g}$ by periodicity of g_{per}. Since $\tilde{g}$ belongs to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, there exists $\tilde{g}_\infty \in \left(L^2(\mathbb{R}^d)\right)^d$ such that $\tau_q \nabla v$ converges in $\mathcal{D}'(\mathbb{R}^d)$ to $\nabla v_\infty = g_{per} + \tilde{g}_\infty$ when $|q| \to \infty$. In addition, considering the limit in equation (2.14), we obtain that v_∞ is a solution to :

$$-\operatorname{div}((a_{per} + \tilde{a}_\infty)\nabla v_\infty) = 0 \quad \text{in } \mathbb{R}^d.$$

Since a satisfies assumption (2.12) and (2.13), the coefficient $a_{per} + \tilde{a}_\infty$ is a bounded and elliptic matrix-valued coefficient. Therefore, the result established in [25, Lemma 1] allows us to conclude that $g_{per} = 0$ and finally, that $\nabla v = \tilde{g} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. Since v is a solution to :

$$- \operatorname{div}((a_{per} + \tilde{a})\nabla v) = 0 \quad \text{in } \mathbb{R}^d,$$

we use Lemma 2.1 to obtain that $\nabla v = 0$ and the uniqueness is proved. □

2.5 Homogenization results and convergence rates

In this section we use the corrector, solution to (2.14) and defined in Theorem 2.1, to establish an homogenization theory similar to that established in [20] for the periodic case with local perturbations. In proposition 2.13 we first study the homogenized equation associated with (2.1) and we conclude showing estimates (2.15) and (2.16) stated in Theorem 2.2.

2.5.1 Homogenization results

To start with, we determine here the limit of the sequence u^ε of solutions to (2.1). In Proposition 2.13 below we show the homogenized equation is actually the diffusion equation (2.3) where the diffusion coefficient a^* is defined by (2.5), that is the homogenized coefficient is the same as in the periodic case when $a = a_{per}$. This phenomenon is similar to the results established in [27] in the case of localized defects of $L^p(\mathbb{R}^d)$. It is a direct consequence of Proposition 2.10 regarding the average of the functions in $\mathcal{B}^2(\mathbb{R}^d)$ which is satisfied by our perturbations. The idea is that, on average, the perturbations belonging to $\mathcal{B}^2(\mathbb{R}^d)$ therefore do not impact the periodic background.

Proposition 2.13. *Assume Ω is an open bounded set of $\mathbb{R}^d$, let $f \in L^2(\Omega)$ and consider the sequence u^ε of solutions in $H^1_0(\Omega)$ to (2.1). Then the homogenized (weak-$H^1(\Omega)$ and strong-$L^2(\Omega)$) limit u^* obtained when $\varepsilon \to 0$ is the solution to (2.3) where the homogenized coefficient is identical to the periodic homogenized coefficient (2.5).*

Proof. We denote $w = (w_{e_i})_{i \in \{1,\dots,d\}}$, the correctors given by Theorem 2.1 for $p = e_i$. The general homogenization theory of equations in divergence form (see for instance [90, Chapter 6, Chapter 13]), gives the convergence, up to an extraction, of u^ε to a function u^* solution to an equation in the form (2.3). In addition, for all $1 \leq i, j \leq d$, the homogenized matrix a^* associated with a is given by :

$$[a^*]_{i,j} = weak \lim_{\varepsilon \to 0} a(./\varepsilon)(I_d + \nabla w(./\varepsilon)),$$

where the weak limit is taken in $L^2(\Omega)^{d \times d}$.

By assumption, we have $a = a_{per} + \tilde{a}$ and we know that $\nabla w_{e_i} = \nabla w_{per,e_i} + \nabla \tilde{w}_{e_i}$ where $\tilde{a} \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^{d \times d}$ and $\nabla \tilde{w}_{e_i} \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^d$. Therefore, Corollary 2.2 ensures that $|\tilde{a}(./\varepsilon)|$ and $|\nabla \tilde{w}_{e_i}(./\varepsilon)|$ converge to 0 for the weak-$\star$ topology of L^∞. Since a_{per} and ∇w_{per} are bounded, we can deduce that :

$$weak \lim_{\varepsilon \to 0} a_{per}(./\varepsilon)\nabla \tilde{w}(./\varepsilon) + \tilde{a}(./\varepsilon)(I_d + \nabla w(./\varepsilon)) = 0.$$

Consequently, we have

$$[a^*]_{i,j} = weak \lim_{\varepsilon \to 0} a_{per}(./\varepsilon)(I_d + \nabla w_{per}(./\varepsilon)) = \left[a^*_{per}\right]_{i,j}.$$

This limit being independent of the extraction, all the sequence u^ε converges to u^* and we have the equality $a^* = a^*_{per}$. □

2.5.2 Approximation of the homogenized solution and quantitative estimates

The existence of the corrector established in Theorem 2.1 allows to consider a sequence of approximated solutions defined by $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^d \partial_i u^* w_i(./\varepsilon)$ where for every i in $\{1, ..., d\}$, we have denoted $w_i = w_{e_i}$. Our aim here is to estimate the accuracy of this approximation for the topology of $H^1(\Omega)$. In particular, we want to prove the convergence to 0 of the sequence R^ε defined by :

$$R^\varepsilon(x) = u^\varepsilon(x) - u^*(x) - \varepsilon \sum_{j=1}^d w_j \left(\frac{x}{\varepsilon}\right) \partial_j u^*(x),$$

and specify the convergence rate in $H^1(\Omega)$.

A classical method in homogenization used to obtain some expected quantitative estimates consists in defining the divergence-free matrix $M_k^i(x) = a^*_{i,k} - \sum_{j=1}^d a_{i,j}(x)(\delta_{j,k} + \partial_j w_k(x))$ and to find a potential B which formally solves $M = \mathrm{curl}(B)$. Knowing that both the coefficient a and ∇w belong to $L^2_{per} + \mathcal{B}^2(\mathbb{R}^d)$, we can split M in two terms and obtain $M = M_{per} + \tilde{M} \in \left(L^2_{per}(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d}$. Therefore, we expect to find a potential of the same form, that is $B = B_{per} + \tilde{B}$. Rigorously, for every $i, j \in \{1, ..., d\}$, we want to solve the equation :

$$-\Delta B_k^{i,j} = \partial_j M_k^i - \partial_i M_k^j \quad \text{in } \mathbb{R}^d. \tag{2.72}$$

The existence of a periodic potential B_{per} solution to $M_{per} = \mathrm{curl}(B_{per})$ is well known since, component by component, M_{per} is divergence-free. Here, the main difficulty is to show the existence of the potential $\tilde{B}$ associated with the $\mathcal{B}^2$-perturbation. This result is given by the following lemma.

Lemma 2.7. *Let* $\tilde{M} = \left(\tilde{M}_k^i\right)_{1\leq i,k\leq d} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d}$ *such that* $\mathrm{div}(\tilde{M}_k) = 0$ *for every* $k \in \{1, ..., d\}$. *Then, the potential* $\tilde{B}_k^{i,j}$ *defined by :*

$$\tilde{B}_k^{i,j}(x) = C(d) \int_{\mathbb{R}^d} \left(\frac{x_i - y_i}{|x-y|^d} \tilde{M}_k^j(y) - \frac{x_j - y_j}{|x-y|^d} \tilde{M}_k^i(y) \right) dy, \tag{2.73}$$

where $C(d) > 0$ *is a constant associated with the unit ball surface of* $\mathbb{R}^d$, *satisfies* $\nabla \tilde{B}_k^{i,j} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$ *and for all* $i, j, k \in \{1, ..., d\}$ *:*

$$-\Delta \tilde{B}_k^{i,j} = \partial_j \tilde{M}_k^i - \partial_i \tilde{M}_k^j, \tag{2.74}$$

$$\tilde{B}_k^{i,j} = -\tilde{B}_k^{j,i}, \tag{2.75}$$

$$\sum_{i=1}^d \partial_i \tilde{B}_k^{i,j} = \tilde{M}_k^j. \tag{2.76}$$

In addition, there exists a constant $C_1 > 0$ which only depends of the ambient dimension d such that :

$$\|\nabla \tilde{B}\|_{\mathcal{B}^2(\mathbb{R}^d)} \leq C_1 \|\tilde{M}\|_{\mathcal{B}^2(\mathbb{R}^d)}. \tag{2.77}$$

Proof. First, for every $i, j, k \in \{1, ..., d\}$, equation (2.74) is equivalent to an equation of the following form :

$$-\Delta \tilde{B}_k^{i,j} = \operatorname{div}\left(\mathcal{M}_k^{i,j}\right),$$

where $\mathcal{M}_k^{i,j}$ is a vector function defined by :

$$\left(\mathcal{M}_k^{i,j}\right)_l = \begin{cases} \tilde{M}_k^i & \text{if } l = j, \\ -\tilde{M}_k^j & \text{if } l = i, \\ 0 & \text{else.} \end{cases}$$

Since $\mathcal{M}_k^{i,j} \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$, the existence of $\tilde{B}$ and estimate (2.77) are given by Lemmas 2.3, 2.4 and 2.5 (here $a_{per} \equiv 1$). Equality (2.75) is a direct consequence of the definition of $\tilde{B}$. Property (2.76) can be easily obtained applying the divergence operator to (2.73). □

Corollary 2.5. *The potential $B = B_{per} + \tilde{B}$, where $\tilde{B}$ is given by Lemma 2.7, is the expected potential solution of (2.72). In addition, the couple (M, B) satisfies the following equalities :*

$$B_k^{i,j} = -B_k^{j,i},$$
$$\sum_{i=1}^{d} \partial_i B_k^{i,j} = M_k^j.$$

Now that existence of the potential B has been dealt with, we can remark that R^ε is a solution to the following equation :

$$-\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla R^\varepsilon\right) = \operatorname{div}(H^\varepsilon) \quad \text{in } \Omega, \tag{2.78}$$

where :

$$H_i^\varepsilon(x) = \varepsilon \sum_{j,k=1}^{d} a_{i,j}\left(\frac{x}{\varepsilon}\right) w_k\left(\frac{x}{\varepsilon}\right) \partial_j \partial_k u^*(x) - \varepsilon \sum_{j,k=1}^{d} B_k^{i,j}\left(\frac{x}{\varepsilon}\right) \partial_j \partial_k u^*(x). \tag{2.79}$$

For a complete proof of equality (2.78), we refer to [20, Proposition 2.5].

To conclude, we have to study the properties of H^ε. In particular, we next prove that both the corrector $\tilde{w}$ and the potential $\tilde{B}$ are bounded. This property is essential for establishing the estimates of Theorem 2.2.

Lemma 2.8. *The corrector $w = (w_i)_{i \in \{1,...,d\}}$ defined by Theorem 2.1 and the potential B solution to (2.72) are in $L^\infty(\mathbb{R}^d)$.*

Proof. First, it is well known that both w_{per} and B_{per} belong to $L^\infty(\mathbb{R}^d)$. Next, for all $i \in \{1, ..., d\}$, $\tilde{w}_i$ solves :

$$-\operatorname{div}\left(a_{per}\nabla \tilde{w}_i\right) = \operatorname{div}\left(\tilde{a}\left(e_i + \nabla w_{per,i} + \nabla \tilde{w}_i\right)\right).$$

We know the gradient of the corrector defined in Theorem 2.1 is in $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. A direct consequence of Assumption (2.13) and Proposition 2.9 ensures that $f = \tilde{a}\left(e_i + \nabla w_{per,i} + \nabla \tilde{w}_i\right)$ belongs to $\left(L^\infty(\mathbb{R}^d) \cap \mathcal{B}^2(\mathbb{R}^d)\right)^d$ and the results of uniqueness and existence established in Lemmas 2.1 and 2.4 imply we have the following representation :

$$\tilde{w}_i(x) = \int_{\mathbb{R}^d} \nabla_y G_{per}(x,y) f(y) dy.$$

We want to prove that the integral is bounded independently of x. We take $x \in \mathbb{R}^d$ and denote p_x the unique element of $\mathcal{P}$ such that $x \in V_{p_x}$. We define $W_{p_x} = W_{2^{p_x}}$ such as in Proposition 2.4 and we split the integral in three parts :

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla_y G_{per}(x,y) f(y) dy = \int_{B_1(x)} \nabla_y G_{per}(x,y) f(y) dy + \int_{W_{p_x}\setminus B_1(x)} \nabla_y G_{per}(x,y) f(y) dy \\ + \int_{\mathbb{R}^d \setminus W_{p_x}} \nabla_y G_{per}(x,y) f(y) dy = I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

We start by finding a bound for $I_1(x)$. To this end, we use Estimate (2.37) for the Green function and we obtain

$$\begin{aligned}|I_1(x)| &\leq \|f\|_{L^\infty(\mathbb{R}^d)} \int_{B_1(x)} |\nabla_y G_{per}(x,y)|\, dy \\ &\leq C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{B_1(x)} \frac{1}{|x-y|^{d-1}} dy \leq C\|f\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

Where C denotes a positive constant independent of x. Indeed, the value of the integral in the last inequality depends only of the dimension d.

Next, using Proposition 2.4, we know there exists $C_1 > 0$ and $C_2 > 0$ independent of x such that $W_{p_x} \subset B_{C_1 2^{p_x}(x)}$ and the number of $q \in \mathcal{P}$ such that $V_q \cap W_{p_x} \neq \emptyset$ is bounded by C_2. Therefore, using the Cauchy-Schwarz inequality, we have :

$$\begin{aligned}|I_2(x)| &\leq \int_{W_{p_x}\setminus B_1(x)} \frac{1}{|x-y|^{(d-1)}} |f(y)| dy \\ &\leq \left(\int_{W_{p_x}\setminus B_1(x)} \frac{1}{|x-y|^{2(d-1)}} dy\right)^{1/2} \left(\int_{W_{p_x}\setminus B_1(x)} |f(y)|^2 dy\right)^{1/2} \\ &\leq C_2 \left(\int_{B_{C_1 2^{p_x}(x)}\setminus B_1(x)} \frac{1}{|x-y|^{2(d-1)}} dy\right)^{1/2} \sup_{p\in\mathcal{P}} \|f\|_{L^2(V_q)}.\end{aligned}$$

In addition since $d > 2$, we have :

$$\int_{B_{C_1 2^{p_x}(x)}\setminus B_1(x)} \frac{1}{|x-y|^{2(d-1)}} dy = \int_{B_{C_1 2^{p_x}(0)}\setminus B_1(0)} \frac{1}{|y|^{2(d-1)}} dy \leq C\left(1 - \frac{1}{2^{|p_x|(d-2)}}\right),$$

and therefore :

$$I_2(x) \leq C\left(1 - \frac{1}{2^{|p_x|(d-2)}}\right)^{1/2} \leq C.$$

Finally, to bound $I_3(x)$ we split the integral on each cell V_q for $q \in \mathcal{P}$. Using the Cauchy-Schwarz inequality, we obtain :

$$\begin{aligned}|I_3(x)| &\leq \sum_{q\in\mathcal{P}} \int_{V_q\setminus W_{p_x}} |\nabla_y G_{per}(x,y) f(y)|\, dy \\ &\leq \sum_{q\in\mathcal{P}} \left(\int_{V_q\setminus W_{p_x}} |\nabla_y G_{per}(x,y)|^2\, dy\right)^{\frac{1}{2}} \left(\int_{V_q\setminus W_{p_x}} |f(y)|^2\, dy\right)^{\frac{1}{2}} \\ &\leq \|f\|_{\mathcal{B}^2(\mathbb{R}^d)} \sum_{q\in\mathcal{P}} \left(\int_{V_q\setminus W_{p_x}} |\nabla_y G_{per}(x,y)|^2\, dy\right)^{\frac{1}{2}}.\end{aligned}$$

We proceed exactly as in the proof of Lemma 2.3 (see the proof of estimate (2.54) for details) to obtain :

$$\begin{aligned}\sum_{q\in\mathcal{P}} \left(\int_{V_q\setminus W_{p_x}} |\nabla_y G_{per}(x,y)|^2\, dy\right)^{\frac{1}{2}} &\leq C \sum_{q\in\mathcal{P}} \left(\int_{V_q\setminus W_{p_x}} \frac{1}{|x-y|^{2(d-1)}} dy\right)^{\frac{1}{2}} \\ &\leq C \sum_{q\in\mathcal{P}} \frac{1}{2^{|q|(d-2)/2}} < \infty.\end{aligned}$$

Finally we have bounded the integral independently of x and we deduce that $\tilde{w}_i \in L^\infty(\mathbb{R}^d)$. With the same method we obtain the same result for $B = B_{per} + \tilde{B}$ which allows us to conclude. □

Remark 2.7. *As in the proofs of Lemmas 2.3 and 2.4, the above proof strongly uses the specific behavior of the Green function G_{per} and our approach consists in showing the convergence of a sum of the form $\displaystyle\sum_{q\in\mathcal{P}} \int_{V_q} \frac{1}{|x-y|^{d-1}} f(y) dy$. Here, we explicitly use the geometric properties satisfied by the 2^p but, once again, this convergence is not specific to the set (2.9) and also holds under the generality of (H1), (H2) and (H3). We refer to Remark 2.4 for additional details.*

Remark 2.8. *The assumption $d > 2$ is essential in the above proof and the boundedness of $\tilde{w}$ in $L^\infty(\mathbb{R}^d)$ may be false if $d = 1$ or $d = 2$. If $d = 1$ we give a counter-example in Remark 2.9. The case $d = 2$ is a critical case and we are not able to conclude. This phenomenon is closely linked to the critical integrability of the function $|x|^{-1}$ in $L^2(\mathbb{R}^2)$.*

We are now able to give a complete proof of Theorem 2.2.

Proof of Theorem 2.2. First, we use the explicit definition of H^ε given by (2.79) and a triangle inequality to obtain the following estimate :

$$\|H^\varepsilon\|_{L^2(\Omega)} \leq \left(1 + \|a\|_{L^\infty(\mathbb{R}^d)}\right) \|D^2 u^*\|_{L^2(\Omega)} \left(\|\varepsilon w(./\varepsilon)\|_{L^\infty(\Omega)} + \|\varepsilon B(./\varepsilon)\|_{L^\infty(\Omega)}\right).$$

Applying Lemma 2.8, we obtain the existence of $C > 0$ independent of ε such that :

$$\|H^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \|D^2 u^*\|_{L^2(\Omega)}. \tag{2.80}$$

Next, we use the two following estimates satisfied by R^ε :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \left(\varepsilon \left(\|w(./\varepsilon)\|_{L^\infty(\Omega)} + \|B(./\varepsilon)\|_{L^\infty(\Omega)}\right) \|f\|_{L^2(\Omega)} + \|H^\epsilon\|_{L^2(\Omega)}\right), \tag{2.81}$$

and for every $\Omega_1 \subset\subset \Omega$:

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \left(\|H^\varepsilon\|_{L^2(\Omega)} + \|R^\varepsilon\|_{L^2(\Omega)}\right), \tag{2.82}$$

where $C_1 > 0$ and $C_2 > 0$ are independent of ε. These estimates are established for instance in [20] where the authors use the elliptic regularity associated with equation (2.78) and the properties of the Green function associated with the operator $-\operatorname{div}(a^*\nabla.)$ on Ω with homogeneous Dirichlet boundary condition. The first estimate is established in the proof of [20, Lemma 4.8] and the second estimate is a classical inequality of elliptic regularity proved in [20, Proposition 4.2] and applied to equation (2.78).

In addition, an application of elliptical regularity to equation (2.3) provides the existence of $C_3 > 0$ such that :

$$\|u^*\|_{H^2(\Omega)} \leq C_3\|f\|_{L^2(\Omega)}. \tag{2.83}$$

To conclude we use Lemma 2.8 to bound w and B and estimates (2.80), (2.81), (2.82) and (2.83). We finally obtain :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|f\|_{L^2(\Omega)},$$

and

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq \tilde{C}\varepsilon\|f\|_{L^2(\Omega)},$$

where C and $\tilde{C}$ are independent of ε. We have proved Theorem 2. □

Remark 2.9. *In the one-dimensional case, that is when $d = 1$, we are not able to conclude in the same way. With an explicit calculation, we obtain :*

$$\begin{aligned}
&(u^\varepsilon)'(x) = \left(a_{per} + \tilde{a}\right)^{-1}(x/\varepsilon)\left(F(x) + C^\varepsilon\right),\\
&(u^*)'(x) = \left(a^*\right)^{-1}\left(F(x) + C^*\right),\\
&w(x) = -x + a^* \int_0^x \frac{1}{a_{per}(y)}dy - a^* \int_0^x \frac{\tilde{a}}{a_{per}\left(a_{per} + \tilde{a}\right)}(y)dy,
\end{aligned}$$

where :

$$\begin{aligned}
F(x) &= \int_0^x f(y)dy,\\
C^\varepsilon &= -\left(\int_0^1 (a_{per} + \tilde{a})^{-1}(y/\varepsilon)\right)^{-1} \int_0^1 (a_{per} + \tilde{a})^{-1}(y/\varepsilon)F(y)dy,\\
C^* &= -\int_0^1 F(y)dy.
\end{aligned}$$

In this case, $w_{per}(x) = -x + a^* \int_0^x \frac{1}{a_{per}(y)}dy$ and $\tilde{w}(x) = -a^* \int_0^x \frac{\tilde{a}}{a_{per}\left(a_{per} + \tilde{a}\right)}(y)dy$ and we can show the corrector w is not necessarily bounded. However, the results of Proposition 2.10, allow us to obtain the following estimate :

$$\|\left(R^\varepsilon\right)'\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\left|\log(\varepsilon)\right|^{\frac{1}{2}}.$$

As an illustration, we can consider $\Omega =]0,1[$, $a_{per} = 1$ *and* $\tilde{a} = \sum_{p \in \mathbb{Z}} \tau_{-2^p}\varphi$, *where* φ *is a positive function of* $\mathcal{D}(\mathbb{R})$, $\|\varphi\|_{L^\infty} = 1$, $\int_{\mathbb{R}} \varphi > 0$ *and* $Supp(\varphi) \in [0, 1/2]$. *With these parameters, for every* $x \in \Omega$, *we have :*

$$|\tilde{w}(x/\varepsilon)| = \int_0^{x/\varepsilon} \frac{\tilde{a}}{1+\tilde{a}}(y)dy \geq \frac{1}{2} \sum_{0 \leq p < [\log_2(x/\varepsilon)]} \int_0^{1/2} \varphi \overset{\varepsilon \to 0}{\longrightarrow} +\infty.$$

And therefore, the corrector is actually not bounded.

Remark 2.10. *The result of Theorem 2.2 ensures that the corrector introduced in Theorem 2.1 allows to precisely describe the behavior of the sequence* u^ε *in* H^1 *using the approximation defined by* $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_i(./\varepsilon)$. *However, since the perturbations of* $\mathcal{B}^2(\mathbb{R}^d)$ *are "small" at the macroscopic scale (in the sense of average given by (2.25)), we can naturally expect that it is also possible to approximate* u^ε *in* H^1 *considering the sequence* $u^{\varepsilon,1}_{per} := u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{per,i}(./\varepsilon)$ *which only uses the periodic part* w_{per} *of our corrector. To this aim, we can show that* $u^\varepsilon - u^{\varepsilon,1}_{per}$ *is solution to*

$$-\operatorname{div}\left(a\left(\frac{\cdot}{\varepsilon}\right)\nabla(u^\varepsilon - u^{\varepsilon,1}_{per})\right) = \operatorname{div}\left(H^\varepsilon_{per}\right) \quad \text{on } \Omega,$$

where the right-hand side

$$H^\varepsilon_{per} := -a\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla\left(u^\varepsilon - u^{\varepsilon,1}\right) + \varepsilon \sum_{i=1}^{d} \nabla \partial_i u^* \tilde{w}_i(./\varepsilon) + \sum_{i=1}^{d} \partial_i u^* \nabla \tilde{w}_i(./\varepsilon)\right), \tag{2.84}$$

strongly converges to 0 in L^2 *when* $\varepsilon \to 0$. *A method similar to that used in the proof of Theorem 2.2 therefore allows to show the convergence to 0 of* $u^\varepsilon - u^{\varepsilon,1}_{per}$ *in* H^1. *It follows, at the macroscopic scale, that the choice of our adapted corrector instead of the periodic corrector seems to be not necessarily relevant in order to calculate an approximation of* u^ε *in* H^1. *However, the choice of the periodic corrector is not adapted if the idea is to approximate the behavior of* u^ε *at the microscopic scale* ε. *Indeed, at this scale, the perturbations of the periodic background can be possibly large and it is necessary to consider a corrector which takes these perturbations into account. Particularly, if* H^ε_{per} *is the function defined by (2.84), we can easily show that* $H^\varepsilon_{per}(\varepsilon.)$ *does not converge to 0 in any ball* B_R *such that* $\varepsilon B_R \subset \Omega$, *which formally reflects a poor quality of the approximation of* u^ε *by* $u^{\varepsilon,1}_{per}$ *at the scale* ε. *This fact particularly motivates the choice of our adapted corrector in order to approximate* u^ε.

2.6 Appendix : The case of $\mathcal{B}^r(\mathbb{R}^d)$, $1 < r < \infty$

The purpose of this section is to generalize the results established above in a context where the perturbation $\tilde{a}$ locally behaves, at the vicinity of the points $x_p \in \mathcal{G}$, as a fixed function of $L^r(\mathbb{R}^d)$ truncated over the domain V_p, where r can denote any Lebesgue exponent in $]1, +\infty[$. In this context, we can naturally generalize the definition (2.10) of the space $\mathcal{B}^2(\mathbb{R}^d)$ and, using the topology of L^r, introduce a collection of spaces $\mathcal{B}^r(\mathbb{R}^d)$ defined by (2.85) in order to describe our perturbations of the periodic background. Exactly as in the case $r = 2$, we consider here a perturbed coefficient a of the form (2.2) which is the sum of a periodic coefficient a_{per} and a defect $\tilde{a}$ in $\mathcal{B}^r(\mathbb{R}^d)^{d\times d}$. Our aim is therefore to establish the homogenization of equation (2.1) in this case, showing the existence of an adapted corrector solution to (2.14) and establishing some convergence rates similar to those of Theorem 2.2. To this end, the idea is to follow the same strategy as in the case $r = 2$. However, several difficulties appear when $r \neq 2$, some results established in our approach for the case $r = 2$ no longer hold and we have to adapt them. In particular, when $r \neq 2$, the space $\mathcal{B}^r$ defined below does not have an "Hilbert" structure induced by the topology of L^2, which prevents us from using some techniques (such as Caccioppoli-type inequalities for example) to prove the uniqueness results of Section 2.4. Moreover, when $r > d$, the decay far from the points 2^p of the functions of $\mathcal{B}^r$ is not sufficient to ensure the convergence of some series involving the Green function G_{per} as in the proofs of Lemmas 2.3, 2.4 and 2.8. Consequently, in this case we have to adapt our approach to prove the existence of a corrector w given in Theorem 2.3 and, in contrast to the case $r = 2$ (see Lemma 2.8), this corrector is not necessarily bounded in $L^\infty(\mathbb{R}^d)$. As we can see in Theorem 2.4 below, this phenomenon implies in particular that the convergence rates of the sequence of remainders R^ε depends on the ratio $\dfrac{r}{d}$.

To start with, we fix $r \in]1, +\infty[$ and we consider the following functional space :

$$\mathcal{B}^r(\mathbb{R}^d) = \left\{ f \in L^r_{unif}(\mathbb{R}^d) \;\middle|\; \exists f_\infty \in L^r(\mathbb{R}^d),\ \lim_{|p|\to\infty} \int_{V_p} |f(x) - \tau_{-p} f_\infty(x)|^r dx = 0 \right\}, \quad (2.85)$$

equipped with the norm :

$$\|f\|_{\mathcal{B}^r(\mathbb{R}^d)} = \|f_\infty\|_{L^r(\mathbb{R}^d)} + \|f\|_{L^r_{unif}(\mathbb{R}^d)} + \sup_{p\in\mathcal{P}} \|f - \tau_{-p} f_\infty\|_{L^r(V_p)}. \quad (2.86)$$

In (2.85), (2.86) we have denoted by :

$$L^r_{unif}(\mathbb{R}^d) = \left\{ f \in L^r_{loc}(\mathbb{R}^d),\ \sup_{x\in\mathbb{R}^d} \|f\|_{L^r(B_1(x))} < \infty \right\},$$

and

$$\|f\|_{L^r_{unif}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \|f\|_{L^r(B_1(x))}.$$

In the sequel we assume again that the ambient dimension d is equal or larger than 3. We consider here a matrix-valued coefficient of the form (2.2) where a_{per} is periodic and $\tilde{a}$ is in $\mathcal{B}^r(\mathbb{R}^d)^{d\times d}$. We also assume that a_{per}, $\tilde{a}$ and the associated L^r-limit matrix-valued function $\tilde{a}_\infty$ satisfy Assumptions (2.12) and (2.13) of coercivity, boundedness and Hölder continuity.

In this study we establish the two theorems below. In Theorem 2.3, we show the existence of an adapted corrector w, strictly sub-linear at infinity, solution to (2.14) and such that $\nabla w \in \left(L^2_{per} + \mathcal{B}^r(\mathbb{R}^d)\right)^d$. In Theorem 2.4, we use this corrector in order to establish the homogenization of equation (2.1). Precisely, we show the convergence to zero in H^1 of the sequence of remainders $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$, where $u^{\varepsilon,1}$ is defined as in (2.6) but using the adapted corrector of Theorem 2.3. We also establish convergence rates for this topology.

Theorem 2.3. *For every $p \in \mathbb{R}^d$, there exists an unique (up to an additive constant) function $\tilde{w}_p \in W^{1,r}_{loc}(\mathbb{R}^d)$ such that $w_p = w_{per,p} + \tilde{w}_p$ is solution to corrector equation (2.14), where $w_{per,p}$ is the unique periodic corrector solution to (2.4) and $\nabla \tilde{w}_p \in \left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.*

Remark 2.11. *We note that, in the case $r \neq 2$, we are actually not able to prove the uniqueness of a corrector w_p such that $\nabla w_p \in \left(L^2_{per}(\mathbb{R}^d) + \mathcal{B}^r(\mathbb{R}^d)\right)^d$. We are only able to prove the uniqueness (up to an additive constant) of a solution of the form $w_p = w_{per,p} + \tilde{w}_p$ where $\nabla \tilde{w}_p \in \mathcal{B}^r(\mathbb{R}^d)^d$. This result is however sufficient to study the homogenization of problem (2.1) stated in the next theorem and studied in Section 2.6.6.*

Theorem 2.4. *Assume that Ω is a $C^{2,1}$-bounded domain and that $r \neq d$. Let $\Omega_1 \subset\subset \Omega$. We define $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon)$ where w_{e_i} is the corrector given by Theorem 2.3 for $p = e_i$ and $u^* \in H^1(\Omega)$ is the solution to (2.3). we define*

$$\nu_r = \min\left(1, \frac{d}{r}\right) \in]0,1], \tag{2.87}$$

and

$$\mu_r = \begin{cases} 0 & \text{if } r < d, \\ \dfrac{1}{r} & \text{else.} \end{cases} \tag{2.88}$$

Then $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ satisfies the following estimates :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \left(\log|\varepsilon|\right)^{\mu_r} \varepsilon^{\nu_r} \|f\|_{L^2(\Omega)}, \tag{2.89}$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \left(\log|\varepsilon|\right)^{\mu_r} \varepsilon^{\nu_r} \|f\|_{L^2(\Omega)}, \tag{2.90}$$

where C_1 and C_2 are two positive constants independent of f and ε.

Here, it is important to note that the convergence rate in H^1 depends on the values of the ambient dimension d and the exponent r. To explain this phenomenon, we remark that the behavior of R^ε in $H^1(\Omega)$ is controlled by the sub-linearity of w, that is by the rate of convergence to 0 of the sequence $\varepsilon w(./\varepsilon)$ when ε tends to 0 (see the proof of Theorem 2.4). Moreover, in Proposition 2.18 and Lemma 2.18 established in the sequel we show that the behavior of w at the infinity actually depends on the integrability of ∇w (and, in a certain sense, on its decrease) far from the points x_p, that is, on the value of r.

Remark 2.12. *The case $r = d$ is a critical case for the study of the corrector. Indeed as we shall see in Sections 2.6.2 and 2.6.6, this phenomenon is closely linked to the critical integrability of the function $|x|^{-1}$ in $L^d(\mathbb{R}^d)$ and, in this case, we do not know if the corrector w_p is necessarily bounded. Since the data are bounded in $L^\infty(\mathbb{R}^d)$, we can however apply the result of the case $r > d$ in order to have the above estimates, in which $\left(\log|\varepsilon|\right)^{\mu_r} \varepsilon^{\nu_r}$ is replaced by $\left(\log|\varepsilon|\right)^{\mu_s} \varepsilon^{\nu_s}$ for every $s > d$.*

In the sequel of this work, our approach is similar to that of the case $r = 2$ and we apply here the following strategy :

1) We first study the structure of the space $\mathcal{B}^r(\mathbb{R}^d)$ in Section 2.6.1 and we show several properties satisfied by its elements. In particular, we claim that the functions of $\mathcal{B}^r(\mathbb{R}^d)$ have an average value equal to zero and we study a property of strict sub-linearity satisfied by the functions with a gradient in this space.

2) In section 2.6.2, we next study the diffusion problem $-\operatorname{div}(a\nabla u) = \operatorname{div}(f)$ when $f \in \mathcal{B}^r(\mathbb{R}^d)^d$ and the coefficient a is periodic. Precisely, we establish the existence of a solution u to (2.35) such that ∇u belongs to $\mathcal{B}^r(\mathbb{R}^d)^d$. To this end, we use the results of Avellaneda and Lin established in [11, 14] regarding the behavior of the Green function associated with the operator $-\operatorname{div}(a_{per}\nabla.)$ on $\mathbb{R}^d$, in order to explicit the gradient of the solution u.
3) In section 2.6.4, we generalize these results in the perturbed periodic context when a is a coefficient of the form (2.2). Here, we first establish the continuity of the operator $\nabla\left(-\operatorname{div}(a\nabla.)\right)^{-1}\operatorname{div}$ from $\mathcal{B}^r(\mathbb{R}^d)^d$ to $\mathcal{B}^r(\mathbb{R}^d)^d$ stated in Lemma 2.14 and, again, we adapt a method presented in [27] using an argument of connexity as in the proof of Lemma 2.6.
4) We finally apply this result in section 2.6.5 in order to prove the existence of the corrector stated in Theorem 2.3 and we use its properties in section 2.6.6 to establish the convergence rates stated in Theorem 2.4.

In order not to repeat what we have done in the case $r = 2$, we choose here to refer the reader to the corresponding propositions as soon as the proofs are similar and, here, we only detail the proofs for which the arguments are specific to the case $r \neq 2$. In particular, we mainly detail the existence results in the periodic case established in Section 2.6.2, the uniqueness results of Section 2.6.3 and the uniform bound satisfied by the corrector in the specific case $r < d$ stated in Proposition 2.18.

2.6.1 Preliminary results

To start with, we establish here several properties regarding the elements of $\mathcal{B}^r(\mathbb{R}^d)$. As in the case $r = 2$, we need to study the behavior of these functions, their average value and the property of sub-linearity, in order to obtain some information satisfied by the corrector w_p (in particular we want to study its decrease at infinity) and to apply it to establish the homogenization of diffusion problem (2.1) and the convergence rates stated in Theorem 2.4.

First of all, in the next three propositions, we naturally generalize the results of the case $r = 2$ regarding the structure of the space $\mathcal{B}^r(\mathbb{R}^d)$: in Proposition 2.14 we claim that $\mathcal{B}^r(\mathbb{R}^d)$ is a Banach space, in Proposition 2.15 we introduce a result of density and, finally, in Proposition 2.16, we give a result regarding the multiplication between two elements of $\mathcal{B}^r(\mathbb{R}^d)$. The proofs of these propositions can be easily adapted from the proofs of the similar propositions established in Section 2.3 (see Propositions 2.7, 2.8 and 2.9).

Proposition 2.14. *The space $\mathcal{B}^r(\mathbb{R}^d)$ equipped with the norm defined by (2.86), is a Banach space.*

Proposition 2.15. *Let $\alpha \in]0,1[$, then $\mathcal{C}^{0,\alpha}(\mathbb{R}^d) \cap \mathcal{B}^r(\mathbb{R}^d)$ is dense in $\left(\mathcal{B}^r(\mathbb{R}^d), \|.\|_{\mathcal{B}^r(\mathbb{R}^d)}\right)$.*

Proposition 2.16. *Let g and h be in $\mathcal{B}^r(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We assume the associated limit L^r function of g, denoted by g_∞, belongs to $L^\infty(\mathbb{R}^d)$, then $hg \in \mathcal{B}^r(\mathbb{R}^d)$.*

We next claim that the functions of $\mathcal{B}^r(\mathbb{R}^d)$ have an average value equal to zero in the sense of (2.91). As in the case $r = 2$, this property is essential and allows us to prove that the homogenized equation (2.3) is actually the same as in the periodic background, that is, when $\tilde{a} = 0$.

Proposition 2.17. *Let $u \in \mathcal{B}^r(\mathbb{R}^d)$. Then, for every $x_0 \in \mathbb{R}^d$:*

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)| dx = 0, \tag{2.91}$$

with the following convergence rate :

$$\frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)| dx \leq C \left(\frac{\log R}{R^d} \right)^{\frac{1}{r}}, \tag{2.92}$$

where $C > 0$ is independent of R and x_0.

Proof. We fix $R > 0$. Using the Hölder inequality, we have :

$$\begin{aligned}\frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)| dx &\leq \frac{1}{|B_R|^{\frac{1}{r}}} \left(\int_{B_R(x_0)} |u(x)|^r dx \right)^{\frac{1}{r}} \\ &= \frac{1}{|B_R|^{\frac{1}{r}}} \left(\sum_{p \in \mathcal{P}} \int_{V_p \cap B_R(x_0)} |u(x)|^r dx \right)^{\frac{1}{r}}.\end{aligned}$$

Since the number of V_p such that $B_R(x_0) \cap V_p \neq \emptyset$ is bounded by $\log(R)$ according to Corollary 2.1, we obtain :

$$\frac{1}{|B_R|} \int_{B_R(x_0)} |u(x)| dx \leq \frac{(\log R)^{\frac{1}{r}}}{|B_R|^{\frac{1}{r}}} \sup_p \|u\|_{L^r(V_p)} \leq C(d) \left(\frac{\log(R)}{R^d} \right)^{\frac{1}{r}} \sup_p \|u\|_{L^r(V_p)}.$$

Here, $C(d)$ depends only on the ambient dimension d. The last inequality yields (2.92) and we conclude the proof. □

Corollary 2.6. *Let $u \in \mathcal{B}^r(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $|u(./\varepsilon)|$ converges to 0 for the weak-$\star$ topology of L^∞ when $\varepsilon \to 0$.*

We also have to study an other fundamental result regarding the strict sublinearity at infinity of a function u such that $\nabla u \in \mathcal{B}^r(\mathbb{R}^d)^d$. Exactly as in the case $r = 2$, this result is key for establishing estimates (2.89) and (2.90). Indeed, as we shall see in Section 2.6.6, the error $u^\varepsilon - u^{\varepsilon,1}$ is actually controlled by the L^∞-norm of the sequence $\varepsilon w_{e_i}(./\varepsilon)$ when $\varepsilon \to 0$.

Proposition 2.18. *Let $u \in W^{1,r}_{loc}(\mathbb{R}^d)$ such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^d$. Then u is strictly sub-linear at infinity and for every $s \in \mathbb{R}$ such that $s > d$ and $s \geq r$, there exists $C > 0$ such that for every $x, y \in \mathbb{R}^d$ with $x \neq y$:*

$$|u(x) - u(y)| \leq C \left|\log(|x-y|)\right|^{\frac{1}{s}} |x-y|^{1-\frac{d}{s}}. \tag{2.93}$$

The proof of estimate (2.93) is an easy adaptation to that of Proposition 2.11 and the reader can see it for details.

To conclude this section, we finally introduce a generalization of Proposition 2.12 in order to obtain an uniform estimate for the integral remainders of the functions in $\mathcal{B}^r(\mathbb{R}^d)$.

Proposition 2.19. *Let f be in $\mathcal{B}^r(\mathbb{R}^d)$ and f_∞ be its associated limit function in $L^r(\mathbb{R}^d)$. For any $\varepsilon > 0$, there exists $R^* > 0$ such that for every $R > R^*$ and every $p, q \in \mathcal{P}$:*

$$\left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-p} f_\infty|^r \right)^{1/r} < \varepsilon,$$

where $B_R(2^q)^c$ denotes the set $\mathbb{R}^d \setminus B_R(2^q)$. Therefore, we have the following limit :

$$\lim_{R \to \infty} \sup_{\substack{(p,q) \in \mathcal{P} \\ p \neq q}} \left(\int_{V_q \cap B_R(2^q)^c} |f - \tau_{-p} f_\infty|^r \right)^{1/r} = 0.$$

2.6.2 Existence results in the periodic problem

In this section, we next turn to the study of the general equation (2.31) in a periodic background, that is for $a = a_{per}$ periodic, when $f \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. In this case, our aim is to use the behaviors (2.37), (2.38) and (2.39) of the Green function associated with the operator $-\operatorname{div}(a_{per}\nabla.)$ on $\mathbb{R}^d$. Exactly as in the case $r = 2$, we first use the asymptotic behavior of the Green function to explicit a solution such that $\sup_{p \in \mathcal{P}} \|\nabla u\|_{L^r(V_p)} < \infty$ and we next prove that this function satisfies $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ as soon as $f \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$.

Lemma 2.9. *Let f such that $\sup_{p \in \mathcal{P}} \|f\|_{L^r(V_p)} < \infty$. Then, there exists a function u in $W^{1,r}_{loc}(\mathbb{R}^d)$ solution to (2.35) such that $\sup_{p \in \mathcal{P}} \|\nabla u\|_{L^r(V_p)} < \infty$ and we have the following estimate :*

$$\sup_{p \in \mathcal{P}} \|\nabla u\|_{L^r(V_p)} \leq C \sup_{p \in \mathcal{P}} \|f\|_{L^r(V_p)}, \tag{2.94}$$

where $C > 0$ is a constant independent of f and u. In addition, if $r < d$, this solution u can be defined by :

$$u = \int_{\mathbb{R}^d} \nabla_y G_{per}(., y).f(y)dy. \tag{2.95}$$

Proof. The method used here is similar to that employed for the proof of Lemma 2.2. We first introduce the Green function G_{per} in order to find a solution to (2.35) in $W^{1,r}_{loc}(\mathbb{R}^d)$ and we show this solution satisfies (2.94).

Step 1 : Definition of a solution u.

In the sequel the letter C denotes a generic constant that may change from one line to another. To start with, for each $q \in \mathcal{P}$, we define the function :

$$u_q = \int_{\mathbb{R}^d} \nabla_y G_{per}(., y) f(y) 1_{V_q}(y) dy.$$

The results of [14] ensure this function is a solution in $H^1_{loc}(\mathbb{R}^d)$ to

$$-\operatorname{div}(a_{per}\nabla u_q) = \operatorname{div}(f 1_{V_q}) \quad \text{in } \mathbb{R}^d,$$

such that $\nabla u_q \in \left(L^r(\mathbb{R}^d)\right)^d$. In particular, [14, Theorem A] gives the existence of a constant $C > 0$ independent of q such that

$$\|\nabla u_q\|_{L^r(\mathbb{R}^d)} \leq C \|\nabla f 1_{V_q}\|_{L^r(\mathbb{R}^d)}.$$

For every $N \in \mathbb{N}^*$, we next define the two following sequences :

$$U_N = \sum_{q \in \mathcal{P},\ |q| \leq N} u_q,$$

and

$$S_N = \nabla U_N = \sum_{q \in \mathcal{P},\ |q| \leq N} \nabla u_q.$$

By linearity, we have that, for every $N \in \mathbb{N}^*$, U_N is a solution to

$$-\operatorname{div}(a_{per} \nabla U_N) = \operatorname{div}\left(f \sum_{q \in \mathcal{P},\ |q| \leq N} 1_{V_q}\right) \quad \text{in } \mathbb{R}^d. \tag{2.96}$$

Here, our aim is to show that S_N admits a limit in $\left(L^r_{loc}(\mathbb{R}^d)\right)^d$ and its limit is the gradient of a solution u to (2.35). For every $q \in \mathcal{P}$, we introduce a set W_q and five constants C_1, C_2, C_3, C_4 and C_5 independent of q and defined by Proposition 2.4 such that :

(i) $V_q \subset W_q$,

(ii) $Diam(W_q) \leq C_1 2^{|q|}$, and $D(V_q, \partial W_q) \geq C_2 2^{|q|}$,

(iii) $\forall s \in \mathcal{P} \setminus \{q\}$, $Dist(2^s, W_q) \geq C_3 2^{|q|}$,

(iv) $\sharp \{s \in \mathcal{P} | V_s \cap W_q \neq \emptyset\} \leq C_4$,

(v) $\forall s \in \mathcal{P} \setminus \{q\}$, $D\left(V_q, V_s \setminus W_q\right) \geq C_5 2^{|s|}$.

For every $q \in \mathcal{P}$ such that $W_p \cap V_q = \emptyset$, we use the behavior (2.39) of the Green function G_{per} and the Hölder inequality to obtain :

$$\|\nabla u_q\|_{L^r(V_p)} \leq C \sup_{q \in \mathcal{P}} \|f\|_{L^r(V_q)} \left(\int_{V_p} \left(\int_{V_q} \frac{1}{|x-y|^{r'd}} dy\right)^{r/r'} dx\right)^{1/r},$$

where $r' = r/(r-1)$. We next use Property (v) of W_p and the fact that $|V_q| \leq C 2^{|q|}$ (consequence of Propositions 2.1 and 2.5). We therefore have :

$$\begin{aligned}
\|\nabla u_q\|_{L^r(V_p)} &\leq C \sup_{q \in \mathcal{P}} \|f\|_{L^r(V_q)} \left(\int_{V_p} \left(\frac{|V_q|}{2^{|q| r' d}}\right)^{r/r'} dx\right)^{1/r} \\
&\leq C \sup_{q \in \mathcal{P}} \|f\|_{L^r(V_q)} |V_p|^{1/r} \frac{1}{2^{|q| d(r'-1)/r'}}.
\end{aligned}$$

Since $r'/(r'-1) = r$, we conclude that :

$$\|\nabla u_q\|_{L^r(V_p)} \leq C \sup_{q \in \mathcal{P}} \|f\|_{L^r(V_q)} |V_p|^{1/r} \frac{1}{2^{|q| d/r}}.$$

In addition, Property (iv) ensures that the number of q such that $W_p \cap V_q \neq \emptyset$ is bounded uniformly with respect to q. Since the sequence $\left(\dfrac{1}{2^{|q|d/r}}\right)_{q\in\mathbb{Z}^d}$ is summable, we finally deduce that :

$$\lim_{N\to\infty} \sum_{q\in\mathcal{P},\ |q|\leq N} \|\nabla u_q\|_{L^r(V_p)} < \infty.$$

Therefore, we have established the absolute convergence of S_N in $\left(L^r(V_p)\right)^d$ for every $p \in \mathcal{P}$, that is the convergence of S_N in $\left(L^r_{loc}(\mathbb{R}^d)\right)^d$. We denote by T the limit of S_N in $\left(L^r_{loc}(\mathbb{R}^d)\right)^d$. Letting N go to the infinity in (2.96), we obtain :

$$-\operatorname{div}\left(a_{per}T\right) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d.$$

We next prove that there exists u in $\mathcal{D}'(\mathbb{R}^d)$ such that $T = \nabla u$ showing that, for every i, j in $\{1, ..., d\}$, we have $\partial_i T_j = \partial_j T_j$ in $\mathcal{D}'(\mathbb{R}^d)$. We denote by $\langle ., .\rangle_{\mathcal{D},\mathcal{D}'}$ the duality bracket in $\mathcal{D}'(\mathbb{R}^d)$ and we have :

$$\begin{aligned}\langle \partial_i T_j, \varphi\rangle_{\mathcal{D},\mathcal{D}'} &= -\langle T_j, \partial_i\varphi\rangle_{\mathcal{D},\mathcal{D}'}\\ &= -\sum_{q\in\mathcal{P}} \langle \partial_j u_q, \partial_i \varphi\rangle_{\mathcal{D},\mathcal{D}'}.\end{aligned}$$

The last equality is justified by the normal convergence of S_N in $L^r_{loc}(\mathbb{R}^d)$. We next use the Schwarz lemma and we have :

$$\begin{aligned}\langle \partial_i T_j, \varphi\rangle_{\mathcal{D},\mathcal{D}'} &= \sum_{q\in\mathcal{P}} \langle \partial_i\partial_j u_q, \varphi\rangle_{\mathcal{D},\mathcal{D}'} = \sum_{q\in\mathcal{P}} \langle \partial_j\partial_i u_q, \varphi\rangle_{\mathcal{D},\mathcal{D}'}\\ &= -\sum_{q\in\mathcal{P}} \langle \partial_i u_q, \partial_j\varphi\rangle_{\mathcal{D},\mathcal{D}'} = \langle \partial_j T_i, \varphi\rangle_{\mathcal{D},\mathcal{D}'}.\end{aligned}$$

Finally, we obtain that $\partial_i T_j = \partial_j T_i$ and, therefore, there exists $u \in \mathcal{D}'(\mathbb{R}^d)$ such that $T = \nabla u$. In addition, u is a solution to (2.35) in $\mathcal{D}'(\mathbb{R}^d)$. Finally, since ∇u belongs to $L^r_{loc}(\mathbb{R}^d)^d$, the result of [39, corollary 2.1] ensures that $u \in W^{1,r}_{loc}(\mathbb{R}^d)$.

Step 2 : The case $r < d$

In this step only, we assume that $r < d$. Our aim here is to show that the sequence U_N also converges in $L^r_{loc}(\mathbb{R}^d)$. We use again the behavior of the Green function and for every $q \in \mathcal{P}$ such that $W_p \cap V_q = \emptyset$, we have :

$$\begin{aligned}\|u_q\|_{L^r(V_p)} &\leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)} \left(\int_{V_p}\left(\int_{V_q} \frac{1}{|x-y|^{r'(d-1)}}dy\right)^{r/r'} dx\right)^{1/r}\\ &\leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)} |V_p|^{1/r} \frac{1}{2^{|q|((r'-1)d/r'-1)}}.\end{aligned}$$

In addition, we remark that $(r'-1)/r' = 1/r$ and we obtain :

$$\|u_q\|_{L^r(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)} |V_p|^{1/r} \frac{1}{2^{|q|(d/r-1)}}.$$

Since $r < d$, we have $d/r > 1$ and we deduce that :

$$\lim_{N\to\infty} \sum_{q\in\mathcal{P},\ |q|\leq N} \|u_q\|_{L^r(V_p)} < \infty.$$

Therefore, we have established the convergence of U_N in $L^p_{loc}(\mathbb{R}^d)$. In addition, using the uniqueness of the limit in $\mathcal{D}'(\mathbb{R}^d)$, we deduce that the function u defined in the first step is actually equal to $\lim_{N\to\infty} U_N = \int_{\mathbb{R}^d} \nabla_y G_{per}(.,y)f(y)dy$.

Step 3 : Proof of estimate (2.94)

Next, we have to establish estimate (2.94). Let p be in $\mathcal{P}$, we start by splitting ∇u in two parts :

$$\begin{aligned}\nabla u &= \sum_{\substack{q\in\mathcal{P},\\ V_q\cap W_p\neq\emptyset}} \nabla u_q + \sum_{\substack{q\in\mathcal{P},\\ V_q\cap W_p=\emptyset}} \nabla u_q \\ &= \nabla I_{1,p}(x) + \nabla I_{2,p}(x).\end{aligned}$$

We denote by $\mathcal{P}_p$ the set of indices $q \in \mathcal{P}$ such that $V_q \cap W_p \neq \emptyset$. First of all, since $\mathcal{P}_p$ is finite according to Property (iv), the function $f \sum_{q\in\mathcal{P}_p} 1_{V_q}$ belongs to $\left(L^r(\mathbb{R}^d)\right)^d$. The results of [14] therefore ensure that $I_{1,p}$ is actually a solution in $\mathcal{D}'(\mathbb{R}^d)$ to :

$$-\operatorname{div}\left(a_{per}\nabla I_{1,p}\right) = \operatorname{div}\left(f \sum_{q\in\mathcal{P}_p} 1_{V_q}\right) \quad \text{in } \mathbb{R}^d,$$

such that $\nabla I_{1,p} \in \left(L^r(\mathbb{R}^d)\right)^d$ and there exists a constant $C > 0$ independent of p and f such that :

$$\|\nabla I_{1,p}\|_{L^r(V_p)} \leq \|\nabla I_{1,p}\|_{L^r(\mathbb{R}^d)} \leq C \left\| f \sum_{q\in\mathcal{P}_p} 1_{V_q}\right\|_{L^r(\mathbb{R}^d)}.$$

Finally, we use a triangle inequality and Property (iv) of W_p to conclude there exists $C > 0$ independent of p such that :

$$\|\nabla I_{1,p}\|_{L^r(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)}. \tag{2.97}$$

Next, for $\nabla I_{2,p}$ we proceed exactly as in the proof of lemma 2.3 (see the proof of estimate (2.54) for details) using Holder inequalities, the asymptotic behavior of G_{per} and properties of W_p in order to show the existence of $C > 0$ such that :

$$\|\nabla I_{2,p}\|_{L^r(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)} \, |V_p|^{1/r} \frac{1}{2^{|p|d/r}}.$$

In addition, since $|V_p| \leq C 2^{|p|d}$, we obtain :

$$\|\nabla I_{2,p}\|_{L^r(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)}. \tag{2.98}$$

Finally, we use a triangle inequality and estimates (2.97) and (2.98) to conclude that :

$$\|\nabla u\|_{L^r(V_p)} \leq \|\nabla I_{1,p}\|_{L^r(V_p)} + \|\nabla I_{2,p}\|_{L^r(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)}.$$

We have established (2.94). □

Now that we have defined a particular solution u in $W^{1,r}_{loc}(\mathbb{R}^d)$ to (2.35), we have to show that the function ∇u belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ as soon as $f \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. This result is given in the following lemma.

Lemma 2.10. *Let $f \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$, then the function u defined in Lemma 2.9 satisfies $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$.*

The proof of this result being extremely similar to the proof of Lemma 2.4 in the case $r = 2$, we choose here not to detail it. The idea is to show that ∇u belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. In order to find its limit function in $\left(L^r(\mathbb{R}^d)\right)^d$ (denoted by ∇u_∞), it is possible to define :

$$\nabla u_\infty = \int_{\mathbb{R}^d} \nabla_y G_{per}(.,y) f_\infty(y) dy,$$

and to prove the convergence of ∇u to ∇u_∞ in the sense of definition (2.85). To this aim, the main idea is to use Proposition 2.19 in order to prove several estimates similar to estimates (2.60) and (2.61) established in the proof of Lemma 2.4.

2.6.3 Uniqueness results

We next turn to the uniqueness of solution u to (2.31) such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. In our approach, such a result of uniqueness is actually essential to prove the existence of a corrector stated in Theorem 2.3. Indeed, as we shall see in the proof of Lemma 2.14, the uniqueness is key to establish the continuity of the operator $\nabla\left(-\operatorname{div}(a\nabla .)\right)^{-1}\operatorname{div}$ from $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. By contrast with the uniqueness results established in Section 2.4 for the space $\mathcal{B}^2(\mathbb{R}^d)$, the topology of L^r used to define the space $\mathcal{B}^r(\mathbb{R}^d)$ for $r \neq 2$ does not allow us to use some "Hilbertian" techniques here. In order to overcome this difficulty, our idea consists in using both the structure (2.2) of the coefficient a and the results established in the previous section in the case $a = a_{per}$. To this aim, we need to first introduce a uniqueness result in the case $a = a_{per}$, given in Lemma 2.11, in order to use it in the proof of Lemma 2.14 regarding the general case when $a \neq a_{per}$.

To start with, we introduce two technical lemmas related to the periodic case when $\tilde{a} = 0$. The first lemma is a result of uniqueness and the second establishes a property similar to a Gagliardo-Nirenberg-Sobolev inequality satisfied by the solutions to equation (2.35).

Lemma 2.11. *Let $r > 1$ and u be a solution in $W^{1,r}_{loc}(\mathbb{R}^d)$ to :*

$$-\operatorname{div}(a_{per}\nabla u) = 0 \quad \text{in } \mathbb{R}^d, \tag{2.99}$$

such that :

$$\sup_{p\in\mathcal{P}} \int_{V_p} |\nabla u|^r < \infty.$$

Then $\nabla u = 0$.

Proof. First of all, since u is a solution in $W^{1,r}_{loc}(\mathbb{R}^d)$ to (2.99) the result of [33, Theorem 1] ensures that u actually belongs to $W^{1,s}_{loc}(\mathbb{R}^d)$ for every $s < \infty$ and is therefore locally Holder continuous according to [75, Section 5]. Next, for every $k \in \mathbb{Z}^d$, we translate equation (2.99) by k and, using the periodicity of a_{per}, we obtain that $u - \tau_k u$ is a solution in $W^{1,r}_{loc}(\mathbb{R}^d)$ to :

$$-\operatorname{div}(a_{per}\nabla(u - \tau_k u)) = 0 \quad \text{in } \mathbb{R}^d.$$

We claim that for every $i \in \{1, ..., d\}$, we have $\sup\limits_{p\in\mathcal{P}} \|u - \tau_{e_i} u\|_{L^r(V_p)} < \infty$. Indeed, for every $x \in \mathbb{R}^d$, we have :

$$u(x + e_i) - u(x) = \int_0^1 \nabla u(x + te_i).e_i dt.$$

For every $p \in \mathcal{P}$, we use the Hölder inequality and the Fubini Theorem to obtain :

$$\begin{aligned}
\left(\int_{V_p} |u(x + e_i) - u(x)|^r \, dx\right)^{1/r} &\leq C_1 \left(\int_{V_p}\int_0^1 |\nabla u(x + te_i)|^r \, dt dx\right)^{1/r} \\
&= C_1 \left(\int_0^1 \int_{V_p} |\nabla u(x + te_i)|^r \, dx dt\right)^{1/r} \\
&\leq C_2 \sup_{p\in\mathcal{P}} \left(\int_{V_p} |\nabla u|^r\right)^{1/r},
\end{aligned}$$

where C_1 and C_2 are two positive constants which only depend on d. Taking the supremum over all $p \in \mathcal{P}$, we obtain the expected result. We can also easily show that, for every $x_0 \in \mathbb{R}^d$, we have

$$\|u - \tau_{e_i} u\|_{W^{1,r}(B_2(x_0))} \leq C_3 \sup_{p\in\mathcal{P}} \left(\|u - \tau_{e_i} u\|_{L^r(V_p)} + \|\nabla u - \tau_{e_i}\nabla u\|_{L^r(V_p)}\right),$$

where $C_3 > 0$ does not depend on x_0. We use again [33, Theorem 1] and we obtain the existence of a constant $C > 0$ such that for every $x_0 \in \mathbb{R}^d$:

$$\|u - \tau_{e_i} u\|_{H^1(B_1(x_0))} \leq C\|u - \tau_{e_i} u\|_{W^{1,r}(B_2(x_0))}.$$

Here, C only depends on the ellipticity constant of a_{per}, on $\|a\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}$, on d and on r. Finally the De Giorgi-Nash inequality (see for example [47, Theorem 8.13 p.176]) ensures the existence of a constant $C > 0$ independent of x_0 such that :

$$\|u - \tau_{e_i} u\|_{L^\infty(B_2(x_0))} \leq C\|u - \tau_{e_i} u\|_{L^2(B_2(x_0))},$$

and we conclude that $u - \tau_{e_i} u$ is bounded in $L^\infty(\mathbb{R}^d)$. The results of [75, Section 6] therefore ensure that $u - \tau_{e_i} u$ is constant.

We next denote by C_i the constant such that $u(x + e_i) - u(x) = C_i$ for every $x \in \mathbb{R}^d$ and we fix $C = (C_i)_{i\in\{1,..,d\}}$. We remark that the function w defined by $w(x) = u(x) - C.x$ is periodic and, therefore, $\nabla u = \nabla w + C$ is also periodic. Since $\sup\limits_{p\in\mathcal{P}} \displaystyle\int_{V_p} |\nabla u|^r < \infty$, we can conclude that $\nabla u = 0$. $\square$

Lemma 2.12. *Assume $r < d$ and let $f \in L^r_{loc}(\mathbb{R}^d)^d$ such that* $\sup_{p\in\mathcal{P}} \int_{V_p} |f|^r < \infty$. *We define* $r^* = \dfrac{dr}{r-d}$. *Then, the function u defined by (2.95) satisfies* $\sup_{p\in\mathcal{P}} \int_{V_p} |u|^{r^*} < \infty$.

Proof. We fix $p \in \mathcal{P}$. We begin by splitting u in two parts such that for every $x \in V_p$, we have :

$$\begin{aligned} u(x) &= \int_{W_p} \nabla_y G_{per}(x,y).f(y)dy + \sum_{q\in\mathcal{P}} \int_{V_q\setminus W_p} \nabla_y G_{per}(x,y).f(y)dy \\ &= I_{1,p}(x) + I_{2,p}(x), \end{aligned}$$

Where W_p is defined by Proposition 2.4. First of all, we use estimate (2.37) and we have :

$$|I_{1,p}(x)| \leq C \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}} |f(y)| 1_{W_p}(y) dy.$$

and

$$|I_{2,p}(x)| \leq C \sum_{q\in\mathcal{P}} \int_{V_q\setminus W_p} \frac{1}{|x-y|^{d-1}} |f(y)| dy.$$

Next, since $1 + \dfrac{1}{r^*} = \dfrac{d-1}{d} + \dfrac{1}{r}$, using the Hardy-Littlewood-Sobolev inequality (see for instance [47, Theorem 7.25 p. 162]), we obtain that $I_{1,p}$ belongs to $L^{r^*}(\mathbb{R}^d)$ and there exists a constant $C > 0$ independent of p such that :

$$\|I_{1,p}\|_{L^{r^*}(\mathbb{R}^d)} \leq C \|f 1_{W_p}\|_{L^r(\mathbb{R}^d)},$$

that is, using the property (iv) of W_p, there exists $C > 0$ such that :

$$\|I_{1,p}\|_{L^{r^*}(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)}. \tag{2.100}$$

Next, using Hölder inequalities, we repeat the different steps of the proof of Lemma 2.9 (see in particular the step 2 regarding the case $r < d$) and we obtain for every x in V_p :

$$\begin{aligned} |I_{2,p}(x)| &\leq C \sum_{q\in\mathcal{P}} \left(\int_{V_q\setminus W_p} \frac{1}{|x-y|^{r'(d-1)}} dy \right)^{1/r'} \left(\int_{V_q\setminus W_p} |f(y)|^r dy \right)^{1/r} \\ &\leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)} \frac{1}{2^{|p|(d/r-1)}}. \end{aligned}$$

Since, $r < d$, we have $d/r > 1$ and therefore :

$$|I_{2,p}(x)| \leq \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)} \frac{1}{2^{|p|(d/r-1)}}.$$

Since $|V_p| \leq C 2^{|p|d}$, we deduce that :

$$\|I_{2,p}\|^{r^*}_{L^{r^*}(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|^{r^*}_{L^r(V_q)} \frac{1}{2^{|p|r^*(d/r-1)}} |V_p| \leq C \sup_{q\in\mathcal{P}} \|f\|^{r^*}_{L^r(V_q)} \frac{2^{|p|d}}{2^{|p|r^*(d/r-1)}}.$$

Next, we remark that $r^* \left(\dfrac{d}{r} - 1\right) = \dfrac{dr}{r-d}\left(\dfrac{d}{r} - 1\right) = d$ and we deduce that :

$$\|I_{2,p}\|_{L^{r^*}(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)}. \tag{2.101}$$

Using a triangle inequality and estimates (2.100) and (2.101), we finally conclude that for every $p \in \mathcal{P}$, we have :

$$\|u\|_{L^{r^*}(V_p)} \leq \|I_{1,p}\|_{L^{r^*}(V_p)} + \|I_{2,p}\|_{L^{r^*}(V_p)} \leq C \sup_{q\in\mathcal{P}} \|f\|_{L^r(V_q)}.$$

Since C is independent of p, we conclude the proof considering the supremum over all $p \in \mathcal{P}$ in the above inequality. □

Using Lemmas 2.11 and 2.12, we are now able to establish the uniqueness of a solution u to (2.31) such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ for every $1 < r < \infty$.

Lemma 2.13. *Let $r > 1$ and $u \in W^{1,r}_{loc}(\mathbb{R}^d)$ be a solution in the sense of distributions to :*

$$-\operatorname{div}(a\nabla u) = 0 \quad \textit{in } \mathbb{R}^d, \tag{2.102}$$

such that $\sup_{p\in\mathcal{P}} \int_{V_p} |\nabla u|^r < \infty$. Then $\nabla u = 0$.

Proof. We first assume that $r > 2$ and we remark that equation (2.102) is equivalent to :

$$-\operatorname{div}\left(a_{per}\nabla u\right) = \operatorname{div}(\tilde{a}\nabla u) \quad \text{in } \mathbb{R}^d. \tag{2.103}$$

We denote $f = \tilde{a}\nabla u$. We know that $\tilde{a}$ belongs to $L^\infty(\mathbb{R}^d)$ and we therefore obtain that $\sup_{p\in\mathcal{P}} \|f\|_{L^r(V_p)} < \infty$. The uniqueness result of Lemma 2.11 ensures that u is actually the solution defined in Lemma 2.9 (up to an additive constant). In addition, since $\sup_{p\in\mathcal{P}} \|\tilde{a}\|_{L^r(V_p)} < \infty$, we use the Cauchy-Schwarz inequality and we obtain that

$$\sup_{p\in\mathcal{P}} \|f\|_{L^{r/2}(V_p)} < \infty.$$

Lemma 2.9 therefore ensures that ∇u is such that $\sup_{p\in\mathcal{P}} \|\nabla u\|_{L^{r/2}(V_p)} < \infty$. We can iterate this argument in order to obtain that $\sup_{p\in\mathcal{P}} \|\nabla u\|_{L^{r_n}(V_p)} < \infty$ where $\dfrac{1}{r_n} = \dfrac{1}{r} + \dfrac{n}{r}$ for every $n \in \mathbb{N}^*$ such that $\dfrac{r}{n} > 1$. We have assumed $r > 2$ and, it is therefore always possible to find $n \in \mathbb{N}$ such that $1 \leq r_n \leq 2$. Thus, since $r_n \leq 2 \leq r$, we deduce that $\sup_{p\in\mathcal{P}} \|\nabla u\|_{L^2(V_p)} < \infty$ by an interpolation result. We finally conclude that $\nabla u = 0$ using the result of Lemma 2.1 in the case $r = 2$.

We next assume that $r \leq 2$. We know that u is a solution to (2.103). Since $r \leq 2 < d$, the function u is actually defined by (2.95) (up to an additive constant) where $f = \tilde{a}\nabla u$.

Lemma 2.12 ensures that $\sup_{p\in\mathcal{P}} \int_{V_p} |u|^{r^*} < \infty$. Now, since $r^* \geq r$, there exists a constant $C > 0$ such that for every $x_0 \in \mathbb{R}^d$, we have :

$$\|u\|_{L^r(B_2(x_0))} \leq C\|u\|_{L^{r^*}(B_2(x_0))},$$

and the properties of the cells V_p ensure the existence of a constant $C_1 > 0$ independent of x_0 such that

$$\|u\|_{L^{r^*}(B_2(x_0))} \leq C_1 \sup_{p\in\mathcal{P}} \int_{V_p} |u|^{r^*}.$$

We deduce the existence of $C_2 > 0$ independent of x_0 such that

$$\|u\|_{W^{1,r}(B_2(x_0))} \leq C_2 \left(\sup_{p\in\mathcal{P}} \left(\int_{V_p} |u|^{r^*} \right)^{1/r^*} + \sup_{p\in\mathcal{P}} \left(\int_{V_p} |\nabla u|^{r} \right)^{1/r} \right).$$

Therefore, since u is solution to (2.102), we repeat the method of the proof of uniqueness of Lemma 2.11 using the result of [33, Theorem 1] and we obtain the existence of a constant $C > 0$ such that for every x_0 in $\mathbb{R}^d$, we have

$$\|u\|_{H^1(B_1(x_0))} \leq C \left(\sup_{p\in\mathcal{P}} \left(\int_{V_p} |u|^{r^*} \right)^{1/r^*} + \sup_{p\in\mathcal{P}} \left(\int_{V_p} |\nabla u|^{r} \right)^{1/r} \right).$$

Using the De Giorgi-Nash inequality, we finally obtain that u belongs to $L^\infty(\mathbb{R}^d)$ and, according to [75, Section 6], u is constant. Finally we have $\nabla u = 0$. □

2.6.4 Existence results in the general problem

In this section, we study equation (2.31) in the case when $\tilde{a} \neq 0$) showing the existence of a solution u to (2.31) such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. To this end, we first prove the continuity of the operator $\nabla\left(-\operatorname{div} a\nabla\right)^{-1}\operatorname{div}$ from $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ in Lemma 2.14. We next use this property in order to generalize the existence result of the periodic context (that is when $\tilde{a} = 0$) applying the same argument as in the case $r = 2$ used in Lemma 2.6.

Lemma 2.14. *There exists a constant $C > 0$ such that for every $f \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ and u solution in $\mathcal{D}'(\mathbb{R}^d)$ to (2.31) with $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$, we have the following estimate :*

$$\|\nabla u\|_{\mathcal{B}^r(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{B}^r(\mathbb{R}^d)}.$$

Proof. We follow here the same pattern as the proof of Lemma 2.5, that is we argue by contradiction using a compactness-concentration method. To this aim, we assume the existence of a sequence f_n in $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ and an associated sequence of solutions u_n such that ∇u_n is in $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ and :

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u_n) = \operatorname{div}(f_n),$$

$$\lim_{n\to\infty} \|f_n\|_{\mathcal{B}^r(\mathbb{R}^d)} = 0,$$

$$\forall n \in \mathbb{N}, \quad \|\nabla u_n\|_{\mathcal{B}^r(\mathbb{R}^d)} = 1.$$

We only give here the main steps of this proof and we refer the reader to the proof of Lemma 2.5 for details.

Step 1 : Compactness-concentration method. Using a property of the supremum, we can find a sequence x_n such that for every $n \in \mathbb{N}$, we have

$$\|\nabla u_n\|_{L^r_{unif}} \geq \|\nabla u_n\|_{L^r(B_1(x_n))} \geq \|\nabla u_n\|_{L^r_{unif}} - \frac{1}{n}.$$

We denote by $\bar{u}_n = \tau_{x_n} u_n$, $\bar{f}_n = \tau_{x_n} f_n$ and $\bar{\tilde{a}}_n = \tau_{x_n} \tilde{a}$ and we have for every $n \in \mathbb{N}$:

$$\|\nabla u_n\|_{L^r_{unif}} \geq \|\nabla \bar{u}_n\|_{L^r(B_1)} \geq \|\nabla u_n\|_{L^r_{unif}} - \frac{1}{n}. \tag{2.104}$$

In addition, for every $n \in \mathbb{N}$, the sequence $\bar{u}_n$ satisfies :

$$-\operatorname{div}\left(\bar{a}_n \nabla \bar{u}_n\right) = \operatorname{div}\left(\bar{f}_n\right) \quad \text{in } \mathbb{R}^d, \tag{2.105}$$

and

$$\|\nabla \bar{u}_n\|_{L^r_{unif}} \leq 1. \tag{2.106}$$

The idea is now to study the behavior of $\nabla \bar{u}_n$ on the compact subset B_1 in order to precise the behavior of ∇u_n in L^r_{unif}.

Step 2 : Study of the limit function when $n \to \infty$. We next use estimate (2.106) and some elliptic regularity properties associated with equation (2.105) to deduce the strong convergence of $\bar{u}_n$ in $W^{1,r}(B_1)$ to a function u solution to :

$$-\operatorname{div}(A\nabla u) = 0 \quad \text{in } \mathbb{R}^d,$$

where :

- $A \in \left(L^r_{per} + \mathcal{B}^r(\mathbb{R}^d)\right)^{d\times d}$, $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ if x_n is bounded,
- $A \in \left(L^r_{per} + L^r(\mathbb{R}^d)\right)^{d\times d}$, $\nabla u \in \left(L^r(\mathbb{R}^d)\right)^d$ if x_n is not bounded.

In both case, using Proposition 2.13 or the results of [27, Proposition 2.1] in the case of local defects in L^r, we deduce that $\nabla u = 0$. Using (2.104), we finally obtain that the sequence ∇u_n converges to 0 in L^r_{unif}.

Step 3 : Convergence of ∇u_n to 0 in $\mathcal{B}^r(\mathbb{R}^d)$ and contradiction. We finally remark that for every n, ∇u_n is a solution to :

$$-\operatorname{div}(a_{per}\nabla u_n) = \operatorname{div}\left(\tilde{a}\nabla u_n + f_n\right).$$

Therefore, using estimate (2.94) established in the periodic case, the uniform convergence of ∇u_n to 0 in L^r_{unif} and the properties of $\tilde{a}$, we finally deduce the convergence of ∇u_n to 0 in $\mathcal{B}^r(\mathbb{R}^d)$ and we obtain a contradiction. $\square$

We next establish a result regarding the regularity of the solutions u to (2.31) such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. As we shall see in the sequel, this result allows to establish Lemma 2.16 using a argument of density induced by Proposition 2.15. Since the coefficient a satisfies the property of regularity (2.13), it also ensures that the gradient of the corrector given by Theorem 2.3 belongs to $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)^d$.

Lemma 2.15. *Assume $r \geq 2$. Let f be in $\left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and $u \in L^1_{loc}(\mathbb{R}^d)$ be a solution to (2.31) such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. Then ∇u belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.*

Proof. First of all, if ∇u belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$, we have in particular that ∇u is in $\left(L^r_{unif}(\mathbb{R}^d)\right)^d$. In addition, since $r \geq 2$, using the Holder inequality, we have $L^2_{unif} \subset L^r_{unif}$ and the existence of a constant $C_1 = C_1(d, r) > 0$ such that :

$$\|\nabla u\|_{L^2_{unif}} \leq C_1 \|\nabla u\|_{L^r_{unif}}.$$

Since $a \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$, a direct consequence of a regularity result established in [15, Theorem 5.19 p.87] gives the existence of $C_2 > 0$ such that for all $x \in \mathbb{R}^d$:

$$\begin{aligned}
\|\nabla u\|_{\mathcal{C}^{0,\alpha}(B_1(x))} &\leq C_2 \left(\|\nabla u\|_{L^2_{unif}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) \\
&\leq C_2(C_1 + 1) \left(\|\nabla u\|_{L^r_{unif}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).
\end{aligned}$$

Since C_1 and C_2 are independent of x, we directly conclude that $\nabla u \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. □

We are now able to conclude the study of equation (2.31) in our particular case. The next Lemma establishes the existence and the uniqueness of a solution u such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ in the case $r \geq 2$.

Lemma 2.16. *Let $f \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ and assume $r \geq 2$. There exists $u \in L^1_{loc}(\mathbb{R}^d)$ solution to (2.31) such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$.*

Proof. The proof of this Lemma is extremely similar to that of Lemma 2.6 for the case $r = 2$. Therefore, in the sequel we only explain the main strategy to prove the existence result. For a generic coefficient a, we define $\mathbf{P}(a)$ the following assertion : "There exists a solution $u \in \mathcal{D}'(\mathbb{R}^d)$ to :

$$-\operatorname{div}(a\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d$$

such that $\nabla u \in \left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$."
For $t \in [0, 1]$, we denote $a_t = a_{per} + t\tilde{a}$ and we define the following set :

$$\mathcal{I} = \{t \in [0, 1] \mid \forall s \in [0, t], \mathbf{P}(a_s) \text{ is true}\}.$$

The idea is to prove that $\mathcal{I}$ is non empty, closed and open for the topology of $[0, 1]$ in order to use an argument of connexity adapted from [27]. First, $\mathcal{I}$ is obviously non empty according to the results of Section 2.6.2. In order to prove that $\mathcal{I}$ is open and closed, we apply exactly the same method as in the proof of Lemma 2.6 using the continuity result of Lemma 2.14 and we conclude. □

Remark 2.13. *Since the coefficient $\tilde{a}$ and its associated limit $\tilde{a}_\infty$ belong to $L^\infty(\mathbb{R}^d)$, we will show in the next section that if $\tilde{a}$ is in $\mathcal{B}^r(\mathbb{R}^d)$ for $r < 2$, then $\tilde{a}$ also belongs to $\mathcal{B}^2(\mathbb{R}^d)$ and the result of Lemma 2.16 is sufficient to establish Theorem 2.3.*

2.6.5 Existence of the corrector

In this section, we finally give a proof of Theorem 2.3. To this end, it is important to note that, for every $p \in \mathbb{R}^d$, corrector equation (2.14) is equivalent to :

$$-\operatorname{div}(a\nabla w_p) = \operatorname{div}(\tilde{a}\,(\nabla w_{per,p} + p)) \quad \text{in } \mathbb{R}^d. \tag{2.107}$$

The idea is therefore to use the results of the previous section showing that the function $\tilde{a}\,(\nabla w_{per,p} + p)$ belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$.

Proof of theorem 2.3. For every $p \in \mathbb{R}^d$, our aim here is to find a function $\tilde{w}_p$ where $\nabla \tilde{w}_p \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ and such that $w_{per,p} + \tilde{w}_p$ is a solution to (2.14). First of all, since $w_{per,p}$ is the periodic corrector solution to (2.4), we have remarked below that the existence of $\tilde{w}_p$ is equivalent to the existence of a solution $\tilde{w}_p$ to (2.107). In addition, under assumption (2.13) of regularity, it is well known that the function $\nabla w_{per,p}$ belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and we can directly show that the periodicity of $\nabla w_{per,p}$ implies that the function $f = \tilde{a}\,(\nabla w_{per,p} + p)$ belongs to $\left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, as soon as $\tilde{a}$ belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^{d\times d}$ and satisfies (2.13). In order to use the results established in Section 2.6.4, two different cases can be distinguished depending on the value of r.

First case : $r \geq 2$

Since f belongs to $\left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, the existence of an unique (up to an additive constant) solution $\tilde{w}_p$ to (2.107) such that $\nabla \tilde{w}_p \in \left(\mathcal{B}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ is a direct consequence of Lemmas 2.15 and 2.16.

Second case : $r < 2$

In this case we remark that $\tilde{a}$ actually belongs to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d}$. Indeed, assumption (2.13) ensures that both $\tilde{a}$ and $\tilde{a}_\infty$, its associated limit function in $L^r(\mathbb{R}^d)$, are bounded in $L^\infty(\mathbb{R}^d)$. Therefore, since $r < 2$, we have $L^r \cap L^\infty \subset L^2 \cap L^\infty$ and we obtain that $\tilde{a}_\infty$ is in $\left(L^2(\mathbb{R}^d)\right)^{d\times d}$. Moreover, for every $q \in \mathcal{P}$, we have

$$\lim_{|q|\to\infty} \|\tilde{a} - \tau_{-q}\tilde{a}_\infty\|^2_{L^2(V_q)} \leq M \lim_{|q|\to\infty} \|\tilde{a} - \tau_{-q}\tilde{a}_\infty\|^r_{L^r(V_q)} = 0,$$

where we have denoted by $M = \left(\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} + \|\tilde{a}_\infty\|_{L^\infty(\mathbb{R}^d)}\right)^{2-r}$. We deduce that $\tilde{a}$ belongs to $\left(\mathcal{B}^2(\mathbb{R}^d)\right)^{d\times d}$ and finally, that $f \in \left(\mathcal{B}^2(\mathbb{R}^d)\right)^d$. The existence result of the case $r = 2$ implies there exists $\tilde{w}_p$, solution to (2.107), such that $\nabla \tilde{w}_p \in \left(\mathcal{B}^2(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. We want to show that $\nabla \tilde{w}_p$ is actually in $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$. First of all, for every $q \in \mathcal{P}$, considering a 2^q-translation of equation (2.107) and using the periodicity of a_{per}, we obtain :

$$-\operatorname{div}((a_{per} + \tau_q\tilde{a})\tau_q(\nabla \tilde{w}_p)) = \operatorname{div}(\tau_q f).$$

Letting $|q|$ go to the infinity in the above equation, we obtain :

$$-\operatorname{div}((a_{per} + \tilde{a}_\infty)\nabla \tilde{w}_{p,\infty}) = \operatorname{div}(f_\infty).$$

Since both $\tilde{a}_\infty$ and f_∞ belong to $L^r(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, a result of uniqueness established in [27, Proposition 2.1] ensures that $\nabla \tilde{w}_{p,\infty} \in \left(L^r(\mathbb{R}^d)\right)^d$. In addition, for every $q \in \mathcal{P}$, we have :

$$\begin{aligned}
\tilde{a}\nabla\tilde{w}_p - \tau_{-q}\left(\tilde{a}_\infty \nabla \tilde{w}_{p,\infty}\right) &= \left(\tilde{a} - \tau_{-q}\tilde{a}_\infty\right)\nabla\tilde{w}_p + \tau_{-q}\tilde{a}_\infty\left(\nabla\tilde{w}_p - \tau_{-q}\nabla\tilde{w}_{p,\infty}\right). \\
&= I_q^1 + I_q^2.
\end{aligned}$$

Since $\nabla \tilde{w}_p$ is bounded in $L^\infty(\mathbb{R}^d)$ and $\tilde{a} \in \left(\mathcal{B}^r(\mathbb{R}^d)\right)^{d\times d}$ we directly obtain :

$$\lim_{|q|\to\infty} \|I_q^1\|_{L^r(V_q)} = 0.$$

Next, we consider $s > 1$ such that $\dfrac{1}{r} = \dfrac{1}{2} + \dfrac{1}{s}$. Such a real s always exists since $r < 2$ and, in particular, we have $s > r$. Using the Hölder inequality and the fact that $\tilde{a}$ belongs to $\left(L^r(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^{d\times d}$, we obtain :

$$\begin{aligned}
\lim_{|q|\to\infty} \|I_q^2\|_{L^r(V_q)} &\leq \|\tilde{a}_\infty\|_{L^s(V_q)} \lim_{|q|\to\infty} \|\nabla \tilde{w}_p - \tau_{-q}\nabla \tilde{w}_{p,\infty}\|_{L^2(V_q)} \\
&\leq \|\tilde{a}_\infty\|_{L^s(\mathbb{R}^d)} \lim_{|q|\to\infty} \|\nabla \tilde{w}_p - \tau_{-q}\nabla \tilde{w}_{p,\infty}\|_{L^2(V_q)} \\
&\leq C\|\tilde{a}_\infty\|_{L^r(\mathbb{R}^d)} \lim_{|q|\to\infty} \|\nabla \tilde{w}_p - \tau_{-q}\nabla \tilde{w}_{p,\infty}\|_{L^2(V_q)} = 0.
\end{aligned}$$

In the last inequality, the constant C only depends on $\|\tilde{a}_\infty\|_{L^\infty(\mathbb{R}^d)}$. We finally obtain that

$$\lim_{|q|\to\infty} \|\tilde{a}\nabla \tilde{w}_p - \tau_{-q}\left(\tilde{a}_\infty \nabla \tilde{w}_{p,\infty}\right)\|_{L^r(V_q)} = 0.$$

In addition, since $\tilde{a}_\infty$ is bounded and $\nabla \tilde{w}_{p,\infty} \in \left(L^r(\mathbb{R}^d)\right)^d$, we decuce that $\tilde{a}_\infty \nabla \tilde{w}_{p,\infty}$ belongs to $\left(L^r(\mathbb{R}^d)\right)^d$. We therefore obtain that $\tilde{a}\nabla \tilde{w}_p$ belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ with $\left(\tilde{a}\nabla \tilde{w}_p\right)_\infty = \tilde{a}_\infty \nabla \tilde{w}_{p,\infty}$. To conclude, we finally remark that equation (2.107) is equivalent to :

$$-\operatorname{div}(a_{per}\nabla \tilde{w}_p) = \operatorname{div}(\tilde{a}\nabla \tilde{w}_p + f).$$

The existence result of Lemma 2.9 and the uniqueness result of Lemma 2.11 in the case $a = a_{per}$ therefore ensure that $\nabla \tilde{w}_p$ belongs to $\left(\mathcal{B}^r(\mathbb{R}^d)\right)^d$ and we can conclude the proof. □

2.6.6 Homogenization results and convergence rates

In this section we generalize the method employed in Section 2.5 in the case $r = 2$ in order to establish the homogenization of equation (2.14) and, in particular, the results of Theorem 2.4. To this end, we use the corrector $w = (w_{e_i})_{i\in\{1,\ldots,d\}}$ given by Theorem 2.3.

A first crucial step is to indentify the limit of the sequence $(u^\varepsilon)_{\varepsilon>0}$ of solutions to (2.1). First of all, Proposition 2.17 ensures that both the coefficient $\tilde{a}$ and the gradient of the corrector $\tilde{w}_i = \tilde{w}_{e_i}$ (for every $i \in \{1, ..., d\}$) have average value zero in the sense (2.91) since they belong to $\mathcal{B}^r(\mathbb{R}^d)$. In particular, for every $1 < r < \infty$, the perturbations of $\mathcal{B}^r(\mathbb{R}^d)$ do not impact the periodic background on average and we can easily generalize the results of Proposition 2.13 to the case $r \neq 2$. Therefore, we obtain that the limit u^* (weak-$H^1(\Omega)$ and strong-$L^2(\Omega)$) of the sequence u^ε is actually a solution to (2.3) where a^* is the same homogenized coefficient as in the periodic case.

In the sequel, we study the behavior of the sequence $u^{\varepsilon,1} = u^* + \varepsilon \displaystyle\sum_{i=1}^d \partial_i u^* w_i(./\varepsilon)$. In particular, we established here the convergence rates stated in Theorem 2.4 regarding the convergence of the sequence $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$. We follow here the method employed in Section

2.5 and we consider the divergence-free matrix defined by $M_k^i = a_{i,k}^* - \sum_{j=1}^{d} a_{i,j}(\delta_{j,k} + \partial_j w_k) \in L^2_{per} + \mathcal{B}^r(\mathbb{R}^d)$. In our case, the existence of a potential of the form $B = B_{per} + \tilde{B} \in L^2_{per}(\mathbb{R}^d) + \mathcal{B}^r(\mathbb{R}^d)$ solution to (2.72) is given by the following Lemma.

Lemma 2.17. *Let $\tilde{M} = \left(\tilde{M}_k^i\right)_{1\leq i,k\leq d} \in \mathcal{B}^r(\mathbb{R}^d)^{d\times d}$ such that $\operatorname{div}(\tilde{M}_k) = 0$, for every $k \in \{1, ..., d\}$. Then there exists a potential $\tilde{B}_k^{i,j} \in W^{1,r}_{loc}(\mathbb{R}^d)$ such that $\nabla \tilde{B} \in \mathcal{B}^r(\mathbb{R}^d)$ and for all $i, j, k \in \{1, ..., d\}$:*

$$\begin{aligned} -\Delta \tilde{B}_k^{i,j} &= \partial_j \tilde{M}_k^i - \partial_i \tilde{M}_k^j, \\ \tilde{B}_k^{i,j} &= -\tilde{B}_k^{j,i}, \\ \sum_{i=1}^{d} \partial_i \tilde{B}_k^{i,j} &= \tilde{M}_k^j. \end{aligned}$$

In addition, there exists a constant $C_1 > 0$ which only depends of the ambient dimension d and such that :

$$\|\nabla \tilde{B}\|_{\mathcal{B}^r(\mathbb{R}^d)} \leq C_1 \|\tilde{M}\|_{\mathcal{B}^r(\mathbb{R}^d)}.$$

This result is actually a consequence of Lemma 2.9. For the details of the proof, we refer the reader to the similar Lemma 2.7 for the case $r = 2$. Now that the existence of B has been dealt with, we need to study the behavior of the sequences $\varepsilon\tilde{w}(./\varepsilon)$ and $\varepsilon\tilde{B}(./\varepsilon)$ in $L^\infty(\mathbb{R}^d)$ when $\varepsilon \to 0$. We shall see that the case $r > d$ is a consequence of estimate (2.93) established in Proposition 2.18. The case $r < d$ is studied in the next lemma.

Lemma 2.18. *Assume $r < d$. Then, the corrector $w = (w_i)_{i\in\{1,...,d\}} := (w_{e_i})_{i\in\{1,...,d\}}$ defined by Theorem 2.3 and the potential B solution to (2.72) are in $L^\infty(\mathbb{R}^d)$.*

Proof. For all $i \in \{1, ..., d\}$, it is important to remark that $\tilde{w}_i$ is also a solution to :

$$-\operatorname{div}\left(a_{per}\nabla\tilde{w}_i\right) = \operatorname{div}\left(\tilde{a}\left(e_i + \nabla w_{per,i} + \nabla\tilde{w}_i\right)\right).$$

We know the gradient of the corrector defined in Theorem 2.3 is in $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$. Next, using Property 2.16, we deduce that $f = \tilde{a}\left(e_i + \nabla w_{per,i} + \nabla\tilde{w}_i\right) \in \left(L^\infty(\mathbb{R}^d) \cap \mathcal{B}^r(\mathbb{R}^d)\right)^d$. Lemmas 2.9 and 2.11 therefore ensure that for every $i \in \{1, ..., d\}$, $\tilde{w}_i$ is defined by :

$$\tilde{w}_i(x) = \int_{\mathbb{R}^d} \nabla_y G_{per}(x, y) f(y) dy.$$

We want to prove that $\tilde{w}_i$ actually belongs to $L^\infty(\mathbb{R}^d)$. In order to prove that the integral is bounded independently of x, we follow step by step the method used in the proof of Lemma 2.8. We first fix $x \in \mathbb{R}^d$ and denote p_x the unique element of $\mathcal{P}$ such that $x \in V_{p_x}$. We define $W_{p_x} = W_{2^{p_x}}$ such as in Proposition 2.4 and we split the integral in three parts :

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla_y G_{per}(x,y) f(y) dy = &\int_{B_1(x)} \nabla_y G_{per}(x,y) f(y) dy + \int_{W_{p_x}\setminus B_1(x)} \nabla_y G_{per}(x,y) f(y) dy \\ &+ \int_{\mathbb{R}^d\setminus W_{p_x}} \nabla_y G_{per}(x,y) f(y) dy = I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Firstly, using estimate (2.37) for the Green function, we obtain

$$|I_1(x)| \leq C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{B_1(x)} \frac{1}{|x-y|^{d-1}} dy \leq C\|f\|_{L^\infty(\mathbb{R}^d)}.$$

Where C denotes a positive constant independent of x.

Next, we know there exist $C_1 > 0$ and $C_2 > 0$ independent of x such that $W_{p_x} \subset B_{C_1 2^{p_x}}(x)$ and the number of $q \in \mathcal{P}$ such that $V_q \cap W_{p_x} \neq \emptyset$ is bounded by C_2 (as a consequence of Proposition 2.4). We therefore use the Hölder inequality and we obtain :

$$\begin{aligned} |I_2(x)| &\leq \int_{W_{p_x} \setminus B_1(x)} \frac{1}{|x-y|^{d-1}} |f(y)| dy \\ &\leq C_2 \left(\int_{B_{C_1 2^{p_x}(x)} \setminus B_1(x)} \frac{1}{|x-y|^{r'(d-1)}} dy \right)^{1/r'} \sup_{p \in \mathcal{P}} \|f\|_{L^r(V_q)}, \end{aligned}$$

where we have denoted by $r' = \dfrac{r}{r-1}$.

In addition, since $r < d$, we have $(r'-1)(d-1) > 1$ and we deduce that :

$$\int_{B_{C_1 2^{p_x}(x)} \setminus B_1(x)} \frac{1}{|x-y|^{r'(d-1)}} dy = \int_{B_{C_1 2^{p_x}(0)} \setminus B_1(0)} \frac{1}{|y|^{r'(d-1)}} dy \leq C \left(1 - \frac{1}{2^{|p_x|((r'-1)(d-1)-1)}} \right).$$

We finally obtain :

$$I_2(x) \leq C \sup_{p \in \mathcal{P}} \|f\|_{L^r(V_q)} \left(1 - \frac{1}{2^{|p_x|((r'-1)(d-1)-1)}} \right)^{1/r'} \leq C \sup_{p \in \mathcal{P}} \|f\|_{L^r(V_q)}.$$

Finally, to bound $I_3(x)$ we split the integral on each cell V_q for $q \in \mathcal{P}$. Using the Hölder inequality, we obtain :

$$\begin{aligned} |I_3(x)| &\leq \sum_{q \in \mathcal{P}} \int_{V_q \setminus W_{p_x}} |\nabla_y G_{per}(x,y) f(y)| \, dy \\ &\leq \|f\|_{\mathcal{B}^r(\mathbb{R}^d)} \sum_{q \in \mathcal{P}} \left(\int_{V_q \setminus W_{p_x}} |\nabla_y G_{per}(x,y)|^{r'} \, dy \right)^{1/r'}. \end{aligned}$$

Using again estimate (2.37), we have :

$$\sum_{q \in \mathcal{P}} \left(\int_{V_q \setminus W_{p_x}} |\nabla_y G_{per}(x,y)|^{r'} \, dy \right)^{\frac{1}{r'}} \leq C \sum_{q \in \mathcal{P}} \left(\int_{V_q \setminus W_{p_x}} \frac{1}{|x-y|^{r'(d-1)}} dy \right)^{1/r'}.$$

Again, for every $q \in \mathcal{P}_{\mathcal{C}}$, we have $|V_q| \leq C 2^{|q|}$, and the properties of W_{p_x} (see proposition 2.4 for details) ensure that $|x-y| \geq C 2^{|q|}$. We deduce that

$$|I_3(x)| \leq C\|f\|_{\mathcal{B}^r(\mathbb{R}^d)} \sum_{q \in \mathcal{P}} \frac{1}{2^{|q|\left(\frac{(r'-1)d}{r'} - 1\right)}} < \infty.$$

Finally, we have bounded $\tilde{w}_i(x)$ independently of x and we deduce that $\tilde{w}_i \in L^\infty(\mathbb{R}^d)$. With the same method we obtain the same result for $B = B_{per} + \tilde{B}$ which allows us to conclude. $\square$

We next remind that for every $\varepsilon > 0$, the function R^ε is a solution to

$$-\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla R^\varepsilon\right) = \operatorname{div}(H^\varepsilon) \quad \text{in } \Omega,$$

where H^ε is defined by (2.79). We are now able to give a complete proof of Theorem 2.4 using the results previously established in this study.

Proof of Theorem 2.4. Here we can exactly repeat the different steps of the proof of Theorem 2.2 in the case $r = 2$ and we obtain the following estimates :

$$\|H^\varepsilon\|_{L^2(\Omega)} \leq C_1 \left(\|\varepsilon w(./\varepsilon)\|_{L^\infty(\Omega)} + \|\varepsilon B(./\varepsilon)\|_{L^\infty(\Omega)}\right) \|f\|_{L^2(\Omega)}, \tag{2.108}$$

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_2 \left(\left(\|\varepsilon w(./\varepsilon)\|_{L^\infty(\Omega)} + \|\varepsilon B(./\varepsilon)\|_{L^\infty(\Omega)}\right) \|f\|_{L^2(\Omega)} + \|H^\varepsilon\|_{L^2(\Omega)}\right), \tag{2.109}$$

and for every $\Omega_1 \subset\subset \Omega$:

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_3 \left(\|H^\varepsilon\|_{L^2(\Omega)} + \|R^\varepsilon\|_{L^2(\Omega)}\right), \tag{2.110}$$

where $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ are independent of ε. We note here that we want to bound $\varepsilon w(./\varepsilon)$ and $\varepsilon B(./\varepsilon)$ in $L^\infty(\mathbb{R}^d)$ in order to establish estimates (2.89) and (2.90). First, it is well known that w_{per} and B_{per} are in $L^\infty(\mathbb{R}^d)$. Secondly, since both $\nabla\tilde{w}$ and $\nabla\tilde{B}$ belong to $\left(\mathcal{B}^r(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)^d$, Proposition 2.18 (for the case $r > d$) and Lemma 2.18 (for the case $r < d$) ensure the existence of a constant $C > 0$ independent of ε such that :

$$\|\varepsilon\tilde{w}(./\varepsilon)\|_{L^\infty(\mathbb{R}^d)} \leq C\left(\log|\varepsilon|\right)^{\mu_r}\varepsilon^{\nu_r}, \tag{2.111}$$

$$\|\varepsilon\tilde{B}(./\varepsilon)\|_{L^\infty(\mathbb{R}^d)} \leq C\left(\log|\varepsilon|\right)^{\mu_r}\varepsilon^{\nu_r}, \tag{2.112}$$

where ν_r and μ_r are defined by (2.87) and (2.88). To conclude, we finally use (2.108), (2.109), (2.110), (2.111) and (2.112) and we obtain :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C\left(\log|\varepsilon|\right)^{\mu_r}\varepsilon^{\nu_r}\|f\|_{L^2(\Omega)},$$

and

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq \tilde{C}\left(\log|\varepsilon|\right)^{\mu_r}\varepsilon^{\nu_r}\|f\|_{L^2(\Omega)},$$

where C and $\tilde{C}$ are independent of ε. We have proved Theorem 2.4.

$\square$

Chapitre 3

Homogénéisation elliptique avec coefficients presque invariants par translation à l'infini

Ce chapitre reproduit un article publié dans Asymptotic Analysis [Gou22b].

On s'intéresse ici à l'homogénéisation de l'équation de diffusion $-\operatorname{div}(a(./\varepsilon)\nabla u^\varepsilon) = f$ quand le coefficient a est presque invariant par translation à l'infini et modélise une géométrie proche d'une géométrie périodique. Cette géométrie est caractérisée par le fait qu'un certain gradient discret du coefficient a appartient à un espace de Lebesgue $L^p(\mathbb{R}^d)$ pour $p \in [1, +\infty[$. Quand $p < d$, on établit une adaptation discrète de l'inégalité de Gagliardo-Nirenberg-Sobolev afin de montrer que le coefficient a appartient à une certaine classe de coefficients périodiques perturbés par un défaut local. On prouve alors l'existence d'un correcteur adapté et on identifie la limite homogénéisée de u^ε. Quand $p \geq d$, on exhibe des coefficients a pour lesquels u^ε possède plusieurs valeurs d'adhérence dans L^2.

Sommaire

Elliptic homogenization with almost translation-invariant coefficients

3.1 Introduction

Our purpose is to address an homogenization problem for a second order elliptic equation in divergence form with highly oscillatory coefficients :

$$\begin{cases} -\operatorname{div}\left(a\left(./\varepsilon\right)\nabla u^{\varepsilon}\right) = f & \text{on } \Omega, \\ u^{\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$), f is a function in $L^2(\Omega)$ and $\varepsilon > 0$ is a small scale parameter. The (matrix-valued) coefficient a is assumed to model a perturbed periodic geometry and to satisfy an almost translation invariance at infinity. Such a property, which will be formalized in the sequel, ensures that the coefficient describes a non-periodic medium with a structure close to that of a periodic medium at infinity. The present work follows up on several previous works [20, 27, 26, 25] where the authors have studied the homogenization of problem (3.1) for non-periodic geometries characterized by a known periodic background perturbed by certain local defects. This structure was generically modeled using a particular class of coefficients of the form $a = a_{per} + \tilde{a}$ where a_{per} is a periodic coefficient and $\tilde{a}$ is a perturbation that in some formal sense vanishes at infinity since it belongs to a Lebesgue space $\left(L^p(\mathbb{R}^d)\right)^{d\times d}$ for $p \in]1,\infty[$. In this paper, we adopt a somewhat more general approach for the study of problem (3.1) in a context of a perturbed periodic geometry, without postulating the specific structure "$a = a_{per} + \tilde{a}$" for the coefficient a. The only assumption that we make on the ambient background, which is the starting point of our study, is an assumption of almost Q-translation invariance at infinity (where $Q =]0,1[^d$ denotes the d-dimensional unit cube) satisfied by a. Typically, such an assumption in dimension $d = 1$ will be expressed as the integrability of the function $\delta a := a(.+1) - a$ at infinity.

To start with, the coefficients a considered are assumed to be elliptic, uniformly bounded and uniformly α-Hölder continuous on $\mathbb{R}^d$:

$$\exists \lambda > 0,\ \forall x, \xi \in \mathbb{R}^d, \quad \lambda|\xi|^2 \leq \langle a(x)\xi, \xi\rangle, \tag{3.2}$$

$$a \in \left(L^\infty(\mathbb{R}^d)\right)^{d\times d}, \tag{3.3}$$

$$a \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}, \qquad \text{for } \alpha \in]0,1[, \tag{3.4}$$

where $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ is the space of functions which are both uniformly bounded and uniformly α-Hölder continuous on $\mathbb{R}^d$, defined by

$$\mathcal{C}^{0,\alpha}(\mathbb{R}^d) = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Assumptions (3.2) and (3.3) are standard for the study of the homogenization problem (3.1). Assumption (3.4) is an additional assumption which is required in our approach to apply some results of elliptic regularity and to use pointwise estimates satisfied by the Green functions associated with equations in divergence form (see for instance [11, 14] in which these assumptions are already made in the case of periodic coefficients). Since assumptions (3.2) and (3.3) imply that $a(./\varepsilon)$ is uniformly elliptic and uniformly bounded in $L^\infty(\Omega)$ with respect to ε, the general homogenization theory of second order elliptic equations in divergence form (3.1) (see [90, Chapter 6, Chapter 13]) shows the existence of an extraction φ such that $u_{\varphi(\varepsilon)}$ converges, strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, to a function u^* when ε converges to 0. The limit function is a solution to an homogenized problem, which is also a second order elliptic equation in divergence form,

$$\begin{cases} -\operatorname{div}(a^*\nabla u^*) = f & \text{on } \Omega, \\ u^*(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

for some matrix-valued coefficient a^* to be determined. In the periodic case, that is (3.1) when $a = a_{per}$ is periodic, it is well-known (see [18, 59]) that the whole sequence u^ε converges to u^* and $(a_{per})^*$ is a constant matrix. The convergence in the $H^1(\Omega)$ norm can be obtained upon introducing a corrector $w_{per,q}$ defined for all q in $\mathbb{R}^d$ as the periodic solution (unique up to the addition of a constant) to :

$$-\operatorname{div}(a_{per}(\nabla w_{per,q} + q)) = 0 \quad \text{on } \mathbb{R}^d.$$

This corrector allows to both make explicit the homogenized coefficient

$$((a_{per})^*)_{i,j} = \int_Q e_i^T a_{per}(y)\left(e_j + \nabla w_{per,e_j}\right) dy, \tag{3.6}$$

(where $(e_i)_{\{1,\dots,d\}}$ denotes the canonical basis of $\mathbb{R}^d$) and define an approximation

$$u^{\varepsilon,1} = u^*(.) + \varepsilon \sum_{i=1}^d \partial_i u^*(.) w_{per,e_i}(./\varepsilon), \tag{3.7}$$

such that $u^{\varepsilon,1} - u^\varepsilon$ strongly converges to 0 in $H^1(\Omega)$ (see [3] for more details).

Our purpose here is to study the possibility to extend the above results to the setting of the non-periodic problem (3.1) when a satisfies assumptions (3.2)-(3.3)-(3.4) and is almost translation invariant at infinity. In our non-periodic case, the main difficulty is that, analogously to the periodic context, the behavior of u^ε is closely linked to the existence of a corrector satisfying a property of strict sub-linearity at infinity, that is a solution, for $q \in \mathbb{R}^d$ fixed, to the corrector equation

$$\begin{cases} -\operatorname{div}\left(a\left(\nabla w_q + q\right)\right) = 0 & \text{on } \mathbb{R}^d, \\ \displaystyle\lim_{|x|\to\infty} \frac{|w_q(x)|}{1+|x|} = 0. \end{cases} \tag{3.8}$$

Here the corrector equation, formally obtained by a two-scale expansion (see again [3] for the details), is defined on the whole space $\mathbb{R}^d$ and cannot be reduced to an equation posed on a bounded domain, as is the case in periodic context in particular. This prevents us from using classical techniques.

3.1.1 Mathematical setting and preliminary approach

In order to formalize our setting of non-periodic coefficients satisfying an almost translation invariance at infinity, we introduce, for every function $g \in L^1_{loc}(\mathbb{R}^d)$, the discrete gradient of g denoted by δg. It is a vector-valued function defined by

$$\delta g := (\delta_i g)_{i\in\{1,\dots,d\}} := (g(.+e_i) - g)_{i\in\{1,\dots d\}} . \tag{3.9}$$

For every $p \in [1, +\infty[$, we also define the set $\mathbf{A}^p$ of locally integrable functions with a discrete gradient in $\left(L^p(\mathbb{R}^d)\right)^d$:

$$\mathbf{A}^p = \left\{ g \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \delta g \in \left(L^p(\mathbb{R}^d)\right)^d \right\}. \tag{3.10}$$

Defined as above, the operator δ measures the deviation of a function g from a Q-periodic function. This discrete gradient has been already used in the literature to study the behavior of solutions to elliptic equations posed in a periodic background, particularly to establish Liouville-type properties in [76] and, more recently, to establish some regularity results satisfied by the solution to $-\operatorname{div}(a_{per}\nabla u) = 0$ in [9, Lemma 3.1]. In our study, the class of coefficients a we consider to model an asymptotically Q-periodic geometry is assumed to satisfy :

$$\exists p \in [1, +\infty[, \ \forall i, j \in \{1, ..., d\}, \quad a_{i,j} \in \mathbf{A}^p. \tag{3.11}$$

Such an assumption ensures in a certain sense that δa vanishes at infinity and, consequently, that the behavior of a is close to that of a Q-periodic coefficient far from the origin (see Figure 3.1 for examples in dimension $d = 1$ and $d = 2$). In addition, although we choose here to consider a specific case in which the coefficient is characterized by a "$\mathbb{Z}^d$-periodicity" at infinity, the results of the present paper can be easily adapted in a context of "T-periodicity" at infinity for any period T (see Remark 3.5). From a practical point of view, for a given medium modeled by a coefficient a, we are aware that the main difficulty is actually to identify the underlying period T that characterizes the behavior of a at infinity. A possible approach to overcome this difficulty consists in performing a spectrum analysis in order to identify the frequency of occurrence of the Dirac delta functions in the Fourier transform of a. We additionally note that, adapting the definition of the discrete gradient (3.9), similar questions to those addressed in the present article may be studied for random coefficients that are stationary at infinity in a sense that has to be made precise. We refer to Section 1.5 for more details.

A preliminary approach to understand the behavior of u^ε and to prove existence of an adapted corrector in our context is to consider a continuous version of (3.11) for which

$$\exists p \in [1, +\infty[, \forall i, j \in \{1, ..., d\}, \quad \nabla a_{i,j} \in \left(L^p(\mathbb{R}^d)\right)^d. \tag{3.12}$$

Since $\delta_k a_{i,j} = \displaystyle\int_0^1 \nabla a_{i,j}(.+te_k).e_k \, dt$ for every i, j, k in $\{1, ..., d\}$, the Hölder inequality actually allows to show that assumption (3.12) is stronger than (3.11). Such an assumption implies the convergence to 0 of ∇a at infinity, that is to say, it models a medium close to an homogeneous medium at infinity. For this particular setting, we can distinguish two cases that depend on the value of the ratio $\dfrac{p}{d}$:

1. The case $p < d$: If we denote by $p^* = \dfrac{pd}{d-p}$ the Sobolev exponent associated with p, a consequence of the Gagliardo-Nirenberg-Sobolev inequality (see for instance [42, Section

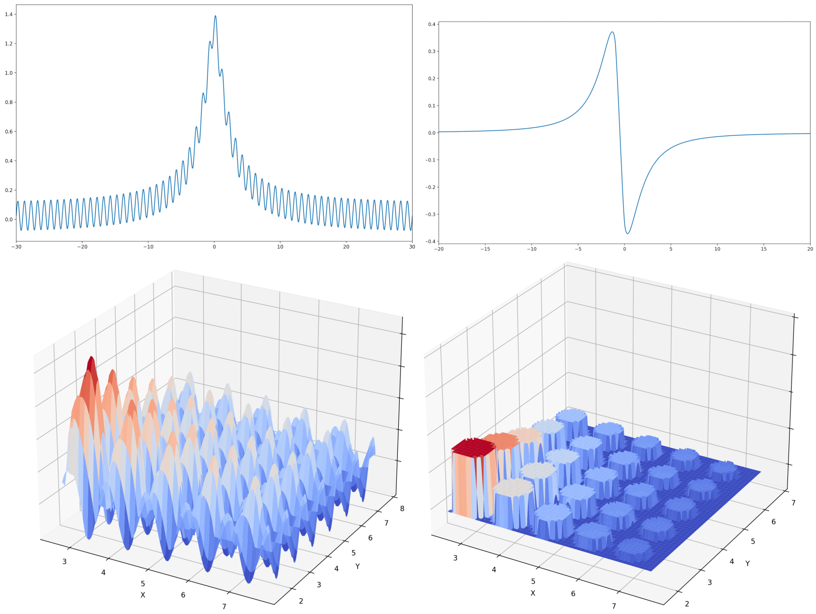

FIGURE 3.1 – Examples of coefficients a satisfying assumption (3.11) in dimension $d = 1$ (left : $a(x)$; right : $\delta a(x)$) and $d = 2$ (left : $a(x,y)$; right : $|\delta_2 a(x,y)|$).

5.6.1]) gives the existence of a constant $c \in \mathbb{R}^{d\times d}$ such that $a - c \in \left(L^{p^*}(\mathbb{R}^d)\right)^{d\times d}$ and the following inequality holds :

$$\|a - c\|_{L^{p^*}(\mathbb{R}^d)} \leq M\|\nabla a\|_{L^p(\mathbb{R}^d)},$$

where M is a constant independent of a. It is therefore possible to split the coefficient a as the sum of a constant and a "local" perturbation. Precisely,

$$a = c + a - c = c + \tilde{a}, \tag{3.13}$$

where $\tilde{a}$ belongs to $\left(L^{p^*}(\mathbb{R}^d)\right)^{d\times d}$ and the setting is that of a periodic geometry (actually an homogeneous background described by the constant c) perturbed by a defect of $L^{p^*}(\mathbb{R}^d)$. Consequently, if a satisfies (3.2), (3.3) and (3.4), our problem is equivalent to a perturbed periodic problem introduced in [20, 27]. In this case the existence of an adapted corrector is established, the gradient of which shares the same structure as the coefficient a : it is a gradient of a periodic function perturbed by a function in $L^{p*}(\mathbb{R}^d)$. The homogenization problem can also be addressed : the whole sequence u^ε converges to u^* and the coefficient a^* can be made explicit.

2. The case $p \geq d$: This case is characterized by a slow decay of ∇a at infinity. Contrary to the case $p < d$, we can show here the existence of coefficients a satisfying (3.12) and such that it is impossible to split a as in (3.13), that is to characterize our particular geometry as a periodic (let alone homogeneous) background perturbed by a local defect. A typical counter example, which will be detailed in Section 3.4, is given by a coefficient a which oscillates very slowly at infinity ($a = 2 + \sin(\ln(1 + |x|)$ in dimension $d = 1$ for example). Far from the origin, such a coefficient locally looks as constant but does not converge at infinity. In this particular case, we can show that the sequence u^ε itself does not converge. We have only the convergence up to an extraction and the sequence admits an infinite number of converging subsequences.

In our discrete case, when δa belongs to $\left(L^p(\mathbb{R}^d)\right)^d$, we therefore expect a similar phenomenon : the convergence of u^ε should depend on the value of the ratio $\frac{p}{d}$, that is, on the type of decay at infinity of the discrete gradient δa.

3.1.2 Main results

In the sequel, we denote by B_R the ball of radius $R > 0$ centered at the origin, by $B_R(x)$ the ball of radius $R > 0$ and center $x \in \mathbb{R}^d$ and by $Q + x := \prod_{i=1}^{d}]x_i, x_i + 1[$, the unit cell translated by a vector $x \in \mathbb{R}^d$. We also denote by $|A|$ the volume of any Borel subset $A \subset \mathbb{R}^d$. In addition, for a normed vector space $(X, \|.\|_X)$ and a matrix-valued function $f \in X^n$, $n \in \mathbb{N}$, we use the notation $\|f\|_X \equiv \sum_{i=1}^{n} \|f_i\|_X$.

Assuming that the coefficient a satisfies (3.2), (3.3), (3.4) and (3.11), the main questions that we examine in this paper are the following : does the whole sequence u^ε converges to u^* (and not only a sub-sequence)? If it is the case, can the diffusion coefficient a^* of the homogenized equation be made explicit? Can we establish the existence of a strictly sub-linear corrector solution to (3.8)?

3.1.2.1 The case $p < d$

When $\delta a \in \left(L^p(\mathbb{R}^d)\right)^d$ for $p < d$, our approach is an adaptation of that of the continuous case which we have just sketched above in Section 3.1.1 : we show that the coefficient a actually models a periodic geometry perturbed by a local defect which, up to a local averaging, belongs to $L^{p^*}(\mathbb{R}^d)$, for $p^* = \frac{pd}{d-p}$ the Sobolev exponent associated with p. To this end, we introduce an operator $\mathcal{M}$ to describe the local averages of a function $f \in L^1_{loc}(\mathbb{R}^d)$ and defined by :

$$\mathcal{M}(f)(z) = \int_{Q+z} f(x)dx.$$

We also introduce the following two functional spaces :

$$\mathcal{E}^p = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \mathcal{M}(|f|) \in L^{p^*}(\mathbb{R}^d) \right\}, \tag{3.14}$$

$$\mathcal{A}^p = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \mathcal{M}(|f|) \in L^{p^*}(\mathbb{R}^d) \text{ and } \delta f \in (L^p(\mathbb{R}^d))^d \right\}, \tag{3.15}$$

equipped with the norms :

$$\|f\|_{\mathcal{E}^p} = \|\mathcal{M}(|f|)\|_{L^{p^*}(\mathbb{R}^d)}, \tag{3.16}$$

$$\|f\|_{\mathcal{A}^p} = \|\mathcal{M}(|f|)\|_{L^{p^*}(\mathbb{R}^d)} + \|\delta f\|_{(L^p(\mathbb{R}^d))^d}. \tag{3.17}$$

We particularly note that the functions in $\mathcal{E}^p$ or $\mathcal{A}^p$ are characterized by the integrability of the local averaging operator $\mathcal{M}$ applied to their absolute value. Our main result regarding the functions of $\mathbf{A}^p$ when $p < d$ is given in the following proposition :

Proposition 3.1. *Assume $p < d$. Let $f \in \mathbf{A}^p$, then there exists a unique periodic function f_{per} such that $f - f_{per} \in \mathcal{E}^p$. In addition, there exists a constant $C > 0$ independent of f such that :*

$$\|f - f_{per}\|_{\mathcal{E}^p} \leq C\|\delta f\|_{L^p(\mathbb{R}^d)}.$$

Proposition 3.1 is a discrete adaptation of the Gagliardo-Nirenberg-Sobolev inequality and Section 3.2 is devoted to its proof. It ensures that every function $f \in \mathbf{A}^p$ is the sum of a periodic function and a "perturbation" of $\mathcal{A}^p$. This result therefore allows to identify a periodic background perturbed by a local defect. Precisely, the coefficient a is of the form $a = a_{per} + \tilde{a}$, that is, it is the sum of a periodic coefficient a_{per}, that will be made explicit in this paper (see Proposition 3.10), and a perturbation denoted by $\tilde{a}$. To address the homogenization problem in this particular perturbed case, we establish the following result :

Theorem 3.1. *Assume a satisfies (3.2), (3.3), (3.4) and (3.11) for $1 < p < d$. We denote by a_{per} the unique periodic coefficient given by Proposition 3.1 such that $\tilde{a} := a - a_{per} \in (\mathcal{A}^p)^{d\times d}$. Let $q \in \mathbb{R}^d$. If $w_{per,q}$ is the periodic solution, unique up to an additive constant, to*

$$- \operatorname{div}\left(a_{per}\left(\nabla w_{per,q} + q\right)\right) = 0 \quad \text{on } \mathbb{R}^d.$$

Then, there exists $\tilde{w}_q \in L^1_{loc}(\mathbb{R}^d)$ solution to

$$\begin{cases} - \operatorname{div}(a(\nabla w_{per,q} + \nabla \tilde{w}_q + q)) = 0 & \text{on } \mathbb{R}^d, \\ \displaystyle\lim_{|x|\to\infty} \frac{|\tilde{w}_q(x)|}{1+|x|} = 0, \end{cases} \tag{3.18}$$

such that $\nabla \tilde{w}_q \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Such a solution $\tilde{w}_q$ is unique up to an additive constant.

In addition, the sequence u^ε of solutions to (3.1) converges, strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to u^* solution to :

$$\begin{cases} - \operatorname{div}(a^*_{per} \nabla u^*) = f & \text{on } \Omega, \\ u^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.19}$$

where a^*_{per} is defined by (3.6).

Theorem 3.1 states the existence of an adapted corrector and shows the convergence of u^ε to an homogenized limit u^*. Similarly to the case of a periodic geometry perturbed by local defects of $L^r(\mathbb{R}^d)$ studied in [20, 27, 26, 25], the gradient of our adapted corrector shares the same structure as the coefficient a : it is the sum of a periodic function and a perturbation in $\mathcal{A}^p$. The perturbations of $\mathcal{A}^p$ does not impact the homogenized solution since the homogenized coefficient is the same as in the periodic problem (3.1) when $a = a_{per}$. In the sequel, we establish Theorem 3.1 in the case where $p < d$ and $p \neq 1$, the case $p = 1$ being specific. Indeed, as we shall see in Section 3.3, our approach is based on the study of the general diffusion equation

$$- \operatorname{div}(a\nabla u) = \operatorname{div}(f) \quad \text{on } \mathbb{R}^d,$$

when a belongs to $\left(L^2_{per}(\mathbb{R}^d) + \mathcal{A}^p\right)^{d\times d}$ (where $L^2_{per}(\mathbb{R}^d)$ denotes the space of locally L^2 periodic functions) and f belongs to $(\mathcal{A}^p)^d$. A key element of this study is a continuity result from $(\mathcal{A}^p)^d$ to $(\mathcal{A}^p)^d$ established in Proposition 3.12 (for periodic coefficients) and in Lemma 3.8 (in the general case) satisfied by the operator $-\nabla\left(- \operatorname{div} a\nabla\right)^{-1} \operatorname{div}$, which is false when $p = 1$ (see Remark 3.6 for a counter-example) and, in this case, we are not able to show the existence of $\tilde{w}_q$ such that $\nabla \tilde{w}_q \in (\mathcal{A}^1)^d$. However, since the coefficient a belongs to $\left(L^\infty(\mathbb{R}^d)\right)^{d\times d}$, assumption (3.11) for $p = 1$ implies that the same assumption is true for every $p > 1$ and Theorem 3.1 gives the existence of an adapted corrector such that $\nabla \tilde{w}_q$ belongs to $(\mathcal{A}^p)^d$ for every $p > 1$.

The existence of an adapted corrector is actually key to establish an homogenization theory in the context of problem (3.5). We use it first in the proof of Theorem 3.1 in order to identify the homogenized equation (3.19). Moreover, if we define an approximation

$$u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon),$$

such as (3.7) in the periodic case but using our adapted corrector, it is also possible to describe the behavior of ∇u^ε in several topologies exactly as in the periodic context. If we denote $R^\varepsilon := u^\varepsilon - u^{\varepsilon,1}$, the results established in the present paper ensure that our setting is covered by the work of [20] which studies problem (3.1) under general assumptions (the existence of a corrector strictly sublinear at infinity in particular) and shows the convergence of R^ε to 0 for the topology of $W^{1,r}$ when $r \geq 2$. Some properties related to the strict sublinearity of our corrector therefore allow to make precise the convergence rate of ∇R^ε (see estimate (3.85)).

We also note that assumption (3.4) regarding the Hölder continuity of the coefficient together with (3.11) implies that $\tilde{a}$ belongs to $\left(L^q(\mathbb{R}^d)\right)^{d\times d}$ for a given exponent $p^* < q$ as a consequence of Proposition 3.7 established in Section 3.2. It follows that [27, 26, 25] actually cover our setting and show the existence of a corrector of the form $w = w_{per} + \tilde{w}$ where $\tilde{w}$ a is solution to (3.18) such that $\nabla \tilde{w} \in \left(L^q(\mathbb{R}^d)\right)^d$ for this particular exponent q. However, the results of Theorem 3.1 are stronger in our approach : it ensures that the perturbed part of our corrector has a gradient in $(\mathcal{A}^p)^d$ and, since $p^* < q$, it provides better properties regarding its integrability at infinity. It is indeed shown in Section 3.3.4 that the theoretical convergence rates of ∇R^ε are improved if we assume $\tilde{a} \in (\mathcal{A}^p)^{d\times d}$ rather than only $\tilde{a} \in \left(L^q(\mathbb{R}^d)\right)^{d\times d}$. Besides proving the stronger results of Theorem 3.1, one contribution of the present study is also to put in place a whole methodological machinery for functions with integrable discrete gradients that allows to obtain homogenization results, similarly but *independently* from the proofs and the arguments conducted in the context of L^q functions. Our aim is, in particular, to highlight the fact that the methodology employed in [27, 26, 25] only requires to know the global behavior of a at infinity, and the non-local control of the averages of $\tilde{a}$ (in contrast to the assumptions of L^q integrability in [27, 26, 25]) given by Proposition 3.1 is sufficient to perform the homogenization of problem (3.1). Although we have not pursued in this direction, we also believe that the so-called large-scale regularity results established in [50] could possibly be adapted to our setting in order to obtain homogenization results for problem (3.1), similar to those of Theorem 3.1 but without assumption of Hölder regularity satisfied by a.

3.1.2.2 The case $p > d$

When $p \geq d$, we show that the homogenization of problem (3.1) is not always possible. More precisely, we exhibit a couple of sequences u^ε that have subsequential limits. Our two counter examples slowly oscillate at infinity (see Figure 3.3 for examples).

Our article is organized as follows. In Section 3.2, we study the properties of the space $\mathbf{A}^p$ in the case $p < d$ and we establish the discrete version of the Gagliardo-Nirenberg-Sobolev inequality stated in Proposition 3.1. In section 3.3, we prove Theorem 3.1. Finally, in Section 3.4, we study the homogenization problem (3.1) in the case $p \geq d$.

3.2 Properties of the functional space $\mathbf{A}^p$, the case $p < d$

Throughout this section, we assume that $d \geq 2$ and that $p \in [1, d[$. We study the properties of the space $\mathbf{A}^p$ defined by (3.10). The main idea is, of course, to see the operator δ as a discrete gradient operator and to draw an analogy between this discrete gradient and the usual continuous gradient ∇. We show that the functions of $\mathbf{A}^p$ satisfies several properties similar to those satisfied by the functions f such that $\nabla f \in \left(L^p(\mathbb{R}^d)\right)^d$ and we establish a discrete variant of the Gagliardo-Nirenberg-Sobolev inequality proving that the functions $f \in \mathbf{A}^p$ satisfy, up to the addition of a periodic function, some properties of integrability. More precisely, we prove that such a function f can be split as the the sum of a periodic function f_{per} and a function $\tilde{f} \in \mathcal{A}^p$, which belongs, up to a local averaging (made precise in formula (3.16) above), to the particular Lebesgue space $L^{p^*}(\mathbb{R}^d)$.

3.2.1 Properties of $\mathcal{E}^p$ and $\mathcal{A}^p$

To start with, we need to introduce several properties satisfied by the spaces $\mathcal{E}^p$ and $\mathcal{A}^p$, respectively defined in (3.14) and (3.15), and we establish some asymptotic properties regarding the average value and the strict sub-linearity of the functions belonging to $\mathcal{A}^p$.

We first claim that the spaces $\mathcal{A}^p$ and $\mathcal{E}^p$ respectively equipped with the norms (3.16) and (3.17) are two Banach spaces.

Proposition 3.2. *The space $\mathcal{E}^p$ equipped with the norm defined by (3.16) is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{E}^p$. Then, there exists a sub-sequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every k, we have $\left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{E}^p} \leq 2^{-k}$. For every $k \in \mathbb{N}$, we define the function $g_k := \mathcal{M}\left(\left|f_{n_{k+1}} - f_{n_k}\right|\right)$. Since

$$\|g_k\|_{L^{p^*}(\mathbb{R}^d)} = \left\|\mathcal{M}\left(\left|f_{n_{k+1}} - f_{n_k}\right|\right)\right\|_{L^{p^*}(\mathbb{R}^d)} = \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{E}^p} \leq 2^{-k},$$

we obtain that the series $\sum\limits_{k\in\mathbb{N}} g_k$ converges normally in $L^{p^*}(\mathbb{R}^d)$ and we denote by g its sum. The function g belongs to $L^{p^*}(\mathbb{R}^d)$ and it is therefore finite almost everywhere, that is, for almost every x, we have

$$g(x) = \sum_{k\in\mathbb{N}} \mathcal{M}\left(\left|f_{n_{k+1}} - f_{n_k}\right|\right)(x) = \sum_{k\in\mathbb{N}} \int_{Q+x} \left|f_{n_{k+1}} - f_{n_k}\right| < \infty.$$

Since for every $x \in \mathbb{R}^d$ the space $L^1(Q+x)$ is a Banach space, we deduce that for almost all x, the series $\sum\limits_{k\in\mathbb{Z}} f_{n_{k+1}} - f_{n_k}$ converges in $L^1(Q+x)$ and, consequently, f_{n_k} converges in $L^1(Q+x)$ when k tends to $+\infty$. Therefore, there exists $f \in L^1_{loc}(\mathbb{R}^d)$ such that f_{n_k} converges to f in $L^1_{loc}(\mathbb{R}^d)$ and f is given by $f = \sum\limits_{k\in\mathbb{N}} \left(f_{n_{k+1}} - f_{n_k}\right) + f_{n_0}$.

Next, we want to show that f_{n_k} converges to f in $\mathcal{E}^p$. Since f_{n_k} converges to f in $L^1_{loc}(\mathbb{R}^d)$, we know that the sequence $\mathcal{M}\left(|f_{n_k}|\right)$ converges pointwise to $\mathcal{M}\left(|f|\right)$. In addition, since we know that $f = \sum_{k \in \mathbb{N}} \left(f_{n_{k+1}} - f_{n_k}\right) + f_{n_0}$ in $L^1_{loc}(\mathbb{R}^d)$, we have for every $K \in \mathbb{N}$:

$$\begin{aligned}\left\| f - f_{n_{K+1}} \right\|_{\mathcal{E}^p} &= \left\| f - \sum_{k=0}^{K} (f_{n_{k+1}} - f_{n_k}) - f_{n_0} \right\|_{\mathcal{E}^p} \\ &= \left\| \sum_{k=K+1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right) \right\|_{\mathcal{E}^p} \leq 2^{-K}.\end{aligned}$$

Taking the limit when $K \to \infty$, we obtain the convergence of f_{n_K} to f in $\mathcal{E}^p$. Therefore, f is an adherent value of the sequence $(f_n)_n$. Since f_n is a Cauchy sequence in $\mathcal{E}^p$, we deduce that the sequence f_n converges to f in $\mathcal{E}^p$. $\square$

In the next Proposition, we also give an useful property satisfied by the sequences that converge in $\mathcal{E}^p$.

Proposition 3.3. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions of $\mathcal{E}^p$ that converges to f in $\mathcal{E}^p$. Then, f_n converges to f in $L^1_{loc}(\mathbb{R}^d)$.*

Démonstration. We fix $R > 0$. For every $n \in \mathbb{N}$, since $|Q| = 1$, we have

$$\int_{B_R} |f_n(x) - f(x)| dx = \int_Q \int_{B_R} |f_n(x) - f(x)| dx\, dy. \tag{3.20}$$

Next, for every $y \in Q$, we have $B_R \subset (B_{4R} + y)$ and it follows

$$\begin{aligned}\int_Q \int_{B_R} |f_n(x) - f(x)| dx\, dy &\leq \int_Q \int_{B_{4R}+y} |f_n(x) - f(x)| dx\, dy \\ &= \int_Q \int_{B_{4R}} |f_n(x+y) - f(x+y)| dx\, dy = \int_{B_{4R}} \mathcal{M}(|f_n - f|)(x) dx.\end{aligned} \tag{3.21}$$

The latter equality above is a consequence of the Fubini Theorem. Using (3.20), (3.21) and the Hölder inequality, we obtain

$$\begin{aligned}\int_{B_R} |f_n(x) - f(x)| dx \leq \int_{B_{4R}} \mathcal{M}(|f_n - f|)(x) dx &\leq |B_{4R}|^{\frac{p^*-1}{p^*}} \|\mathcal{M}(|f_n - f|)\|_{L^{p^*}(B_{4R})} \\ &\leq |B_{4R}|^{\frac{p^*-1}{p^*}} \|f_n - f\|_{\mathcal{E}^p} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

We deduce that f_n converges to f in $L^1(B_R)$ for every $R > 0$ and we can conclude. $\square$

Remark 3.1. *For every $R > 0$ and $f \in \mathcal{E}^p$, we note that the above proof actually shows that*

$$\|f\|_{L^1(B_R)} \leq |B_{4R}|^{\frac{p^*-1}{p^*}} \|f\|_{\mathcal{E}^p}.$$

Using the fact that $\mathcal{E}^p$ is a Banach space, we now establish that $\mathcal{A}^p$ is also a Banach space.

Corollary 3.1. *The space $\mathcal{A}^p$ equipped with the norm $\|.\|_{\mathcal{A}^p}$ defined by (3.17) is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}^p$. Since $\mathcal{A}^p \subset \mathcal{E}^p$, Proposition 3.2 implies that f_n converges to a function f in $\mathcal{E}^p$. Since $L^p(\mathbb{R}^d)$ is a Banach space, the sequence $(\delta f_n)_{n\in\mathbb{N}}$ converges to a function g in $L^p(\mathbb{R}^d)$. Our aim is to show that $g = \delta f$. Proposition 3.3 ensures that f_n converges to f in $L^1_{loc}(\mathbb{R}^d)$, thus that δf_n converges to δf in $L^1_{loc}(\mathbb{R}^d)$. Using the uniqueness of the limit in $L^1_{loc}(\mathbb{R}^d)$, we obtain that $g = \delta f$. Therefore f belongs to $\mathcal{A}^p$ and we can conclude. $\square$

Our next aim is to study the asymptotic behavior of sufficiently regular functions of $\mathcal{A}^p$. We begin by proving that uniformly continuous functions in $\mathcal{A}^p$ vanish at infinity.

Proposition 3.4. *Let f be an uniformly continuous function on $\mathbb{R}^d$ such that $\mathcal{M}(|f|)(x)$ vanishes at infinity. Then $\lim\limits_{|x|\to\infty} f(x) = 0$.*

Proof. We argue by contradiction and assume that f does not converge to 0 at infinity. Then, there exists $\varepsilon > 0$ such that for every $R > 0$, there exists $x_R \in \mathbb{R}^d$ with $|x_R| > R$ and $|f(x_R)| > \varepsilon$. f being uniformly continuous, there exists $\delta > 0$ such that for every $R > 0$ and for every $y \in B_\delta(x_R)$, we have $|f(y)| > \dfrac{\varepsilon}{2}$. Since $\lim\limits_{|x|\to\infty} \mathcal{M}(|f|)(x) = 0$, there exists $R_0 > 0$ such that $|x| > R_0$ implies $\mathcal{M}(|f|)(x) < \dfrac{|B_\delta|}{2}\varepsilon$. On the other hand,

$$\mathcal{M}(|f|)(x_{R_0}) \geq \int_{B_\delta(x_{R_0})} |f(z)|dz \geq \frac{|B_\delta|}{2}\varepsilon.$$

Since $|x_{R_0}| > R_0$, we have a contradiction. $\square$

From the previous proposition, we deduce the following corollary.

Corollary 3.2. *Let $f \in \mathcal{E}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ for $\alpha \in]0,1[$, then $\lim\limits_{|x|\to\infty} f(x) = 0$.*

Proof. Since $f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, the function $\mathcal{M}(|f|)$ also belongs to $C^{0,\alpha}(\mathbb{R}^d)$ and we have

$$|\mathcal{M}(|f|)(x) - \mathcal{M}(|f|)(z)| \leq \|f\|_{C^{0,\alpha}(\mathbb{R}^d)}|x-z|^\alpha,$$

for every $x, z \in \mathbb{R}^d$. In addition, since $\mathcal{M}(|f|) \in L^{p^*}(\mathbb{R}^d)$, it follows that $\lim\limits_{|x|\to\infty} \mathcal{M}(|f|)(x) = 0$ and we conclude using Proposition 3.4. $\square$

The next proposition regards the average value of the functions in $\mathcal{A}^p$ and is actually key for the homogenization of problem (3.1). Indeed, as stated in Corollary 3.3, it implies a weak convergence to 0 of the sequence $(|f(./\varepsilon)|)_{\varepsilon>0}$, which means in a certain sense that a perturbation of $\mathcal{A}^p$ has no macroscopic impact on the ambient background. This property shall be particularly useful to identify the homogenized coefficient a^* associated with problem (3.1).

Proposition 3.5. *Let $f \in \mathcal{A}^p$. Then, $\lim\limits_{R\to\infty} \dfrac{1}{|B_R|}\displaystyle\int_{B_R(x_0)} |f| = 0$ for every $x_0 \in \mathbb{R}^d$ and we have the following convergence rate :*

$$\frac{1}{|B_R|}\int_{B_R(x_0)} |f| \leq \frac{C}{R^{d/p^*}}, \tag{3.22}$$

where $C > 0$ is independent of R and x_0.

Proof. Let $R > 0$ and $x_0 \in \mathbb{R}^d$. Since $|Q| = 1$, we have :

$$\int_{B_R(x_0)} |f|(x)dx = \int_Q \int_{B_R(x_0)} |f|(x)dxdz = \int_Q \int_{B_R(x_0)-z} |f|(x+z)dxdz.$$

In addition, for R large enough and for every $z \in Q$, we have $B_R(x_0) - z \subset B_{2R}(x_0)$. Using the Fubini theorem, we therefore obtain

$$\int_Q \int_{B_R(x_0)-z} |f|(x+z)dxdz \leq \int_Q \int_{B_{2R}(x_0)} |f|(x+z)dxdz = \int_{B_{2R}(x_0)} \mathcal{M}(|f|)(x)dx.$$

The Hölder inequality next gives

$$\frac{1}{|B_R|} \int_{B_{2R}(x_0)} \mathcal{M}(|f|) \leq \frac{|B_{2R}|^{1/(p^*)'}}{|B_R|} \|f\|_{\mathcal{E}^p} = C(d) \frac{1}{R^{d/p^*}} \|f\|_{\mathcal{E}^p},$$

where $(p^*)'$ is the conjugate exponent of p^* and $C(d) > 0$ depends only on the ambient dimension d. We obtain (3.22). $\square$

Corollary 3.3. *Let $u \in \mathcal{A}^p \cap L^\infty(\mathbb{R}^d)$, then the sequence $(|u(./\varepsilon)|)_{\varepsilon>0}$ converges to 0 for the weak-* topology of $L^\infty(\mathbb{R}^d)$ as ε vanishes.*

Proof. We fix $R > 0$ and we begin by considering $\varphi = 1_{B_R(x_0)}$ for $x_0 \in \mathbb{R}^d$. For every $\varepsilon > 0$, we have :

$$\left| \int_{\mathbb{R}^d} |u(x/\varepsilon)| \varphi(x) dx \right| = \int_{B_R(x_0)} |u(x/\varepsilon)| \, dx = \varepsilon^d \int_{B_{R/\varepsilon}(x_0/\varepsilon)} |u(y)| \, dy.$$

We therefore use Proposition 3.5 and we obtain

$$\left| \int_{\mathbb{R}^d} |u(x/\varepsilon)| \varphi(x) dx \right| \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

We conclude using the density of simple functions in $L^1(\mathbb{R}^d)$. $\square$

We next show that every function with a gradient in $\left(\mathcal{A}^p \cap L^\infty(\mathbb{R}^d)\right)^d$ is strictly sub-linear at infinity.

Proposition 3.6. *Let $\frac{d}{2} < p < d$ and $u \in L^1_{loc}(\mathbb{R}^d)$ such that $\nabla u \in \left(\mathcal{E}^p \cap L^\infty(\mathbb{R}^d)\right)^d$. Then u is strictly sub-linear at infinity. More precisely, we have*

$$\frac{|u(x)|}{1+|x|} = O\left(\frac{1}{|x|^{d/p^*}}\right). \tag{3.23}$$

Proof. Let $x \in \mathbb{R}^d$ such that $x \neq 0$. We denote by $r = |x|$ and we have

$$\begin{aligned} |u(x) - u(0)| &\leq \int_{B_r(x)} \frac{|\nabla u(w)|}{|x-w|^{d-1}} dw + \int_{B_r} \frac{|\nabla u(w)|}{|w|^{d-1}} dw \\ &= \int_Q \left(\int_{B_r(x-z)} \frac{|\nabla u(w+z)|}{|x-w-z|^{d-1}} dw + \int_{B_r(-z)} \frac{|\nabla u(w+z)|}{|w+z|^{d-1}} dw \right) dz. \end{aligned} \tag{3.24}$$

The first inequality above is established for instance in [42, p.266] in the proof of the Morrey's inequality ([42, Theorem 4 p.266]). We next remark that $B_r(x-z) \subset B_{2r}(x)$ for every r sufficiently large and every $z \in Q$. For $z \in Q$, since $|z| \leq \sqrt{d}$, there also exists a constant $C > 0$ independent of x and z such that $\dfrac{1}{|x-w-z|^{d-1}} \leq C\dfrac{1}{|x-w|^{d-1}}$ for every $w \in B_{2r}(x) \setminus B_{2\sqrt{d}}(x)$. We deduce

$$\int_{B_r(x-z)\setminus B_{2\sqrt{d}}(x)} \frac{|\nabla u(w+z)|}{|x-w-z|^{d-1}} dw \leq C \int_{B_{2r}(x)} \frac{|\nabla u(w+z)|}{|x-w|^{d-1}} dw.$$

Since $p > \frac{d}{2}$, we also have $p^* > d$ and $\dfrac{(d-1)p^*}{p^*-1} < d$. Using the Fubini theorem and the Hölder inequality, it therefore follows

$$\begin{aligned}\int_Q \int_{B_r(x-z)\setminus B_{2\sqrt{d}}(x)} \frac{|\nabla u(w+z)|}{|x-w-z|^{d-1}} dw\, dz &\leq C \left(\int_{B_{2r}(x)} \frac{1}{|x-w|^{(d-1)\frac{p^*}{p^*-1}}} dw \right)^{\frac{p^*-1}{p^*}} \|\nabla u\|_{\mathcal{E}^p} \\ &= C_1 r^{1-\frac{d}{p^*}} \|\nabla u\|_{\mathcal{E}^p},\end{aligned}$$

where C_1 is a constant that depends only on p^* and the dimension d. We also have

$$\begin{aligned}\int_Q \int_{B_{2\sqrt{d}}(x)} \frac{|\nabla u(w+z)|}{|x-w-z|^{d-1}} dw\, dz &\leq \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \int_Q \int_{B_{2\sqrt{d}}} \frac{1}{|w+z|^{d-1}} dw\, dz \\ &\leq \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \int_{B_{3\sqrt{d}}} \frac{1}{|w|^{d-1}} dw,\end{aligned}$$

and, if we denote $C_2 = \max\left(C_1, \displaystyle\int_{B_{3\sqrt{d}}} \frac{1}{|w|^{d-1}} dw\right)$, we obtain

$$\int_Q \int_{B_r(x-z)} \frac{|\nabla u(w+z)|}{|x-w-z|^{d-1}} dw\, dz \leq C_2 \left(r^{1-\frac{d}{p^*}} \|\nabla u\|_{\mathcal{E}^p} + \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \right). \tag{3.25}$$

We can similarly show that

$$\int_Q \int_{B_r(-z)} \frac{|\nabla u(w+z)|}{|w+z|^{d-1}} dw\, dz \leq C_2 \left(r^{1-\frac{d}{p^*}} \|\nabla u\|_{\mathcal{E}^p} + \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \right). \tag{3.26}$$

Since $\dfrac{d}{p^*} < 1$, estimates (3.24), (3.25) and (3.26) finally show the existence of a constant $M > 0$ which depends only on u, p and d such that

$$\frac{|u(x) - u(0)|}{|x|} \leq \frac{M}{|x|^{d/p^*}},$$

when $|x|$ is sufficiently large. We both obtain the strict sub-linearity at infinity of u and estimate (3.23). $\square$

We conclude this section proving that Hölder-continuous functions of $\mathcal{E}^p$ actually belong to $L^q(\mathbb{R}^d)$ for some Lebesgue exponent $q \geq p$.

Proposition 3.7. *Let $\alpha \in]0,1[$ and $p \geq 1$. Then the set $\left\{f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d) \;\middle|\; \mathcal{M}(|f|) \in L^p(\mathbb{R}^d)\right\}$ is a subset of $L^q(\mathbb{R}^d)$ for every $q \geq \mathbf{q} := \dfrac{p(\alpha+d)-d}{\alpha}$. In addition, the inclusion does not hold if $q < \mathbf{q}$.*

Proof. We show the proposition in dimension $d=1$ for clarity, the proof for higher dimensions is a simple adaptation. Let $f \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ such that $\mathcal{M}(|f|) \in L^p(\mathbb{R})$. For every $N \in \mathbb{Z}$ and $k \in \left\{0, ..., 2^{|N|}-1\right\}$, we denote $Q_{N,k} := \left[N + \dfrac{k}{2^{|N|}} \;;\; N + \dfrac{k+1}{2^{|N|}}\right]$, $\beta_{N,k} := \max\limits_{x \in Q_{N,k}} |f(x)|$ and $x_{N,k} := \operatorname*{argmax}\limits_{x \in Q_{N,k}} |f(x)|$. Using Corollary 3.2, we know that $\sup\limits_{k \in \left\{0,...,2^{|N|}-1\right\}} \beta_{N,k}$ converges to 0 when $N \to \infty$. Since $f \in \mathcal{C}^{0,\alpha}(\mathbb{R})$, there exists $C > 0$ such that for every N and every $k \in \left\{0, ..., 2^{|N|}-1\right\}$, we have $|f(y)| \geq \dfrac{\beta_{N,k}}{2}$ for all $y \in [x_{N,k} - C(\beta_{N,k})^{\frac{1}{\alpha}}, x_{N,k} + C(\beta_{N,k})^{\frac{1}{\alpha}}]$. For every $x \in \left[N, N+\dfrac{1}{2}\right]$, we have $\left[N+\dfrac{1}{2}, N\right] \subset [x, x+1]$ and we deduce :

$$\int_x^{x+1} |f(y)| dy \geq \int_{N+\frac{1}{2}}^{N} |f(y)| dy \geq C \sum_{k=2^{|N|-1}}^{2^{|N|}-1} (\beta_{N,k})^{\frac{1}{\alpha}} \beta_{N,k}.$$

Therefore,

$$\frac{C^p}{2} \sum_{N \in \mathbb{Z}} \sum_{k=2^{|N|-1}}^{2^{|N|}-1} (\beta_{N,k})^{p\left(1+\frac{1}{\alpha}\right)} \leq \sum_{N \in \mathbb{Z}} \int_N^{N+\frac{1}{2}} \left| \int_x^{x+1} |f(y)| dy \right|^p dx \leq \int_{\mathbb{R}} |\mathcal{M}(|f|)|^p < +\infty.$$

We similarly obtain $\sum\limits_{N \in \mathbb{Z}} \sum\limits_{k=0}^{2^{|N|-1}} (\beta_{N,k})^{p\left(1+\frac{1}{\alpha}\right)} < +\infty$. For every $q \geq \dfrac{p(1+\alpha)-1}{\alpha}$, we have

$$\int_{\mathbb{R}} |f(x)|^q dx \leq \sum_{N \in \mathbb{Z}} \sum_{k=0}^{2^{|N|}-1} \int_{Q_{N,k}} (\beta_{N,k})^q = \sum_{N \in \mathbb{Z}} \sum_{k=0}^{2^{|N|}-1} \frac{1}{2^{|N|}} (\beta_{N,k})^q.$$

If we distinguish the two cases $(\beta_{N,k})^{\frac{1}{\alpha}} \leq \dfrac{1}{2^{|N|}}$ and $(\beta_{N,k})^{\frac{1}{\alpha}} \geq \dfrac{1}{2^{|N|}}$, we obtain the following bound :

$$\int_{\mathbb{R}^d} |f(x)|^q dx \leq \sum_{N \in \mathbb{Z}} \sum_{k=0}^{2^{|N|}-1} \left(\frac{1}{2^{(q\alpha+1)|N|}} + (\beta_{N,k})^{q+\frac{1}{\alpha}} \right) = \sum_{N \in \mathbb{Z}} \left(\frac{1}{2^{q\alpha|N|}} + \sum_{k=0}^{2^{|N|}-1} (\beta_{N,k})^{q+\frac{1}{\alpha}} \right).$$

Since $q + \dfrac{1}{\alpha} \geq p\left(1+\dfrac{1}{\alpha}\right)$, we conclude that $\displaystyle\int_{\mathbb{R}^d} |f|^q < +\infty$.

In order to show that the inclusion does not hold when $q < \dfrac{p(1+\alpha)-1}{\alpha}$, we denote $a_k = \dfrac{1}{\ln(k)\sqrt{k}}$ for $k \in \mathbb{N} \setminus \{1, 2\}$ and consider the function

$$f(x) = \sum_{k \geq 2} \left(-\frac{\sqrt{k}}{k^{\frac{\alpha}{2}}} |x-k| + \frac{1}{\ln(k) k^{\frac{\alpha}{2}}} \right) 1_{[-a_k, a_k]}(x-k),$$

where 1_A denotes the characteristic function of a subset A. We can easily show that $f \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ and $\mathcal{M}(|f|)$ belongs to $L^p(\mathbb{R})$ for $p = \dfrac{2}{\alpha+1}$ whereas $f \notin L^q(\mathbb{R})$ if

$$q < \frac{1}{\alpha} = \frac{p(1+\alpha)-1}{\alpha}.$$

□

Remark 3.2. *Without assumption of Hölder continuity, we have the existence of functions f such that $\mathcal{M}(|f|) \in L^p(\mathbb{R}^d)$ for $p > 1$ whereas f does not belong to any $L^q(\mathbb{R}^d)$. Consider indeed, for $d = 1$,*

$$f(x) = \sum_{k\geq 2} \left(\frac{1}{\ln(k)} - \frac{\sqrt{k}}{\ln(k)} |x-k| \right) 1_{\left[-\frac{1}{\sqrt{k}}, -\frac{1}{\sqrt{k}}\right]}(x-k).$$

This function, composed of a sum of "bumps" centered at the integers $k \geq 2$, does not belong to any $L^q(\mathbb{R})$ for $q \geq 1$ due to its logarithmic decrease at infinity. However, a simple calculation allows to show that $\mathcal{M}(|f|)$ belongs to $L^p(\mathbb{R}^d)$, for every $p > 2$, because of a classical regularizing property of the local averaging.

3.2.2 Discrete variant of the Gagliardo-Nirenberg-Sobolev inequality

In this section we show that, still under the assumption $p < d$, it is possible to obtain a bound on the local average $\mathcal{M}(|f|)$ of a function $f \in \mathcal{A}^p$ using bounds on its discrete derivative δf. To this end, we establish a discrete variant of the Gagliardo-Nirenberg-Sobolev inequality (see for instance [42, Section 5.6.1] for the classical inequality) adapting its proof in our discrete setting. We begin by showing the result for the functions of $L^p(\mathbb{R}^d)$ in Proposition 3.8 below. Our aim will be next to extend this result to $\mathcal{A}^p$ arguing by density.

Proposition 3.8. *There exists a constant $C > 0$ such that for every $f \in L^p(\mathbb{R}^d)$, we have :*

$$\|\mathcal{M}\left(|f|\right)\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\delta f\|_{L^p(\mathbb{R}^d)}. \tag{3.27}$$

Proof. **Step 1**. We first prove the result for the space of continuous compactly supported functions, denoted by $\mathcal{C}^0_c(\mathbb{R}^d)$ in the sequel. Let $f \in \mathcal{C}^0_c(\mathbb{R}^d)$. For every $i \in \{1, ..., d\}$, we remark that :

$$\partial_i \mathcal{M}\left(|f|\right)(x) = \int_{Q_i + \tilde{x}_i} \delta_i |f(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_d)| d\tilde{y}_i,$$

where $Q_i = \prod_{j\in\{1,...,d\}\setminus\{i\}}]0,1[$ and $\tilde{x}_i = (x_1, .., x_{i-1}, x_{i+1}, ..., x_d)$. In addition, using a triangle inequality, we have $|\delta_i |f(y)|| \leq |\delta_i f(y)|$ for every i. We next denote by p' the conjugate exponent associated with p. Successively using the Hölder inequality, a change of variable and the Fubini theorem, we obtain :

$$\begin{aligned}
\int_{\mathbb{R}^d} |\partial_i \mathcal{M}\left(|f|\right)(x)|^p dx &\leq |Q_i + \tilde{x}_i|^{p/p'} \int_{\mathbb{R}^d} \int_{Q_i+\tilde{x}_i} |\delta_i f(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_d)|^p d\tilde{y}_i dx \\
&= \int_{\mathbb{R}^d} \int_{Q_i} |\delta_i f(z_1 + x_1, ..., z_{i-1} + x_{i-1}, x_i, z_{i+1} + x_{i+1}, ..., z_d + x_d)|^p d\tilde{z}_i dx \\
&= |Q_i| \int_{\mathbb{R}^d} |\delta_i f(x)|^p dx = \|\delta_i f\|^p_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

We therefore deduce
$$\|\nabla \mathcal{M}(|f|)\|_{L^p(\mathbb{R}^d)} \leq \|\delta f\|_{L^p(\mathbb{R}^d)}.$$
Since $f \in \mathcal{C}_c^0(\mathbb{R}^d)$, we have $\mathcal{M}(|f|) \in \mathcal{C}_c^1(\mathbb{R}^d)$. The classical Gagliardo-Nirenberg-Sobolev inequality allows to conclude to the existence of a constant $C > 0$ independent of f such that
$$\|\mathcal{M}(|f|)\|_{L^{p^*}(\mathbb{R}^d)} \leq C\|\delta f\|_{L^p(\mathbb{R}^d)}.$$

Step 2. Now that the case of compactly supported function has been dealt with, we can generalize for $L^p(\mathbb{R}^d)$ using a density result. Let $f \in L^p(\mathbb{R}^d)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{C}_c^0(\mathbb{R}^d)$ that converges to f in $L^p(\mathbb{R}^d)$. We can easily show that δf_n converges to δf in this space. In the first step, we have shown the existence of $C > 0$ such that for every $n, m \in \mathbb{N}$, we have
$$\|f_n\|_{\mathcal{E}^p} \leq C\|\delta f_n\|_{L^p(\mathbb{R}^d)}, \tag{3.28}$$
$$\|f_n - f_m\|_{\mathcal{E}^p} \leq C\|\delta f_n - \delta f_m\|_{L^p(\mathbb{R}^d)}. \tag{3.29}$$
Since (δf_n) converges in $\left(L^p(\mathbb{R}^d)\right)^d$, it is a Cauchy sequence in this space and, using (3.29), we deduce that f_n is a Cauchy sequence in the Banach space $\mathcal{E}^p$ and f_n therefore converges to f in this space. We finally take the limit in (3.28) when $n \to \infty$ and we obtain (3.27). □

We next prove a discrete version of Schwarz lemma for the functions of $L^1_{loc}(\mathbb{R}^d)$ (see [86, Theorem VI, p.59] for the classical version). More precisely, we show that if a vectored-valued function T satisfies some discrete Cauchy equations in the sense of (3.30), then there exists a function $u \in L^1_{loc}$ such that T is the discrete gradient of u. In the sequel of this section we shall use this result in order to establish some density properties in $\mathcal{A}^p$.

Proposition 3.9. *Let $T \in \left(L^1_{loc}(\mathbb{R}^d)\right)^d$ such that for every $i, j \in \{1, ..., d\}$, we have :*
$$\delta_j T_i = \delta_i T_j. \tag{3.30}$$
Then, there exists $u \in L^1_{loc}(\mathbb{R}^d)$ such that $T = \delta u$.

Proof. For clarity, we only show this result in the case $d = 2$, the proof for higher dimensions is similar. In the sequel, for $y \in \mathbb{R}$, we denote by $[y] \in \mathbb{Z}$ the integer part of y and by $\{y\} \in [0, 1[$ its fractional part. We begin by considering u defined by :
$$u(x) = \sum_{n=0}^{[x_1]-1} T_1\left((\{x_1\}, x_2) + ne_1\right) + \sum_{m=0}^{[x_2]-1} T_2\left((\{x_1\}, \{x_2\}) + me_2\right),$$
for almost all $x = (x_1, x_2) \in \mathbb{R}^2$. Since T_i belongs to $L^1_{loc}(\mathbb{R}^d)$ for every $i \in \{1, 2\}$, we clearly have $u \in L^1_{loc}(\mathbb{R}^d)$. We next show that $\delta u = T$. Indeed, we have
$$\begin{aligned}\delta_1 u(x) = u(x + e_1) - u(x) &= \sum_{n=0}^{[x_1]} T_1\left((\{x_1\}, x_2) + ne_1\right) - \sum_{n=0}^{[x_1]-1} T_1\left((\{x_1\}, x_2) + ne_1\right) \\ &= T_1\left((\{x_1\}, x_2) + [x_1]e_1\right) = T_1(x).\end{aligned}$$

We can similarly show that :

$$\delta_2 u(x) = \sum_{n=0}^{[x_1]-1} \delta_2 T_1 \left((\{x_1\}, x_2) + ne_1\right) + T_2\left((\{x_1\}, x_2)\right).$$

Since $\delta_2 T_1 = \delta_1 T_2$, we deduce :

$$\sum_{n=0}^{[x_1]-1} \delta_2 T_1 \left((\{x_1\}, x_2) + ne_1\right) = \sum_{n=0}^{[x_1]-1} \delta_1 T_2 \left((\{x_1\}, x_2) + ne_1\right) = T_2(x) - T_2\left((\{x_1\}, x_2)\right).$$

We finally obtain that $\delta_2 u(x) = T_2(x)$, and we can conclude that $T = \delta u$. $\square$

In the sequel, for every $R > 0$, we denote

$$Q_R = \left\{ x \in \mathbb{R}^d \;\middle|\; \max_{i \in \{1,\dots,d\}} |x_i| < R \right\}.$$

The next lemma is a discrete version of a particular case of the Poincaré-Wirtinger inequality (see for example [42, Section 5.8.1] for the classical version). This result is a technical tool that will allow us to establish our discrete Gagliardo-Nirenberg-Sobolev inequality stated in Proposition 3.1.

Lemma 3.1. *Assume $d \geq 2$. Let $p \in [1, +\infty[$ and f be in $\mathbf{A}^p$, the set defined by (3.10). For every $N \in \mathbb{N}^*$, we denote $A_N = Q_{2N} \setminus Q_N$ and $\mathcal{I}_N = \left\{ k \in \mathbb{Z}^d \;\middle|\; Q + k \subset A_N \right\}$. We consider*

$$f_{per,N}(x) = \frac{1}{\sharp \mathcal{I}_N} \sum_{k \in \mathcal{I}_N} f(x+k) \quad \text{for } x \in Q, \tag{3.31}$$

(where $\sharp B$ denotes the cardinality of a discrete set B) which we extend by periodicity. Then there exists a constant $C > 0$ independent of N and f such that :

$$\|f - f_{per,N}\|_{L^p(A_N)} \leq CN \|\delta f\|_{L^p(Q_{6N} \setminus Q_N)}.$$

Proof. For clarity, we show the result in the case $d = 2$, the proof in higher dimensions is similar. First of all, we remark that for every $k \in \mathcal{I}_N$, we have

$$N \leq \max(|k_1|, |k_2|) < 2N.$$

We next estimate the L^p-norm of $f - f_{per,N}$:

$$\|f - f_{per,N}\|^p_{L^p(A_N)} = \sum_{q \in \mathcal{I}_N} \int_{Q+q} |f - f_{per,N}|^p = \sum_{q \in \mathcal{I}_N} \int_Q \left| \frac{1}{\sharp \mathcal{I}_N} \sum_{k \in \mathcal{I}_N} f(x+q) - f(x+k) \right|^p dx.$$

Using the Holder inequality, we therefore obtain :

$$\|f - f_{per,N}\|^p_{L^p(A_N)} \leq \frac{1}{\sharp \mathcal{I}_N} \sum_{q \in \mathcal{I}_N} \sum_{k \in \mathcal{I}_N} \int_Q |f(x+q) - f(x+k)|^p \, dx. \tag{3.32}$$

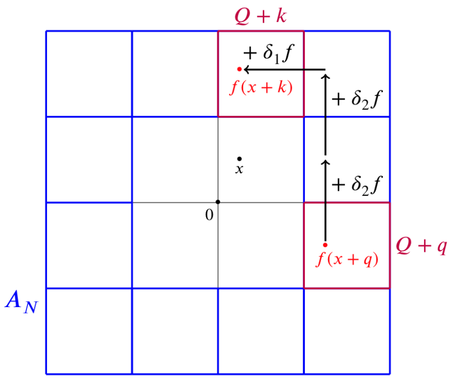

FIGURE 3.2 – Illustration of the relation between $f(x+q)$, $\delta_i f$ and $f(x+k)$ in dimension $d=2$.

Our aim is now to study $|f(x+q)-f(x+k)|$ for every $q,k \in \mathcal{I}_N$ and $x \in Q$ splitting $f(x+q)-f(x+k)$ as a sum of several translations of the functions $\delta_i f$, where $i \in \{1,2\}$. Our approach, which is illustrated in figure 3.2, consists in considering a discrete path of A_N connecting $x+k$ and $x+q$ and, next, in iterating the relation :

$$f(x+ne_j) = f(x) + \sum_{m=0}^{n-1} \delta_j f(x+me_j), \quad \forall n \in \mathbb{N}. \tag{3.33}$$

In the sequel, we use the convention $\sum_{m=0}^{n} := \sum_{m=n}^{0}$ when $n \leq 0$. We consider two different cases.

Case 1 : There exists $i \in \{1,2\}$ **such that** $|k_i| \geq N$ **and** $|q_i| \geq N$. Without loss of generality, we can assume that $i=1$. For every $x \in Q$, using (3.33) for $j=2$ we have :

$$f(x+q) = f(x+(q_1, 2N-1)) - \sum_{l=0}^{2N-2-q_2} \delta_2 f(x+q+le_2). \tag{3.34}$$

We can again iterate (3.33) and we have :

$$\begin{aligned} f\left(x+(q_1,2N-1)\right) = & \ f\left(x+(k_1,2N-1)\right) \\ & - \sum_{m=0}^{k_1-q_1-1} \delta_1 f(x+(q_1,2N-1)+me_1), \end{aligned} \tag{3.35}$$

$$f(x+(k_1,2N-1)) = f(x+k) - \sum_{n=0}^{k_2-2N} \delta_2 f(x+(k_1,2N-1)+ne_2). \tag{3.36}$$

Using (3.34), (3.35) and (3.36), we therefore obtain :

$$\begin{aligned} f(x+k)-f(x+q) = & \sum_{l=0}^{2N-2-q_2} \delta_2 f(x+q+le_2) + \sum_{m=0}^{k_1-q_1-1} \delta_1 f(x+(q_1,2N-1)+me_1) \\ & + \sum_{n=0}^{k_2-2N} \delta_2 f(x+(k_1,2N-1)+ne_2). \end{aligned}$$

Since each sum in the right-hand side of the above equality contains at most $4N$ terms, the Hölder inequality implies that :

$$
\begin{aligned}
|f(x+q) - f(x+k)|^p \leq C N^{p-1} & \sum_{l=0}^{2N-2-q_2} |\delta_2 f(x+q+le_2)|^p \\
& + \sum_{m=0}^{k_1-q_1-1} |\delta_1 f(x+(q_1, 2N-1)+me_1)|^p \\
& + C N^{p-1} \sum_{n=0}^{k_2-2N} |\delta_2 f(x+(k_1, 2N-1)+ne_2)|^p,
\end{aligned}
$$

where $C > 0$ depends only on d and p. We now insert our last inequality in (3.32), for every $|k_1| > N$ and $|q_1| > N$. We consequently have to study the following three sums :

$$
S_1 := \frac{1}{\sharp \mathcal{I}_N} \sum_{q \in \mathcal{I}_N,\, |q_1| \geq N} \; \sum_{k \in \mathcal{I}_N,\, |k_1| \geq N} \int_Q C N^{p-1} \sum_{l=0}^{2N-2-q_2} |\delta_2 f(x+q+le_2)|^p dx,
$$

$$
S_2 := \frac{1}{\sharp \mathcal{I}_N} \sum_{q \in \mathcal{I}_N,\, |q_1| \geq N} \; \sum_{k \in \mathcal{I}_N,\, |k_1| \geq N} \int_Q C N^{p-1} \sum_{m=0}^{k_1-q_1-1} |\delta_1 f(x+(q_1, 2N-1)+me_1)|^p dx,
$$

$$
S_3 := \frac{1}{\sharp \mathcal{I}_N} \sum_{q \in \mathcal{I}_N,\, |q_1| \geq N} \; \sum_{k \in \mathcal{I}_N,\, |k_1| \geq N} \int_Q C N^{p-1} \sum_{n=0}^{k_2-2N} |\delta_2 f(x+(k_1, 2N-1)+ne_2)|^p dx.
$$

Here we only study S_1, the method to estimate S_2 and S_3 being extremely similar. First, since $|q_2| \leq 2N-1$, we have

$$
\begin{aligned}
S_1 & \leq \frac{1}{\sharp \mathcal{I}_N} \sum_{q \in \mathcal{I}_N,\, |q_1| \geq N} \; \sum_{k \in \mathcal{I}_N,\, |k_1| \geq N} \int_Q C N^{p-1} \sum_{l=-4N}^{4N} |\delta_2 f(x+q+le_2)|^p dx \\
& = \frac{C N^{p-1}}{\sharp \mathcal{I}_N} \sum_{l=-4N}^{4N} \; \sum_{k \in \mathcal{I}_N,\, |k_1| \geq N} \; \sum_{q \in \mathcal{I}_N,\, |q_1| \geq N} \int_{Q+q+le_2} |\delta_2 f(x)|^p dx.
\end{aligned}
\tag{3.37}
$$

In addition, for every $q \in \mathcal{I}_N$ such that $|q_1| \geq N$ and every $l \in \{-4N, ..., 4N\}$, we have that $N \leq \max_{i \in \{1,2\}} |(q+le_2)_i| < 6N$, that is :

$$
\sum_{q \in \mathcal{I}_N,\, |q_1| \geq N} \int_{Q+q+le_2} |\delta_2 f(x)|^p dx \leq \int_{Q_{6N} \setminus Q_N} |\delta_2 f|^p. \tag{3.38}
$$

Since $\sharp I_N \sim K N^2$, where $K > 0$ is a constant independent of N, we have

$$
\frac{1}{\sharp \mathcal{I}_N} \sum_{l=-4N}^{4N} \; \sum_{k \in \mathcal{I}_N, |k_1| \geq N} 1 = 0(N).
$$

Using (3.37) and (3.38), we obtain the existence of a constant $\tilde{C} > 0$ independent of N such that :

$$
S_1 \leq \tilde{C} N^p \int_{Q_{6N} \setminus Q_N} |\delta_2 f|^p.
$$

With exactly the same method, it is possible to establish similar bounds for the sums S_2 and S_3, which show that

$$\begin{aligned}\frac{1}{\sharp\mathcal{I}_N}\sum_{q\in\mathcal{I}_N,\ |q_1|\geq N}\ \sum_{k\in\mathcal{I}_N,\ |k_1|\geq N}\int_Q |f(x+q)-f(x+k)|^p\,dx \\ \leq 3\tilde{C}N^p\int_{Q_{6N}\setminus Q_N}\sum_{i\in\{1,2\}}|\delta_i f|^p.\end{aligned} \tag{3.39}$$

Case 2 : There exists $i,j\in\{1,2\}$**,** $i\neq j$**, such that** $|q_i|\geq N$ **and** $|k_j|\geq N$. We assume that $i=1$ and $j=2$, the idea being identical if $i=2$ and $j=1$. Proceeding exactly as in the first case, it is possible to show that

$$\begin{aligned}|f(x+q)-f(x+k)|^p \leq CN^{p-1}\sum_{l=0}^{k_2-q_2-1}|\delta_2 f(x+q+le_2)|^p \\ + CN^{p-1}\sum_{m=0}^{k_1-q_1-1}|\delta_2 f(x+(q_1,k_2)+me_1)|^p.\end{aligned}$$

Since $|q_1|\geq N$, et $|k_2|\geq N$, we remark that each point of the form $x+q+le_2$ or $x+(q_1,k_2)+me_1$ belongs to A_N. Some estimates similar to those established in the first case allow to obtain

$$\begin{aligned}\frac{1}{\sharp\mathcal{I}_N}\sum_{q\in\mathcal{I}_N,\ |q_1|\geq N}\ \sum_{k\in\mathcal{I}_N,\ |k_2|\geq N}\int_Q |f(x+q)-f(x+k)|^p\,dx \\ \leq \tilde{C}N^p\sum_{i\in\{1,2\}}\int_{Q_{6N}\setminus Q_N}|\delta_i f|^p.\end{aligned} \tag{3.40}$$

Using finally (3.32) and inequalities (3.39), (3.40) established in the two different cases, we have

$$\begin{aligned}\|f-f_{per,N}\|^p_{L^p(A_N)} &\leq \sum_{i,j\in\{1,2\}}\frac{1}{\sharp\mathcal{I}_N}\sum_{q\in\mathcal{I}_N,\ |q_i|\geq N}\ \sum_{k\in\mathcal{I}_N,\ |k_j|\geq N}\int_Q |f(x+q)-f(x+k)|^p\,dx \\ &\leq \tilde{C}N^p\sum_{i\in\{1,2\}}\int_{Q_{6N}\setminus Q_N}|\delta_i f|^p.\end{aligned}$$

We have therefore established the existence of a constant $C>0$ independent of f and N such that :

$$\|f-f_{per,N}\|_{L^p(A_N)} \leq CN\|\delta f\|_{L^p(Q_{6N}\setminus Q_N)}.$$

□

Remark 3.3. *For clarity, we have chosen to show the inequality of Lemma 3.1 only for the particular sets* A_N *but the proof could be adapted to any connected set. On the other hand, for non-connected set, this result does not hold. In particular Lemma 3.1 is not true in dimension* $d=1$*. As a counter-example, we can consider the function* f *such that* $f(x)=2$ *if* $x<0$ *and* $f(x)=1$ *else. Such a function satisfies* $\delta f=0$ *on* $\mathbb{R}\setminus[-1,1]$ *but, for every* $N\in\mathbb{N}^*$*,* $f-f_{N,per}$ *does not vanish on* $A_N=]-2N,-N]\cup[N,2N[$*. Indeed,* $f-f_{N,per}(x)$ *is equal to* $\frac{1}{2}$ *if* $x\in]-2N,-N]$ *and* $-\frac{1}{2}$ *if* $x\in[N,2N[$*.*

We are finally able to prove the discrete version of the Gagliardo-Nirenberg-Sobolev inequality stated in Proposition 3.1.

Proof of Proposition 3.1. We begin by establishing the density of $L^p(\mathbb{R}^d)$ in $\mathbf{A}^p$, equipped with the semi-norm $\|f\| = \|\delta f\|_{\left(L^p(\mathbb{R}^d)\right)^d}$. We adapt step by step the method used in [83, Theorem 2.1] which studies the continuous case $\nabla f \in \left(L^p(\mathbb{R}^d)\right)^d$. We fix $f \in L^1_{loc}(\mathbb{R}^d)$ such that $\delta f \in \left(L^p(\mathbb{R}^d)\right)^d$. For every $R > 0$, we consider $\chi_R \in \mathcal{D}(\mathbb{R}^d)$, a positive function such that

$$Supp(\chi_R) \subset Q_{\frac{5R}{3}}, \quad \chi_R \equiv 1 \text{ in } Q_{\frac{4R}{3}}, \quad \|\chi_R\|_{L^\infty(\mathbb{R}^d)} = 1, \quad \|\nabla \chi_R\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{R}. \tag{3.41}$$

Let $N \in \mathbb{N}^*$. We denote $A_N = Q_{2N} \setminus Q_N$ and we consider $f_{per,N}$ the Q-periodic function defined by (3.31) in Lemma 3.1. We introduce the sequence

$$f_N = \chi_N \left(f - f_{per,N}\right).$$

For every $N \in \mathbb{N}$ and $i \in \{1, ..., d\}$, using a triangle inequality, we have

$$\|\delta_i f_N - \delta_i f\|_{L^p(\mathbb{R}^d)} \leq \|\delta_i \chi_N \left(f - f_{per,N}\right)\|_{L^p(\mathbb{R}^d)} + \|(1-\chi_N)\delta_i f\|_{L^p(\mathbb{R}^d)}.$$

Since $\delta f \in \left(L^p(\mathbb{R}^d)\right)^d$, we have

$$\|(1-\chi_N)\delta_i f\|_{L^p(\mathbb{R}^d)} \leq \|\delta_i f\|_{L^p(\mathbb{R}^d \setminus Q_N)} \underset{N \to +\infty}{\longrightarrow} 0.$$

Since χ_N is supported in $Q_{\frac{5N}{3}}$ and $\chi_N \equiv 1$ on $Q_{\frac{4N}{3}}$, $\delta \chi_N$ is supported in A_N for N sufficiently large. Lemma 3.1 yields the existence of a constant $C > 0$ such that for every $N \in \mathbb{N}^*$, we have :

$$\|f - f_{per,N}\|_{L^p(A_N)} \leq CN \|\delta f\|_{L^p(\mathbb{R}^d \setminus Q_N)}. \tag{3.42}$$

We next use the mean-value inequality so that, for every $i \in \{1, ..., d\}$ and $N \in \mathbb{N}$:

$$\|\delta_i \chi_N\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla \chi_N\|_{\left(L^\infty(\mathbb{R}^d)\right)^d} \leq \frac{1}{N}. \tag{3.43}$$

Using (3.42) and (3.43), we obtain :

$$\|\delta_i \chi_N \left(f - f_{per,N}\right)\|_{L^p(\mathbb{R}^d)} \leq C\|\delta f\|_{L^p(\mathbb{R}^d \setminus Q_N)} \underset{N \to +\infty}{\longrightarrow} 0.$$

We conclude that the sequence δf_N converges to δf in $L^p(\mathbb{R}^d)$.

We next show that the sequence f_N is a Cauchy sequence in $\mathcal{E}^p$. Let M and N in $\mathbb{N}$. Since $f_N - f_M \in L^p(\mathbb{R}^d)$, we know from Proposition 3.8 the existence of a constant $C > 0$ independent of N and M such that :

$$\|f_N - f_M\|_{\mathcal{E}^p} \leq C\|\delta f_N - \delta f_M\|_{L^p(\mathbb{R}^d)}.$$

We have proved that δf_N converges in $\left(L^p(\mathbb{R}^d)\right)^d$, it is therefore a Cauchy sequence for the L^p-norm and we can deduce that f_N is also a Cauchy sequence in the Banach space $\mathcal{E}^p$. Thus, there exists $g \in \mathcal{E}^p$ such that f_N converges to g in $\mathcal{E}^p$. We use Proposition 3.3 and we obtain that f_N also converges to g in $L^1_{loc}(\mathbb{R}^d)$. Since δf_N converges to δf in L^p, the uniqueness of the limit in $L^1_{loc}(\mathbb{R}^d)$ shows that

$$\delta g = \lim_{N\to\infty} \delta f_N = \delta f.$$

We deduce that $\delta(f-g) = 0$, that is, $f_{per} := f - g$ is a Q-periodic function. To conclude, we again use Proposition 3.8 to obtain

$$\|f_N\|_{\mathcal{E}^p} \leq C\|\delta f_N\|_{L^p(\mathbb{R}^d)}.$$

We pass to the limit in this inequality and we obtain

$$\|f - f_{per}\|_{\mathcal{E}^p} = \|g\|_{\mathcal{E}^p} \leq C\|\delta f\|_{L^p(\mathbb{R}^d)}.$$

There remains to show the uniqueness of f_{per}. We assume there exists $f_{per,1}$ and $f_{per,2}$ two periodic functions such that both $\tilde{f}_1 := f - f_{per,1}$ and $\tilde{f}_2 := f - f_{per,2}$ belong to $\mathcal{E}^p$. Thus, we have $\mathcal{M}\left(|f_{per,1} - f_{per,2}|\right) = \mathcal{M}\left(|\tilde{f}_2 - \tilde{f}_1|\right) \in L^{p^*}(\mathbb{R}^d)$. Since $|f_{per,1} - f_{per,2}|$ is a periodic function, $\mathcal{M}\left(|f_{per,1} - f_{per,2}|\right)$ is constant and is therefore equal to 0. □

From the previous proof, we deduce the next corollary, which will be useful in the next Section.

Corollary 3.4. *Let $f \in \mathcal{A}^p$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of $L^p(\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{A}^p} = 0.$$

In addition, if $f \in L^\infty(\mathbb{R}^d)$, the sequence $(f_n)_{n\in\mathbb{N}}$ can be choosen such that for every $n \in \mathbb{N}$,

$$\|f_n\|_{L^\infty(\mathbb{R}^d)} \leq 2\|f\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. In the proof of Proposition 3.1, we have established the existence of a sequence f_n of $L^p(\mathbb{R}^d)$-functions and the existence of a Q-periodic function f_{per} such that f_n converges to $f - f_{per}$ in $\mathcal{E}^p$. Since f and $f - f_{per}$ both belong to $\mathcal{E}^p$, we clearly have $f_{per} = 0$. We can conclude that f_n converges to f in $\mathcal{E}^p$. Now, we assume that $f \in L^\infty(\mathbb{R}^d)$. We have shown in the proof of Proposition 3.1, that the sequence $(f_n)_{n\in\mathbb{N}^*}$ can be defined by $f_n = \chi_n\left(f - f_{per,n}\right)$, where $f_{per,n}$ is the periodic function given by (3.31) and χ_n satisfies (3.41). We clearly have

$$\|f_{per,n}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}.$$

Finally, since $\|\chi_n\|_{L^\infty(\mathbb{R}^d)} \leq 1$, we obtain

$$\|f_n\|_{L^\infty(\mathbb{R}^d)} \leq 2\|f\|_{L^\infty(\mathbb{R}^d)}$$

using a triangle inequality. □

To conclude this section, we show that it is possible to describe the periodic function f_{per} given in Proposition 3.1 when the function $f \in \mathcal{A}^p$ is assumed uniformly Hölder-continuous.

Proposition 3.10. *Let $f \in \mathbf{A}^p \cap L^\infty(\mathbb{R}^d)$ for $1 \le p < d$. Then the unique Q-periodic function f_{per} such that $f - f_{per} \in \mathcal{E}^p$ given by Proposition 3.1 is equal to*

$$f_{per} = \lim_{N \to \infty} \frac{1}{\sharp I_N} \sum_{k \in I_N} f(. + k), \tag{3.44}$$

where $I_N = \left\{k \in \mathbb{Z}^d \;\middle|\; |k| \le N\right\}$ for every $N \in \mathbb{N}^*$. In addition, if there exists $\alpha \in]0,1[$ such that $f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, then $f_{per} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$.

Proof. We define $\tilde{f} = f - f_{per}$. We first show that if $f \in L^\infty(\mathbb{R}^d)$, then $\mathcal{M}(|\tilde{f}|)$ is Lipschitz continuous on $\mathbb{R}^d$. To this end, we remark that for every $i \in \{1, ..., d\}$, we have

$$\partial_i \mathcal{M}(|\tilde{f}|)(x) = \int_{Q_i + \tilde{x}_i} \delta_i |\tilde{f}(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_d)| d\tilde{y}_i,$$

where $Q_i = \prod_{j \in \{1,...,d\} \setminus \{i\}}]0,1[$ and $\tilde{x}_i = (x_1, .., x_{i-1}, x_{i+1}, ..., x_d)$. Using a triangle inequality we have $\delta_i |\tilde{f}| \le |\delta_i \tilde{f}|$ and, since $\delta_i \tilde{f} = \delta_i f$ and $\|\delta f\|_{L^\infty(\mathbb{R}^d)} \le 2\|f\|_{L^\infty(\mathbb{R}^d)}$, we can bound the integral uniformly with respect to x in the previous equality and we have :

$$\left\|\partial_i \mathcal{M}(|\tilde{f}|)\right\|_{L^\infty(\mathbb{R}^d)} \le 2\|f\|_{L^\infty(\mathbb{R}^d)}.$$

It follows that $\nabla \mathcal{M}(|\tilde{f}|)$ belongs to $\left(L^\infty(\mathbb{R}^d)\right)^d$ and we deduce that $\mathcal{M}(|\tilde{f}|)$ is Lipschitz-continuous. Moreover, since $\mathcal{M}(|\tilde{f}|)$ belongs to $L^{p^*}(\mathbb{R}^d)$, we have that $\lim_{|x| \to \infty} \mathcal{M}(|\tilde{f}|)(x) = 0$, and for every $x \in \mathbb{R}^d$, the Cesàro mean of the sequence $\left(\mathcal{M}(|\tilde{f}|)(x+k)\right)_{k \in \mathbb{Z}^d}$ is equal to 0. We therefore obtain that

$$\lim_{N \to \infty} \frac{1}{\sharp I_N} \sum_{k \in I_N} \mathcal{M}(|\tilde{f}|)(x+k) = 0.$$

Consequently, for every $x \in \mathbb{R}^d$ and $N \in \mathbb{N}^*$, we have

$$\begin{aligned} \int_{Q+x} \left| \frac{1}{\sharp I_N} \sum_{k \in I_N} f(y+k) - f_{per}(y) \right| dy &= \int_{Q+x} \left| \frac{1}{\sharp I_N} \sum_{k \in I_N} \tilde{f}(y+k) \right| dy \\ &\le \frac{1}{\sharp I_N} \sum_{k \in I_N} \mathcal{M}(|\tilde{f}|)(x+k) \overset{N \to \infty}{\longrightarrow} 0, \end{aligned}$$

and deduce $\lim_{N \to \infty} \frac{1}{\sharp I_N} \sum_{k \in I_N} f(. + k) = f_{per}$ in $L^1_{loc}(\mathbb{R}^d)$. If we now assume that $f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ for $\alpha \in]0,1[$, we have for every $N \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^d$:

$$\left| \frac{1}{\sharp I_N} \sum_{k \in I_N} f(x+k) \right| \le \|f\|_{L^\infty(\mathbb{R}^d)}, \tag{3.45}$$

$$\left| \frac{1}{\sharp I_N} \sum_{k \in I_N} f(x+k) - \frac{1}{\sharp I_N} \sum_{k \in I_N} f(y+k) \right| \le \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} |x-y|^\alpha. \tag{3.46}$$

In addition, up to an extraction, the sequence $\left(\frac{1}{\sharp I_N}\sum_{k\in I_N} f(.+k)\right)_{N\in\mathbb{N}^*}$ converges to f_{per} almost everywhere and, considering the limit when $N\to\infty$ in (3.46) and (3.45), we obtain for almost all $x,y\in\mathbb{R}^d$:

$$|f_{per}(x)|\leq \|f\|_{L^\infty(\mathbb{R}^d)} \quad \text{and} \quad |f_{per}(x)-f_{per}(y)|\leq \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}|x-y|^\alpha.$$

□

Remark 3.4. *More generally, in the proof of Proposition 3.10 we have actually shown that if $f=f_{per}+\tilde{f}$ where f_{per} is periodic and $\lim\limits_{|x|\to\infty}\mathcal{M}(|\tilde{f}|)=0$, then f_{per} is necessarily given by* (3.44).

Remark 3.5. *All the results established in this section can be easily adapted in a context of T-periodicity at infinity, for any period $T=(T_1,T_2,...,T_d)$, considering*

$$\delta_T f := (f(.+T_i)-f)_{i\in\{1,...,d\}},$$

instead of δf and

$$\mathcal{M}_T(|f|)(x) := \int_{\prod_{i=1}^d]0,T_i[} |f(y+x)|dy,$$

instead of $\mathcal{M}(|f|)$.

3.3 The homogenization problem when $p<d$

In this section we study homogenization problem (3.1) when the coefficient a satisfies assumptions (3.2), (3.3), (3.4) and (3.11) for $1<p<d$ and we prove Theorem 3.1 in this case. As in the previous section, our assumption $p<d$ of course requires that $d\geq 2$. Since $p<d$, Proposition 3.1 gives the existence of two matrix-valued functions $a_{per}\in\left(L^2_{per}\right)^{d\times d}$ and $\tilde{a}\in(\mathcal{A}^p)^{d\times d}$ such that $a=a_{per}+\tilde{a}$ and $\|\tilde{a}\|_{\mathcal{E}^p}\leq C\|\delta a\|_{L^p(\mathbb{R}^d)}$, where $C>0$ is a constant independent of a. We are therefore indeed studying a problem of perturbed periodic geometry in the presence of a local defect $\tilde{a}$, which is, up to a local averaging, a matrix-valued function with components in $L^{p^*}(\mathbb{R}^d)$. We note that assumptions (3.2), (3.4) and Proposition 3.10 ensure that the coefficients a_{per} and $\tilde{a}$ also satisfy the following two properties of ellipticity and regularity :

$$\exists\lambda>0,\ \forall x,\xi\in\mathbb{R}^d \quad \lambda|\xi|^2\leq\langle a_{per}(x)\xi,\xi\rangle, \tag{3.47}$$

$$a_{per},\tilde{a}\in\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}. \tag{3.48}$$

In order to study the corrector equation, we adapt the method introduced in [27]. We remark that (3.18) is equivalent to $-\operatorname{div}(a\nabla\tilde{w}_q)=\operatorname{div}(\tilde{a}(\nabla w_{per,q}+q))$. Under assumption (3.48), elliptic regularity theory (see for instance [47, Theorem 5.19 p.87]) also implies that $\nabla w_{per,q}\in\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Thus, $f=\tilde{a}(\nabla w_{per}+q)$ belongs to $\left(\mathcal{E}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and, since $\nabla w_{per,q}$ is periodic, we have $\delta f\in\left(L^p(\mathbb{R}^d)\right)^{d\times d}$. To prove Theorem 3.1, it is therefore sufficient to study the more general problem :

$$-\operatorname{div}(a\nabla u)=\operatorname{div}(f) \quad \text{on } \mathbb{R}^d, \tag{3.49}$$

for every $f\in\left(\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.

3.3.1 Preliminary regularity result

We begin by establishing a regularity result for the solutions u to (3.49) such that ∇u belongs to $(\mathcal{A}^p)^d$. We need to introduce the space

$$L^2_{unif}(\mathbb{R}^d) = \left\{ f \in L^2_{loc}(\mathbb{R}^d) \;\middle|\; \sup_{x\in\mathbb{R}^d} \|f\|_{L^2(B_1(x))} < \infty \right\},$$

equipped with $\|f\|_{L^2_{unif}} = \sup_{x\in\mathbb{R}^d} \|f\|_{L^2(B_1(x))}$. We have :

Lemma 3.2. *There exists $C > 0$ such that, for every $0 < R < 1$ and $f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$,*

$$\|f\|_{L^2_{unif}(\mathbb{R}^d)} \leq C \left(\frac{1}{|B_R|} \|\mathcal{M}(|f|)\|_{L^2_{unif}(\mathbb{R}^d)} + R^\alpha \|f\|_{C^{0,\alpha}(\mathbb{R}^d)} \right). \tag{3.50}$$

Proof. We denote $B^+_R(x) = B_R(x) \cap (Q + x)$ and $\fint_{B^+_R(x)} = \frac{1}{|B^+_R(x)|} \int_{B^+_R(x)}$, for $R > 0$ and $x \in \mathbb{R}^d$. For every $x_0 \in \mathbb{R}^d$ and $0 < R < 1$, we have :

$$\begin{aligned}
\int_{B_1(x_0)} \left| f(x) - \fint_{B^+_R(x)} f(y)dy \right|^2 dx &\leq \int_{B_1(x_0)} \left| \fint_{B^+_R(x)} |f(x) - f(y)|dy \right|^2 dx \\
&\leq \|f\|^2_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \int_{B_1(x_0)} \left| \fint_{B^+_R(x)} |x-y|^\alpha dy \right|^2 dx \\
&\leq \|f\|^2_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} R^{2\alpha} |B_1|.
\end{aligned}$$

Using a triangle inequality, we therefore deduce :

$$\begin{aligned}
\|f\|_{L^2(B_1(x_0))} &\leq \left\| f - \fint_{B^+_R(.)} f \right\|_{L^2(B_1(x_0))} + \left\| \fint_{B^+_R(.)} |f| \right\|_{L^2(B_1(x_0))} \\
&\leq R^\alpha |B_1|^{1/2} \|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} + \left\| \fint_{B^+_R(.)} |f| \right\|_{L^2(B_1(x_0))}.
\end{aligned}$$

Since $B^+_R(x) \subset Q + x$, we obtain

$$\|f\|_{L^2(B_1(x_0))} \leq R^\alpha |B_1|^{1/2} \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} + \frac{1}{|B_R|} \|\mathcal{M}(|f|)\|_{L^2_{unif}(\mathbb{R}^d)}.$$

Taking the supremum over all x_0 yields (3.50) which concludes the proof. ☐

Lemma 3.2 is now useful to establish :

Proposition 3.11. *There exists a constant $C > 0$ such that for every $f \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and $u \in H^1_{loc}(\mathbb{R}^d)$ solution to (3.49) with $\nabla u \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, we have :*

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C \left(\|\mathcal{M}(|\nabla u|)\|_{L^2_{unif}} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \right).$$

Proof. Since u is a solution to equation (3.49) where the coefficient a satisfies assumption (3.48), we know from [47, Theorem 5.19 p.87] (see also [46, Theorem 3.2 p.88]) there exists a constant $C > 0$, such that for every $x_0 \in \mathbb{R}^d$, we have :

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(B_1(x_0))} \leq C\left(\|\nabla u\|_{L^2(B_4(x_0))} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) \leq C\left(C_1\|\nabla u\|_{L^2_{unif}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right),$$

where $C_1 > 0$ depends only on the dimension d. Since the right-hand side in the previous inequality is independent of x_0, there exists a constant $C_2 > 0$ such that

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C_2\left(\|\nabla u\|_{L^2_{unif}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).$$

For every $0 < R < 1$, we use this inequality and Lemma 3.2 to obtain the existence of a constant $C_3 > 0$, independent of R, u and f, such that,

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C_3\left(\frac{1}{|B_R|}\|\mathcal{M}(|\nabla u|)\|_{L^2_{unif}(\mathbb{R}^d)} + R^\alpha\|\nabla u\|_{C^{0,\alpha}(\mathbb{R}^d)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^d)}\right).$$

It remains to choose R such that $R^\alpha C_3 < 1$ to conclude. □

3.3.2 Well-posedness for (3.49) when the coefficient is periodic

We next study equation (3.49) when the coefficient a is periodic, that is when $\tilde{a} = 0$. For every $f \in \mathcal{A}^p$, we prove the existence and uniqueness of a solution u to :

$$-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(f) \quad \text{on } \mathbb{R}^d, \tag{3.51}$$

such that $\nabla u \in (\mathcal{A}^p)^d$. Adapting a method introduced in [27], this result is the first step to study (3.49). We begin with existence of the solution.

Proposition 3.12. *Assume a_{per} satisfies (3.47) and (3.48). Let $f \in (\mathcal{A}^p)^d$ for $p \in]1, d[$. Then, there exists a solution $u \in L^1_{loc}(\mathbb{R}^d)$ to (3.51) such that $\nabla u \in (\mathcal{A}^p)^d$. In addition, there exists a constant $C_1 > 0$ independent of u and f such that*

$$\|\nabla u\|_{\mathcal{A}^p} \leq C_1\|f\|_{\mathcal{A}^p}. \tag{3.52}$$

If we additionally assume that $f \in \left(L^\infty(\mathbb{R}^d)\right)^d$, then $\nabla u \in \left(L^2_{unif}(\mathbb{R}^d)\right)^d$ and there exists $C_2 > 0$ independent of f and u such that :

$$\|\nabla u\|_{L^2_{unif}(\mathbb{R}^d)} \leq C_2\left(\|f\|_{\mathcal{E}^p} + \|f\|_{L^\infty(\mathbb{R}^d)}\right). \tag{3.53}$$

We will need to introduce the Green function G_{per} associated with $-\operatorname{div}(a_{per}\nabla .)$ on $\mathbb{R}^d$ defined as the unique solution to

$$\begin{cases} -\operatorname{div}_x\left(a_{per}(x)\nabla_x G_{per}(x,y)\right) = \delta_y(x) & \text{in } \mathcal{D}'(\mathbb{R}^d), \\ \lim\limits_{|x-y|\to\infty} G_{per}(x,y) = 0. \end{cases}$$

In order to define a solution to (3.51), we will use several pointwise estimates established in [14, Section 2] and satisfied by G_{per} on the whole space $\mathbb{R}^d$. Indeed, we know there exist $C_1 > 0$ and $C_2 > 0$ such that for every $x, y \in \mathbb{R}^d$ with $x \neq y$, it holds :

$$|\nabla_y G_{per}(x,y)| \leq C_1\frac{1}{|x-y|^{d-1}}, \tag{3.54}$$

$$|\nabla_x\nabla_y G_{per}(x,y)| \leq C_3\frac{1}{|x-y|^{d}}. \tag{3.55}$$

Proof. **Step 1 : Existence of a solution**. Corollary 3.4 gives the existence of a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\left(L^p(\mathbb{R}^d)\right)^d$ that converges to f in $(\mathcal{A}^p)^d$. For every $n \in \mathbb{N}$, the results of [14] establish the existence of a solution, unique up to an additive constant, u_n in $L^1_{loc}(\mathbb{R}^d)$ to :

$$-\operatorname{div}(a_{per}\nabla u_n) = \operatorname{div}(f_n) \quad \text{on } \mathbb{R}^d, \tag{3.56}$$

and such that $\nabla u_n \in \left(L^p(\mathbb{R}^d)\right)^d$. In addition, using the periodicity of a_{per}, we apply the operator δ_i to equation (3.56), and we obtain

$$-\operatorname{div}(a_{per}\delta_i\nabla u_n) = \operatorname{div}(\delta_i f_n),$$

for every $i \in \{1, ..., d\}$. Since $\delta_i\nabla u_n$ and $\delta_i f_n$ both belong to $\left(L^p(\mathbb{R}^d)\right)^d$, the continuity result of [14, Theorem A] yields the existence of a constant $C_1 > 0$ independent of n such that

$$\|\delta\nabla u_n\|_{L^p(\mathbb{R}^d)} \leq C_1\|\delta f_n\|_{L^p(\mathbb{R}^d)}.$$

From Proposition 3.8 we infer the existence of $C_2 > 0$ independent of n such that :

$$\|\nabla u_n\|_{\mathcal{A}^p} \leq C_2\|\delta\nabla u_n\|_{L^p(\mathbb{R}^d)} \leq C_1C_2\|\delta f_n\|_{L^p(\mathbb{R}^d)}. \tag{3.57}$$

Likewise, for every $m, n \in \mathbb{N}$, the function $u_n - u_m$ is a solution to

$$-\operatorname{div}(a_{per}(\nabla u_n - \nabla u_m)) = \operatorname{div}(f_n - f_m).$$

Since $f_n - f_m$ and $\nabla(u_n - u_m)$ both belong to $\left(L^p(\mathbb{R}^d)\right)^d$, we similarly obtain :

$$\|\nabla u_n - \nabla u_m\|_{\mathcal{A}^p} \leq C_1C_2\|\delta f_n - \delta f_m\|_{L^p(\mathbb{R}^d)}.$$

Since $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{A}^p$, it is a Cauchy sequence in this space and the previous inequality shows $(\nabla u_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $(\mathcal{A}^p)^d$. We therefore obtain the existence of $T \in (\mathcal{A}^p)^d$ such that ∇u_n converges to T in $(\mathcal{A}^p)^d$. Using Proposition 3.3, we have that ∇u_n also converges to T in $\left(L^1_{loc}(\mathbb{R}^d)\right)^d$ and the Schwarz Lemma shows the existence of $u \in L^1_{loc}(\mathbb{R}^d)$ such that $T = \nabla u$. Finally, taking the limit when $n \to \infty$ in (3.56) and (3.57), we obtain that u is a solution to (3.51) such that :

$$\|\nabla u\|_{\mathcal{A}^p} \leq C_2\|\delta f\|_{L^p(\mathbb{R}^d)} \leq C_2\|f\|_{\mathcal{A}^p}.$$

Step 2 : Proof of estimate (3.53). We now additionally assume that $f \in \left(L^\infty(\mathbb{R}^d)\right)^d$. Corollary 3.4 gives the existence of a sequence $(f_n)_{n\in\mathbb{N}}$ of $\left(L^p(\mathbb{R}^d)\right)^d$ such that f_n converges to f in $(\mathcal{A}^p)^d$ and such that for every $n \in \mathbb{N}$, we have :

$$\|f_n\|_{L^\infty(\mathbb{R}^d)} \leq 2\|f\|_{L^\infty(\mathbb{R}^d)}.$$

Exactly as in step 1, we denote by u_n, the unique solution (up to an additive constant) to (3.56) such that $\nabla u_n \in \left(L^p(\mathbb{R}^d)\right)^d$. We fix $x_0 \in \mathbb{R}^d$ and our aim is to show that the norm of ∇u_n in $L^2(B_1(x_0))$ is uniformly bounded with respect to n and x_0. We begin by splitting ∇u_n in two parts. We write $\nabla u_n = \nabla u_{n,1} + \nabla u_{n,2}$ where $u_{n,1}$ is the unique solution (up to an additive constant) to

$$-\operatorname{div}(a_{per}\nabla u_{n,1}) = \operatorname{div}\left(f_n 1_{B_{4\sqrt{d}}(x_0)}\right) \quad \text{on } \mathbb{R}^d,$$

such that $\nabla u_{n,1} \in \left(L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\right)^d$ and $u_{n,2}$ is the unique solution (again up to an additive constant) to

$$-\operatorname{div}(a_{per}\nabla u_{n,2}) = \operatorname{div}\left(f_n\left(1 - 1_{B_{4\sqrt{d}}(x_0)}\right)\right) \quad \text{on } \mathbb{R}^d,$$

such that $\nabla u_{n,2} \in \left(L^p(\mathbb{R}^d)\right)^d$. Since $f_n 1_{B_{4\sqrt{d}}(x_0)}$ belongs to $\left(L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\right)^d$, the existence of $u_{n,1}$ is established in [14, Theorem A], and we have the existence of a constant $C_1 > 0$ independent of n and x_0 such that :

$$\|\nabla u_{n,1}\|_{L^2(\mathbb{R}^d)} \leq C_1 \|f_n 1_{B_{4\sqrt{d}}(x_0)}\|_{L^2(\mathbb{R}^d)}. \tag{3.58}$$

Similarly, since $f_n\left(1 - 1_{B_{4\sqrt{d}}(x_0)}\right)$ belongs to $\left(L^p(\mathbb{R}^d)\right)^d$, the existence of $u_{n,2}$ is also given in [14, Theorem A] and we have

$$\nabla u_{n,2} = \int_{\mathbb{R}^d} \nabla_x \nabla_y G_{per}(.,y)\left(f_n\left(1 - 1_{B_{4\sqrt{d}}(x_0)}\right)\right)(y)dy.$$

We note that the equality $\nabla u_n = \nabla u_{n,1} + \nabla u_{n,2}$ holds as a consequence of the uniqueness of a solution to (3.56) with a gradient in $\left(L^p(\mathbb{R}^d)\right)^d$. We next respectively estimate the norm of $\nabla u_{n,1}$ and $\nabla u_{n,2}$ in $L^2(B_1(x_0))$. First, using (3.58), we have

$$\|\nabla u_{n,1}\|_{L^2(B_1(x_0))} \leq C_1 \left\|f_n 1_{B_{4\sqrt{d}}(x_0)}\right\|_{L^2(\mathbb{R}^d)} \leq C_1 |B_{4\sqrt{d}}|^{1/2} \|f_n\|_{L^\infty(\mathbb{R}^d)}. \tag{3.59}$$

In order to estimate the L^2-norm of $\nabla u_{n,2}$, we use the behavior (3.55) of the Green function G_{per} and we obtain the existence of a constant $C > 0$ such that for every $x \in B_1(x_0)$, we have

$$|\nabla u_{n,2}(x)| \leq C \int_{\mathbb{R}^d \setminus B_{4\sqrt{d}}(x_0)} \frac{1}{|x-y|^d}|f_n(y)|dy.$$

Since $|Q| = 1$, using a change of variables, we have

$$I(x) := \int_{\mathbb{R}^d \setminus B_{4\sqrt{d}}(x_0)} \frac{1}{|x-y|^d}|f_n(y)|dy = \int_Q \int_{\mathbb{R}^d \setminus B_{4\sqrt{d}}(z+x-x_0)} \frac{1}{|y+z|^d}|f_n(x-y-z)|dydz.$$

We note that for every $z \in Q$, $|z| \leq \sqrt{d}$ and $|x - x_0| \leq 1 \leq \sqrt{d}$, and it follows

$$I(x) \leq \int_Q \int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}} \frac{1}{|y+z|^d}|f_n(x-y-z)|dydz.$$

Next, for every $y \in \mathbb{R}^d \setminus B_{2\sqrt{d}}$, we have $|z| \leq \dfrac{1}{2}|y|$ and we use a triangle inequality to deduce $|z+y| \geq |y| - |z| \geq \frac{1}{2}|y|$, and

$$\int_Q \int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}} \frac{1}{|y+z|^d}|f_n(x-y-z)|dydz \leq 2^d \int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}} \frac{1}{|y|^d} \int_{Q+y} |f_n(x-z)|dzdy.$$

We use the Hölder inequality and obtain,

$$I(x) \leq 2^d \left(\int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}} \frac{1}{|y|^{(p^*)'d}}dy\right)^{1/(p^*)'} \|\mathcal{M}(|f_n|)\|_{L^{p^*}(\mathbb{R}^d)}.$$

Here, we have denoted by $(p^*)'$ the conjugate Lebesgue exponent associated with p^*. We have finally proved that for every $x \in B_1(x_0)$,

$$|\nabla u_{n,2}(x)| \leq CI(x) \leq A\|f_n\|_{\mathcal{E}^p}, \tag{3.60}$$

where $A = C2^d \left(\displaystyle\int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}} \frac{1}{|y|^{(p^*)'d}} dy\right)^{1/(p^*)'}$ is clearly independent of x_0 and f. We integrate (3.60) on $B_1(x_0)$ and we obtain the existence of a constant $C_2 > 0$, independent of n, x_0 and f, and such that :

$$\|\nabla u_{n,2}\|_{L^2(B_1(x_0))} \leq C_2\|f_n\|_{\mathcal{E}^p}. \tag{3.61}$$

Since $\nabla u_n = \nabla u_{n,1} + \nabla u_{n,2}$, we use (3.59) and (3.61) :

$$\|\nabla u_n\|_{L^2(B_1(x_0))} \leq \|\nabla u_{n,1}\|_{L^2(B_1(x_0))} + \|\nabla u_{n,2}\|_{L^2(B_1(x_0))} \leq C\left(\|f_n\|_{L^\infty(\mathbb{R}^d)} + \|f_n\|_{\mathcal{E}^p}\right),$$

where $C > 0$ is independent of n, x_0 and f. Since the previous inequality holds for every $x_0 \in \mathbb{R}^d$, ∇u_n is bounded in $\left(L^2_{loc}(\mathbb{R}^d)\right)^d$ and, up to an extraction, it weakly converges to a function $v \in \left(L^2_{loc}(\mathbb{R}^d)\right)^d$. We recall that ∇u_n also converges to ∇u in $\left(L^1_{loc}(\mathbb{R}^d)\right)^d$, it follows that $v = \nabla u$. In addition the L^2 norm being lower semi-continuous, for every $x_0 \in \mathbb{R}^d$ we obtain :

$$\|\nabla u\|_{L^2(B_1(x_0))} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^2(B_1(x_0))} \leq C\left(\|f\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{\mathcal{E}^p}\right).$$

We finally take the supremum over all the points $x_0 \in \mathbb{R}^d$ and we obtain (3.53). □

Remark 3.6. *For $p = 1$, estimate (3.52) stated in Proposition 3.12 does not hold. For $d = 2$ say, we can consider $a_{per} = I_2$ and*

$$f(x) := \left(-\frac{x_2}{|x|^2 \ln(|x|)^2} \arctan\left(\frac{x_2}{x_1}\right), \frac{x_1}{|x|^2 \ln(|x|)^2} \arctan\left(\frac{x_2}{x_1}\right)\right),$$

where we have denoted $x = (x_1, x_2)$. A solution to (3.51) is $u(x) = \ln(\ln(|x|))$. When $|x| \to \infty$ we can show that

$$|f(x)| = O\left(\frac{1}{|x|\ln(|x|)^2}\right) \in L^2(\mathbb{R}^2) = L^{1^*}(\mathbb{R}^2),$$

and

$$|\delta f_1(x)| = O\left(\frac{1}{|x|^2 \ln(|x|)^2}\right) \in L^1(\mathbb{R}^2), \qquad |\delta f_2(x)| = O\left(\frac{1}{|x|^2 \ln(|x|)^2}\right) \in L^1(\mathbb{R}^2).$$

Consequently f belongs to $(\mathcal{A}^1)^2$. However $|\delta\partial_1 u(x_1, x_2)| \sim \dfrac{C}{|x|^2 \ln|x|} \notin L^1(\mathbb{R}^2)$ and ∇u does not belong to $(\mathcal{A}^1)^2$. This is of course related to the fact that the operator $-\nabla\Delta^{-1}\operatorname{div}$ is not continuous from $\left(L^1(\mathbb{R}^d)\right)^d$ to $\left(L^1(\mathbb{R}^d)\right)^d$. We only have continuity from L^1 to weak $-L^1$ (see [73, Section 7.3] for the details).

Remark 3.7. *The property $f \in (\mathcal{E}^p)^d$ in (3.53) is required to obtain the uniform estimate in $L^2_{unif}(\mathbb{R}^d)$ satisfied by ∇u. When f only belongs to $\left(L^\infty(\mathbb{R}^d)\right)^d$, (3.51) may possibly have no solution with a gradient in $L^2_{unif}(\mathbb{R}^d)$. Consider indeed for $d = 2$,*

$$f(x) = f(x_1, x_2) = \left(-2\frac{x_1^2}{|x|^2}, -2\frac{x_1 x_2}{|x|^2}\right) \in \left(L^\infty(\mathbb{R}^2)\right)^2.$$

Then $u(x) = x_1 \ln(|x|)$ satisfies $-\Delta u = \operatorname{div}(f)$, while $\nabla u \notin \left(L^2_{unif}(\mathbb{R}^2)\right)^2$ and has logarithmic growth.

We next deal with uniqueness of the solution.

Lemma 3.3. *Assume a_{per} satisfies (3.47) and (3.48). Let $u \in L^1_{loc}(\mathbb{R}^d)$ be a solution in $\mathcal{D}'(\mathbb{R}^d)$ to :*

$$-\operatorname{div}(a_{per}\nabla u) = 0 \quad \text{on } \mathbb{R}^d, \tag{3.62}$$

such that $\nabla u \in (\mathcal{A}^p)^d$ for $p \in]1, d[$. Then $\nabla u = 0$.

Proof. For every $i \in \{1, ..., d\}$, we consider a translation by e_i of equation (3.62) and we subtract it from the original equation. The periodicity of a_{per} implies $-\operatorname{div}(a_{per}\nabla \delta_i u) = 0$. Since we have assumed $\nabla u \in (\mathcal{A}^p)^d$, we know that $\nabla \delta_i u$ belongs to $\left(L^p(\mathbb{R}^d)\right)^d$ and the uniqueness result established in [27, Proposition 2.1] for solutions with gradient in $\left(L^p(\mathbb{R}^d)\right)^d$ therefore implies that $\nabla \delta_i u = 0$. It follows that ∇u is Q-periodic and, consequently, the function $\mathcal{M}(|\nabla u|)$ is constant. Since by assumption $\mathcal{M}(|\nabla u|)$ belongs to $L^{p^*}(\mathbb{R}^d)$, we obtain $\mathcal{M}(|\nabla u|) = 0$ which shows that $\nabla u = 0$. □

Corollary 3.5. *Let $p \in]1, d[$. There exists $C > 0$ such that for every $f \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and u solution to (3.51) such that $\nabla u \in (\mathcal{A}^p)^d$, we have :*

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C\left(\|\mathcal{M}(|f|)\|_{L^{p^*}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).$$

Proof. Using both Proposition 3.12 and the uniqueness of Lemma 3.3, we know that ∇u belongs to $\left(L^2_{unif}(\mathbb{R}^d)\right)^d$ and the elliptic regularity theory (see for instance the results of [47, Theorem 5.19 p.87]) implies that $\nabla u \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. The inequality is therefore a direct consequence of Proposition 3.11 and estimate (3.53). □

In the sequel of this section, we study the specific case where $1 < p < \frac{d}{2}$, that is when $p^* < d$. We can then show some additional properties satisfied by u solution to (3.51). We successively show, respectively in Lemma 3.4 and Lemma 3.5 that, up to an additive constant u belongs to $\mathcal{E}^{p^*}$ and it is uniformly bounded as soon as f belongs to $\left(L^\infty(\mathbb{R}^d)\right)^d$. To this end, we first need to recall the Hardy-Littlewood-Sobolev (see for instance [47, Theorem 7.25 p. 162]).

Proposition 3.13 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \alpha < d$. We define*

$$I(f)(x) := \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} f(y)dy.$$

Let $p, q > 1$ such that $1 + \dfrac{1}{q} = \dfrac{\alpha}{d} + \dfrac{1}{p}$. Then, there exists $C > 0$ such that for every $f \in L^p(\mathbb{R}^d)$ we have :

$$\|I(f)\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

We may now prove that the unique (up to an additive constant) solution u to (3.51) such that $\nabla u \in (\mathcal{A}^p)^d$ can be made explicit using the Green function G_{per}.

Lemma 3.4. *Assume that $1 < p < \frac{d}{2}$ and let $f \in (\mathcal{A}^p)^d$, then the solution u (unique up to an additive constant) to (3.51) such that $\nabla u \in (\mathcal{A}^p)^d$ is given by*

$$u = Tf := \int_{\mathbb{R}^d} \nabla_y G_{per}(., y) f(y) dy. \tag{3.63}$$

In addition, we have $u \in \mathcal{E}^{p^*}$.

Proof. We begin by showing that the function Tf given by (3.63) is well-defined in $L^1_{loc}(\mathbb{R}^d)$ if $f \in (\mathcal{A}^p)^d$ and satisfies $\mathcal{M}(|Tf|) \in L^{p^{**}}(\mathbb{R}^d)$. Using estimate (3.54), we know there exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$, we have :

$$|Tf(x)| \leq C \int_{\mathbb{R}^d} \frac{1}{|y|^{d-1}} |f|(x-y) dy.$$

For every $z \in \mathbb{R}^d$, we integrate the previous inequality with respect to $x \in Q + z$ and we use the Fubini Theorem to obtain :

$$\mathcal{M}\left(|Tf|\right)(z) \leq C \int_{\mathbb{R}^d} \frac{1}{|y|^{d-1}} \int_{Q+z} |f|(x-y) dx dy = C \frac{1}{|.|^{d-1}} * \mathcal{M}(|f|).$$

Since $1 + \dfrac{1}{p^{**}} = \dfrac{d-1}{d} + \dfrac{1}{p^*}$, the Hardy-Littlewood-Sobolev inequality therefore shows the existence of a constant $C > 0$, such that :

$$\|\mathcal{M}\left(|Tf|\right)\|_{L^{p^{**}}(\mathbb{R}^d)} \leq C \|\mathcal{M}\left(|f|\right)\|_{L^{p^*}(\mathbb{R}^d)}. \tag{3.64}$$

In particular $\mathcal{M}\left(|Tf|\right)$ belongs to $L^{p^{**}}(\mathbb{R}^d)$ and is therefore finite for almost every $x \in \mathbb{R}^d$. We deduce that Tf is well-defined in $L^1_{loc}(\mathbb{R}^d)$.

We next show that, up to an additive constant, we have $u = Tf$. We first recall that in the proof of Proposition 3.12, we have considered a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\left(L^p(\mathbb{R}^d)\right)^d$ that converges to f in $\mathcal{A}^p$ and an associated sequence of functions $(u_n)_{n\in\mathbb{N}}$, solutions to (3.56) with a gradient in $\left(L^p(\mathbb{R}^d)\right)^d$, that converges in $L^1_{loc}(\mathbb{R}^d)$ to u solution to (3.51) such that $\nabla u \in (\mathcal{A}^p)^d$. We claim that u_n is actually defined, up to an additive constant, by

$$u_n = Tf_n := \int_{\mathbb{R}^d} \nabla_y G_{per}(., y).f_n(y) dy.$$

Using estimate (3.54), we indeed know there exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$,

$$|Tf_n| \leq C \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}} |f_n|(y) dy.$$

Since f_n belongs to $\left(L^p(\mathbb{R}^d)\right)^d$ for every $n \in \mathbb{N}$ and $1 + \dfrac{1}{p^*} = \dfrac{d-1}{d} + \dfrac{1}{p}$, we know from the Hardy-Littlewood-Sobolev inequality that Tf_n belongs to $L^{p^*}(\mathbb{R}^d)$. In addition, the results established in [14, Theorem A] implies that Tf_n is a solution to (3.56) such that $\nabla Tf_n \in \left(L^p(\mathbb{R}^d)\right)^d$. A solution to (3.56) with a gradient in $\left(L^p(\mathbb{R}^d)\right)^d$ being unique up to an additive constant, we conclude that $\nabla u_n = \nabla Tf_n$. We therefore obtain that $u_n = Tf_n$ up to an additive constant. Exactly as in the proof of inequality (3.64), the Hardy-Littlewood-Sobolev inequality gives the existence of a constant $C > 0$ independent of n such that

$$\|\mathcal{M}\left(|u_n - Tf|\right)\|_{L^{p^{**}}(\mathbb{R}^d)} = \|\mathcal{M}\left(|Tf_n - Tf|\right)\|_{L^{p^{**}}(\mathbb{R}^d)} \leq C\|\mathcal{M}\left(|f_n - f|\right)\|_{L^{p^*}(\mathbb{R}^d)} \underset{n\to\infty}{\longrightarrow} 0.$$

Finally, using Proposition 3.3 we know that u_n converges to T_f in $L^1_{loc}(\mathbb{R}^d)$. The uniqueness of the limit in $L^1_{loc}(\mathbb{R}^d)$ allows to conclude that $u = T_f$. □

Lemma 3.5. *Assume $1 < p < \frac{d}{2}$ and let $f \in \left(\mathcal{A}^p \cap L^\infty(\mathbb{R}^d)\right)^d$. Then, the function u defined by (3.63) belongs to $L^\infty(\mathbb{R}^d)$.*

Proof. We begin by considering $x_0 \in \mathbb{R}^d$ and we split $u(x_0)$ in two parts as follows :

$$\begin{aligned} u(x_0) &= \int_{\mathbb{R}^d \setminus B_{3\sqrt{d}}(x_0)} \nabla_y G_{per}(x_0, y) f(y) dy + \int_{B_{3\sqrt{d}}(x_0)} \nabla_y G_{per}(x_0, y) f(y) dy \\ &= I_1(x_0) + I_2(x_0). \end{aligned}$$

We want to bound both $|I_1(x_0)|$ and $|I_2(x_0)|$ uniformly with respect to x_0. Estimate (3.54) gives $C > 0$ independent of x_0 such that :

$$|I_1(x_0)| \leq C \int_{\mathbb{R}^d \setminus B_{3\sqrt{d}}(x_0)} \frac{1}{|x_0 - y|^{d-1}} |f(y)| dy.$$

Since $|Q| = 1$, we have by integrating the previous inequality :

$$|I_1(x_0)| \leq C \int_Q \int_{\mathbb{R}^d \setminus B_{3\sqrt{d}}(x_0 - z)} \frac{1}{|x_0 - y - z|^{d-1}} |f(y+z)| dydz.$$

For every $z \in Q$ and $y \in \mathbb{R}^d \setminus B_{3\sqrt{d}}(x_0 - z)$, since $|z| < \sqrt{d}$ and $|x_0 - y - z| \geq 3\sqrt{d}$, we have $|x_0 - y| = |x_0 - y - z + z| \geq 2\sqrt{d}$. It follows that $\mathbb{R}^d \setminus B_{3\sqrt{d}}(x_0 - z) \subset \mathbb{R}^d \setminus B_{2\sqrt{d}}(x_0)$. We also have $|z| \leq \sqrt{d} \leq \dfrac{1}{2}|x_0 - y|$ which gives $|x_0 - y - z| \geq \frac{1}{2}|x_0 - y|$. Using respectively the Fubini theorem and the Hölder inequality, we deduce :

$$\begin{aligned} |I_1(x_0)| &\leq 2^{d-1} C \int_Q \int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}(x_0)} \frac{1}{|x_0 - y|^{d-1}} |f(y+z)| dydz \\ &\leq 2^{d-1} C \left(\int_{\mathbb{R}^d \setminus B_{2\sqrt{d}}} \frac{1}{|y|^{(d-1)(p^*)'}} dy \right)^{1/(p^*)'} \|\mathcal{M}\left(|f|\right)\|_{L^{p^*}(\mathbb{R}^d)}. \end{aligned}$$

Here we have denoted by $(p^*)' = \dfrac{pd}{d(p-1)+p}$, the conjugate exponent associated with p^*. The integral of the right-hand term being finite as soon as $(d-1)\dfrac{pd}{d(p-1)+p} > d$, that is as soon as $p < \dfrac{d}{2}$, we have finally bounded $|I_1(x_0)|$ uniformly with respect to x_0.

Next, in order to bound $|I_2(x_0)|$, we again use (3.54) :

$$|I_2(x_0)| \leq C \int_{B_{3\sqrt{d}}(x_0)} \frac{1}{|x_0 - y|^{d-1}} |f(y)| dy \leq C \left(\int_{B_{3\sqrt{d}}} \frac{1}{|y|^{d-1}} dy \right) \|f\|_{L^\infty(\mathbb{R}^d)} < \infty.$$

The right-hand side in the latter inequality being independent of x_0, we conclude the proof. □

3.3.3 Well posedness in the non-periodic setting

In this section we return to the non-periodic problem (3.49), when $a = a_{per} + \tilde{a}$ and the perturbation $\tilde{a} \in (\mathcal{A}^p)^d$ of the periodic geometry does not necessarily vanish. We assume it satisfies the regularity assumption (3.48). We again adapt a method introduced in [27] which consists to, first, establish the continuity of operator $\nabla(-\operatorname{div} a\nabla)^{-1}\operatorname{div}$ from $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, and, second, to use both this continuity result and a connectedness argument to extend the results established in the periodic case $a = a_{per}$ to the general case. In order to show the continuity result (established in Lemma 3.8 below), we need to first introduce a preliminary result when the perturbation $\tilde{a}$ is sufficiently small and next a uniqueness result regarding the solutions u to (3.49) such that $\nabla u \in (\mathcal{A}^p)^d$, respectively in Lemma 3.6 and Lemma 3.7.

Lemma 3.6. *Let a_{per} be a Q-periodic matrix-valued function satisfying (3.47) and (3.48). Then, for every $r \in]1, +\infty[$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, every $f \in \left(L^r(\mathbb{R}^d)\right)^d$ and every matrix-valued coefficient $\tilde{a} \in \left(L^\infty(\mathbb{R}^d)\right)^{d\times d}$ satisfying $\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} < \varepsilon$, equation (3.49) with $a = a_{per} + \tilde{a}$ admits a unique (up to an additive constant) solution u such that $\nabla u \in \left(L^r(\mathbb{R}^d)\right)^d$.*

Proof. We begin by remarking that the existence and uniqueness of such a solution is equivalent to the existence and uniqueness of a solution u to $-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(f + \tilde{a}\nabla u)$. We apply a fixed-point method on $\mathbb{R}^d$, considering $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 = 0$ and for every $n \in \mathbb{N}$, u_{n+1} is solution to :

$$-\operatorname{div}(a_{per}\nabla u_{n+1}) = \operatorname{div}(f + \tilde{a}\nabla u_n) \quad \text{on } \mathbb{R}^d, \tag{3.65}$$

such that $\nabla u_{n+1} \in \left(L^r(\mathbb{R}^d)\right)^d$. Since, for every $n \in \mathbb{N}$, the function $F_n := f + \tilde{a}\nabla u_n$ belongs to $\left(L^r(\mathbb{R}^d)\right)^d$, the results of [14, Theorem A] show the sequence u_n is well-defined. Since, likewise for every $n \in \mathbb{N}^*$, the function $u_{n+1} - u_n$ is solution to $-\operatorname{div}(a_{per}(\nabla(u_{n+1} - u_n))) = \operatorname{div}(\tilde{a}(\nabla u_n - \nabla u_{n-1}))$, the result of [14, Theorem A] also yields a constant $C > 0$ independent of $\tilde{a}$ and n such that

$$\begin{aligned}\|\nabla u_{n+1} - \nabla u_n\|_{L^r(\mathbb{R}^d)} &\leq C\|\tilde{a}(\nabla u_n - \nabla u_{n-1})\|_{L^r(\mathbb{R}^d)} \\ &\leq C\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)}\|\nabla u_n - \nabla u_{n-1}\|_{L^r(\mathbb{R}^d)}.\end{aligned} \tag{3.66}$$

Therefore, if

$$\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} < \frac{1}{C}, \tag{3.67}$$

the sequence ∇u_n is a Cauchy sequence in $\left(L^r(\mathbb{R}^d)\right)^d$ and it converges to a gradient ∇u in $\left(L^r(\mathbb{R}^d)\right)^d$. Passing to the limit in the distribution sense in (3.65), we obtain that ∇u is solution to (3.49). To prove uniqueness, we consider u^1 and u^2 two solutions to (3.49) such that ∇u^1 and ∇u^2 belongs to $\left(L^r(\mathbb{R}^d)\right)^d$ and we have that $u^1 - u^2$ is solution to $-\operatorname{div}(a_{per}(\nabla(u^1 - u^2))) = \operatorname{div}(\tilde{a}(\nabla u^1 - \nabla u^2))$. Estimate (3.66) implies

$$\|\nabla u^1 - \nabla u^2\|_{L^r(\mathbb{R}^d)} \leq C\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)}\|\nabla u^1 - \nabla u^2\|_{L^r(\mathbb{R}^d)},$$

which, given (3.67), shows $\nabla u^1 = \nabla u^2$. $\square$

Lemma 3.7. *Let $a_{per} \in \left(L^2_{per}(\mathbb{R}^d)\right)^{d\times d}$ and $\tilde{a} \in (\mathcal{A}^p)^{d\times d}$ for $p \in]1, d[$. Assume that $a = a_{per} + \tilde{a}$ satisfies (3.2), (3.47) and (3.48). Let $u \in L^1_{loc}(\mathbb{R}^d)$ solution in $\mathcal{D}'(\mathbb{R}^d)$ to :*

$$-\operatorname{div}(a\nabla u) = 0 \quad on\ \mathbb{R}^d, \tag{3.68}$$

such that $\nabla u \in \left(\mathcal{A}^p \cap L^\infty(\mathbb{R}^d)\right)^d$. Then $\nabla u = 0$.

Proof. **Step 1 : Truncation of** $\tilde{a}$. For every $R > 0$, we consider $\chi_R \in \mathcal{D}'(\mathbb{R}^d)$ a non-negative function such that $\mathrm{Supp}(\chi_R) \subset B_{R+1}$, $\chi_{R|B_R} \equiv 1$, $\|\chi_R\|_{L^\infty(\mathbb{R}^d)} = 1$ and $\|\nabla\chi_R\|_{L^\infty(\mathbb{R}^d)} \leq C_0$, where $C_0 > 0$ is a constant independent of R. In the sequel, we denote $\tilde{a}_R = \chi_R\tilde{a}$ and $\tilde{a}_R^C = (1-\chi_R)\tilde{a}$. We next consider the following equation :

$$-\operatorname{div}((a_{per} + \tilde{a}_R^C)\nabla v) = \operatorname{div}(\tilde{a}_R\nabla u) \quad \text{on } \mathbb{R}^d. \tag{3.69}$$

Since u is solution to (3.68), $v \equiv u$ is clearly solution to (3.69).

Step 2 : Study of a particular solution to (3.69). Since $\tilde{a}_R$ is compactly supported and $\nabla u \in \left(L^\infty(\mathbb{R}^d)\right)^d$, the function $\tilde{a}_R\nabla u$ belongs to $\left(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\right)^d$. In addition, using Corollary 3.2, we know that $\tilde{a}(x)$ converges to 0 when $|x| \to \infty$ and for every $\varepsilon > 0$, there exists $R_0 > 0$ such that for every $R > R_0$, we have :

$$\left\|\tilde{a}_R^C\right\|_{L^\infty(\mathbb{R}^d)} < \varepsilon. \tag{3.70}$$

Thus, using Lemma 3.6, we obtain that for every R large enough, there exists a solution v_R to (3.69) such that $\nabla v_R \in \left(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\right)^d$. In addition, since ∇v_R belongs to $\left(L^2(\mathbb{R}^d)\right)^d$, we have for every $x \in \mathbb{R}^d$:

$$\int_{Q+x} |\nabla v_R| \leq \left(\int_{Q+x} |\nabla v_R|^2\right)^{1/2} \leq \|\nabla v_R\|_{L^2(\mathbb{R}^d)}.$$

Consequently, $\mathcal{M}(|\nabla v_R|)$ is uniformly bounded with respect to x and belongs to $L^2_{unif}(\mathbb{R}^d)$. Since $\tilde{a}_R\nabla u \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, the regularity result of Proposition 3.11 gives that ∇v_R belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. We next prove the existence of $R > 0$ such that $\nabla u = \nabla v_R$.

Step 3 : Existence of R **such that** $\nabla u = \nabla v_R$. We know that $\nabla u \in \left(L^\infty(\mathbb{R}^d)\right)^d$ and Proposition 3.11 therefore shows that ∇u belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. In the sequel, we denote $w = u - v_R$. Since $\nabla v_R \in \left(L^p(\mathbb{R}^d)\right)^d \subset (\mathcal{A}^p)^d$, we have $\nabla w \in (\mathcal{A}^p)^d$. In addition, w is solution to $-\operatorname{div}((a_{per} + \tilde{a}_R^C)\nabla w) = 0$ or equivalently, a solution to :

$$-\operatorname{div}(a_{per}\nabla w) = \operatorname{div}(\tilde{a}_R^C\nabla w) \quad \text{on } \mathbb{R}^d. \tag{3.71}$$

We have $\tilde{a}_R^C \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$ and $\nabla w \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, a short calculation allows to show that $\tilde{a}_R^C\nabla w$ also belongs to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. We next remark that for every $\alpha \in]0,1[$, we have $\mathcal{C}^{0,\alpha}(\mathbb{R}^d) \subset \mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)$. We apply the estimate of Corollary 3.5 to equation (3.71) and we obtain the existence of a constant $C > 0$ independent of w, R and $\tilde{a}$ such that :

$$\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} \leq C\left(\left\|\mathcal{M}\left(|\tilde{a}_R^C\nabla w|\right)\right\|_{L^{p^*}(\mathbb{R}^d)} + \|\tilde{a}_R^C\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)}\right). \tag{3.72}$$

Our aim is now to estimate each norm of the right-hand side in the previous inequality. Let $\varepsilon > 0$. Since $\tilde{a} \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$ and $\mathcal{M}(|\tilde{a}|) \in L^{p^*}(\mathbb{R}^d)$, there exists $R_1 > 0$ such that for every $R > R_1$ we have $\|\mathcal{M}(|\tilde{a}|)\|_{L^{p^*}(\mathbb{R}^d\setminus B_R)} \leq \varepsilon$. It follows :

$$\left\|\mathcal{M}\left(\left|\tilde{a}_R^C\nabla w\right|\right)\right\|_{L^{p^*}(\mathbb{R}^d)} \leq \|\mathcal{M}\left(|\tilde{a}\nabla w|\right)\|_{L^{p^*}(\mathbb{R}^d\setminus B_R)} \leq \|\mathcal{M}\left(|\tilde{a}|\right)\|_{L^{p^*}(\mathbb{R}^d\setminus B_R)}\|\nabla w\|_{L^\infty(\mathbb{R}^d)}.$$

If we therefore consider $R \geq R_1$, we obtain :

$$\left\|\mathcal{M}\left(\left|\tilde{a}_R^C\nabla w\right|\right)\right\|_{L^{p^*}(\mathbb{R}^d)} \leq \varepsilon\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)}. \tag{3.73}$$

In the sequel, for $\beta \in]0,1[$ we denote $[f]_{\mathcal{C}^{0,\beta}(\mathbb{R}^d)} = \sup\limits_{x,y\in\mathbb{R}^d, x\neq y} \dfrac{|f(x)-f(y)|}{|x-y|^\beta}$. We next remark that :

$$\left\|\tilde{a}_R^C\nabla w\right\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} \leq 2\left\|\tilde{a}_R^C\right\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)}\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)}. \tag{3.74}$$

Since $\tilde{a} \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$, for every $x, y \in \mathbb{R}^d$ we have using (3.70) :

$$\left|\tilde{a}_R^C(x) - \tilde{a}_R^C(y)\right| \leq \sqrt{2}\|\tilde{a}_R^C\|_{L^\infty(\mathbb{R}^d)}^{1/2}\left|\tilde{a}_R^C(x) - \tilde{a}_R^C(y)\right|^{1/2} \leq \sqrt{2\varepsilon}\left\|\tilde{a}_R^C\right\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}^{1/2}|x-y|^{\alpha/2}.$$

In addition, for every $R > 0$, we have :

$$\begin{aligned}\left|\tilde{a}_R^C(x) - \tilde{a}_R^C(y)\right| &\leq |\tilde{a}(x)-\tilde{a}(y)|\,|1-\chi_R(x)| + |\chi_R(x)-\chi_R(y)|\,|\tilde{a}(y)| \\ &\leq \left(\|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\|1-\chi_R\|_{L^\infty(\mathbb{R}^d)} + \|\chi_R\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\|\tilde{a}\|_{L^\infty(\mathbb{R}^d)}\right)|x-y|^\alpha.\end{aligned}$$

Since χ_R and $\nabla\chi_R$ are uniformly bounded with respect to R in $L^\infty(\mathbb{R}^d)$, we deduce there exists a constant $C > 0$ independent of R such that $\left\|\tilde{a}_R^C\right\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C$. It follows that, if $R > R_0$, where R_0 is such that (3.70) is satisfied for every $R > R_0$, then $[\tilde{a}_R^C]_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} \leq C\sqrt{\varepsilon}$. Using (3.70), we deduce that $\|\tilde{a}_R^C\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} \leq C\sqrt{\varepsilon}$, and as a consequence of (3.74),

$$\left\|\tilde{a}_R^C\nabla w\right\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} \leq 2C\sqrt{\varepsilon}\,\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)}. \tag{3.75}$$

Finally, if $R > \max(R_0, R_1)$, we can insert (3.73) and (3.75) in (3.72), and we obtain the existence of a constant $C > 0$ independent of R and ε such that :

$$\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} \leq C\sqrt{\varepsilon}\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)}.$$

If ε is small enough, we have $C\sqrt{\varepsilon} < 1$, and we obtain $\|\nabla w\|_{\mathcal{C}^{0,\alpha/2}(\mathbb{R}^d)} = 0$. We conclude that $\nabla w = 0$, that is $\nabla u = \nabla v_R \in \left(L^2(\mathbb{R}^d)\right)^d$.

Step 4 : Conclusion. In the previous step we have established $\nabla u = \nabla v_R \in \left(L^2(\mathbb{R}^d)\right)^d$. Since $p < d$ and a is uniformly bounded and elliptic according to assumptions (3.2)-(3.3), the result of uniqueness of [25, Lemma 1] for solution u to (3.68) with a gradient in $\left(L^p(\mathbb{R}^d)\right)^d$ shows that $\nabla u = 0$. □

We are now in position to establish the continuity of the operator $\nabla\left(-\operatorname{div}(a\nabla.)\right)^{-1}\operatorname{div}$ from $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.

Lemma 3.8. *Let $a_{per} \in \left(L^2_{per}(\mathbb{R}^d)\right)^{d\times d}$ and $\tilde{a} \in (\mathcal{A}^p)^{d\times d}$ for $p \in]1,d[$. Assume that $a = a_{per} + \tilde{a}$ satisfies (3.2), (3.47) and (3.48). There exists a constant $C > 0$ such that for every $f \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and u solution to $-\operatorname{div}(a\nabla u) = \operatorname{div}(f)$ on $\mathbb{R}^d$ with $\nabla u \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, we have :*

$$\|\nabla u\|_{\mathcal{A}^p} + \|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C \left(\|f\|_{\mathcal{A}^p} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).$$

Proof. We argue by contradiction. We assume the existence of two sequences u_n and f_n such that for every $n \in \mathbb{N}$ we have $\nabla u_n, f_n \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and :

$$-\operatorname{div}(a\nabla u_n) = \operatorname{div}(f_n), \tag{3.76}$$

$$\|\nabla u_n\|_{\mathcal{A}^p} + \|\nabla u_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 1, \tag{3.77}$$

$$\lim_{n\to\infty} \|f_n\|_{\mathcal{A}^p} + \|f_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 0. \tag{3.78}$$

Since ∇u_n is bounded uniformly with respect to n for the topology of $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, the Arzela-Ascoli theorem shows the uniform convergence of ∇u_n (up to an extraction) on every compact of $\mathbb{R}^d$ to a gradient $\nabla u \in \left(L^\infty(\mathbb{R}^d)\right)^d$. Consequently, if we consider the limit in (3.76) when $n \to \infty$, we obtain that ∇u is solution to $-\operatorname{div}(a\nabla u) = 0 \quad \text{on } \mathbb{R}^d$.

We next claim that ∇u belongs to $(\mathcal{A}^p)^d$. Indeed, since $\delta\nabla u_n$ is uniformly bounded with respect to n in $\left(L^p(\mathbb{R}^d)\right)^{d\times d}$ for $p \in]1,d[$, it weakly converges (up to an extraction) in this space and its limit is equal to $\delta\nabla u$ due to the uniqueness of the limit in the distribution sense. Moreover, since $p > 1$, the L^p norm is lower semi-continuous and we have

$$\|\delta\nabla u\|_{L^p(\mathbb{R}^d)} \leq \liminf_{n\to\infty} \|\delta\nabla u_n\|_{L^p(\mathbb{R}^d)} = 1.$$

We also know that ∇u_n uniformly converges on every compact of $\mathbb{R}^d$, and consequently, that $\mathcal{M}(|\nabla u_n|)$ converges pointwise to $\mathcal{M}(|\nabla u|)$. Using the Fatou lemma, we obtain :

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{M}(|\nabla u|)(z)|^{p^*} dz &= \int_{\mathbb{R}^d} \liminf_{n\to\infty} |\mathcal{M}(|\nabla u_n|)(z)|^{p^*} dz \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} |\mathcal{M}(|\nabla u_n|)(z)|^{p^*} dz \leq 1.\end{aligned}$$

It follows that ∇u belongs to $\left(\mathcal{A}^p \cap L^\infty(\mathbb{R}^d)\right)^d$ and the uniqueness result of Lemma 3.7 implies that $\nabla u = 0$. Our aim is now to prove that $\lim\limits_{n\to\infty} \|\nabla u_n\|_{\mathcal{A}^p} + \|\nabla u_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 0$, in order to reach a contradiction. We first remark that (3.76) is equivalent to :

$$-\operatorname{div}(a_{per}\nabla u_n) = \operatorname{div}(\tilde{a}\nabla u_n + f_n) \quad \text{on } \mathbb{R}^d. \tag{3.79}$$

Since ∇u_n and $\tilde{a}$ both belong to L^∞, we can easily show that $\tilde{a}\nabla u_n$ belongs to $\mathcal{A}^p$. Consequently, Proposition 3.12 gives the existence of a constant $C > 0$ independent of n such that :

$$\|\nabla u_n\|_{\mathcal{A}^p} \leq C \left(\|\tilde{a}\nabla u_n\|_{\mathcal{A}^p} + \|f_n\|_{\mathcal{A}^p}\right). \tag{3.80}$$

We fix $\varepsilon > 0$. Since $\tilde{a}$ is Hölder-continuous according to assumption (3.48), Corollary 3.2 gives the existence of $R_1 > 0$ such that

$$\|\tilde{a}\|_{L^\infty(\mathbb{R}^d \setminus B_{R_1})} < \frac{\varepsilon}{2}.$$

In addition, since $\delta\tilde{a} \in \left(L^p(\mathbb{R}^d)\right)^{d\times d\times d}$ and ∇u_n is uniformly bounded for the norm of L^∞ with respect to n, there exists $R_2 > 0$ such that for every $n \in \mathbb{N}$:

$$\|\delta_i\tilde{a}\|_{L^p(\mathbb{R}^d\setminus B_{R_2})}\|\nabla u_n\|_{L^\infty(\mathbb{R}^d)} < \frac{\varepsilon}{2}.$$

We next denote $R = \max(R_1, R_2)$. We have proved that ∇u_n uniformly converges to 0 on every compact of $\mathbb{R}^d$. Therefore, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have :

$$\|\nabla u_n\|_{L^\infty(B_{R+1})}\left(\|\tilde{a}\|_{L^p(B_R)} + 2\|\tilde{a}\|_{\mathcal{A}^p}\right) < \frac{\varepsilon}{2}.$$

For every $n \geq N$ and $i \in \{1, ..., d\}$, we have :

$$\|\delta_i\left(\tilde{a}\nabla u_n\right)\|_{L^p(\mathbb{R}^d)} \leq \|\delta_i\tilde{a}\tau_{e_i}\nabla u_n\|_{L^p(\mathbb{R}^d)} + \|\tilde{a}\delta_i\nabla u_n\|_{L^p(\mathbb{R}^d)}. \tag{3.81}$$

We next prove that the right-hand side of the previous inequality converges to 0 when $n \to \infty$. We have :

$$\begin{aligned}\|\delta_i\tilde{a}\tau_{e_i}\nabla u_n\|_{L^p(\mathbb{R}^d)} &\leq \|\delta_i\tilde{a}\tau_{e_i}\nabla u_n\|_{L^p(B_R)} + \|\delta_i\tilde{a}\tau_{e_i}\nabla u_n\|_{L^p(\mathbb{R}^d\setminus B_R)} \\ &\leq \|\tilde{a}\|_{\mathcal{A}^p}\|\nabla u_n\|_{L^\infty(B_{R+1})} + \|\delta_i\tilde{a}\|_{L^p(\mathbb{R}^d\setminus B_R)}\|\nabla u_n\|_{L^\infty(\mathbb{R}^d)} < \varepsilon.\end{aligned}$$

Since the parameter ε can be chosen arbitrarily small, it follows that

$$\lim_{n\to\infty}\|\delta_i\tilde{a}\tau_{e_i}\nabla u_n\|_{L^p(\mathbb{R}^d)} = 0.$$

Similarly :

$$\|\tilde{a}\delta_i\nabla u_n\|_{L^p(\mathbb{R}^d)} \leq 2\|\tilde{a}\|_{L^p(B_R)}\|\nabla u_n\|_{L^\infty(B_{R+1})} + \|\tilde{a}\|_{L^\infty(\mathbb{R}^d\setminus B_R)}\|\nabla u_n\|_{\mathcal{A}^p} < \varepsilon,$$

and $\lim\limits_{n\to\infty}\|\tilde{a}\delta_i\nabla u_n\|_{L^p(\mathbb{R}^d)} = 0$. Using (3.81), we therefore obtain that $\|\tilde{a}\nabla u_n\|_{\mathcal{A}^p}$ converges to 0 when $n \to \infty$. In addition, assumption (3.78) ensures that f_n converges to 0 in $(\mathcal{A}^p)^d$ and, according to inequality (3.80), we obtain that $\lim\limits_{n\to\infty}\|\nabla u_n\|_{\mathcal{A}^p} = 0$.

The last step of the proof consists in showing that $\lim\limits_{n\to\infty}\|\nabla u_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 0$. Since ∇u_n is solution to (3.79), the estimate established in Proposition 3.12 shows the existence of $C > 0$ such that for every n,

$$\|\nabla u_n\|_{L^2_{unif}} \leq C\left(\|\tilde{a}\nabla u_n\|_{\mathcal{A}^p} + \|\tilde{a}\nabla u_n\|_{L^\infty(\mathbb{R}^d)}\right).$$

A method similar to that presented above allows us to show that $\|\nabla u_n\|_{L^2_{unif}}$ converges to 0 when $n \to \infty$. The regularity estimate of Corollary 3.5 and assumption (3.78) therefore show that $\lim\limits_{n\to\infty}\|\nabla u_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 0$. We finally reach a contradiction with (3.77) and we conclude the proof. □

We are in position to prove the main result of this section, that is the existence and uniqueness of a solution to (3.49).

Lemma 3.9. *Let $a_{per} \in \left(L^2_{per}(\mathbb{R}^d)\right)^{d\times d}$ and $\tilde{a} \in (\mathcal{A}^p)^{d\times d}$ for $p \in]1, d[$. Assume that the coefficient $a = a_{per} + \tilde{a}$ satisfies (3.2), (3.47) and (3.48). Let $f \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, then, there exists an unique, up to an additive constant, function $u \in L^1_{loc}(\mathbb{R}^d)$ solution in $\mathcal{D}'(\mathbb{R}^d)$ to (3.49) such that $\nabla u \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.*

Proof. We use here a method introduced in [27, proof of Proposition 2.1] using the connectedness of the set $[0,1]$. In the sequel, we denote $a_s = a_{per} + s\tilde{a}$ for every $s \in [0,1]$ and we consider the following assertion $\mathcal{P}(s)$ = "for every $f \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, there exists a unique, up to an additive constant, function $u \in L^1_{loc}(\mathbb{R}^d)$ solution to $-\operatorname{div}(a_s \nabla u) = \operatorname{div}(f)$, in $\mathcal{D}'(\mathbb{R}^d)$ and such that $\nabla u \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$". We define the set

$$\mathcal{I} = \{s \in [0,1] \mid \mathcal{P}(s) \text{ is true}\}.$$

Our aim is to prove that $s = 1$ belongs to $\mathcal{I}$ establishing that $\mathcal{I}$ is non empty, open and closed for the topology of $[0,1]$.

Step 1 : $\mathcal{I}$ is non empty. The results of Proposition 3.12 and Corollary 3.5 show that $s = 0$ belongs to $\mathcal{I}$.

Step 2 : $\mathcal{I}$ is open. We assume there exists $s \in \mathcal{I}$ and we will show the existence of $\varepsilon > 0$ such that $[s, s+\varepsilon]$ is included in $\mathcal{I}$, that is that there exists u solution to :

$$-\operatorname{div}((a_{per} + (s+\varepsilon)\tilde{a})\nabla u) = \operatorname{div}(f), \tag{3.82}$$

when ε is sufficiently small. We first remark that the above equation is equivalent to :

$$-\operatorname{div}((a_{per} + s\tilde{a})\nabla u) = \operatorname{div}(\varepsilon\tilde{a}\nabla u + f).$$

A simple calculation allows to show that if ∇u belongs to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, then $\varepsilon\tilde{a}\nabla u$ also belongs to this space. Therefore, the existence and uniqueness of a solution u to (3.82) is equivalent to the existence and uniqueness of a solution to the fixed-point problem

$$\nabla u = \Phi_s(\varepsilon\tilde{a}\nabla u + f),$$

where Φ_s is the linear application $\nabla\left(-\operatorname{div}(a_s\nabla .)\right)^{-1}\operatorname{div}$ defined from $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Since $s \in \mathcal{I}$, Φ_s is well defined and the result of Lemma 3.8 ensures that it is continuous for the norm $\|.\|_{\mathcal{A}^p} + \|.\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}$. We claim that the application $g \to \Phi_s(\varepsilon\tilde{a}g + f)$ is a contraction if ε is sufficiently small. First, if g_1 and g_2 are two functions of $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and if we denote $\nabla v_1 = \Phi_s(\varepsilon\tilde{a}g_1 + f)$ and $\nabla v_2 = \Phi_s(\varepsilon\tilde{a}g_2 + f)$, $v_1 - v_2$ is solution to

$$-\operatorname{div}(a_s\nabla(v_1 - v_2)) = \operatorname{div}(\varepsilon\tilde{a}(g_1 - g_2)).$$

The continuity estimate of Lemma 3.8 therefore shows the existence of a constant $C > 0$ independent of g_1, g_2 and ε such that :

$$\|\nabla v_1 - \nabla v_2\|_{\mathcal{A}^p} + \|\nabla v_1 - \nabla v_2\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C\varepsilon\left(\|\tilde{a}(g_1 - g_2)\|_{\mathcal{A}^p} + \|\tilde{a}(g_1 - g_2)\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).$$

In addition, we have :

$$\|\tilde{a}(g_1 - g_2)\|_{\mathcal{A}^p} \leq \left(\|\tilde{a}\|_{\mathcal{A}^p} + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right)\left(\|g_1 - g_2\|_{\mathcal{A}^p} + \|g_1 - g_2\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right),$$

and :

$$\|\tilde{a}(g_1 - g_2)\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq 2\|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\|g_1 - g_2\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}.$$

Thus, if ε satisfies $3C\left(\|\tilde{a}\|_{\mathcal{A}^p} + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right)\varepsilon < 1$, the operator $g \to \Phi_s(\varepsilon\tilde{a}g + f)$ is a contraction. Since $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ equipped with the associated norm is a Banach space, we can use the Banach fixed-point theorem. We deduce the existence and the uniqueness of a solution to (3.82).

Step 3 : $\mathcal{I}$ is closed. We assume the existence of a sequence (s_n) of $\mathcal{I}$ that converges to $s \in [0,1]$. We want to show that s belongs to $\mathcal{I}$. Let $f \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. By assumption, for every $n \in \mathbb{N}$, there exists $u_n \in L^1_{loc}(\mathbb{R}^d)$ solution to :

$$-\operatorname{div}(a_{s_n}\nabla u_n) = \operatorname{div}(f), \tag{3.83}$$

such that $\nabla u_n \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. For every $n \in \mathbb{N}$, we use Lemma 3.8 to obtain the existence of a constant $C_n > 0$ such that :

$$\|\nabla u_n\|_{\mathcal{A}^p} + \|\nabla u_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C_n \left(\|f\|_{\mathcal{A}^p} + \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).$$

We first assume that C_n is uniformly bounded with respect to n. in this case, ∇u_n is uniformly bounded with respect to n in $\left(L^\infty(\mathbb{R}^d)\right)^d$. Up to an extraction, the sequences ∇u_n converges to a gradient ∇u for the weak-* topology of L^∞. In addition, $a_{s_n} = a_{per} + s_n\tilde{a}$ uniformly converges to a_s. We can consider the limit when $n \to \infty$ in (3.83) and we obtain that u is solution to $-\operatorname{div}(a_s\nabla u) = \operatorname{div}(f)$.

We next show that ∇u_n is a Cauchy sequence in $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ in order to conclude that ∇u also belongs to this space. Indeed, for every $m, n \in \mathbb{N}$, $u_n - u_m$ is solution to :

$$-\operatorname{div}(a_s(\nabla u_n - \nabla u_m)) = \operatorname{div}((a_{s_n} - a_s)\nabla u_n - (a_{s_m} - a_s)\nabla u_m).$$

Since for every $n \in \mathbb{N}$, we have $(a_{s_n} - a_s)\nabla u_n \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, Lemma 3.8 gives the existence of $\tilde{C} > 0$ independent of n and m such that :

$$\begin{aligned}\|\nabla u_n - \nabla u_m\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} &\leq \tilde{C}\left(\|(a_{s_n} - a_s)\nabla u_n\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} + \|(a_{s_m} - a_s)\nabla u_m\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) \\ &= \left(|s_n - s|\|\tilde{a}\nabla u_n\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} + |s_m - s|\|\tilde{a}\nabla u_m\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right),\end{aligned}$$

where we have denoted $\|.\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = \|.\|_{\mathcal{A}^p} + \|.\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}$. Since for every n, ∇u_n is bounded in $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ uniformly with respect to n and since s_n converges to s, we deduce that ∇u_n is a Cauchy sequence in $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. This space being a Banach space, we have $\nabla u \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. The uniqueness result being established in Lemma 3.7, we therefore obtain that $s \in \mathcal{I}$.

To conclude this step, we have to show that C_n is uniformly bounded with respect to n. To this end, we assume the existence of two sequences (f_n) and (∇u_n) of $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ such that $\lim\limits_{n\to\infty} \|f_n\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 0$, $\|\nabla u_n\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 1$ and $-\operatorname{div}(a_{s_n}\nabla u_n) = \operatorname{div}(f_n)$ on $\mathbb{R}^d$. We remark that for every n, we have $-\operatorname{div}(a_s\nabla u_n) = \operatorname{div}((s - s_n)\tilde{a}\nabla u_n + f_n)$. Since ∇u_n is bounded for the norm of $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ and that s_n converges to s, we deduce that $\lim\limits_{n\to\infty} \|(s - s_n)\tilde{a}\nabla u_n\|_{\mathcal{A}^p\cap\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = 0$. We conclude the proof exactly as in the proof of Lemma 3.8.

Step 4 : Conclusion. We have established that $\mathcal{I}$ is non-empty, open, and closed for the topology of $[0,1]$. The connectedness of this set therefore gives $\mathcal{I} = [0,1]$. In particular, $1 \in \mathcal{I}$ and we can conclude. $\square$

Proposition 3.14. *Assume $1 < p < \frac{d}{2}$. Then, under the assumptions of Lemma 3.9, the unique solution u to (3.49) such that $\nabla u \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ satisfies $u \in L^\infty(\mathbb{R}^d)$.*

Proof. We remark that (3.49) is equivalent to $-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(g)$ where $g := f + \tilde{a}\nabla u$. Since $\tilde{a}$ and ∇u both belongs to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, we can easily show that g also belongs to $\left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Given $1 < p < \frac{d}{2}$, we conclude that $u \in L^\infty(\mathbb{R}^d)$ using both Lemma 3.5 and the uniqueness result of Lemma 3.3. □

3.3.4 Existence of an adapted corrector and homogenization results

The well-posedness of (3.49) now allows for a proof of Theorem 3.1. We first establish the existence of a corrector adapted to our particular problem (3.18) and we next use it to identify the limit of the sequence u^ε, solution to (3.1).

Proof of Theorem 3.1. As a consequence of Proposition 3.10, we have that a_{per} and $\tilde{a}$ satisfy the properties of ellipticity (3.47) and regularity (3.48). We next remark that (3.18) is equivalent to

$$-\operatorname{div}(a\nabla \tilde{w}_q) = \operatorname{div}(\tilde{a}(q + \nabla w_{per,q})),$$

and we denote $f = \tilde{a}(q + \nabla w_{per,q})$. Since a_{per} belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$, a classical regularity property of elliptic equations shows that $\nabla w_{per,q}$ belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Using the periodicity of $\nabla w_{per,q}$, we also have $f \in (\mathcal{A}^p)^d$. In addition, since $\tilde{a}$ belongs to $\left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$, we deduce that $f \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$. Existence and uniqueness (up to an additive constant) of $\tilde{w}_q$ solution to (3.18) such that $\nabla \tilde{w}_q \in \left(\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$ are therefore implied by Lemma 3.9. The strict sub-linearity at infinity of $\tilde{w}_q$ is a consequence of Proposition 3.6.

Next we denote $w = (w_{per,e_i} + \tilde{w}_{e_i})_{i\in\{1,\dots,d\}}$, where $\tilde{w}_{e_i}$ is the corrector solution to (3.18) when $q = e_i$. The general homogenization theory for equations in divergence form (see for example [90, Chapter 6, Chapter 13]) shows that, up to an extraction, the sequence u^ε converges (strongly in L^2, weakly in H^1) to a function $u^* \in H^1_0(\Omega)$ solution to

$$-\operatorname{div}(a^*\nabla u^*) = f.$$

For every $1 \leq i,j \leq d$, the homogenized matrix-valued coefficient a^* associated with a is given by

$$[a^*]_{i,j} = \text{weak} \lim_{\varepsilon\to 0} a(./\varepsilon)(I_d + \nabla w(./\varepsilon)),$$

where the weak limit is considered in $L^2(\Omega)^{d\times d}$. Since $\tilde{a}$ and ∇w_{e_i} both belong to $\mathcal{A}^p \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, Corollary 3.3 implies the convergence to 0 of $|\tilde{a}|(./\varepsilon)$ and $|\nabla \tilde{w}_{e_i}|(./\varepsilon)$ when $\varepsilon \to 0$ for the weak-* topology of $L^\infty(\Omega)$. In particular, we have for every $\varphi \in (\mathcal{D}(\mathbb{R}))^d$:

$$\left| \int_\Omega \tilde{a}(x/\varepsilon)\nabla \tilde{w}_{e_i}(x/\varepsilon).\varphi(x)dx \right| \leq \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \int_\Omega |\nabla \tilde{w}_{e_i}(x/\varepsilon)|.|\varphi(x)|dx \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

It follows $\text{weak} \lim\limits_{\varepsilon\to 0} \tilde{a}(./\varepsilon)\nabla \tilde{w}_{e_i}(./\varepsilon) = 0$. We similarly have $\text{weak} \lim\limits_{\varepsilon\to 0} \tilde{a}(./\varepsilon)\nabla w_{per,e_i}(./\varepsilon) = 0$ and $\text{weak} \lim\limits_{\varepsilon\to 0} a_{per}(./\varepsilon)\nabla \tilde{w}_{e_i}(./\varepsilon) = 0$. Since $a = a_{per} + \tilde{a}$ and $w_{e_i} = w_{per,e_i} + \tilde{w}_{e_i}$, we obtain :

$$[a^*]_{i,j} = \text{weak} \lim_{\varepsilon\to 0} a_{per}(./\varepsilon)(I_d + \nabla w_{per}(./\varepsilon)) = \left[a^*_{per}\right]_{i,j}.$$

This limit being independent of the extraction, we deduce that the whole sequence u^ε converges to u^* and $a^* = a^*_{per}$. $\square$

We conclude this section with a discussion regarding the rates of convergence of u^ε to u^*. Similarly to the periodic case and in order to make precise the behavior of ∇u^ε, we can consider a sequence of remainders R^ε using the adapted corrector of Theorem 3.1 and defined by

$$R^\varepsilon(x) = u^\varepsilon(x) - u^*(x) - \varepsilon \sum_{j=1}^{d} \partial_j u^*(x) w_{e_j}\left(\frac{x}{\varepsilon}\right).$$

The homogenization results we have established in Theorem 3.1 and, more generally, the results of Section 3.3, allow to use the general results of [20], which performs a study of homogenization problem (3.1) under general assumptions, provided one has sufficient regularity of the coefficient a and the existence of a corrector with a prescribed rate of strict sub-linearity at infinity. More precisely, if $r \geq 2$, $f \in L^r(\Omega)$ and Ω is a $\mathcal{C}^{2,1}$ domain, [20, Theorem 1.5] shows estimates of the form :

$$\|\nabla R^\varepsilon\|_{L^r(\Omega_1)} \leq C\varepsilon^\beta \|f\|_{L^r(\Omega)}, \tag{3.84}$$

for every $\Omega_1 \subset\subset \Omega$ and where the value of β is related to the decreasing rate of $\varepsilon w_{e_i}\left(\frac{\cdot}{\varepsilon}\right)$. In our particular setting, we obtain (3.84) with $\beta = \mu$, where

$$\mu := \begin{cases} \dfrac{d}{p^*} & \text{if } p > \dfrac{d}{2}, \\ 1 & \text{if } p < \dfrac{d}{2}, \end{cases} \tag{3.85}$$

is obtained as a direct consequence of Propositions 3.6 and 3.14. On the other hand, We recall that Proposition 3.7 shows that $\tilde{a}$ also belongs to $\left(L^q(\mathbb{R}^d)\right)^{d\times d}$ for $q = \dfrac{p^*(\alpha+d)-d}{\alpha}$ and the results of [27, 26, 25] and [20, Theorem 1.2] give (3.84) with

$$\beta = \nu := \begin{cases} \dfrac{d}{q} & \text{if } q > d, \\ 1 & \text{if } q < d. \end{cases}$$

Since $q > p^*$, a simple calculation shows that μ is always larger than ν and the theoretical convergence rates are significantly improved when $\delta a \in \left(L^p(\mathbb{R}^d)\right)^{d\times d}$. We point out that this improvement is particularly relevant if one is interested in fine convergence properties of u^ε, that is for the topology of $W^{1,r}$ when r is large, at which scale the local perturbations of $\mathcal{A}^p$ affect the periodic background. This comparison therefore shows the interest of the specific study performed in the present article.

3.4 The homogenization problem when $p \geq d$

We devote this section to the homogenization problem (3.1) when $p \geq d$. In this case, the behavior of the functions of $\mathbf{A}^p$ can be very different from the case $p < d$. A Gagliardo-Nirenberg-Sobolev type inequality such as that of Proposition 3.1 does not hold. We exhibit two counter-examples of coefficient a satisfying assumptions (3.2), (3.3), (3.4) and (3.11), respectively for $p > d$ and $p = d$, but for which a can not be split as the sum of a periodic coefficient and a perturbation integrable at infinity. The reason is, the decay of δa at infinity may be too slow to ensure the existence of a periodic limit of a at infinity. We illustrate the phenomenon with two coefficients a respectively in dimension $d = 1$ and $d = 2$ which slowly oscillate at infinity, typically as $\sin(\ln(x))$ or $\sin(\ln(\ln(x)))$. For such coefficients, we show that the homogenization of problem (3.1) is not possible, since such u^ε has subsequences converging to different limits.

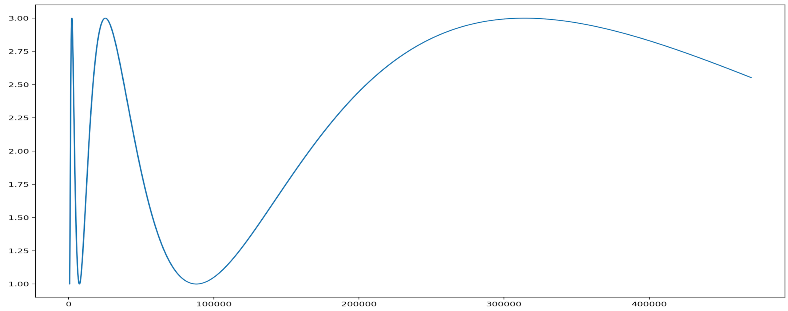

FIGURE 3.3 – Example of coefficient a with slow oscillations at infinity in dimension $d = 1$.

3.4.1 Counter-example for $d = 1, p > 1$

To start with, we study a case where $a \in \mathbf{A}^p$ for $d = 1$ and $p > 1$. We define $a(x) = 2 + \sin(\ln(1 + |x|))$, for every $x \in \mathbb{R}$. It is clear that this coefficient satisfies assumptions (3.2), (3.3) and (3.4). We claim that $\delta a \in L^p(\mathbb{R})$ for every $p > 1$. There indeed exists a constant C such that for every $x \in \mathbb{R}$ with $|x| > 1$,

$$|a'(x)| = \left|\frac{\cos(\ln(1 + |x|))}{1 + |x|}\right| \leq \frac{C}{|x|}.$$

The mean value theorem then shows that, for $|x| > 1$,

$$|\delta a(x)| = |\sin(\ln(1 + |x|)) - \sin(\ln(1 + |x + 1|))| \leq \frac{C}{|x|},$$

from which, we deduce, as announced above, that $\delta a \in L^p(\mathbb{R})$ for every $p > 1$.

We then consider the homogenization problem (3.1) for $\Omega =]1, 2[$, that is

$$\begin{cases} \dfrac{d}{dx}\left(a(x/\varepsilon)\dfrac{d}{dx}u^\varepsilon\right) = f \quad \text{on }]1, 2[, \\ u^\varepsilon(1) = u^\varepsilon(2) = 0. \end{cases} \tag{3.86}$$

Our aim is to establish the existence of two sub-sequences $(\varepsilon_n^1)_{n\in\mathbb{N}}$ and $(\varepsilon_n^2)_{n\in\mathbb{N}}$ such that $u^{\varepsilon_n^1} \to u^{*,1}$ and $u^{\varepsilon_n^2} \to u^{*,2}$ in $L^2(\Omega)$ when $\varepsilon_n^1, \varepsilon_n^2 \to 0$ and such that $u^{*,1} \neq u^{*,2}$. To this end, for $n \in \mathbb{N}$, we define $\varepsilon_n^1 = \exp(-2n\pi)$ and $\varepsilon_n^2 = \exp(-(2n + 1)\pi)$. For $x \in]1, 2[$, we have

$$\begin{aligned} a\left(\frac{x}{\varepsilon_n^1}\right) = 2 + \sin\left(\ln\left(1 + \frac{x}{\varepsilon_n^1}\right)\right) &= 2 + \sin\left(2\pi n + \ln\left(\varepsilon_n^1 + x\right)\right) \\ &= 2 + \sin\left(\ln\left(\varepsilon_n^1 + x\right)\right). \end{aligned}$$

Therefore, since ε_n^1 converges to 0 when $n \to \infty$, $a(x/\varepsilon_n^1) = 2 + \sin\left(\ln\left(\varepsilon_n^1 + x\right)\right)$ converges uniformly to $a^{*,1}(x) = 2 + \sin(\ln(x))$ on $]1,2[$. Since a satisfies (3.2) and (3.3), $u^{\varepsilon_n^1}$ is bounded in $H^1(\Omega)$ and, up to an extraction, it weakly converges to a function $u^{*,1}$ in $H^1(\Omega)$. Thus, for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$\lim_{n\to\infty} \int_\Omega a(x/\varepsilon_n^1) \frac{du^{\varepsilon_n^1}}{dx}(x)\frac{d\varphi}{dx}(x)dx = \int_\Omega a^{1,*}(x)\frac{du^{*,1}}{dx}(x)\frac{d\varphi}{dx}(x)dx.$$

We obtain that $u^{1,*}$ is solution in $H_0^1(\Omega)$ to

$$-\frac{d}{dx}\left(a^{*,1}\frac{d}{dx}u^{*,1}\right) = f \quad \text{on }]1,2[.$$

We may similarly show that $a(x/\varepsilon_n^2) = 2 - \sin\left(\ln\left(\varepsilon_n^2 + x\right)\right)$ converges uniformly to

$$a^{*,2}(x) = 2 - \sin(\ln(x))$$

on $]1,2[$ and that $u^{\varepsilon_n^2}$ weakly converges in $H^1(\Omega)$ (up to an extraction) to $u^{2,*}$, solution in $H_0^1(\Omega)$ to

$$-\frac{d}{dx}\left(a^{*,2}\frac{d}{dx}u^{*,2}\right) = f \quad \text{on }]1,2[.$$

To conclude, we show that $u^{*,1} \neq u^{*,2}$. Indeed, if we assume that $u^{*,1} = u^{*,2} = u^*$ we have,

$$\frac{d}{dx}\left((a^{*,1} - a^{*,2})\frac{d}{dx}u^*\right) = 0 \quad \text{on }]1,2[.$$

We use u^* as a test function and, since $a^{*,1} - a^{*,2} = 2\sin(\ln(.))$, we obtain :

$$2\int_\Omega \sin(\ln(x))\left|\frac{d}{dx}u^*\right|^2 = 0.$$

We remark that for every $x \in]1,2[$, $\sin(\ln(x)) > 0$ and obtain that $\frac{d}{dx}u^* = 0$ on $]1,2[$. Since $u^* \in H_0^1(\Omega)$, it follows that $u^* = 0$ and we reach a contradiction as soon as $f \neq 0$.

We conclude with the following three remarks :

1. For every $y \in [0, 2\pi]$, we could also consider the sub-sequence $\varepsilon_n = \exp(-2n\pi - y)$ and, as above, we could obtain that u^{ε_n} converges to u^* solution to $-\frac{d}{dx}\left(a^*\frac{d}{dx}u^*\right) = f$ on $]1,2[$, where $a^* = 2 + \sin(y + \ln(x))$. Therefore u^* actually has an infinite number of adherent values.

2. Unlike the periodic case, that is when $a = a_{per}$ is periodic, the coefficients a^* of the homogenized equation that we obtain here (which depends on a chosen extraction) are not constant.

3. For the specific coefficient a chosen, a property similar to that of Proposition 3.1 cannot hold. Actually, if a were on the form $a = a_{per} + \tilde{a}$ with a_{per} periodic and $\tilde{a}$ a function vanishing at infinity, we would be able to homogenize problem (3.86). Indeed, some explicit calculations give

$$u^\varepsilon(x) = -\int_1^x \frac{1}{a(y/\varepsilon)}F(y)dy + C^\varepsilon \int_1^x \frac{1}{a(y/\varepsilon)}dy,$$

where $F(x) = \int_1^x f(x)dx$ and $C^\varepsilon = \left(\int_1^2 \frac{1}{a(y/\varepsilon)}dy\right)^{-1} \int_1^2 \frac{1}{a(y/\varepsilon)}F(y)dy$.

Since $\lim_{|x|\to\infty} |\tilde{a}(x)| = 0$, we can show that $|\tilde{a}(./\varepsilon)| \xrightarrow{\varepsilon\to 0} 0$ in $L^\infty(\Omega) - *$. If we remark that

$$\frac{1}{a} = \frac{1}{a_{per}} - \frac{\tilde{a}}{a_{per}(\tilde{a} + a_{per})},$$

it follows that $\frac{1}{a}$ converges to $\left\langle \frac{1}{a_{per}} \right\rangle = (a^*_{per})^{-1}$ for the weak-* topology of $L^\infty(\Omega)$ (where $\langle . \rangle$ denotes the average value of a periodic function). Therefore the limit u^* of u^ε can be made explicit :

$$u^*(x) = -(a^*_{per})^{-1} \int_1^x F(y)dy + (x-1)(a^*_{per})^{-1} \int_1^2 F(y)dy,$$

which is the unique solution in $H^1_0(\Omega)$ to $-a^*_{per}(u^*)'' = f$. In this case, the convergence of the whole sequence u^ε to u^* would be in contradiction with the results obtained above.

3.4.2 Counter-example for $d = 2$, $p = 2$

We next study the case where $a \in \mathbf{A}^p$ for $p = d$, more specifically when $p = d = 2$. We define $a(x) = 2 + \sin(\ln(1 + \ln(1 + |x|)))$, for every $x \in \mathbb{R}^d$, which satisfies assumptions (3.2), (3.3) and (3.4). We remark there exists a constant C such that for every $x \in \mathbb{R}^2$ with $|x| > 1$,

$$|\nabla a(x)| \leq \frac{C}{\ln(|x|)|x|},$$

and the mean value theorem implies

$$|\delta a(x)| \leq \frac{C}{\ln(|x|)|x|}$$

which provides that $\delta a \in L^2(\mathbb{R}^2)^2$.

We then consider the homogenization problem (3.1) on the annulus

$$\Omega = \left\{ x \in \mathbb{R}^d \;\middle|\; 1 < |x| < 2 \right\}.$$

We again intend to establish the existence of two sub-sequences $(\varepsilon^1_n)_{n\in\mathbb{N}}$ and $(\varepsilon^2_n)_{n\in\mathbb{N}}$ such that $u^{\varepsilon^1_n} \to u^{*,1}$ and $u^{\varepsilon^2_n} \to u^{*,2}$ in $L^2(\Omega)$ when $\varepsilon^1_n, \varepsilon^2_n \to 0$ and such that $u^{*,1} \neq u^{*,2}$ thereby proving that homogenization does not hold in this setting. For $n \in \mathbb{N}$, we define $\varepsilon^1_n = \exp\left(-\left(\exp(2n\pi)\right)\right)$ and $\varepsilon^2_n = \exp\left(-\exp\left(\left((4n+1)\frac{\pi}{2}\right)\right)\right)$. We have for every $x \in \Omega$:

$$\begin{aligned} a\left(\frac{x}{\varepsilon^1_n}\right) &= 2 + \sin\left(\ln\left(1 + \ln\left(1 + \frac{x}{\varepsilon^1_n}\right)\right)\right) \\ &= 2 + \sin\left(2\pi n + \ln\left(1 + \exp(-2n\pi)\left(1 + \ln\left(\varepsilon^1_n + x\right)\right)\right)\right) \\ &= 2 + \sin\left(\ln\left(1 + \exp(-2n\pi)\left(1 + \ln\left(\varepsilon^1_n + x\right)\right)\right)\right), \end{aligned}$$

and we can deduce that $a\left(\frac{\cdot}{\varepsilon_n^1}\right)$ uniformly converges to $a^{*,1} \equiv 2$ on Ω. Since a satisfies (3.2) and (3.3), $u^{\varepsilon_n^1}$ is bounded in $H^1(\Omega)$ and, up to an extraction, it weakly converges to a function $u^{*,1}$ in $H^1(\Omega)$. Thus, for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$\lim_{n\to\infty} \int_\Omega a(x/\varepsilon_n^1)\nabla u^{\varepsilon_n^1}(x)\nabla\varphi(x)dx = \int_\Omega 2\nabla u^{*,1}(x)\nabla\varphi(x)dx.$$

We obtain that $u^{1,*}$ is solution in $H_0^1(\Omega)$ to

$$\begin{cases} -2\Delta u^{*,1} = f & \text{on } \Omega, \\ u^{*,1} = 0 & \text{on } \partial\Omega. \end{cases}$$

We similarly have $a\left(\frac{x}{\varepsilon_n^2}\right) = 2 + \cos\left(\ln\left(1 + \exp\left((-(4n+1)\frac{\pi}{2}\right)(1 + \ln(\varepsilon_n^2 + x))\right)\right)$ and $a\left(\frac{\cdot}{\varepsilon_n^2}\right)$ converges uniformly to $a^{*,2} \equiv 3$ on Ω. Therefore, $u^{\varepsilon_n^2}$ weakly converges in $H^1(\Omega)$ (up to an extraction) to $u^{2,*}$, solution in $H_0^1(\Omega)$ to

$$\begin{cases} -3\Delta u^{*,2} = f & \text{on } \Omega, \\ u^{*,2} = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly $u^{*,1} \neq u^{*,2}$ as soon as $f \neq 0$ and we can conclude that u^ε does not converge in $L^2(\Omega)$.

Chapitre 4

Homogénéisation elliptique pour une classe de potentiels oscillants non-périodiques

Ce chapitre reproduit une prépublication [GLB22] écrite en collaboration avec Claude Le Bris.

On s'intéresse ici à un problème d'homogénéisation pour l'équation elliptique d'ordre deux $-\Delta u^\varepsilon + \dfrac{1}{\varepsilon} V(./\varepsilon) u^\varepsilon + \nu u^\varepsilon = f$ quand le potentiel V rapidement oscillant appartient à une classe particulière de potentiels non-périodiques. On montre l'existence d'un correcteur adapté et on prouve la convergence de u^ε vers sa limite homogénéisée.

Sommaire

Linear ellitptic homogenization for a class of highly oscillating non-periodic potentials

4.1 Introduction

Our purpose is to homogenize the stationary Schrödinger equation :

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{1}{\varepsilon} V(./\varepsilon) u^\varepsilon + \nu u^\varepsilon = f & \text{on } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$), f is a function in $L^2(\Omega)$, $\varepsilon > 0$ is a small scale parameter, $V \in L^\infty(\mathbb{R}^d)$ is an highly oscillatory non-periodic potential that models a perturbed periodic geometry and $\nu \in \mathbb{R}^d$ is a fixed scalar. The potential V is assumed to have a vanishing "average" in the following sense :

$$\lim_{\varepsilon \to 0} V(./\varepsilon) = 0 \quad \text{in } L^\infty(\mathbb{R}^d) - \star, \tag{4.2}$$

which is a necessary assumption to expect the convergence of u^ε to a non trivial function u^* when ε converges to 0 due to the exploding term $\dfrac{1}{\varepsilon} V(./\varepsilon)$ in (4.1). In order to avoid some technical details, we also assume that Ω is sufficiently regular, say $\mathcal{C}^1$.

When the potential $V = V_{per}$ is periodic, the homogenization problem (4.1) is well known, see for instance [18, Chapter 1, Section 12]. The behavior of u^ε is then described using a corrector w_{per}, that is the periodic solution (unique up to the addition of an irrelevant constant) to

$$\Delta w_{per} = V_{per} \quad \text{on } \mathbb{R}^d. \tag{4.3}$$

More precisely, u^ε converges, strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, to the unique solution u^* in $H^1(\Omega)$ to

$$\begin{cases} -\Delta u^* + (V_{per})^* u^* + \nu u^* = f & \text{on } \Omega, \\ u^* = 0 & \text{on } \partial\Omega, \end{cases}$$

where the homogenized potential $(V_{per})^*$ is the scalar constant defined by

$$(V_{per})^* = \langle V_{per} w_{per} \rangle = -\langle |\nabla w_{per}|^2 \rangle, \tag{4.4}$$

the rightmost equality being a consequence of the virial theorem applied to the setting of (4.3), in that case a simple application of the Green formula.

We note that the well-posedness of (4.1) requires an additional assumption. In [18], the homogenization of (4.1) is performed under the (sufficient) assumption

$$\mu_1 + \langle V_{per} w_{per} \rangle + \nu > 0, \tag{4.5}$$

where μ_1 is the first eigenvalue of $-\Delta$ on $H^1_0(\Omega)$. The existence and uniqueness of u^ε, for ε sufficiently small, is then a consequence of the convergence of λ_1^ε, the first eigenvalue of the operator $-\Delta + \dfrac{1}{\varepsilon} V(./\varepsilon) + \nu$ with homogeneous Dirichlet boundary conditions, to $\mu_1 + \langle w_{per} V_{per} \rangle + \nu$.

One can remark that assumption (4.5) only depends on the properties of the potential V_{per} since w_{per} is solution to (4.3).

The behavior of u^ε can also be made explicit using the periodic corrector. The sequence of remainders

$$R^\varepsilon = u^\varepsilon - u^* - \varepsilon u^* w_{per}(./\varepsilon)$$

is shown to strongly converge to 0 in $H^1(\Omega)$. The convergence of the eigenvalues of $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon) + \nu$ is also studied in [94, 34]. Respectively denoting by λ_l^ε and λ_l^* the lowest l^{th} eigenvalue (counting multiplicities) of $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon) + \nu$ and $-\Delta + \langle w_{per}V_{per}\rangle + \nu$, the results of [94, Theorem 1.4] show the existence of a constant $C > 0$ such that $|\lambda_l^\varepsilon - \lambda_l^*| \leq C|\lambda_l^*|^{3/2}\varepsilon$. All these results are essentially established using two specific properties of the corrector w_{per} : the strict sublinearity at infinity, that is the uniform convergence to 0 of $\varepsilon w_{per}(./\varepsilon)$, and the weak convergence of the periodic function $|\nabla w_{per}(./\varepsilon)|^2$ to its average. For completeness, let us also mention that both elliptic and parabolic variants of (4.1) have also been studied in [4],[69, Chapter 1, Section5] and were alternatively considered in the case where the potential V is random and stationary, see for instance [15, 17, 56].

The aim of the present contribution is to extend the results of the elliptic periodic case to an elliptic deterministic non-periodic setting that models a periodic geometry perturbed by a certain class of so-called defects which we make precise in the next section.

4.1.1 The non-periodic case : mathematical setting and assumptions

Throughout this work, we denote by B_R the ball of radius $R > 0$ centered at the origin and by $B_R(x)$ the ball of radius R and center $x \in \mathbb{R}^d$, by $|A|$ the volume of any measurable subset $A \subset \mathbb{R}^d$ and by $A/\varepsilon = \left\{ \frac{x}{\varepsilon} \;\middle|\; x \in A \right\}$ the scaling of A by $\varepsilon > 0$. In addition, for a normed vector space $(X, \|.\|_X)$ and a d-dimensional vector $f \in X^d$, we use the notation $\|f\|_X \equiv \sum_{i=1}^{d} \|f_i\|_X$.

The class of potentials V we consider in this paper consists of those that read as

$$V(x) = g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(x - k - Z_k), \tag{4.6}$$

where g_{per} is a Q-periodic function (where $Q =]0,1[^d$ denotes the unit cell of $\mathbb{R}^d$), φ belongs to $\mathcal{D}(\mathbb{R}^d)$, the space of smooth compactly supported functions, and $Z = (Z_k)_{k\in\mathbb{Z}^d}$ is a vector-valued sequence that satisfies

$$Z \in \left(l^\infty(\mathbb{Z}^d)\right)^d. \tag{4.7}$$

In a certain sense, the potential V is a perturbation of the periodic potential

$$V_{per} = g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(x - k) \tag{4.8}$$

by the "defect"

$$\tilde{V} = \sum_{k\in\mathbb{Z}^d} \left(\varphi(x - k - Z_k) - \varphi(x - k)\right). \tag{4.9}$$

A simple calculation shows that assumption (4.2) satisfied by V is actually equivalent to

$$\int_{\mathbb{R}^d} \varphi = -\langle g_{per} \rangle, \tag{4.10}$$

where $\langle g_{per} \rangle$ is the average of the Q-periodic function g_{per}. In this work, we additionally assume that

$$g_{per} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d), \text{ for some } \alpha \in]0,1[, \tag{4.11}$$

where $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ denotes the space of α-Hölder continuous functions. This assumption ensures that the periodic corrector w_{per}, solution to $\Delta w_{per} = V_{per}$, belongs to $L^\infty(\mathbb{R}^d)$, as a consequence of the Schauder regularity theory.

The specific form (4.6) of the potential V is inspired by the work [22], related to the minimization of the energy of an infinite non-periodic system of particles. In this work were introduced several algebras generated by functions of the form $\sum_{k\in\mathbb{Z}^d} \varphi(x - X_k)$ together with general assumptions related to the distribution of the points X_k. For some particular sets $\{X_k\}_{k\in\mathbb{Z}^d}$, this setting has been employed as a *motivation* to introduce several linear elliptic homogenization problems for *local* perturbations of a periodic geometry in [20, 27, 26, 25]. In these cases, $X_k = k + Z_k$, where Z_k is a compactly supported sequence. Stochastic cases with stationary coefficients were also considered in [23] using sets of random points $X_k(\omega)$. Although the homogenization of (4.1) in the whole generality of the sequence X_k introduced in [22] is to date an open mathematical question, the aim of the present contribution is to introduce a somewhat general case of sequences of the form $X_k = k + Z_k$ that model *non-local* perturbations of a periodic setting when Z_k does not vanish or only slowly vanishes at infinity, and for which the homogenization problem can be addressed. This work is also motivated by the hope to establish a theory of homogenization for the diffusion operator $-\operatorname{div}(a(./\varepsilon)\nabla)$, for a coefficient of the form $\sum_{k\in\mathbb{Z}^d} \varphi(x - X_k)$, or of a related general form. To some extent, equation (4.1) is a *bilinear* version (V *multiplies* the unknown function u^ε) of the diffusion equation $-\operatorname{div}(a(./\varepsilon)\nabla u^\varepsilon) = f$ where a and u^ε are, on the other hand, *fully* entangled.

One of the main difficulties in the non-periodic setting (4.6) is the study of the corrector equation

$$\Delta w = V \quad \text{on } \mathbb{R}^d, \tag{4.12}$$

which, in sharp contrast with the periodic setting, cannot be reduced to an equation posed on a bounded domain. In particular, this prevents us from using classical techniques (the Lax-Milgram lemma for instance) to solve this equation. A natural approach to show the existence of a solution to (4.12) consists in finding a solution of the form $w = G * V$, where G denotes the Green function associated with Δ on $\mathbb{R}^d$. For $d \geq 3$, it is well-known that G is of the form $G(x) = C(d)\dfrac{1}{|x|^{d-2}}$, where the constant $C(d)$ only depends on the ambient dimension, and the difficulty for solving our problem is therefore threefold. First, the existence of such a solution w has to be established, the definition of $G * V$ being not obvious since the potential V is non-periodic and is only known to belong to $L^\infty(\mathbb{R}^d)$. Second, we have to establish its strict sub-linearity at infinity in the sense that $\varepsilon w(./\varepsilon)$ converges to 0 on Ω. We shall recall that the weak convergence of $\nabla w(./\varepsilon)$ to 0 is actually sufficient to show the latter property (see Lemma 4.4). Third, we have to rigorously establish that the homogenized potential appearing in (4.4) is the weak limit of $|\nabla w|^2(./\varepsilon)$ when ε converges to 0. In particular, for our purpose, we have to prove some bounds, uniform with respect to ε and satisfied by ∇w on Ω/ε, at least in L^2.

To address the question related to the existence of a solution w to (4.12), our approach first consists in using the specific structure of V, that is a perturbation of the periodic potential (4.8) by (4.9) in order to find a corrector of the form

$$w = w_{per} + \tilde{w}, \tag{4.13}$$

where $\Delta w_{per} = V_{per}$ and $\nabla\tilde{w}$ formally reads as

$$\nabla\tilde{w} = \sum_{k\in\mathbb{Z}^d} \nabla G * \left(\varphi\left(. - k - Z_k\right) - \varphi(. - k)\right). \tag{4.14}$$

Since $\nabla G(x-k)$ behaves as $\dfrac{1}{|x-k|^{d-1}}$ at infinity, obtaining in the right hand side of (4.14) a series that normally converges requires to increase by more than one the exponent in the rate of decay with respect to k for large k. At the very least, a decay in $\dfrac{1}{|k|^d}$, that is a critical decay in the ambient dimension d, is sufficient. But then $\nabla\tilde{w}$ will be expected to only be a BMO function (where BMO denotes the space of functions with bounded mean oscillations, see [88, Chapter IV] for instance) and not an L^∞ function, which will in turn generate some additional technical difficulties in how ∇w is employed in the homogenization proof. We will return to this below, in Sections 4.2.2 and 4.3.

For the time being, let us observe that the gain in the rate of decay must come from a suitable combination of, on the one hand, the properties of the function φ, or functions constructed from φ, appearing in the right hand side of (4.14), and, on the other hand, of the properties of the sequence Z. One possible approach to realize the difficulty is to perform a formal Taylor expansion in the right-hand side of (4.14). Then we have

$$\nabla\tilde{w} = \sum_{k\in\mathbb{Z}^d} \nabla G * \left(-Z_k.\nabla\varphi(. - k) + \int_0^1 (1-t) Z_k^T D^2\varphi(. - k - tZ_k) Z_k dt\right). \tag{4.15}$$

In (4.15), it is immediately realized that the remainder term of order two yields an absolutely converging contribution to the construction of $\nabla\tilde{w}$. This term indeed only contains second derivatives of φ, which, in (4.15), give by integration by parts third-order derivatives D^3G which all decay like $\dfrac{1}{|x|^{d+1}}$. The only possibly delicate term is the leftmost, linear term on the right-hand side of (4.15), for which only one level, and not two levels, of derivation are gained. Put differently, the key point is the consideration of the equation

$$\Delta w_1 = \sum_{k\in\mathbb{Z}^d} Z_k.\nabla\varphi(. - k), \tag{4.16}$$

which is related to the convergence of the sum

$$\sum_{k\in\mathbb{Z}^d} \nabla G * \left(Z_k.\nabla\varphi(. - k)\right). \tag{4.17}$$

For this purpose, a possible additional assumption is that the integral of φ is required to vanish. We note in passing that this algebraic condition is not a matter of a simple renormalization, since in our setting and except under trivial circumstances, there does not exist any partition of unity, that is, any function $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\displaystyle\int_{\mathbb{R}^d} \chi = 1$ and $1 \equiv \displaystyle\sum_{k\in\mathbb{Z}^d} \chi(. - k - Z_k)$ (we will return to this later in Section 4.3.3). Then the first term of (4.15) also gives an absolutely converging series (see Lemma 4.1 for details), and thus, by linearity and combination of all terms, $\nabla\tilde{w}$ is shown to exist. In addition, it is indeed a L^∞ function.

Another, alternative, possible option is to assume that the Cesàro means of Z rapidly vanish, that is, for every $R > 0$, $x_0 \in \mathbb{R}^d$, $\displaystyle\frac{\varepsilon^d}{|B_R|} \sum_{k\in B_R(x_0)/\varepsilon} Z_k$ rapidly converges to 0 as ε vanishes, in which case the convergence of (4.17) in L^∞ can be established (see Proposition 4.1). For example, it is the case when Z_k itself rapidly vanishes at infinity. We will elaborate upon this particular assumption below.

Yet another option is to assume that Z is the discrete gradient of a sequence $(T_k)_{k\in\mathbb{Z}^d}$, that is $(Z_k)_i = T_{k+e_i} - T_k$ for every $i \in \{1, ...d\}$, such that $|T_k| = O\left(|k|^\alpha\right)$ for $\alpha \in [0, 1[$. In that case again, an integration by parts, this time a discrete one, yields the absolute convergence of (4.17).

Our only take-away message to the reader is, here, that various combinations of assumptions may be considered and although we briefly consider some of the above specific cases in Section 4.3.3, the main purpose of the present contribution is to establish the existence of a corrector adapted to our setting (4.1)-(4.6) under a set of "general" assumptions satisfied by the pair (φ, Z) that are as weak as possible. In particular, our aim is to homogenize equation (4.1) even if $\displaystyle\int_{\mathbb{R}^d} \varphi \neq 0$ and Z_k does not vanish by any mean at infinity. As specified above, for such a general setting, we will only be able to show the convergence of (4.17) in a somewhat weak sense using the properties of the operator $T : f \mapsto \nabla^2 G * f$ which, as a particular element of the class of Calderón-Zygmund operators, is known to be only continuous from $L^\infty(\mathbb{R}^d)$ to $BMO(\mathbb{R}^d)$.

In any event, once the gradient (4.14) is shown to exist in a suitable functional space (namely L^∞ or BMO), we have to use the corrector function w defined by (4.13) for the homogenization process. This second step brings additional constraints on our input parameters φ and Z. In the first place, $\nabla w(./\varepsilon)$ must weakly converges to 0 as ε vanishes in order for w to be strictly sublinear at infinity. Furthermore, simply considering the periodic setting and the expression (4.4) of the homogenized coefficient in that setting, we anticipate that we will have to make sure that the weak limit of $|\nabla w(./\varepsilon)|^2$ exists. These two conditions (strict sublinearity of w at infinity and weak convergence of $|\nabla w(./\varepsilon)|^2$) bring additional constraints on ∇w itself besides the only convergence of the sum (4.14). These constraints are again related to, and can be expressed with, suitable properties of the function φ and the sequence Z. Actually, since we intend to be as general as possible regarding the function φ, we will only impose constraints on Z. The first of these constraints, meant to ensure the strict-sublinearity of w, has already been mentioned above for a different purpose. It is related to the average of Z. As for the weak convergence of $|\nabla w(./\varepsilon)|^2$, it is intuitive to realize that *correlations* of the sequence Z will matter.

Given the above general and somewhat vague considerations, we now make precise the detailed properties of the sequence Z that we will assume throughout our work. The necessity of such assumptions will be motivated in Section 4.2.1 with an illustrative one-dimensional example for which the corrector can be explicitly determined.

Our first assumption is related to the strict sublinearity at infinity of the corrector. We assume that Z has an average, that is there exists a constant $\langle Z\rangle \in \mathbb{R}^d$ such that

$$\forall R>0, x_0 \in \mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^d}{|B_R|} \sum_{k\in B_R(x_0)/\varepsilon} Z_k = \langle Z\rangle. \tag{A1}$$

We note that Assumption (A1) is stronger that the existence of a simple average of Z_k in the sense that $\displaystyle\lim_{R\to\infty} \frac{1}{|B_R|} \sum_{k\in B_R(x_0)} Z_k$ exists. Here, since $B_R(x_0)/\varepsilon = B_{\frac{R}{\varepsilon}}\left(\frac{x_0}{\varepsilon}\right)$, the center of the ball of radius $\dfrac{R}{\varepsilon}$ may depends on ε and (A1) actually provides a certain uniformity of the convergence with respect to this center.

The next three assumptions regard the auto-correlations of Z and are related to the specific sum (4.17) that we need to manipulate. Denoting by $\overline{Z}_k := Z_k - \langle Z\rangle$, we assume the existence of a family of constants, denoted by $\mathcal{C}_{l,i,j}$ for every $l\in\mathbb{Z}^d$ and $i,j\in\{1,...,d\}$, such that

$$\bullet\ \forall R>0, x_0\in\mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^d}{|B_R|} \sum_{k\in B_R(x_0)/\varepsilon} (\overline{Z}_k)_i(\overline{Z}_k)_j = \mathcal{C}_{l,i,j}. \tag{A2.a}$$

$$\bullet \begin{cases} \exists\delta:\mathbb{R}^+\to\mathbb{R}^+,\ \forall i,j\in\{1,...,d\},\ \forall R>0, \forall x_0\in\mathbb{R}^d, \exists\,\gamma:\mathbb{R}^+\to\mathbb{R}^+, \\ \displaystyle\sup_{|l|\le\frac{1}{\delta(\varepsilon)}} \left| \frac{\varepsilon^d}{|B_R|} \sum_{k\in B_R(x_0)/\varepsilon} (\overline{Z}_k)_i(\overline{Z}_{k+l})_j - \mathcal{C}_{l,i,j} \right| \le \gamma(\varepsilon), \\ \displaystyle\lim_{\varepsilon\to 0}\gamma(\varepsilon)|\ln(\varepsilon)| = 0, \\ \displaystyle\lim_{\varepsilon\to 0}\varepsilon^{-1}\delta(\varepsilon) = 0. \end{cases} \tag{A2.b}$$

$$\bullet\ \forall i,j\in\{1,...d\}, \\ x\mapsto \sum_{|l|\le L} \mathcal{C}_{l,i,j}\left(\partial_i\partial_j G * \varphi\right)(x-l) \text{ converges in } L^1_{loc}(\mathbb{R}^d) \text{ when } L\to+\infty. \tag{A2.c}$$

We shall see in Section 4.3 that Assumptions (A2.a) (A2.b) and (A2.c) will allow us to establish the weak-convergence of $|\nabla w_1(./\varepsilon)|^2$, where w_1 is solution to (4.16) with a gradient of the form (4.17). As we have just sketched above, equation (4.16) will indeed be key in our present study. We also note that in Assumption (A2.b), the rate $|\ln(\varepsilon)|$ is related to the decreasing rate of the second derivatives of G.

As specified, (A2.a) (A2.b) and (A2.c) will only be sufficient to study the solution to equation (4.16) for which the right hand side is linear with respect to Z. However, in our original problem, the potential (4.6) is actually nonlinear with respect to Z and we shall see that this nonlinearity implies that the convergence of $|\nabla w(./\varepsilon)|^2$, where w will be given by (4.13), requires an additional strong assumption related to the second-order correlations of Z. We therefore assume that

$$\forall F\in\mathcal{C}^0(\mathbb{R}^d\times\mathbb{R}^d),\ \forall l\in\mathbb{Z}^d,\ \exists C_{F,l}\in\mathbb{R},\ \forall R>0,\forall x_0\in\mathbb{R}^d, \\ \lim_{\varepsilon\to 0}\frac{\varepsilon^d}{|B_R|}\sum_{k\in B_R(x_0)/\varepsilon} F(Z_k,Z_{k+l}) = C_{F,l}. \tag{A3}$$

Considering respectively $F(x,y) = x_i$ and $F(x,y) = (x_i - \langle Z \rangle_i)(y_j - \langle Z \rangle_j)$ for every $i, j \in \{1, ..., d\}$, Assumption (A3) of course implies (A1) and (A2.a) but we chose here to consider these assumptions separately for a pedagogic purpose.

We give several examples of sequences Z satisfying assumptions (A1) to (A3) in Section 4.2.3. We refer to Figure 4.1 for an illustration in dimension $d = 2$ of two sequences that satisfy our assumptions respectively in a case of local perturbation $\left(\lim_{|k|\to\infty} Z_k = 0\right)$ and of non-local perturbation $\left(\lim_{|k|\to\infty} Z_k \neq 0\right)$.

We also note that, if V is a potential of the form (4.6) and Z satisfies (A1), then we have $V = g_{per} + \sum_{k\in\mathbb{Z}^d} \overline{\varphi}(. - k - \overline{Z}_k)$ which is also of the form (4.6) where $\overline{\varphi} = \varphi(. - \langle Z \rangle)$ and $\overline{Z_k} = Z_k - \langle Z \rangle$ has a vanishing average. Consequently, for simplicity, we will sometimes assume $\langle Z \rangle = 0$ in (A1) without loss of generality.

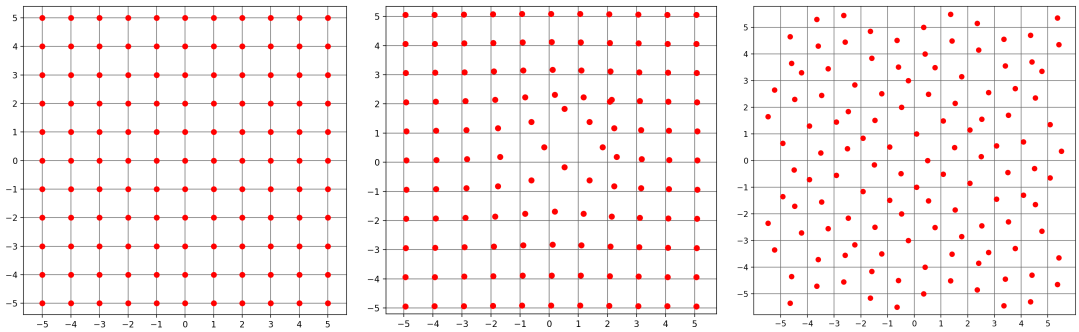

FIGURE 4.1 – Illustration of $X_k = k + Z_k$ for admissible sequences Z_k ($d = 2$).
Left. Reference periodic case : $Z_k = 0$. **Center**. Local perturbation : $\lim_{|k|\to\infty} Z_k = 0$.
Right. Non-local perturbations.

4.1.2 Main results

Our main result regarding the existence of a corrector adapted to our setting is stated in the next theorem when $d \geq 2$. The case of dimension $d = 1$ may be addressed by analytical arguments that are briefly presented in Section 4.2.1.

Theorem 4.1. *Assume $d \geq 2$. Assume that V is a potential of the form (4.6), where $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and g_{per} is a periodic function satisfying (4.11), which also satisfies (4.2). Assume that Z satisfies (4.7), (A1), (A2.a), (A2.b), (A2.c), and (A3).*

Then, for every $R > 0$ and every $\varepsilon > 0$, there exists $W_{\varepsilon,R} \in L^1_{loc}(\mathbb{R}^d)$ solution to

$$\Delta W_{\varepsilon,R} = V \quad \text{on } B_{R/\varepsilon}, \tag{4.19}$$

such that $\left(\nabla W_{\varepsilon,R}(./\varepsilon)\right)_{\varepsilon>0}$ is bounded in $\left(L^p(B_R)\right)^d$ for every $p \in [1, +\infty[$ and

$$\begin{cases} \nabla W_{\varepsilon,R}(./\varepsilon) \xrightarrow{\varepsilon\to 0} 0 \quad \text{weakly in } L^p(B_R),\ \forall p \in [1,+\infty[, \\ \varepsilon W_{\varepsilon,R}(./\varepsilon) \xrightarrow{\varepsilon\to 0} 0 \quad \text{strongly in } L^\infty(B_R), \\ \exists \mathcal{M} \in \mathbb{R}, \quad |\nabla W_{\varepsilon,R}|^2(./\varepsilon) \xrightarrow{\varepsilon\to 0} \mathcal{M} \quad \text{weakly in } L^p(B_R),\ \forall p \in [1,+\infty[. \end{cases} \tag{4.20}$$

We point out one *major* difference from the periodic case : the corrector is "ε-dependent". It however depends on ε in a very specific manner which we may make entirely explicit. We shall indeed see that it is of the form $W_{\varepsilon,R} = w - x.C_{\varepsilon,R}$ where w is a particular solution of (4.19) with a gradient in $\left(BMO(\mathbb{R}^d)\right)^d$ and $C_{\varepsilon,R}$ is a constant related to the average of ∇w on $B_{R/\varepsilon}$. In fact, $W_{\varepsilon,R}$ is not even a solution to $\Delta w = V$ on the whole space $\mathbb{R}^d$ but only on the ball $B_{R/\varepsilon}$. This corrector is however sufficient to address the homogenization problem (4.1) since one eventually only needs to evaluate the corrector on the domain Ω/ε. As specified above, the proof of Theorem 4.1 uses the specific structure of V, that is the sum of the periodic potential V_{per} and of a perturbation $\tilde{V}$ respectively defined by (4.8) and (4.9).

Our methodology to study the perturbation term $\tilde{V}$ is detailed in Section 4.2 and consists in performing a first order Taylor expansion of the potential $\tilde{V}$ with respect to Z_k with the aim to solve both the equation induced by the first order term (linear with respect to Z) and the equation corresponding to the remainder of the expansion exactly suggested by our decomposition (4.15). We shall see that the difficulties to solve these two equations are very different in nature and, as we have sketched in Section 4.1.1, are related to the properties of the derivatives of the Green function G.

We now turn to the study of the convergence of u^ε which is next addressed in two steps. Similarly to the periodic case, the existence of a corrector stated in Theorem 4.1 first allows to establish the well-posedness of (4.1) when

$$\mu_1 - \mathcal{M} + \nu > 0. \tag{4.21}$$

This result is a consequence of our Proposition 4.4 in Section 4.5.1 (the results of which are only based upon those of Theorem 4.1) which shows that the first eigenvalue λ_1^ε of $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon) + \nu$ with homogeneous Dirichlet boundary conditions converges to $\mu_1 - \mathcal{M} + \nu$ when ε vanishes as is the case in the periodic setting. Under assumption (4.21), we next homogenize (4.1) and show the strong convergence of u^ε to the function u^* in $L^2(\Omega)$, solution to

$$\begin{cases} -\Delta u^* - \mathcal{M} u^* + \nu u^* = f & \text{on } \Omega, \\ u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.22}$$

This result is established in Proposition 4.5 in Section 4.5.2. The behavior of u^ε in $H^1(\Omega)$ is given in the following theorem :

Theorem 4.2 (Homogenization results). *Assume, as in Theorem 4.1, that $d \geq 2$, V is a potential of the form (4.6) that satisfies (4.2)-(4.11), Z satisfies (4.7), (A1), (A2.a), (A2.b), (A2.c), and (A3) and denote $W_{\varepsilon,\Omega} = W_{\varepsilon,R}$ where $R = Diam(\Omega)$. Assume also (4.21) is satisfied.*

Then, for ε sufficiently small, there exists an unique solution $u^\varepsilon \in H_0^1(\Omega)$ to (4.1). In addition, the sequence

$$R^\varepsilon = u^\varepsilon - u^* - \varepsilon u^* W_{\varepsilon,\Omega}(./\varepsilon) \tag{4.23}$$

strongly converges to 0 in $H^1(\Omega)$ as ε vanishes.

In a second step, following a method introduced in [94] in the periodic case, we use the convergence of the operator $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon) + \nu$ that holds under the particular assumption (4.21) to study the convergence of *all* its eigenvalues. This allows us to establish the following generalization of Theorem 4.2 :

Theorem 4.3. *Assume, as in Theorems 4.1 and 4.2, that $d \geq 2$, V is a potential of the form (4.6) that satisfies (4.2)-(4.11), Z satisfies (4.7), (A1), (A2.a), (A2.b), (A2.c), and (A3) but, instead of (4.21), assume $\mu_l - \mathcal{M} + \nu \neq 0$ for all $l \in \mathbb{N}^*$, where μ_l is the l^{th} eigenvalue (counting multiplicities) of -Δ on Ω with homogeneous Dirichlet boundary conditions. Then the conclusions of Theorem 4.2 hold true.*

Given the convergence of all the eigenvalues of $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon)$ established in Proposition 4.6, the proof of Theorem 4.3 is actually a simple adaptation of the proof of Theorem 4.2. Although this result also holds in the periodic case, to the best of our knowledge it has never been explicitly stated in the literature and that is why we prove it in the sequel.

Our article is organized as follows. We begin by presenting our approach to solve equation (4.19) and by collecting some preliminary results in Section 4.2. In that section, we also give several examples of admissible sequences Z that satisfy Assumptions (A1) to (A3). Sections 4.3 and 4.4 are next devoted to the study of the corrector equation and the proof of Theorem 4.1. Finally, in Section 4.5, we use the corrector to show Theorems 4.2 and 4.3.

4.2 Preliminaries

Our twofold purpose here is to motivate our assumptions (A1) through (A3) and to emphasize the difficulties related to the corrector equation (4.12). We first address a particular illustrative case in dimension $d = 1$ and we next introduce our approach to solve (4.12) for higher dimensions. We conclude this section with some examples of sequences Z satisfying our assumptions.

4.2.1 A one-dimensional setting

We briefly study here an illustrative one-dimensional setting in order to motivate our Assumptions (A1), (A2.a) and (A3). When $d = 1$, one can remark that (A2.b) and (A2.c) are not needed. These two assumptions are indeed related to the specific behavior of $\nabla^2 G$ in higher dimensions, in particular, to the lack of continuity of $f \mapsto \nabla^2 G * f$ from $\left(L^\infty(\mathbb{R}^d)\right)^d$ to $\left(L^\infty(\mathbb{R}^d)\right)^d$ when $d > 1$.

We recall here that we consider a potential V of the form (4.6) when g_{per} is periodic and $\varphi \in \mathcal{D}(\mathbb{R})$. For simplicity, we assume that $V_{per} = 0$ in (4.8), that is, $g_{per} = -\sum_{k\in\mathbb{Z}} \varphi(. - k)$. For every $k \in \mathbb{Z}$, we evidently have

$$\varphi(x - k - Z_k) - \varphi(x - \ k) = -Z_k \int_0^1 \varphi'(x - k - tZ_k)dt.$$

In this specific case, the derivative of the solution w to (4.12), which reads here as $w'' = V$, is explicitly given by

$$w'(x) = -\sum_{k\in\mathbb{Z}} Z_k \int_0^1 \varphi(x - k - tZ_k)dt + C,$$

where C is a constant.

We first investigate the strict sublinearity of w at infinity. Up to the addition of a constant, we have $w(x) = \int_0^x w'(y)dy$. The strict sublinearity of w at infinity is therefore equivalent to the fact that w' has a vanishing average in the following sense :

$$\forall x \in \mathbb{R}^*, \quad \lim_{\varepsilon \to 0} \frac{\varepsilon}{|x|} \int_0^{x/\varepsilon} w'(y)dy = 0.$$

Since Z_k is assumed to satisfy (4.7) and φ is compactly supported, we have for ε sufficiently small :

$$\begin{aligned}\int_0^{x/\varepsilon} \sum_{k\in\mathbb{Z}} Z_k \int_0^1 \varphi(z-k-tZ_k)dtdz &= \sum_{k=0}^{[x/\varepsilon]} Z_k \int_0^1 \int_{\mathbb{R}} \varphi(z)dzdt + O(1) \\ &= \sum_{k=0}^{[x/\varepsilon]} Z_k \int_{\mathbb{R}} \varphi(z)dz + O(1),\end{aligned} \tag{4.24}$$

where we have denoted by $[x] \in \mathbb{Z}$ the integer part of x. We have then to distinguish two cases :

a) If $\int_{\mathbb{R}} \varphi = 0$, we have $\lim_{\varepsilon\to 0} \frac{\varepsilon}{|x|} \int_0^{x/\varepsilon} w'(y)dy = 0$. In this case, the choice $C = 0$ is the unique value of C that allows for the strict sublinearity of w at infinity and no additional assumption is required for Z.

b) If $\int_{\mathbb{R}} \varphi \neq 0$, the convergence of $\frac{\varepsilon}{|x|} \int_0^{x/\varepsilon} w'(y)dy$ is equivalent to the convergence of $\sum_{k=0}^{[x/\varepsilon]} Z_k$ to a constant $\langle Z\rangle \in \mathbb{R}^d$ independent of $x \in \mathbb{R}$. We can actually show that this property is equivalent to (A1). The choice $C = -\langle Z\rangle \int_{\mathbb{R}} \varphi$ therefore allows for the strict sublinearity of w in this case.

We next study the weak convergence of $|w'(./\varepsilon)|^2$. To this aim, we consider the integral $\int_{\mathbb{R}} g(x)|w'(x/\varepsilon)|^2 dx$ for every characteristic function $g = 1_{B_M(x_0)}$, for $M > 0$ and $x_0 \in \mathbb{R}$. We can show that the convergence of this quantity as $\varepsilon \to 0$ is equivalent to the convergence of

$$S^\varepsilon = \sum_{l\in\mathbb{Z}} \left(\varepsilon \sum_{k\in B_M(x_0)/\varepsilon} Z_k Z_{k+l} \int_{\mathbb{R}} \varphi(x) \int_0^1 \int_0^1 \varphi(x-l+tZ_k-\tilde{t}Z_{k+l})d\tilde{t}\, dt\, dx \right), \tag{4.25}$$

and if S^ε converges, we have $\lim_{\varepsilon\to 0} S^\varepsilon = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} g(x)|w'(x/\varepsilon)|^2 dx$. This equivalence can be obtained with several changes of variables and using that φ is compactly supported. We skip its proof for brevity. We therefore remark, in the right-hand side of (4.25), that a specific assumption regarding the correlations of the sequence Z_k is required to obtain the convergence of S^ε. For instance, if we define

$$F_l(y,z) = yz \int_{\mathbb{R}} \varphi(x) \int_0^1 \int_0^1 \varphi(x-l+ty-\tilde{t}z)d\tilde{t}\, dt\, dx, \quad \text{for } l \in \mathbb{Z},$$

we have $S^\varepsilon = \sum_{l\in\mathbb{Z}} \left(\varepsilon \sum_{k\in B_M(x_0)/\varepsilon} F_l(Z_k, Z_{k+l}) \right)$ and Assumption (A3) gives the existence of a sequence $(C_{F_l,l})_{l\in\mathbb{Z}}$ of constants that only depend on φ and such that

$$\lim_{\varepsilon\to 0} S^\varepsilon = |B_M| \sum_{l\in\mathbb{Z}} C_{F_l,l} = \left(\int_{\mathbb{R}} g \right) \sum_{l\in\mathbb{Z}} C_{F_l,l}.$$

We note that the sum is well defined since the number of terms such that $C_{F_l,l} \neq 0$ is finite, given the compact support of φ.

This one-dimensional example therefore suffices to show that we need two specific properties regarding the distribution of the sequence Z, namely :
(i) the existence of an average to have the strict sublinearity at infinity of the corrector, which assumption is the point of (A1), and
(ii) an assumption regarding the correlations of Z to ensure the weak convergence of the sequence $|\nabla w(./\varepsilon)|^2$, which is the point of (A3) (and of (A2.a) which is implied by (A3)).

We however note that if $\int_{\mathbb{R}} \varphi = 0$, the existence of an average for Z is not required to obtain the strict sublinearity of w at infinity. This phenomenon will also be true for higher dimensions (see Section 4.3.3).

4.2.2 Taylor expansion of V

As specified in the introductory section, the study of the gradient of a solutions to (4.12) is related to the convergence of sums of the form $\sum_{k\in\mathbb{Z}^d} \nabla (G * \psi_k)$, where ψ_k is a compactly supported function for every $k \in \mathbb{Z}^d$ that depends on φ (or its derivatives) and on Z_k. For $\psi \in \mathcal{D}(\mathbb{R}^d)$, it is therefore necessary to first recall the behavior of $u = G * \psi$, solution to $\Delta u = \psi$. The following elementary lemma recalls the answer to this question when the first or the first two moments of ψ vanish. It will be essential throughout our study.

Lemma 4.1. *Assume $d \geq 2$. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx = 0$. Then, if we denote $u = G * \psi$, there exists a constant $C_1 > 0$ such that for every $x \in \mathbb{R}^d$*

$$|\nabla u(x)| \leq \frac{C_1}{1+|x|^d}. \tag{4.26}$$

If we additionally assume that $\int_{\mathbb{R}^d} x\psi(x)dx = 0$, there exists a constant $C_2 > 0$ such that

$$|\nabla u(x)| \leq \frac{C_2}{1+|x|^{d+1}}. \tag{4.27}$$

Proof. For $d \geq 2$ and every $i \in \{1, ..., d\}$, we know that $G \in L^1_{loc}(\mathbb{R}^d)$ and $\partial_i G(x) = C(d)\dfrac{x_i}{|x|^d} \in L^1_{loc}(\mathbb{R}^d)$, where $C(d)$ only depends on the ambient dimension d. In particular, since ψ is compactly supported, $u = G * \psi$ is well-defined and, for every $i \in \{1, ..., d\}$,

$$\partial_i u = \int_{\mathbb{R}^d} C(d) \frac{x_i - y_i}{|x-y|^d} \psi(y)dy.$$

We denote by $A > 0$ a radius such that the support of ψ is included in B_A. An asymptotic expansion when $|x| >> A$ shows that, for every $y \in B_A$,

$$\frac{1}{|x-y|^d} = \frac{1}{(|x|^2 - 2\langle x,y\rangle + |y|^2)^{d/2}} = \frac{1}{|x|^d}\left(1 + d\frac{\langle x,y\rangle}{|x|^2}\right) + O\left(\frac{1}{|x|^{d+2}}\right),$$

the remainder of this expansion being uniformly bounded with respect to $y \in B_A$. Since ψ is supported in B_A, we can perform this expansion in the integral above when $|x| >> A$. Since $\displaystyle\int_{\mathbb{R}^d} \psi = 0$, we deduce

$$\begin{aligned}\int_{B_A} \frac{x_i - y_i}{|x-y|^d}\psi(y)dy &= \int_{B_A}\left(\frac{x_i}{|x|^d} + d\frac{x_i\langle x,y\rangle}{|x|^{d+2}} - \frac{y_i}{|x|^d}\right)\psi(y)dy + O\left(\frac{1}{|x|^{d+1}}\right)\\ &= \int_{B_A}\left(d\frac{x_i\langle x,y\rangle}{|x|^{d+2}} - \frac{y_i}{|x|^d}\right)\psi(y)dy + O\left(\frac{1}{|x|^{d+1}}\right)\\ &= O\left(\frac{1}{|x|^d}\right).\end{aligned}$$

We therefore obtain (4.26). In addition, when $\displaystyle\int_{\mathbb{R}^d} x\psi = 0$, we also have

$$\frac{1}{|x|^d}\int_{B_A} y_i\psi(y)dy = 0,$$

and

$$\int_{B_A} \frac{x_i\langle x,y\rangle}{|x|^{d+2}}\psi(y)dy = \frac{x_i}{|x|^{d+2}}\left\langle x, \int_{B_A} y\psi(y)dy\right\rangle = 0.$$

We obtain $\displaystyle\int_{B_A} \frac{x_i - y_i}{|x-y|^d}\psi(y)dy = O\left(\frac{1}{|x|^{d+1}}\right)$, which shows (4.27).

$\square$

We now present our approach to solve (4.12), or more precisely (4.19). The specific structure of V first allows us to perform a Taylor expansion :

$$\begin{aligned}V(x) &= g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(x-k) - Z_k.\nabla\varphi(x-k)\\ &\quad + \int_0^1 (1-t)Z_k^T D^2\varphi(x-k-tZ_k)Z_k dt,\end{aligned}$$

as already briefly mentioned in (4.15). By linearity, the corrector equation can therefore be split into three different equations :

(a) $\Delta w_{per} = V_{per}$, where $V_{per} = g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(.-k)$ is a Q-periodic function such that $\langle V_{per}\rangle = 0$ as a consequence of (4.10).

(b) $\Delta w_1 = V_1$, where $V_1 = \sum_{k\in\mathbb{Z}^d} Z_k.\nabla\varphi(.-k)$ is the first order term of the Taylor expansion and is linear with respect to Z.

(c) $\Delta w_2 = V_2$, where $V_2 = \sum_{k\in\mathbb{Z}^d} \int_0^1 (1-t) Z_k^T D^2\varphi(.-k-tZ_k) Z_k dt$ is the remainder of the Taylor expansion and is nonlinear with respect to Z.

In the sequel, the proof of Theorem 4.1 will consequently consist in solving each of these equations and in showing the expected properties of weak convergence satisfied by the gradient of their solution. Each of these equations is put under the form $\Delta u = \sum_{k\in\mathbb{Z}^d} \psi_k$ for a specific potential where $\psi_k \in \mathcal{D}(\mathbb{R}^d)$ depends on φ, k and Z_k. In particular, the difficulties to study the gradients of their solution will be related to the convergence of the sum $\sum_{k\in\mathbb{Z}^d} \nabla (G \star \psi_k)$ and are various in nature.

Since V_{per} is a periodic potential with a vanishing average, the existence of a periodic solution w_{per}, therefore strictly sublinear at infinity, is well known. So (a) is easily solved.

The third equation (c) is associated with the remainder of the Taylor expansion. As informally announced in the introduction, Lemma 4.1 shows that the presence of high order derivatives of φ in V_2 ensures the normal convergence of the sum $\sum_{k\in\mathbb{Z}^d} \nabla \left(G * D^2\varphi\right)(.-k)$. This allows to make explicit ∇w_2 and to show the weak convergence of $\nabla w_2(./\varepsilon)$ to 0. However, the nonlinearity of V_2 with respect to Z requires a strong assumption such as (A3) to obtain the expected convergence of $|\nabla w_2(./\varepsilon)|^2$. This problem is addressed in Section 4.4.

As for the second equation (b), the gradient of a solution to $\Delta w_1 = V_1$, associated with the first order term of the Taylor expansion, is related to the potential convergence of the sum $\sum_{k\in\mathbb{Z}^d} \nabla (G * Z_k.\nabla\varphi)(.-k)$. Since the integral of $\nabla\varphi$ vanishes, Lemma 4.1 also implies that the function $\nabla (G * Z_k.\nabla\varphi)(.-k)$ generically formally behaves as $\dfrac{Z_k}{|x-k|^d}$. Most of our arguments to come therefore consist in showing first that the specific properties of the Calderón-Zygmund operator $T : f \mapsto \nabla^2 G * f$, particularly its continuity from $L^\infty(\mathbb{R}^d)$ to $BMO(\mathbb{R}^d)$, allow to prove the convergence of this sum in $BMO(\mathbb{R}^d)$ and, secondly, that the properties of $BMO(\mathbb{R}^d)$ together with our assumptions (A1), (A2.a), (A2.b) and (A2.c) ensure its weak-convergence on B_R/ε, provided we subtract a certain ε-dependent constant. More precisely, in Proposition 4.2 which is established in Section 4.3, we shall obtain the expected weak-convergences for some $\nabla \tilde{W}_{\varepsilon,R} = \nabla w_1 - \fint_{B_R/\varepsilon} \nabla w_1$. This is sufficient to finally construct the ε-dependent corrector of Theorem 4.1 and perform the homogenization of problem (4.1).

4.2.3 Examples of suitable sequences Z

We give here some examples of sequences Z that satisfy Assumptions (A1) to (A3). We begin by proving a technical property related to Assumption (A2.c).

Proposition 4.1. *Assume $d \geq 2$. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and Z be a sequence of $l^\infty(\mathbb{Z}^d)$. Assume there exists $\alpha > 1$ and $C > 0$ such that for every $R > 0$ and $x_0 \in \mathbb{R}^d$,*

$$\left| \frac{1}{|B_R|} \sum_{k\in B_R(x_0)} Z_k \right| \leq \frac{C}{(\ln(1+R))^\alpha}. \tag{4.28}$$

Then, for every $i, j \in \{1, ..., d\}$, the sequence of functions $x \mapsto \sum_{|k| \leq M} Z_k(\partial_i \partial_j G * \varphi)(x-k)$ converges in $L^\infty_{loc}(\mathbb{R}^d)$ when $M \to \infty$.

Proof. In this proof we denote $\mathcal{K}_{i,j} = (\partial_i \partial_j G * \varphi)$ for every $i, j \in \{1, ..., d\}$. We define $|x|_\infty = \sup_{i \in \{1,...,d\}} |x_i|$ and $Q_M = \left\{x \in \mathbb{R}^d \,\middle|\, |x|_\infty < M\right\}$ for $M > 0$. We first show the result for the subsequence $(M_N)_{N\in\mathbb{N}} = (N^2)_{N\in\mathbb{N}}$. We remark that if $\mathcal{Q}_l = \prod_{i=1}^{d} [|l_i|^2, |l_i+1|^2]$ for $l \in \mathbb{Z}^d$, then $Q_{N^2} = \bigcup_{n=0}^{N} \bigcup_{|l|_\infty = n} \mathcal{Q}_l$. Since $(l_i+1)^2 - l_i^2 = O(l_i)$ for every $i \in \{1, ..., d\}$, we have that $|\mathcal{Q}_l| = O(n^d)$ when $|l|_\infty = n$. We next consider a compact subset $K \subset \mathbb{R}^d$ and we fix $x \in K$. For $l \in \mathbb{Z}^d$ and $k \in \mathcal{Q}_l$, we also define $\mathcal{M}_l(Z) = \frac{1}{|\mathcal{Q}_l|} \sum_{q \in \mathcal{Q}_l} Z_q$ and $\omega_{l,k} = \frac{1}{|\mathcal{Q}_l|} \sum_{q \in \mathcal{Q}_l} (Z_k - Z_q)$ and we have

$$\begin{aligned}
\sum_{|k|_\infty \leq N^2} Z_k \mathcal{K}_{i,j}(x-k) &= \sum_{n=0}^{N} \sum_{|l|_\infty = n} \sum_{k \in \mathcal{Q}_l} Z_k \mathcal{K}_{i,j}(x-k) \\
&= \sum_{n=0}^{N} \sum_{|l|_\infty = n} \sum_{k \in \mathcal{Q}_l} \omega_{l,k} \mathcal{K}_{i,j}(x-k) + \sum_{n=0}^{N} \sum_{|l|_\infty = n} \sum_{k \in \mathcal{Q}_l} \mathcal{M}_l(Z) \mathcal{K}_{i,j}(x-k) \\
&= S_N^1(x) + S_N^2(x).
\end{aligned}$$

We remark that $\sum_{k \in \mathcal{Q}_l} \omega_{l,k} = 0$. For every $l \in \mathbb{Z}^d$, we consider a point k_l of $\mathcal{Q}_l \cap \mathbb{Z}^d$ that can be chosen arbitrarily and we deduce

$$S_N^1(x) = \sum_{n=0}^{N} \sum_{|l|_\infty = n} \sum_{k \in \mathcal{Q}_l \setminus \{k_l\}} \omega_{l,k} \left(\mathcal{K}_{i,j}(x-k) - \mathcal{K}_{i,j}(x-k_l)\right).$$

When $|l|$ is sufficiently large, using that φ is compactly supported, we have for $k \in \mathcal{Q}_l$:

$$\mathcal{K}_{i,j}(x-k) - \mathcal{K}_{i,j}(x-k_l) = \int_{\mathbb{R}^d} \left(\partial_i \partial_j G(x-k-y) - \partial_i \partial_j G(x-k_l-y)\right) \varphi(y) dy.$$

Moreover, for every $u, v \in \mathbb{R}^d$ such that $|u| > 2|v|$, the result of [47, Lemma 7.18 p.151] implies the existence of a constant $C_1 > 0$ such that $|\partial_i \partial_j G(u-v) - \partial_i \partial_j G(u)| \leq C_1 \frac{|v|}{|u|^{d+1}}$. We deduce the existence of a constant $C_2 > 0$ such that for $n \in \mathbb{N}$ sufficiently large, $l \in \mathbb{Z}^d$ such that $|l|_\infty = n$ and $k \in \mathcal{Q}_l$,

$$|\mathcal{K}_{i,j}(x-k) - \mathcal{K}_{i,j}(x-k_l)| \leq C_2 \frac{|k-k_l|}{n^{2(d+1)}} \leq C_2 \frac{Diam(\mathcal{Q}_l)}{n^{2(d+1)}}.$$

Since $(l_i+1)^2 - l_i^2 = 2l_i + 1$ for every $i \in \{1, ...d\}$, we can show the existence of $C_3 > 0$ such that for every $n \in \mathbb{N}$ and $l \in \mathbb{Z}^d$ such that $|l|_\infty = n$, the diameter of $\mathcal{Q}_l$ is bounded by $C_3 n$ and we obtain

$$|\mathcal{K}_{i,j}(x-k) - \mathcal{K}_{i,j}(x-k_l)| \leq C_2 C_3 \frac{1}{n^{2d+1}}. \tag{4.29}$$

In addition, since $Z \in l^\infty(\mathbb{Z}^d)$, $\omega_{l,k}$ is uniformly bounded with respect to l and k. We also know that the number of $l \in \mathbb{Z}^d$ such that $|l|_\infty = n$ is a $O(n^{d-1})$. Using also that $|\mathcal{Q}_l| = O(n^d)$, (4.29) therefore gives the existence of $C > 0$ independent of n, l and k such that :

$$\left| \sum_{|l|_\infty = n} \sum_{k \in \mathcal{Q}_l \setminus \{k_l\}} \omega_{l,k} \left(\mathcal{K}_{i,j}(x-k) - \mathcal{K}_{i,j}(x-k_l) \right) \right| \leq \sup_{l,k} |\omega_{l,k}| \sum_{|l|_\infty = n} C_2 C_3 \frac{|\mathcal{Q}_l|}{n^{2d+1}} \leq C \frac{1}{n^2}. \tag{4.30}$$

It follows that the sum S_N^1 normally converges in $L^\infty_{loc}(\mathbb{R}^d)$.

We next study the convergence of S_N^2. If we denote $x_l = \left(l_i^2 + l_i + \frac{1}{2} \right)_{i \in \{1,\ldots,d\}}$, we have $\mathcal{Q}_l = \prod_{i=1}^d \left[-l_i - \frac{1}{2}, l_i + \frac{1}{2} \right] + x_l$. Since $\mathcal{K}_{i,j} = \partial_i \left(G * \partial_j \varphi \right)$ and $\int_{\mathbb{R}^d} \partial_j \varphi = 0$, Lemma 4.1 shows the existence of $C_4 > 0$ such that $|\mathcal{K}_{i,j}(x)| \leq \frac{C_4}{1 + |x|^d}$. A direct consequence of (4.28) also yields the existence of $C_5 > 0$ such that for every $l \in \mathbb{Z}^d$, we have

$$|\mathcal{M}_l(Z)| = \left| \frac{1}{|\mathcal{Q}_l|} \sum_{q \in \mathcal{Q}_l} Z_q \right| \leq \frac{C_5}{\ln(1 + |l|)^\alpha}.$$

We obtain

$$\left| \sum_{|l|_\infty = n} \sum_{k \in \mathcal{Q}_l} \mathcal{M}_l(Z) \mathcal{K}_{i,j}(x-k) \right| \leq C \sum_{|l|_\infty = n} \frac{|\mathcal{Q}_l|}{n^{2d} \ln(1+n)^\alpha} \leq \tilde{C} \frac{1}{n \ln(1+n)^\alpha}, \tag{4.31}$$

where C and $\tilde{C}$ are independent of n. Since $\alpha > 1$, we deduce that the sum S_N^2 normally converges in $L^\infty_{loc}(\mathbb{R}^d)$. We have finally established the convergence of $\sum_{|k| \leq N^2} Z_k \mathcal{K}_{i,j}(x-k)$ in $L^\infty_{loc}(\mathbb{R}^d)$ when $N^2 \to \infty$.

To conclude, for $M > 0$, we denote by $[M]$ the integer part of M. We have

$$\sum_{|k| \leq M} Z_k \mathcal{K}_{i,j}(x-k) = \sum_{[\sqrt{M}]^2 < |k| \leq M} Z_k \mathcal{K}_{i,j}(x-k) + \sum_{|k| \leq [\sqrt{M}]^2} Z_k \mathcal{K}_{i,j}(x-k).$$

We have shown above the convergence of the second sum of the right-hand term. For the other sum, we first remark that $[\sqrt{M}]^2 \overset{M \to \infty}{\sim} M$ and $[\sqrt{M}]^2 \leq M < [\sqrt{M}]^2 + 1$, which imply that $\left| B_M \setminus B_{[\sqrt{M}]^2} \right| = O(M^{d-1})$. Using again that $|\mathcal{K}_{i,j}(x)| \leq \frac{C_4}{1+|x|^d}$, we have the existence of $C > 0$ such that for every x in a compact subset K :

$$\begin{aligned} \left| \sum_{[\sqrt{M}]^2 < |k| \leq M} Z_k \mathcal{K}_{i,j}(x-k) \right| &\leq C_4 \sum_{[\sqrt{M}]^2 < |k| \leq M} \frac{|Z_k|}{|x-k|^d} \\ &\leq C \|Z\|_{l^\infty(\mathbb{Z}^d)} \sum_{[\sqrt{M}]^2 < |k| \leq M} \frac{1}{M^d} = O(M^{-1}) \overset{M \to \infty}{\longrightarrow} 0. \end{aligned}$$

We can conclude that $\sum_{|k|\leq M} Z_k \mathcal{K}_{i,j}(x-k)$ converges in $L^\infty_{loc}(\mathbb{R}^d)$ when $M \to \infty$. $\square$

Remark 4.1. *Assumption (4.28) has two important parts : a logarithmic rate of convergence for the sequence of the partial averages and the uniformity of this rate with respect to the center x_0 of the ball $B_R(x_0)$. They ensure the convergence of the two sums that appear respectively in the rightmost inequality of (4.30) and in the rightmost inequality of (4.31). Actually, we can remark that the result of Proposition 4.1 still holds under the weaker assumption*

$$\left| \frac{1}{|B_R|} \sum_{k \in B_R(R^2 x_0)} Z_k \right| \leq \frac{C_K}{(\ln(1+R))^\alpha},$$

for every x_0 belonging to a compact subset of $\mathbb{R}^d$ denoted by K in the above inequality.

We are now in position to introduce several examples of sequences Z that satisfy our assumptions.

To start with, we mention for completeness two *elementary* examples for which the homogenization of (4.1)-(4.6) can be easily addressed : the case of periodic sequences and the case of local defects such that Z_k *rapidly* decreases when $|k| \to \infty$. The first setting is, of course, periodic homogenization theory. We define the second setting as that for which $Z \in \left(l^p(\mathbb{Z}^d)\right)^d$ for some $p \in [1, +\infty[$. The potentials V_1 and V_2 respectively defined in (b) and (c) are then both in $L^p(\mathbb{R}^d)$ (as a consequence of a discrete Young-type inequality together with the fact that $Z \in \left(l^\infty(\mathbb{Z}^d)\right)^d$) and the existence of a corrector is then provided by the classical properties of the Laplace operator. We now check in details our general assumptions cover these two settings.

1) Periodic sequences :

If Z_k is periodic, $F(Z_k, Z_{k+l})$ is also periodic for every continuous function F and it is well-known that, for every $R > 0$ and $x_0 \in \mathbb{R}^d$, the sequence $\left(\frac{\varepsilon^d}{|B_R|} \sum_{k \in B_R(x_0)/\varepsilon} F(Z_k, Z_{k+l}) \right)_{\varepsilon > 0}$ converges to the average of $F(Z_k, Z_{k+l})$ (in the sense of periodic sequence) and the convergence rate is at least ε uniformly with respect to R, x_0 and l. Z therefore satisfies Assumptions (A1), (A2.a), (A2.b) and (A3). In addition, since the sequence $(Z_{k+l} - \langle Z \rangle)_{l \in \mathbb{Z}^d}$ is periodic with a vanishing average for every $k \in \mathbb{Z}^d$, the sequence $C_{l,i,j}$ given in (A2.a) is also periodic with a vanishing average. Using again that the convergence rate of $\frac{\varepsilon^d}{|B_R|} \sum_{l \in B_R(x_0)/\varepsilon} C_{l,i,j}$ is at least ε uniformly with respect to x_0, Assumption (A2.c) is therefore a consequence of Proposition 4.1.

2) Sequences $Z \in \left(l^p(\mathbb{Z}^d)\right)^d$, for some $p \in [1, +\infty[$:

In this case, since $\lim_{|k|\to\infty} Z_k = 0$, the sequence $\frac{\varepsilon^d}{|B_R|} \sum_{k \in B_R(x_0)/\varepsilon} F(Z_k, Z_{k+l})$ converges to $F(0,0)$ for every continuous function F and we have (A1), (A2.a) and (A3). Moreover, since

$C_{l,i,j} = 0$ (with the notation of (A2.a)), Z clearly satisfies (A2.c) and, using that $Z \in \left(l^p(\mathbb{Z}^d)\right)^d$ and the Hölder inequality, we can easily show that (A2.b) holds with $\delta(\varepsilon) = 0$ and $\gamma(\varepsilon) = O(\varepsilon^{d/p})$.

We next introduce several examples of sequences Z that model both local and non-local perturbations and for which the homogenization of (4.1) with potentials of the form (4.6) will be addressed in the present work.

3) Sequences Z that only *slowly* converge to 0 when $|k| \to \infty$:

These are sequences such that, as in the previous example, (A1), (A2.a), (A2.c) and (A3) are satisfied since $\lim\limits_{|k|\to\infty} Z_k = 0$ and for which, moreover, Assumption (A2.b) holds, since we may ensure that $|Z_k| = O\left(\ln(|k|)^{-\alpha}\right)$ for $\alpha > \dfrac{1}{2}$. We however may take such sequences such that $Z \notin \left(l^p(\mathbb{Z}^d)\right)^d$ for any $p \in [1, +\infty[$.

4) Deterministic approximations of random variables :

Such sequences are deterministic sequences Z that share the property of i.i.d sequences of random variables and are commonly used to simulate random processes. They are some low-discrepancy sequences. We refer to [41, 45, 63] for an overview of the theory of deterministic approximation of random sequences.
A particular example in dimension $d = 1$ is given by $(Z_k)_{k\in\mathbb{N}} = \{k\,\theta^p\}$ (where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$) for a fixed integer $p \geq 2$ and for almost all irrational number $\theta \in \mathbb{R}$, see [45, Section 4] for details. Such a sequence is not periodic, not even almost periodic. It is dense in $[0, 1]$ and simulates a realization of uniform distribution on $[0, 1]$. More precisely, the results of [37, 66] and [63, Theorem 5.1 p.41] ensure that

$$\forall F \in \mathcal{C}^0(\mathbb{R}), \quad \lim_{\varepsilon\to 0} \frac{\varepsilon}{R} \sum_{k=[x_0/\varepsilon]}^{[R/\varepsilon + x_0/\varepsilon]} F(Z_k) = \int_0^1 F(t)dt,$$

$$\forall G \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R}), \forall l \neq 0,\ \lim_{\varepsilon\to 0} \frac{\varepsilon}{R} \sum_{k=[x_0/\varepsilon]}^{[R/\varepsilon + x_0/\varepsilon]} G(Z_k, Z_{k+l}) = \int_0^1 \int_0^1 G(t,u)dtdu.$$

This show that our Assumptions (A1),(A2.a) and (A3) are satisfied. In particular, for (A2.a), we have

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{R} \sum_{k=[x_0/\varepsilon]}^{[R/\varepsilon + x_0/\varepsilon]} Z_k Z_{k+l} = 0,$$

which directly implies (A2.c). A deterministic equivalent of the law of the iterated logarithm (which can be established using the results and the methods introduced in [58], [41, Theorem 1.193, p.198] and [63, Chapter 2]) also ensures

$$\sup_{l\neq 0} \left| \frac{\varepsilon}{R} \sum_{k=[x_0/\varepsilon]}^{[R/\varepsilon + x_0/\varepsilon]} Z_k Z_{k+l} \right| \leq C \frac{\sqrt{\varepsilon}}{\ln(\ln(\varepsilon))},$$

and implies that Assumption (A2.b) is satisfied with $\delta(\varepsilon) = 0$ and $\gamma(\varepsilon) = \dfrac{\sqrt{\varepsilon}}{\ln(\ln(\varepsilon))}$. All these results can, of course, be generalized in higher dimensions considering the vectors

$$(\{kd\theta^p\}, \{(kd+1)\theta^p\}, ..., \{(kd+d-1)\theta^p\})_{k\in\mathbb{N}},$$

for $p > 2d$, such as in [45, Theorem 21].

5) Some other non periodic sequences that do not vanish at infinity :

We give an example of such a sequence in dimension $d = 2$:

$$Z_{(k_1,k_2)} = \left(\cos(\sqrt{2}k_1), \sin(\sqrt{2}k_2)\right).$$

In this case, for every $a, b \in \mathbb{Z}$ $N \in \mathbb{N}$, we have

$$\frac{1}{4N^2}\sum_{k_1=-N+a}^{k_1=N+a}\sum_{k_2=-N+b}^{k_2=N+b}\cos(\sqrt{2}k_1) = \frac{\cos(\sqrt{2}a)\sin\left(\dfrac{2N+1}{\sqrt{2}}\right)}{2N\sin\left(\dfrac{1}{\sqrt{2}}\right)},$$

and

$$\frac{1}{4N^2}\sum_{k_1=-N+a}^{k_1=N+a}\sum_{k_2=-N+b}^{k_2=N+b}\sin(\sqrt{2}k_2) = \frac{\sin(\sqrt{2}b)\sin\left(\dfrac{2N+1}{\sqrt{2}}\right)}{2N\sin\left(\dfrac{1}{\sqrt{2}}\right)}.$$

This implies that (A1) is satisfied by Z_k and there exists $M > 0$ such that for every $R > 0$ and $x_0 \in \mathbb{R}^d$,

$$\left|\frac{1}{|B_R|}\sum_{k\in B_R(x_0)} Z_k\right| \leq \frac{M}{R}.$$

Similarly, a direct calculation that we omit here shows the existence of $M_1 > 0$ and of a family of constants $C_{l,i,j}(n,m) \in \mathbb{R}$ for $l \in \mathbb{Z}^2$, $i, j \in \{1,2\}$, $n, m \in \mathbb{N}$, such that, for every $x_0 \in \mathbb{R}^2$ and $R > 0$, we have

$$\left|\frac{1}{|B_R|}\sum_{k\in B_R(x_0)} (Z_k)_i^n (Z_{k+l})_j^m - C_{l,i,j}(n,m)\right| \leq \frac{M_1}{R}. \tag{4.32}$$

The $C_{l,i,j}(n,m)$ are linear combinations of $\left(\cos(h\sqrt{2}l_1)\right)_{h\in\mathbb{N}}$, $\left(\cos(h\sqrt{2}l_2)\right)_{h\in\mathbb{N}}$, $\left(\sin(h\sqrt{2}l_1)\right)_{h\in\mathbb{N}}$ and $\left(\sin(h\sqrt{2}l_2)\right)_{h\in\mathbb{N}}$. In particular, for every $l \in \mathbb{Z}^2$:

$$C_l = C_l(1,1) = \begin{bmatrix} \dfrac{\cos(\sqrt{2}l_1)}{2} & 0 \\ 0 & \dfrac{\cos(\sqrt{2}l_2)}{2} \end{bmatrix}.$$

This implies that Z_k satisfies (A2.a) and (A2.b) and, using the density of the polynomial functions in the set of continuous functions, (4.32) also shows (A3). In addition, a direct calculation again shows the existence of $M_2 > 0$ such that, for every $R > 0$ and $x_0 \in \mathbb{R}^2$, we have

$$\left| \frac{1}{|B_R|} \sum_{k \in B_R(x_0)} C_l \right| \leq \frac{M_2}{R}.$$

Proposition 4.1 then ensures that Assumption (A2.c) is also satisfied.

4.3 Corrector equation : the first-order equation (b)

This section is devoted to the linear equation :

$$\Delta w_1 = \sum_{k \in \mathbb{Z}^d} Z_k.\nabla\varphi = \operatorname{div}(\mathcal{V}), \tag{4.33}$$

where we have denoted by

$$\mathcal{V} = \sum_{k \in \mathbb{Z}^d} Z_k \, \varphi(x-k). \tag{4.34}$$

The existence of a solution to (4.33), which is the equation obtained for the subproblem (b) in our decomposition of Section 4.2.2 of our original corrector equation (4.12), is related to the convergence of $\sum_{k \in \mathbb{Z}^d} \nabla^2 G * (Z_k \varphi(.-k))$ and thus to the continuity of the Riesz operator $Tf = \nabla^2 G * f$ from $\left(L^\infty(\mathbb{R}^d)\right)^d$ to the space $BMO(\mathbb{R}^d)$ of functions with bounded mean oscillations. We will not solve (4.33) itself but we will find a solution on every ball of radius $\frac{1}{\varepsilon}$. To study the properties of weak convergence satisfied by the gradient of this solution, we next use several properties of $BMO(\mathbb{R}^d)$ together with the specific properties of the sequence Z ensured by assumptions (4.7), (A1), (A2.a), (A2.b) and (A2.c). In addition, although we consider here a generic pair (φ, Z) where Z is only assumed to satisfy (4.7), (A1), (A2.a), (A2.b) and (A2.c), there exist several specific choices of function φ and sequence Z_k for which the study of (4.33) is actually simpler. We give some examples of such simpler settings in Section 4.3.3.

4.3.1 Some preliminary results

Here, we introduce several preliminary results related to the study of equation (4.33). These results are elementary and classical but we include them here for completeness. Our approach being based on the continuity of the Riesz operator from $\left(L^\infty(\mathbb{R}^d)\right)^d$ to $\left(BMO(\mathbb{R}^d)\right)^d$, we begin with two preliminary technical lemmas related to the functions of $BMO(\mathbb{R}^d)$: Lemma 4.2 regards the harmonic functions with a gradient in $\left(BMO(\mathbb{R}^d)\right)^d$ and Lemma 4.3 shows a property of convergence satisfied by the functions that weakly converge to 0 in $BMO(\mathbb{R}^d)$. In Lemma 4.4 below, we also recall a classical result regarding the strict sublinearity at infinity of functions u such that $\nabla u(./\varepsilon)$ weakly converges to 0 as ε vanishes. We finally conclude this section with Lemma 4.5 which is related to the weak convergence of the functions defined as in (4.34) and is the direct generalization of the calculation (4.24) performed for $d = 1$.

Lemma 4.2. *Let $v \in L^1_{loc}(\mathbb{R}^d)$ be a solution to $\Delta v = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ such that $\nabla v \in \left(BMO(\mathbb{R}^d)\right)^d$. Then ∇v is constant.*

Proof. Differentiating the equation, for every $i \in \{1, ...d\}$, we have $\Delta \partial_i v = 0$. Since $\partial_i v \in BMO(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, $\partial_i v$ is an harmonic tempered distribution and it is therefore a polynomial function (see [5, Example 4.11 p.142] for instance). The only polynomials that belong to $BMO(\mathbb{R}^d)$ being the constants, we can conclude. □

Lemma 4.3. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of $BMO(\mathbb{R}^d)$ such that v_n converges to 0 for the weak$-\star$ topology of $BMO(\mathbb{R}^d)$. Then, for every compactly supported function $g \in L^\infty(\mathbb{R}^d)$ such that $\displaystyle\int_{\mathbb{R}^d} g = 1$, we have*

$$v_n - \int_{\mathbb{R}^d} g v_n \overset{n\to\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$, we have :

$$\begin{aligned}\left\langle v_n - \int_{\mathbb{R}^d} g v_n, \phi \right\rangle_{\mathcal{D}',\mathcal{D}} &= \int_{\mathbb{R}^d} v_n \phi - \left(\int_{\mathbb{R}^d} \phi\right) \int_{\mathbb{R}^d} v_n g \\ &= \int_{\mathbb{R}^d} v_n \left(\phi - g \int \phi\right).\end{aligned}$$

We note that $\psi = \phi - g \displaystyle\int_{\mathbb{R}^d} \phi$ belongs to $L^\infty(\mathbb{R}^d)$, is compactly supported and its integral vanishes. Therefore (see [47, Section 6.4.1]), ψ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$, that is to the predual of $BMO(\mathbb{R}^d)$ (see [88, Chapter IV] for details). Since, by assumption, v_n converges to 0 for the weak-$\star$ topology of BMO, it follows that the right-hand side in the latter equality converges to 0 when $n \to \infty$ and we conclude. □

Lemma 4.4. *Let $u \in L^1_{loc}(\mathbb{R}^d)$ such that u(0) = 0 and denote by $v_\varepsilon = \varepsilon u(./\varepsilon)$. Assume that ∇v_ε converges to 0 in $\mathcal{D}'(\mathbb{R}^d)$ when $\varepsilon \to 0$ and that there exists $R > 0$ and $p > d$ such that ∇v_ε is bounded in $L^p(B_{4R})$, uniformly with respect to ε. Then*

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty(B_R)} = 0.$$

Proof. The Morrey inequality (see for instance [42, p.268]) gives the existence of a constant $C > 0$ independent of ε such that for every y, x in B_{2R}

$$|v_\varepsilon(x) - v_\varepsilon(y)| \leq C|x-y|^{1-d/p} \|\nabla v_\varepsilon\|_{L^p(B_{4R})}.$$

Therefore, since ∇v_ε is bounded in $L^p(B_{4R})$ uniformly with respect to ε for $p > d$, the sequence v_ε is also bounded in $L^\infty(B_{2R})$ and equicontinuous on B_{2R}, both uniformly with respect to ε. Thus, the Arzela-Ascoli theorem shows that the sequence v_ε, up to an extraction, converges uniformly on every compact of B_{2R} to some function v. Since ∇v_ε converges to 0 in $\mathcal{D}'(\mathbb{R}^d)$, v is constant and, since $v_\varepsilon(0) = 0$ for every $\varepsilon > 0$, we necessarily have $v = 0$. Finally $v = 0$ is the only adherent value of v_ε in $L^\infty(B_R)$ and we can conclude that the whole sequence v_ε converges to 0 in $L^\infty(B_R)$. □

Lemma 4.5. *Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, Z satisfying (4.7) and (A1) and $\mathcal{V}$ be a function of the form (4.34). Then $\mathcal{V}(./\varepsilon)$ weakly converges to $\langle Z\rangle \displaystyle\int_{\mathbb{R}^d} \varphi$ in $L^\infty(\mathbb{R}^d) - \star$ as ε vanishes.*

Proof. For $M > 0$ and $x_0 \in \mathbb{R}^d$, we first introduce $g = 1_{B_M(x_0)}$, the characteristic function of $B_M(x_0)$. We have

$$\langle \mathcal{V}(./\varepsilon), g\rangle_{L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d)} = \varepsilon^d \sum_{k\in\mathbb{Z}^d} Z_k \int_{B_M(x_0)/\varepsilon} \varphi(x-k)dx.$$

We denote by $A > 0$ a radius such that φ is supported in B_A. We note that, if $k \notin B_{M/\varepsilon+A}(x_0/\varepsilon)$, we have $\varphi(x-k) = 0$ for every $x \in B_M(x_0)/\varepsilon$ and, if $k \in B_{M/\varepsilon-A}(x_0/\varepsilon)$, we have

$$\int_{B_M(x_0)/\varepsilon} \varphi(x-k)dx = \int_{\mathbb{R}^d} \varphi(x)dx.$$

Since $Z \in \left(l^\infty(\mathbb{Z}^d)\right)^d$ and the number of indices $k \in \mathbb{Z}^d$ such that k belongs to the set $B_{M/\varepsilon+A}(x_0/\varepsilon) \setminus B_{M/\varepsilon-A}(x_0/\varepsilon)$ is bounded by $C\varepsilon^{1-d}$, where C only depends on d, M and x_0, we deduce

$$\langle \mathcal{V}(./\varepsilon), g\rangle_{L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d)} = \varepsilon^d \sum_{k\in B_M(x_0)/\varepsilon} Z_k \int_{B_A} \varphi(x)dx + O(\varepsilon). \tag{4.35}$$

Assumption (A1) therefore shows that

$$\lim_{\varepsilon\to 0} \langle \mathcal{V}(./\varepsilon), g\rangle_{L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d)} = \left(\langle Z\rangle \int_{B_A} \varphi\right) |B_M(x_0)| = \left(\langle Z\rangle \int_{B_A} \varphi\right) \int_{\mathbb{R}^d} g.$$

Since g is an arbitrary characteristic function, we conclude using the density of simple (a.k.a step) functions in $L^1(\mathbb{R}^d)$. □

Remark 4.2. *When $\displaystyle\int_{\mathbb{R}^d} \varphi = 0$, we note that the above proof (equality (4.35) in particular) also shows that $\mathcal{V}(./\varepsilon)$ weakly converges to 0 without assuming (A1).*

4.3.2 Existence result

We next turn to the main proposition of this section that shows the existence of an ε-dependent solution to (4.33) on $B_{R/\varepsilon}$ for every fixed $R > 0$. In the sequel, we use the notation $\displaystyle⨍_A = \frac{1}{|A|}\int_A$ for every Borel subset A of $\mathbb{R}^d$.

Proposition 4.2. *Assume $d \geq 2$. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and Z satisfying (4.7), (A1), (A2.a), (A2.b) and (A2.c). Let $\mathcal{V}$ be a function of the form (4.34). We denote by G be the Green function associated with Δ on $\mathbb{R}^d$ and by $u_i = G * \partial_i\varphi$ for $i \in \{1, ..., d\}$. Let $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\varepsilon\to 0} \varepsilon^{-1}\beta(\varepsilon) = 0$. For every $R > 0$ and $\varepsilon > 0$, we define,*

$$\tilde{w}_\varepsilon = \sum_{i\in\{1,...,d\}} \sum_{k\in B_{\frac{1}{\beta(\varepsilon)}}} (Z_k)_i u_i(. - k),$$

and

$$\tilde{W}_{\varepsilon,R} = \tilde{w}_\varepsilon - x.\fint_{B_{4R}} \nabla \tilde{w}_\varepsilon(y/\varepsilon)dy. \tag{4.36}$$

Then, when ε *is sufficiently small,* $\tilde{W}_{\varepsilon,R}$ *is a solution to*

$$\Delta \tilde{W}_{\varepsilon,R} = \operatorname{div}(\mathcal{V}) \quad \text{on } B_{R/\varepsilon}, \tag{4.37}$$

such that $\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon) \in L^p(B_R)$ *for every* $p \in [1,+\infty[$ *and*

$$\begin{cases} \left(\nabla W_{\varepsilon,R}(./\varepsilon)\right)_{\varepsilon>0} \text{ is bounded in } L^p(B_R),\ \forall p \in [1,+\infty[, & (4.38)\\ \nabla \tilde{W}_{\varepsilon,R}(./\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{weakly in } L^p(B_R),\ \forall p \in [1,+\infty[, & (4.39)\\ \varepsilon \tilde{W}_{\varepsilon,R}(./\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{strongly in } L^\infty(B_R). & (4.40)\end{cases}$$

In addition, if we denote by $\mathcal{C}_{l,i,j}$ *the constant defined by Assumption (A2.a) for* $l \in \mathbb{Z}^d$ *and* $i,j \in \{1,...,d\}$*, and by* $\tilde{\mathcal{M}}$ *the constant*

$$\tilde{\mathcal{M}} = \sum_{i,j\in\{1,...,d\}} \sum_{l\in\mathbb{Z}^d} \mathcal{C}_{l,i,j} \int_{\mathbb{R}^d} \varphi(x)\partial_i u_j(x-l)dx,$$

then,

$$|\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)|^2 \overset{\varepsilon\to 0}{\longrightarrow} \tilde{\mathcal{M}} \quad \text{weakly in } L^p(B_R) \text{ for every } p \in [1,+\infty[. \tag{4.41}$$

Proof. The proof is rather lengthy and proceeds in four steps. Here we assume that $\langle Z\rangle = 0$ in (A1). We indeed recall that we can always assume that Z has a vanishing average without loss of generality.

Step 1 : Proof of (4.38). For every $\varepsilon > 0$, we denote $\mathcal{V}_\varepsilon = \sum_{k\in B_{\frac{1}{\beta(\varepsilon)}}} Z_k\,\varphi(.-k)$. For every $R > 0$, since $\beta(\varepsilon) = o(\varepsilon)$ and φ is compactly supported, we have $\mathcal{V}_\varepsilon = \mathcal{V}$ on $B_{R/\varepsilon}$ when ε is sufficiently small. It follows that $\tilde{w}_\varepsilon = G * \operatorname{div}(\mathcal{V}_\varepsilon)$ is a solution to (4.37) in $\mathcal{D}'(B_{R/\varepsilon})$. We next remark that

$$\nabla \tilde{w}_\varepsilon = \nabla G * \operatorname{div}(\mathcal{V}_\varepsilon) = \int_{\mathbb{R}^d} \nabla^2 G(.-y)\mathcal{V}_\varepsilon(y)dy =: T\mathcal{V}_\varepsilon.$$

It is well known that the operator $T : f \mapsto Tf$ is continuous from $\left(L^\infty(\mathbb{R}^d)\right)^d$ to $\left(BMO(\mathbb{R}^d)\right)^d$ (see [88, Section 4.2]), that is, there exists a constant $C > 0$ independent of R and ε such that :

$$\|\nabla \tilde{w}_\varepsilon\|_{BMO(\mathbb{R}^d)} \leq C\|\mathcal{V}_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C\|\mathcal{V}\|_{L^\infty(\mathbb{R}^d)}. \tag{4.42}$$

In addition, for every $p \in [1,+\infty[$, the John-Nirenberg inequality (see for instance [88, p.144]) yields the existence of a constant $c_p > 0$, independent of ε, such that

$$\sup_{M>0} \left(\fint_{B_M} \left|\nabla \tilde{w}_\varepsilon(x) - \fint_{B_M} \nabla \tilde{w}_\varepsilon(y)dy\right|^p dx\right)^{1/p} \leq c_p \|\nabla \tilde{w}_\varepsilon\|_{BMO(\mathbb{R}^d)}. \tag{4.43}$$

Using (4.42) and (4.43), we obtain

$$\left(\int_{B_{4R}}\left|\nabla\tilde{W}_{\varepsilon,R}(x/\varepsilon)\right|^p dx\right)^{1/p} = \left(\varepsilon^d\int_{B_{4R/\varepsilon}}\left|\nabla\tilde{w}_\varepsilon(x) - \fint_{B_{4R/\varepsilon}}\nabla\tilde{w}_\varepsilon(y)dy\right|^p dx\right)^{1/p}$$
$$\leq |B_{4R}|^{1/p}c_p\|\nabla\tilde{w}_\varepsilon\|^p_{BMO(\mathbb{R}^d)}$$
$$\leq |B_{4R}|^{1/p}c_pC\|\mathcal{V}\|_{L^\infty(\mathbb{R}^d)}.$$

We deduce that $\nabla\tilde{W}_{\varepsilon,R}(./\varepsilon)$ is uniformly bounded in $L^p(B_{4R})$ with respect to ε.

Step 2 : Proof of (4.39). We begin by showing that $\nabla\tilde{W}_{\varepsilon,R}(./\varepsilon)$ converges to 0 in $\mathcal{D}'(\mathbb{R}^d)$. Using (4.42), we know that $\nabla\tilde{w}_\varepsilon(./\varepsilon)$ is uniformly bounded in $\left(BMO(\mathbb{R}^d)\right)^d$ with respect to ε. Up to an extraction, it therefore converges for the weak-$\star$ topology of $\left(BMO(\mathbb{R}^d)\right)^d$ when $\varepsilon\to 0$. Its limit is also a gradient and we denote it by ∇v. Since $\tilde{w}_\varepsilon$ is solution to $\Delta\tilde{w}_\varepsilon = \operatorname{div}(\mathcal{V}_\varepsilon)$ in $\mathcal{D}'(\mathbb{R}^d)$, we have for every $\phi\in\mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\nabla\tilde{w}_\varepsilon(./\varepsilon).\nabla\phi = \int_{\mathbb{R}^d}\mathcal{V}_\varepsilon(./\varepsilon).\nabla\phi. \tag{4.44}$$

We note that Assumption (A1) with $\langle Z\rangle = 0$ and Lemma 4.5 ensure that $\mathcal{V}_\varepsilon(./\varepsilon)$ converges to 0 for the weak-$\star$ topology of $\left(L^\infty(\mathbb{R}^d)\right)^d$. Since every function $\chi\in\mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\chi = 0$ belongs to the Hardy space $\mathcal{H}^1$, the predual of BMO, we can pass to the limit in (4.44) when $\varepsilon\to 0$ and it follows

$$\int_{\mathbb{R}^d}\nabla v.\nabla\phi = 0.$$

Therefore, v is solution to $\Delta v = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ such that ∇v belongs to $\left(BMO(\mathbb{R}^d)\right)^d$. Lemma 4.2 therefore implies that ∇v is constant. The space $BMO(\mathbb{R}^d)$ being the quotient space of functions $g\in L^1_{loc}(\mathbb{R}^d)$ such that $\|g\|_{BMO(\mathbb{R}^d)} < +\infty$ modulo the space of constant functions, we have actually shown that ∇v is equal to 0 in $\left(BMO(\mathbb{R}^d)\right)^d$. We deduce that 0 is the only adherent value of $\nabla\tilde{w}_{\varepsilon,R}(./\varepsilon)$ in $BMO(\mathbb{R}^d)-\star$, and a compactness argument shows that the whole sequence $\nabla\tilde{w}_\varepsilon(./\varepsilon)$ converges to 0 for this topology. Lemma 4.3 next shows that

$$\partial_i\tilde{w}_\varepsilon(./\varepsilon) - \int_{\mathbb{R}^d}g_i\partial_i\tilde{w}_\varepsilon(y/\varepsilon)dy \overset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

for every compactly supported function $g\in\left(L^\infty(\mathbb{R}^d)\right)^d$ such that $\int_{\mathbb{R}^d}g_i = 1$ for all $i\in\{1,...,d\}$. Considering $g_i = \frac{1}{|B_{4R}|}1_{B_{4R}}$ for every $i\in\{1,...,d\}$, we obtain that $\nabla\tilde{W}_{\varepsilon,R}(./\varepsilon)$ converges to 0 in $\mathcal{D}'(\mathbb{R}^d)$. Using the density of $\mathcal{D}'(B_R)$ in $L^p(B_R)$ for every $p\in[1,+\infty[$, we finally deduce the weak convergence of $\nabla\tilde{W}_{\varepsilon,R}(./\varepsilon)$ to 0 in $L^p(B_R)$.

Step 3 : Proof of (4.40). We have shown that $\nabla\tilde{W}_{\varepsilon,R}(./\varepsilon)$ is uniformly bounded in $L^p(B_{4R})$ for every $p\in[1,+\infty[$. We note that we can always consider a solution $\tilde{W}_{\varepsilon,R}$ such that $\tilde{W}_{\varepsilon,R}(0) = 0$ without altering the properties of $\nabla\tilde{W}_{\varepsilon,R}$. The uniform convergence of $\varepsilon\tilde{W}_{\varepsilon,R}(./\varepsilon)$

to 0 in B_R is therefore a consequence of Lemma 4.4.

Step 4 : Proof of (4.41). For every $\phi \in \mathcal{D}'(B_R)$ and every $\varepsilon > 0$, multiplying (4.37) by $\chi_\varepsilon(x) = \tilde{W}_{\varepsilon,R}(x/\varepsilon)\phi(x)$ and integrating, we obtain

$$\int_{B_R} |\nabla \tilde{W}_{\varepsilon,R}|^2(./\varepsilon)\phi = \int_{B_R} \mathcal{V}(./\varepsilon).\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)\phi + \int_{B_R} \varepsilon \tilde{W}_{\varepsilon,R}(./\varepsilon)\left(\mathcal{V}(./\varepsilon) - \nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)\right).\nabla\phi.$$

Since $\varepsilon \tilde{W}_{\varepsilon,R}(./\varepsilon)$ converges to 0 in $L^\infty(B_R)$ and $\mathcal{V}(./\varepsilon) - \nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)$ is bounded in $L^2(B_R)$, uniformly with respect to ε, we have

$$\lim_{\varepsilon\to 0} \int_{B_R} \varepsilon \tilde{W}_{\varepsilon,R}(./\varepsilon)\left(\mathcal{V}(./\varepsilon) - \nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)\right).\nabla\phi = 0.$$

It is therefore sufficient to show the weak convergence of $\mathcal{V}(./\varepsilon).\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)$ to obtain the weak convergence of $|\nabla \tilde{W}_{\varepsilon,R}|^2(./\varepsilon)$ and, in this case, we have :

$$\text{weak}\lim_{\varepsilon\to 0} |\nabla \tilde{W}_{\varepsilon,R}|^2(./\varepsilon) = \text{weak}\lim_{\varepsilon\to 0} \mathcal{V}(./\varepsilon).\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon). \tag{4.45}$$

We thus study the sequence $\mathcal{V}(./\varepsilon).\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)$ when $\varepsilon \to 0$. We consider $0 < M \leq R$ and $x_0 \in \mathbb{R}^d$ such that $B_M(x_0) \subset B_R$ and we denote by $g = 1_{B_M(x_0)}$ the characteristic function of $B_M(x_0)$. We next introduce

$$\alpha(\varepsilon) = \frac{\varepsilon^d}{|B_M|} \sum_{k\in B_M(x_0)/\varepsilon} Z_k,$$

and we define $\eta(\varepsilon) = \max(\varepsilon\alpha(\varepsilon), \beta(\varepsilon), \delta(\varepsilon), \varepsilon^2)$, where $\delta(\varepsilon)$ is defined in (A2.b). In particular, Assumptions (A1) with $\langle Z\rangle = 0$ and (A2.b) show that $\lim_{\varepsilon\to 0} \varepsilon^{-1}\eta(\varepsilon) = 0$. We also denote

$$\tilde{w}_\varepsilon^\eta = \sum_{i\in\{1,\dots,d\}} \sum_{k\in B_{\frac{1}{\eta(\varepsilon)}}} (Z_k)_i u_i(. - k).$$

We now study the convergence of $\langle \mathcal{V}(./\varepsilon).\nabla W_{\varepsilon,R}(./\varepsilon), g\rangle = \int_{B_R} \mathcal{V}(x/\varepsilon).\nabla W_{\varepsilon,R}(x/\varepsilon)g(x)dx$ splitting this quantity into three terms. We indeed have $\langle \mathcal{V}(./\varepsilon).\nabla W_{\varepsilon,R}(./\varepsilon), g\rangle = I^\varepsilon - J^\varepsilon + K^\varepsilon$, where

$$I^\varepsilon = \langle \mathcal{V}(./\varepsilon).\nabla \tilde{w}_\varepsilon^\eta(./\varepsilon), g\rangle,$$
$$J^\varepsilon = \langle \mathcal{V}(./\varepsilon), g\rangle. \fint_{B_{4R/\varepsilon}} \nabla \tilde{w}_\varepsilon^\eta(y)dy,$$
$$K^\varepsilon = \left\langle \mathcal{V}(./\varepsilon).\left(\nabla(\tilde{w}_\varepsilon - \tilde{w}_\varepsilon^\eta)(./\varepsilon) - \fint_{B_{4R/\varepsilon}} \nabla(\tilde{w}_\varepsilon - \tilde{w}_\varepsilon^\eta)(y)\right), g\right\rangle.$$

In the sequel we denote by C constants independent of ε which may vary from one line to another.

Substep 4.1 : Convergence of I^ε. We remark that $I^\varepsilon = \sum_{i,j\in\{1,\dots,d\}} I^\varepsilon_{i,j}$ where

$$I^\varepsilon_{i,j} = \varepsilon^d \sum_{k\in\mathbb{Z}^d} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} (Z_k)_i (Z_l)_j \int_{B_M(x_0)/\varepsilon} \varphi(x-k)\partial_i u_j(x-l)dx. \tag{4.46}$$

We first study the convergence of the sequence :

$$\tilde{I}^\varepsilon_{i,j} = \varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}} (Z_k)_i (Z_{k+l})_j \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx, \tag{4.47}$$

where $D = |x_0| + M > 0$. In the sequel we shall show that $\lim_{\varepsilon\to 0} \tilde{I}^\varepsilon_{i,j} = \lim_{\varepsilon\to 0} I^\varepsilon_{i,j}$. We have

$$\tilde{I}^\varepsilon_{i,j} = M_1^\varepsilon + M_2^\varepsilon,$$

where

$$M_1^\varepsilon = \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}} \varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \left((Z_k)_i(Z_{k+l})_j - |B_M|\,\mathcal{C}_{l,i,j}\right) \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx,$$
$$M_2^\varepsilon = \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}} |B_M|\,\mathcal{C}_{l,i,j} \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx.$$

Since $u_j = G * \partial_j\varphi$, Lemma 4.1 shows that $|\partial_i u_j(x)| \leq \dfrac{C}{1+|x|^d}$ for every $x\in\mathbb{R}^d$. We next use Assumption (A2.b) to obtain

$$|M_1^\varepsilon| \leq C\gamma(\varepsilon) \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}} \int_{B_A} |\varphi(x)\partial_i u_j(x-l)|dx \leq C\,\gamma(\varepsilon) \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} \frac{1}{1+|l|^d} \leq C\gamma(\varepsilon)|\ln\left(\eta(\varepsilon)\right)|.$$

Since $|\ln(\eta(\varepsilon))| \leq 2|\ln(\varepsilon)|$, it follows $|M_1^\varepsilon| \leq C\gamma(\varepsilon)|\ln(\varepsilon)|$ which shows the convergence of M_1^ε to 0 due to assumption (A2.b).

We recall that $\partial_i u_j = \partial_i\partial_j G * \varphi$ and, using assumption (A2.c), we can also consider the limit in M_2^ε to obtain

$$M_2^\varepsilon \xrightarrow{\varepsilon\to 0} |B_M| \sum_{l\in\mathbb{Z}^d} \mathcal{C}_{l,i,j} \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx = \left\langle \sum_{l\in\mathbb{Z}^d} \mathcal{C}_{l,i,j} \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx, g \right\rangle.$$

We have proved that

$$\sum_{i,j\in\{1,\dots,d\}} \tilde{I}^\varepsilon_{i,j} \xrightarrow{\varepsilon\to 0} \left\langle \tilde{\mathcal{M}}, g \right\rangle.$$

We next show that $\lim\limits_{\varepsilon\to 0} \tilde{I}^{\varepsilon}_{i,j} = \lim\limits_{\varepsilon\to 0} I^{\varepsilon}_{i,j}$ (where $I^{\varepsilon}_{i,j}$ and $\tilde{I}^{\varepsilon}_{i,j}$ are respectively given by (4.46) and (4.47)). We denote by $A > 0$ a radius such that $\operatorname{Supp}(\varphi) \subset B_A$. We can remark that if $k \notin B_{M/\varepsilon+A}(x_0/\varepsilon)$, we have $\displaystyle\int_{B_M(x_0)/\varepsilon} \varphi(x-k)\partial_i u_j(x-l)dx = 0$ and if $k \in B_{M/\varepsilon-A}(x_0/\varepsilon)$,

$$\int_{B_M(x_0)/\varepsilon} \varphi(x-k)\partial_i u_j(x-l)dx = \int_{\mathbb{R}^d} \varphi(x)\partial_i u_j(x-l+k)dx.$$

We next denote $\mathbf{C}_\varepsilon = B_{\frac{M}{\varepsilon}+A}(\frac{x_0}{\varepsilon}) \setminus B_{\frac{M}{\varepsilon}-A}(\frac{x_0}{\varepsilon})$ and, using $|\partial_i u_j(x)| \leq C(1+|x|^d)^{-1}$, we have

$$\begin{aligned}
&\left|\varepsilon^d \sum_{k\in\mathbf{C}_\varepsilon} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} (Z_k)_i(Z_l)_j \int_{B_{\frac{M}{\varepsilon}}(\frac{x_0}{\varepsilon})-k} \varphi(x)\partial_i u_j(x-l+k)dx\right| \\
&\qquad\qquad \leq \varepsilon^d \|Z\|^2_{l^\infty} \|\varphi\|_{L^\infty(\mathbb{R}^d)} \sum_{k\in\mathbf{C}_\varepsilon} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} \int_{B_A} |\partial_i u_j(x-l+k)|\, dx. \\
&\qquad\qquad \leq C\varepsilon^d \sum_{k\in\mathbf{C}_\varepsilon} |\ln(\eta(\varepsilon))|.
\end{aligned}$$

Since $|\mathbf{C}_\varepsilon| = O\left(\varepsilon^{1-d}\right)$ and $|\ln(\eta(\varepsilon))| \leq 2|\ln(\varepsilon)|$, we obtain

$$\left|\varepsilon^d \sum_{k\in\mathbf{C}_\varepsilon} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} (Z_k)_i(Z_l)_j \int_{B_M(x_0)/\varepsilon-k} \varphi(x)\partial_i u_j(x-l+k)dx\right| \leq C\varepsilon\,|\ln(\varepsilon)| \overset{\varepsilon\to 0}{\to} 0.$$

It follows that

$$\begin{aligned}
I^{\varepsilon}_{i,j} &= \varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} (Z_k)_i(Z_l)_j \int_{B_A} \varphi(x)\partial_i u_j(x-l+k)dx + O\left(\varepsilon|\ln(\varepsilon)|\right) \\
&= \varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}}} (Z_k)_i(Z_{k+l})_j \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx + O\left(\varepsilon|\ln(\varepsilon)|\right) \\
&= \qquad\qquad \mathcal{N}^{\varepsilon}_{i,j} \qquad\qquad + O\left(\varepsilon|\ln(\varepsilon)|\right).
\end{aligned}$$

We recall that $D = M + |x_0|$ and for every $k \in B_M(x_0)/\varepsilon$, we have $|k| \leq \frac{D}{\varepsilon}$. Since $\eta(\varepsilon) = o\left(\varepsilon\right)$, we have for ε sufficiently small :

$$B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}} \subset B_{\frac{1}{\eta(\varepsilon)}}(-k) \subset B_{\frac{1}{\eta(\varepsilon)}+\frac{D}{\varepsilon}} \quad \text{and} \quad \left(B_{\frac{1}{\eta(\varepsilon)}}(-k) \setminus B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}\right) \subset \left(B_{\frac{1}{\eta(\varepsilon)}+\frac{D}{\varepsilon}} \setminus B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}\right).$$

We next split $\mathcal{N}^{\varepsilon}_{i,j}$ in two parts :

$$\begin{aligned}
\mathcal{N}^{\varepsilon}_{i,j} &= \varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}} (Z_k)_i(Z_{k+l})_j \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx \\
&\quad + \varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in\left(B_{\frac{1}{\eta(\varepsilon)}}(-k)\setminus B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}\right)} (Z_k)_i(Z_{k+l})_j \int_{B_A} \varphi(x)\partial_i u_j(x-l)dx \\
&= \tilde{I}^{\varepsilon}_{i,j} + R^{\varepsilon}.
\end{aligned}$$

We have

$$
\begin{aligned}
|R^\varepsilon| &\leq C\|\varphi\|_{L^\infty(\mathbb{R}^d)}\|Z\|_{l^\infty}^2\varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in\left(B_{\frac{1}{\eta(\varepsilon)}}(-k)\setminus B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}\right)} \frac{1}{1+|l|^d} \\
&\leq C\varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} \sum_{l\in\left(B_{\frac{1}{\eta(\varepsilon)}+\frac{D}{\varepsilon}}\setminus B_{\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}}\right)} \frac{1}{1+|l|^d} \\
&\leq C\left(\ln\left(\frac{1}{\eta(\varepsilon)}+\frac{D}{\varepsilon}\right)-\ln\left(\frac{1}{\eta(\varepsilon)}-\frac{D}{\varepsilon}\right)\right) \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}
$$

Since $I_{i,j}^\varepsilon = \tilde{I}_{i,j}^\varepsilon + R^\varepsilon + O(\varepsilon|\ln(\varepsilon)|)$, we finally conclude that $\lim\limits_{\varepsilon\to 0} I_{i,j}^\varepsilon = \lim\limits_{\varepsilon\to 0}\tilde{I}_{i,j}^\varepsilon$.

Substep 4.2 : Convergence of J^ε. We claim that J^ε converges to 0 when $\varepsilon\to 0$. We indeed remark that

$$
\langle \mathcal{V}(./\varepsilon), g\rangle = \varepsilon^d \sum_{k\in\mathbb{Z}^d} Z_k \int_{B_M(x_0)/\varepsilon} \varphi(x-k)dx.
$$

As in the previous substep, if $k\notin B_{M/\varepsilon+A}(x_0/\varepsilon)$, we have $\varphi(x-k)=0$ for every $x\in B_M(x_0)/\varepsilon$ and if $k\in B_{M/\varepsilon-A}(x_0/\varepsilon)$, we have $\displaystyle\int_{B_M(x_0)/\varepsilon}\varphi(x-k)dx = \int_{\mathbb{R}^d}\varphi(x)dx$. Since the number of $k\in\mathbb{Z}^d$ such that $k\in B_{M/\varepsilon+A}(x_0/\varepsilon)$ and $k\notin B_{M/\varepsilon-A}(x_0/\varepsilon)$ is proportional to ε^{1-d}, we have

$$
|\langle \mathcal{V}(./\varepsilon), g\rangle| = \left|\varepsilon^d \sum_{k\in\frac{B_M(x_0)}{\varepsilon}} Z_k \int_{B_A}\varphi(x)dx + O(\varepsilon)\right| \leq C(\alpha(\varepsilon)+\varepsilon). \tag{4.48}
$$

In addition, since $\nabla\tilde{w}_\varepsilon^\eta$ is uniformly bounded in $BMO(\mathbb{R}^d)$ as a consequence of (4.42), we have (see [88, Chapter IV, Section 1]) :

$$
\left| ⨍_{B_{4R/\varepsilon}} \nabla\tilde{w}_\varepsilon^\eta(y)dy - ⨍_{B_{\frac{1}{\eta(\varepsilon)}}} \nabla\tilde{w}_\varepsilon^\eta(y)dy\right| \leq C\left|\ln\left(\frac{\eta(\varepsilon)}{\varepsilon}\right)\right|.
$$

We recall that $\eta(\varepsilon)\geq\max(\epsilon\alpha(\varepsilon),\varepsilon^2)$, that is $\left|\ln\left(\frac{\eta(\varepsilon)}{\varepsilon}\right)\right| \leq \min(|\ln(\alpha(\varepsilon))|,|\ln(\varepsilon)|)$ when ε is sufficiently small. Using (4.48), we obtain

$$
\begin{aligned}
|\langle \mathcal{V}(./\varepsilon), g\rangle| &\left| ⨍_{B_{4R/\varepsilon}} \nabla\tilde{w}_\varepsilon^\eta(y)dy - ⨍_{B_{\frac{1}{\eta(\varepsilon)}}} \nabla\tilde{w}_\varepsilon^\eta(y)dy\right| \\
&\leq C(\alpha(\varepsilon)+\varepsilon)\min(|\ln(\alpha(\varepsilon))|,|\ln(\varepsilon)|) \xrightarrow{\varepsilon\to 0} 0.
\end{aligned} \tag{4.49}
$$

Next, since $\nabla \tilde{w}_\varepsilon^\eta = \nabla^2 G * \left(\sum_{k \in B_{\frac{1}{\eta(\varepsilon)}}} Z_k \varphi(. - k) \right) =: T \left(\sum_{k \in B_{\frac{1}{\eta(\varepsilon)}}} Z_k \varphi(. - k) \right)$, the continuity from $\left(L^2(\mathbb{R}^d)\right)^d$ to $\left(L^2(\mathbb{R}^d)\right)^d$ of the operator T (see [47, Section 7.2.3]) yields a constant $C > 0$ such that :

$$\|\nabla \tilde{w}_\varepsilon^\eta\|_{L^2(\mathbb{R}^d)} \leq C \left\| \sum_{k \in B_{\frac{1}{\eta(\varepsilon)}}} Z_k \varphi(. - k) \right\|_{L^2(\mathbb{R}^d)} \leq C \frac{1}{(\eta(\varepsilon))^{d/2}} \|\mathcal{V}\|_{L^\infty(\mathbb{R}^d)}.$$

The Cauchy-Schwarz inequality therefore gives

$$\left| \fint_{B_{\frac{1}{\eta(\varepsilon)}}} \nabla \tilde{w}_\varepsilon^\eta(y) dy \right| \leq \left| \fint_{B_{\frac{1}{\eta(\varepsilon)}}} |\nabla \tilde{w}_\varepsilon^\eta(y)|^2 dy \right|^{1/2} \leq C \left(\eta(\varepsilon)\right)^{d/2} \|\nabla \tilde{w}_\varepsilon^\eta\|_{L^2(\mathbb{R}^d)} \leq C. \tag{4.50}$$

Using (4.48) and (4.50), we deduce that :

$$|\langle \mathcal{V}(./\varepsilon), g \rangle| \left| \fint_{B_{\frac{1}{\eta(\varepsilon)}}} \nabla \tilde{w}_\varepsilon^\eta(y) dy \right| \leq C(\alpha(\varepsilon) + \varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} 0. \tag{4.51}$$

To conclude, we use a triangle inequality to bound J^ε :

$$\begin{aligned} |J^\varepsilon| \leq |\langle \mathcal{V}(./\varepsilon), g \rangle| & \left(\left| \fint_{B_{4R/\varepsilon}} \nabla \tilde{w}_\varepsilon^\eta(y) dy - \fint_{B_{\frac{1}{\eta(\varepsilon)}}} \nabla \tilde{w}_\varepsilon^\eta(y) dy \right| \right) \\ & + |\langle \mathcal{V}(./\varepsilon), g \rangle| \left| \fint_{B_{\frac{1}{\eta(\varepsilon)}}} \nabla \tilde{w}_\varepsilon^\eta(y) dy \right|. \end{aligned}$$

(4.49) and (4.51) finally show that $J^\varepsilon \to 0$ when $\varepsilon \to 0$.

Substep 4.3 : Convergence of K^ε. Using that φ is supported in B_A and proceeding as in the previous steps, we can show that the convergence of K^ε is equivalent to the convergence of

$$\tilde{K}_{i,j}^\varepsilon = \varepsilon^d \sum_{k \in \frac{B_M(x_0)}{\varepsilon}} \sum_{\frac{1}{\eta(\varepsilon)} < |l-k| < \frac{1}{\beta(\varepsilon)}} (Z_k)_i (Z_{k+l})_j \int_{B_A} \varphi(x) \left(\partial_i u_j(x - l) - \fint_{B_{4R/\varepsilon}} \partial_i u_j(y - l - k) dy \right) dx, \tag{4.52}$$

for every $i, j \in \{1, ..., d\}$. Since $u_j = G * \partial_j \varphi$, we remark that for every $x \in B_A$, $y \in B_{4R/\varepsilon}$, $k \in B_M(x_0)/\varepsilon$ and l such that $|l - k| > \frac{1}{\eta(\varepsilon)}$, we have

$$\begin{aligned} \partial_i u_j(x - l) - \partial_i u_j(y - l - k) &= \int_{B_A} \left(\partial_i \partial_j G(z - x + l) - \partial_i \partial_j G(z - y + l + k) \right) \varphi(z) dz \\ &= \int_{B_A} \left(\partial_i \partial_j G(z - x + l) - \partial_i \partial_j G(l) + \partial_i \partial_j G(l) - \partial_i \partial_j G(z - y + l + k) \right) \varphi(z) dz. \end{aligned}$$

The results of [47, Lemma 7.18 p.151] yield the existence of $C > 0$ such that for every $u, v \in \mathbb{R}^d$ with $|u| > 2|v|$, we have

$$|\partial_i\partial_j G(u-v) - \partial_i\partial_j G(u)| \leq C\frac{|v|}{|u|^{d+1}}.$$

Since $\eta(\varepsilon) = o(\varepsilon)$, for every $z \in B_A$ and x, y, k, l as above, we have $|l| > 2|z-y+k|$ and $|l| > 2|z-x|$ when ε is sufficiently small. It follows

$$\begin{aligned} \fint_{B_{4R/\varepsilon}} |\partial_i u_j(x-l) - \partial_i u_j(y-l-k)|\, dy &\leq C \fint_{B_{4R/\varepsilon}} \left(\frac{|x|}{|l|^{d+1}} + \frac{|y|}{|l|^{d+1}}\right) dy \\ &\leq C\frac{\varepsilon^{-1}}{|l|^{d+1}}. \end{aligned} \tag{4.53}$$

If we now insert (4.53) into (4.52), since Z_k and φ are uniformly bounded, we obtain

$$\left|\tilde{K}^{\varepsilon}_{i,j}\right| \leq C\varepsilon^{-1} \sum_{\frac{1}{\eta(\varepsilon)}<|l|} \frac{1}{|l|^{d+1}} \leq C\varepsilon^{-1}\eta(\varepsilon).$$

By definition of $\eta(\varepsilon)$, we have $\lim\limits_{\varepsilon\to 0} \varepsilon^{-1}\eta(\varepsilon) = 0$. We therefore conclude that $\lim\limits_{\varepsilon\to 0} K^{\varepsilon} = 0$.

Substep 4.3 : Conclusion. In the three substeps above, we have finally shown that

$$\langle \mathcal{V}(./\varepsilon).\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon), g\rangle \overset{\varepsilon\to 0}{\longrightarrow} \left\langle \tilde{\mathcal{M}}, g\right\rangle,$$

for every characteristic function g defined on B_R. Using (4.45) and the density of simple functions in $L^p(B_R)$ for every $p > 1$, we deduce that

$$\text{weak}\lim_{\varepsilon\to 0} |\nabla \tilde{W}_{\varepsilon,R}|^2(./\varepsilon) = \text{weak}\lim_{\varepsilon\to 0} \mathcal{V}.\nabla\tilde{W}_{\varepsilon,R}(./\varepsilon) = \tilde{\mathcal{M}},$$

which concludes the proof of Proposition 4.2. □

Remark 4.3. *Given the only assumptions (A1)-(A2.a)-(A2.b)-(A2.c), we cannot expect to show a uniform bound with respect to ε for $\nabla \tilde{w}_{\varepsilon}(./\varepsilon)$ without subtracting a constant that depends on ε as in (4.36). For $d = 3$, consider indeed $Z_k = \dfrac{1}{\ln(2+|k|)}(1,1,1)$ that satisfies our assumptions. When $\displaystyle\int_{\mathbb{R}^3}\varphi \neq 0$, $\partial_j u_i(x)$ behaves as $\dfrac{1}{|x|^3}$ at infinity and $\displaystyle\sum_{|k|\leq \frac{1}{\beta(\varepsilon)}} (Z_k)_i \partial_j u_i(x/\varepsilon)$ growths at least as $|\ln(\ln(\beta(\varepsilon))|$ on every compact. This phenomenon is related to the non continuity of the operator $T : f \mapsto \nabla^2 G * f$ from $\left(L^\infty(\mathbb{R}^d)\right)^d$ to $\left(L^\infty(\mathbb{R}^d)\right)^d$.*

Remark 4.4. *In Proposition 4.2, we note that the choice $\beta(\varepsilon) = 0$ is also admissible since the bound given by (4.42) allows to show that the sum $\displaystyle\sum_{k\in\mathbb{Z}^d}(Z_k)_i\nabla u_i(. - k)$ makes sense in $\left(BMO(\mathbb{R}^d)\right)^d$. However, we have shown that Proposition 4.2 holds under the less restrictive assumption $\lim\limits_{\varepsilon\to 0}\varepsilon^{-1}\beta(\varepsilon) = 0$, that is for approximation of the full sum.*

4.3.3 Some particular cases

We introduce here some particular settings in terms of the function φ and the sequence Z for which the study of (4.33) is simpler and several stronger results can be shown. In particular, we show that under additional assumptions on φ and Z, the sum $\displaystyle\sum_{k\in\mathbb{Z}^d}(Z_k)_i\nabla u_i(.-k)$ converges in $\left(L^\infty_{loc}(\mathbb{R}^d)\right)^d$ for all $1\leq i\leq d$, which shows the existence of a solution w_1 to (4.33) that is independent of ε.

1) *Vanishing integral* $\displaystyle\int_{\mathbb{R}^d}\varphi=0.$

Given this assumption, the Green formula shows $\displaystyle\int_{\mathbb{R}^d}x\partial_i\varphi = 0$ and, for every $i\in\{1,...,d\}$, $\nabla u_i=\nabla\left(G*\partial_i\varphi\right)$ satisfies $|\nabla u_i(x)|\leq\dfrac{M}{1+|x|^{d+1}}$ as a consequence of Lemma 4.1. It follows that the sums $\displaystyle\sum_{k\in\mathbb{Z}^d}(Z_k)_i\nabla u_i(.-k)$ normally converge in $\left(L^\infty(\mathbb{R}^d)\right)^d$, for all $1\leq i\leq d$. The Schwarz lemma ensures its limit is a gradient, which shows the existence of w_1, solution to (4.33) on $\mathbb{R}^d$ such that $\nabla w_1\in\left(L^\infty(\mathbb{R}^d)\right)^d$ (and not only in $\left(BMO(\mathbb{R}^d)\right)^d$).

Since $\mathcal{V}(./\varepsilon)$ also weakly converges to 0 as ε vanishes without any assumption on Z (see Remark 4.2), we have $\displaystyle\lim_{\varepsilon\to 0}\nabla w_1(./\varepsilon)=0$ in $L^\infty(\mathbb{R}^d)-\star$. In addition, Assumption (A2.a) is sufficient to show the weak convergence of $|\nabla w_1(./\varepsilon)|^2$ to a constant. For the proof of these assertions, we refer to the study of equation (4.54) in Section 4.4.2, where we perform a similar proof.

2) *Existence of a partition of unity.*

If there exists a function $\chi\in\mathcal{D}(\mathbb{R}^d)$ such that $\displaystyle\int_{\mathbb{R}^d}\chi=1$ and $\displaystyle\sum_{k\in\mathbb{Z}^d}\chi(.-k-Z_k)=1$, the potential V defined by (4.6) satisfies $V=W_{per}+W$, where $W_{per}=g_{per}+\displaystyle\int_{\mathbb{R}^d}\varphi$ is periodic with a vanishing average as of consequence of (4.10) and $W=\displaystyle\sum_{k\in\mathbb{Z}^d}\psi(.-k-Z_k)$ where $\psi=\varphi-\left(\displaystyle\int_{\mathbb{R}^d}\varphi\right)\chi\in\mathcal{D}(\mathbb{R}^d)$ and $\displaystyle\int_{\mathbb{R}^d}\psi=0$. The first setting **1)** above thus shows the existence of a corrector independent of ε. However, the existence of a partition of unity χ only exceptionally exists for a sequence Z. For instance, a counter-example is given by $Z_0\neq 0$ and $Z_k=0$ if $k\neq 0$, which clearly satisfies (A1) to (A3). If there exists $\chi\in\mathcal{D}(\mathbb{R}^d)$ such that $1\equiv\displaystyle\sum_{k\in\mathbb{Z}^d}\chi(.-k-Z_k)$, using that χ is compactly supported, we have

$$1=\lim_{|x|\to\infty}\chi(x+Z_0)+\sum_{k\in\mathbb{Z}^d}\chi(x-k)-\chi(x)=\lim_{|x|\to\infty}\sum_{k\in\mathbb{Z}^d}\chi(x-k).$$

For every $x\in\mathbb{R}^d$, we obtain $\displaystyle\sum_{k\in\mathbb{Z}^d}\chi(x-k)=1$ by periodicity. It follows $\chi=\chi(.+Z_0)$, which is a contradiction since $Z_0\neq 0$ and $\chi\neq 0$ is compactly supported.

3) *Additional assumptions related to the distribution of Z.*
As in (4.28), if the convergence of $\dfrac{\varepsilon^d}{|B_R|} \displaystyle\sum_{k \in B_R(x_0)/\varepsilon} Z_k$ to 0 is sufficiently fast, uniformly with respect to the center x_0, Proposition 4.1 shows the convergence of the sum

$$\sum_{k \in \mathbb{Z}^d} (Z_k)_i \partial_i \partial_j G \star \varphi(x-k).$$

In this case, the subtraction of a ε-dependent constant C_ε as in the proof of Proposition 4.2 is therefore not required to obtain the existence of a solution w_1 to (4.37).

On the other hand, the convergence of the sum could also be established if Z_k is the discrete gradient of a sequence $(T_k)_{k \in \mathbb{Z}^d}$, that is, if $(Z_k)_i = \delta_i T_k = T_{k+e_i} - T_k$ for every $i \in \{1, ...d\}$. Formally, the idea would be to perform a summation by parts to obtain

$$\sum_{k \in \mathbb{Z}^d} (Z_k)_i \partial_i \partial_j G \star \varphi(x-k) \sim \sum_{k \in \mathbb{Z}^d} \frac{\delta_i T_k}{|x-k|^d} \sim \sum_{k \in \mathbb{Z}^d} T_k \left(\frac{1}{|x-k+e_i|^d} - \frac{1}{|x-k|^d} \right).$$

Since

$$\frac{1}{|x-k+e_i|^d} - \frac{1}{|x-k|^d} \leq \frac{C}{|x-k|^{d+1}},$$

we could show the convergence as soon as $|T_k| = O\left(|k|^\alpha\right)$ for $\alpha \in [0,1[$. This property is again very specific and, in particular, Z is a discrete gradient if and only if it satisfies a discrete Cauchy equation given by $\delta_i (Z_k)_j = \delta_j (Z_k)_j$ for every $i, j \in \{1, ...d\}$, which is not always true for a generic vector-valued sequence.

4.4 Corrector equation : the full equation (4.12)

In this section we return to our original problem (4.12) when V is the general potential given by (4.6) and we prove Theorem 4.1. The existence of a corrector is performed in two steps. In Proposition 4.3, we first prove the existence of a specific solution to equation (4.54) associated with the nonlinear higher order terms of the Taylor expansion. We next conclude with the proof of Theorem 4.1.

4.4.1 Preliminary properties of convergence

We establish here a preliminary property related to the weak convergence of the functions of the form $\displaystyle\sum_{k \in \mathbb{Z}^d} G(Z_k, x)$ when each $G(Z_k, x)$ behaves as $\dfrac{1}{|x|^{d+\alpha}}$ at infinity, for $\alpha > 0$. To this end, we first consider compactly supported functions in Lemma 4.6 and we conclude in Lemma 4.8 using an argument of density given in Lemma 4.7.

Lemma 4.6. *Assume Z satisfies (4.7) and (A3). Let $G(x,y)$ and $H(x,y)$ be two continuous functions on $\mathbb{R}^d \times \mathbb{R}^d$, compactly supported with respect to y. We denote by*

$$g(x) = \sum_{k,l \in \mathbb{Z}^d} G(Z_k, x-k) H(Z_l, x-l) \in L^\infty(\mathbb{R}^d).$$

Then, when ε *converges to 0,* $g(./\varepsilon)$ *weakly converges to* $\kappa = \sum_{l\in\mathbb{Z}^d} C_{F_l,l}$ *in* $L^\infty(\mathbb{R}^d) - \star$*, where* $C_{F_l,l}$ *is the constant given by Assumption (A3) for* $F_l(x,y) = \int_{\mathbb{R}^d} G(x,z)H(y,z-l)dz$*.*

Proof. For $R>0$ and $x_0 \in \mathbb{R}^d$, we consider $h = 1_{B_R(x_0)}$ and we first show that

$$\lim_{\varepsilon\to 0}\langle g(./\varepsilon), h\rangle_{L^\infty,L^1} = \kappa \int_{\mathbb{R}^d} h.$$

For every $\varepsilon > 0$, we have

$$\langle g(./\varepsilon), h\rangle_{L^\infty,L^1} = \sum_{k,l\in\mathbb{Z}^d} \varepsilon^d \int_{B_{\frac{R}{\varepsilon}}\left(\frac{x_0}{\varepsilon}\right)} G(Z_k, x-k)H(Z_l, x-l)dx.$$

We consider a radius $A>0$ such that $G(x,y) = H(x,y) = 0$ for every $x\in\mathbb{R}^d$ and $y\notin B_A$. If $k \notin B_{\frac{x_0}{\varepsilon}+A}(\frac{x_0}{\varepsilon})$, we clearly have $\int_{B_{\frac{R}{\varepsilon}}\left(\frac{x_0}{\varepsilon}\right)} G(Z_k, x-k)H(Z_l, x-l)dx = 0$. On the other hand, if $k\in B_{\frac{x_0}{\varepsilon}-A}(\frac{x_0}{\varepsilon})$, we have

$$\int_{B_{\frac{R}{\varepsilon}}\left(\frac{x_0}{\varepsilon}\right)} G(Z_k, x-k)H(Z_l, x-l)dx = \int_{\mathbb{R}^d} G(Z_k, x)H(Z_l, x-l+k)dx.$$

Using that $\left|B_{\frac{x_0}{\varepsilon}-A}\left(\frac{x_0}{\varepsilon}\right)\setminus B_{\frac{x_0}{\varepsilon}}\left(\frac{x_0}{\varepsilon}\right)\right| = O(\varepsilon^{1-d})$ and the change of variable $l\mapsto l-k$, it follows that

$$\langle g(./\varepsilon), h\rangle_{L^\infty,L^1} = \sum_{l\in\mathbb{Z}^d}\int_{\mathbb{R}^d} \varepsilon^d \sum_{k\in B_{\frac{R}{\varepsilon}}\left(\frac{x_0}{\varepsilon}\right)} G(Z_k, x)H(Z_{k+l}, x-l)dx + O(\varepsilon).$$

Assumption (A3) therefore shows that

$$\lim_{\varepsilon\to 0}\langle g(./\varepsilon), h\rangle_{L^\infty,L^1} = \kappa|B_R| = \kappa\int_{\mathbb{R}^d} h.$$

We conclude using the density of simple functions in $L^1(\mathbb{R}^d)$. □

The next Lemma is an elementary result of density. We skip its proof for the sake of brevity.

Lemma 4.7. *Let* $(F_k)_{k\in\mathbb{Z}^d} \in \left(L^\infty(\mathbb{R}^d)\right)^{\mathbb{Z}^d}$ *such there exists* $\alpha>0$ *and* $C>0$ *satisfying*

$$|F_k(x)| \leq \frac{C}{1+|x|^{d+\alpha}} \quad \forall k\in\mathbb{Z}^d,\ \forall x\in\mathbb{R}^d.$$

Then the sequence $f_N = \sum_{k\in\mathbb{Z}^d} F_k(.-k)1_{B_N}(.-k)$ *converges to* $f = \sum_{k\in\mathbb{Z}^d} F_k(.-k)$ *in* $L^\infty(\mathbb{R}^d)$ *when* N *tends to* $+\infty$*.*

As a direct consequence of Lemmas 4.6 and 4.7, we therefore obtain :

Lemma 4.8. *Assume Z satisfies (4.7) and (A3). Let $G(x,y)$ and $H(x,y)$ be two continuous functions on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

$$\exists \alpha, \beta > 0,\ \forall x, y \in \mathbb{R}^d, \quad |G(x,y)| \leq \frac{M_1}{1+|y|^{d+\alpha}}, \quad |H(x,y)| \leq \frac{M_2}{1+|y|^{d+\beta}},$$

where $M_1 > 0$ and $M_2 > 0$ are two constants independent of x and y. We denote

$$g(x) = \sum_{k,l \in \mathbb{Z}^d} G(Z_k, x-k) H(Z_l, x-l) \in L^\infty(\mathbb{R}^d).$$

Then, the conclusion of Lemma 4.6 holds true.

4.4.2 Existence result for equation (c)

We are now in point to study the equation

$$\Delta w_2 = \sum_{k \in \mathbb{Z}^d} Z_k^T \left(\int_0^1 (1-t) D^2 \varphi(.-k-tZ_k) dt \right) Z_k. \tag{4.54}$$

Using Lemma 4.1, the existence of w_2 is actually easier to establish than that of w_1 solution to (4.33). On the other hand, the nonlinearity with respect to Z on the right-hand side of (4.54) requires (A3) to show the convergence of $|\nabla w_2|^2(./\varepsilon)$.

Proposition 4.3. *Assume $d \geq 2$. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and Z satisfying (4.7) and (A3). For $i, j \in \{1, ..., d\}$, we denote $u_{i,j} = G * \partial_i \partial_j \varphi$. Then there exists a solution $w_2 \in L^1_{loc}(\mathbb{R}^d)$ to (4.54) such that*

$$\nabla w_2 = \sum_{i,j \in \{1,...,d\}} \sum_{k \in \mathbb{Z}^d} (Z_k)_j (Z_k)_i \int_0^1 (1-t) \nabla u_{i,j}(.-k-tZ_k) dt.$$

In addition, $\nabla w_2 \in \left(L^\infty(\mathbb{R}^d)\right)^d$ and, when ε converges to 0, $\nabla w_2(./\varepsilon)$ weakly converges to 0 in $L^\infty(\mathbb{R}^d) - \star$, $\varepsilon w_2(./\varepsilon)$ strongly converges to 0 in $L^\infty_{loc}(\mathbb{R}^d)$ and there exists $\mathcal{M}_2 \in \mathbb{R}$ such that

$$|\nabla w_2|^2(./\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} \mathcal{M}_2 \quad \text{in } L^\infty(\mathbb{R}^d) - \star.$$

Proof. We first remark that an application of the Green formula shows $\displaystyle\int_{\mathbb{R}^d} \partial_i \partial_j \varphi(x) dx = 0$ and $\displaystyle\int_{\mathbb{R}^d} x \partial_i \partial_j \varphi(x) dx = 0$ for every $i, j \in \{1, ..., d\}$. Lemma 4.1 therefore ensures the existence of $M > 0$ such that

$$|\nabla u_{i,j}(x)| \leq \frac{M}{1+|x|^{d+1}} \quad \forall x \in \mathbb{R}^d. \tag{4.55}$$

For every $N \in \mathbb{N}$, we denote

$$S_N = \sum_{i,j \in \{1,...,d\}} \sum_{|k| \leq N} (Z_k)_j (Z_k)_i \int_0^1 (1-t) u_{i,j}(.-k-tZ_k) dt,$$

solution to

$$\Delta S_N = \sum_{|k| \leq N} Z_K^T \left(\int_0^1 (1-t) D^2 \varphi(.-k-tZ_k) dt \right) Z_k. \tag{4.56}$$

Since Z belongs to $\left(l^\infty(\mathbb{Z}^d)\right)^d$ and $\nabla u_{i,j}$ satisfies (4.55), the sum ∇S_N normally converges in $L^\infty_{loc}(\mathbb{R}^d)$ and its limit is a gradient ∇w_2 as a consequence of the Schwarz lemma. Passing to the limit in (4.56) when $N \to \infty$, we show that w_2 is a solution to (4.54). Moreover, for every $q \in \mathbb{Z}^d$, we have

$$\nabla w_2(.+q) = \sum_{i,j\in\{1,\dots,d\}} \sum_{k\in\mathbb{Z}^d} (Z_{k-q})_j (Z_{k-q})_i \int_0^1 (1-t)\nabla u_{i,j}(.-k-tZ_{k-q})dt.$$

Again, using (4.55) and the fact that Z_k is bounded uniformly with respect to k, we obtain

$$\forall q \in \mathbb{Z}^d, \quad \|\nabla w_2(.+q)\|_{L^\infty(B_2)} \leq C(d)\|Z\|^2_{l^\infty} \sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^{d+1}},$$

where $C(d) > 0$ depends only on the dimension d. Since the right-hand side is independent of q, we deduce that ∇w_2 belongs to $\left(L^\infty(\mathbb{R}^d)\right)^d$.

We have shown that the sequence $\nabla w_2(./\varepsilon)$ is uniformly bounded with respect to ε in $\left(L^\infty(\mathbb{R}^d)\right)^d$. Up to an extraction, it therefore converges to a gradient ∇v for the weak-$\star$ topology of $\left(L^\infty(\mathbb{R}^d)\right)^d$ when $\varepsilon \to 0$. Moreover, since w_2 is solution to (4.54), we have for every $\varepsilon > 0$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \nabla w_2(x/\varepsilon).\nabla\psi(x)dx \\
&\qquad = -\int_{\mathbb{R}^d} \left(\sum_{i,j} \sum_{k\in\mathbb{Z}^d} (Z_k)_i (Z_k)_j \int_0^1 (1-t)\partial_j\partial_i\varphi(x/\varepsilon - k - tZ_k)dt \right) \psi(x)dx. \\
&\qquad = \int_{\mathbb{R}^d} \left(\sum_{i,j} \sum_{k\in\mathbb{Z}^d} (Z_k)_i (Z_k)_j \int_0^1 (1-t)\partial_i\varphi(x/\varepsilon - k - tZ_k)dt \right) \partial_j\psi(x)dx.
\end{aligned}$$

We can next show that $\displaystyle\sum_{k\in\mathbb{Z}^d} (Z_k)_i (Z_k)_j \int_0^1 (1-t)\partial_i\varphi(x/\varepsilon - k - tZ_k)dt$ weakly converges to 0 using that $\displaystyle\int_{\mathbb{R}^d} \partial_i\varphi = 0$ for every $i \in \{1,\dots,d\}$ and proceeding exactly as in the proof of Lemma 4.5. When $\varepsilon \to 0$, it follows that

$$\int_{\mathbb{R}^d} \nabla v.\nabla\psi = 0,$$

that is $\Delta v = 0$ in $\mathcal{D}'(\mathbb{R}^d)$. Since $\nabla v \in \left(L^\infty(\mathbb{R}^d)\right)^d$, we obtain $\nabla v = 0$. We deduce that $\nabla w(./\varepsilon)$ converges to 0 in $L^\infty(\mathbb{R}^d) - \star$ and, as a consequence of Lemma 4.4, that $\varepsilon w_2(./\varepsilon)$ converges to 0 in $L^\infty_{loc}(\mathbb{R}^d)$.

Finally, since

$$\nabla w_2(x) = \sum_{i,j\in\{1,\dots,d\}} \sum_{k\in\mathbb{Z}^d} (Z_k)_j (Z_k)_i \int_0^1 (1-t)\nabla u_{i,j}(x-k-tZ_k)dt,$$

and $\nabla u_{i,j}$ satisfies (4.55), the weak convergence of $|\nabla w_2(./\varepsilon)|^2$ to a constant is a direct consequence of (4.7), (A3) and Lemma 4.8. $\square$

4.4.3 Proof of Theorem 4.1

Problems (4.37) and (4.54) being solved, we are in position to prove Theorem 4.1.

Proof of Theorem 4.1. For every $\varepsilon > 0$ and $R > 0$, we define $W_{\varepsilon,R} = w_{per} - \tilde{W}_{\varepsilon,R} + w_2$ where w_{per} is the periodic solution (unique up to the addition of a constant) to

$$\Delta w_{per} = g_{per} + \sum_{k \in \mathbb{Z}^d} \varphi(. - k) = V_{per},$$

and $\tilde{W}_{\varepsilon,R}$ and w_2 are respectively defined by Proposition 4.2 and Proposition 4.3. By linearity, $W_{\varepsilon,R}$ is a solution to (4.19). In addition, since ∇w_{per} is the gradient of a periodic function, we have $\langle \nabla w_{per} \rangle = 0$ and, by periodicity, we know that w_{per} is strictly sublinear at infinity. The properties of $W_{\varepsilon,R}$ and w_2 given respectively by Proposition 4.2 and Proposition 4.3 therefore ensure that $\nabla W_{\varepsilon,R}(./\varepsilon)$ weakly converges to 0 in $L^p(B_R)$ for every $p \in [1, +\infty[$ and that $\|\varepsilon W_{\varepsilon,R}(./\varepsilon)\|_{L^\infty(B_R)}$ converges to 0 as ε vanishes. We next turn to the weak convergence of $|\nabla W_{\varepsilon,R}(./\varepsilon)|^2$. We first remark that

$$|\nabla W_{\varepsilon,R}|^2 = |\nabla w_{per}|^2 + |\nabla \tilde{W}_{\varepsilon,R}|^2 + |\nabla w_2|^2 - 2\nabla w_{per}.\nabla \tilde{W}_{\varepsilon,R} - 2\nabla w_2.\nabla \tilde{W}_{\varepsilon,R} + 2\nabla w_{per}.\nabla w_2.$$

As a consequence of the periodicity of ∇w_{per} and the results of Propositions 4.2 and 4.3, we already know that $|\nabla w_{per}(./\varepsilon)|^2$, $|\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)|^2$ and $|\nabla w_2(./\varepsilon)|^2$ weakly converge as ε vanishes. We have to show the convergence of the rightmost three terms. We only prove here the convergence of $\nabla w_2(./\varepsilon).\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)$, the proof for the other terms being extremely similar. We denote by $\mathcal{V}$ the function defined in (4.34). For $\phi \in \mathcal{D}(B_R)$, multiplying (4.37) by $\chi(x) = w_2(x/\varepsilon)\phi(x)$ and integrating, we have

$$\int_{B_R} \nabla \tilde{W}_{\varepsilon,R}(./\varepsilon).\nabla w_2(./\varepsilon)\phi = -\varepsilon \int_{B_R} w_2(./\varepsilon)\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon).\nabla \phi + \int_{B_R} \mathcal{V}(./\varepsilon).\nabla w_2(./\varepsilon)\phi$$
$$+ \varepsilon \int_{B_R} w_2(./\varepsilon)\mathcal{V}(./\varepsilon).\nabla \phi.$$

We know that $\nabla \tilde{W}_{\varepsilon,R}(./\varepsilon)$ is bounded in $L^2(B_R)$, uniformly with respect to ε, that $\varepsilon w_2(./\varepsilon)$ uniformly converges to 0 on B_R and that $\mathcal{V}$ is bounded in $\left(L^\infty(\mathbb{R}^d)\right)^d$. It follows that

$$\lim_{\varepsilon \to 0} \int_{B_R} \nabla \tilde{W}_{\varepsilon,R}(./\varepsilon).\nabla w_2(./\varepsilon)\phi = \lim_{\varepsilon \to 0} \int_{B_R} \mathcal{V}(./\varepsilon).\nabla w_2(./\varepsilon)\phi.$$

In addition, we have

$$\mathcal{V}.\nabla w_2 = \sum_{i,j,n \in \{1,\dots,d\}} \sum_{k,l \in \mathbb{Z}^d} (Z_l)_n (Z_k)_j (Z_k)_i \varphi(. - l) \int_0^1 (1-t)\partial_n u_{i,j}(. - k - tZ_k)dt,$$

where $u_{i,j} = G * \partial_i \partial_j \varphi$ and $|u_{i,j}(x)| \leq \dfrac{M}{1 + |x|^{d+1}}$ (where $M > 0$) as a consequence of Lemma 4.1. Under assumptions (4.7) and (A3), Lemma 4.8 therefore shows the existence of a constant C such that $\tilde{V}(./\varepsilon).\nabla w_2(./\varepsilon)$ converges to $C \in \mathbb{R}$ for the weak-$\star$ topology of $L^\infty(B_R)$ and we can conclude. $\square$

Corollary 4.1. *Under the assumptions of Theorem 4.1, the sequence $V(./\varepsilon)W_{\varepsilon,R}(./\varepsilon)$ is bounded in $W^{-1,p}(B_R)$, uniformly with respect to $\varepsilon > 0$ and for every $p \in]1, +\infty[$. In addition, $V(./\varepsilon)W_{\varepsilon,R}(./\varepsilon)$ weakly converges in $W^{-1,p}(B_R)$ to $-\mathcal{M}$ as ε vanishes.*

Proof. For every $\phi \in \mathcal{D}(B_R)$ and every $\varepsilon > 0$, multiplying (4.19) by $\chi = W_{\varepsilon,R}(./\varepsilon)\phi$ and integrating, we obtain :

$$\int_{B_R} W_{\varepsilon,R}(./\varepsilon)V(./\varepsilon)\phi = -\int_{B_R} \varepsilon W_{\varepsilon,R}(./\varepsilon)\nabla W_{\varepsilon,R}(./\varepsilon).\nabla\phi - \int_{B_R} |\nabla W_{\varepsilon,R}(./\varepsilon)|^2\phi \tag{4.57}$$

For every $q \in [1, +\infty[$, since $\varepsilon W_{\varepsilon,R}(./\varepsilon)$ and $\nabla W_{\varepsilon,R}(./\varepsilon)$ are both bounded in $L^q(B_R)$, uniformly with respect to ε, the Hölder inequality gives the existence of a constant $C > 0$ independent of ε such that for every $p \in]1, +\infty[$,

$$\left| \int_{B_R} W_{\varepsilon,R}(./\varepsilon)V(./\varepsilon)\phi \right| \leq C\|\phi\|_{W^{1,p'}(B_R)},$$

where we have denoted by p' the conjugate exponent of p. It follows that $W_{\varepsilon,R}(./\varepsilon)V(./\varepsilon)$ is uniformly bounded in $W^{-1,p}(B_R)$. Since $\varepsilon W_{\varepsilon,R}(./\varepsilon)$ converges to 0 in $L^\infty(B_R)$ as ε vanishes and $|\nabla W_{\varepsilon,R}(./\varepsilon)|^2$ weakly converges to $-\mathcal{M}$ in $L^p(B_R)$ for every $p \in]1, +\infty[$, the weak convergence of $W_{\varepsilon,R}(./\varepsilon)V(./\varepsilon)$ to $-\mathcal{M}$ in $W^{-1,p}$ is therefore a consequence of (4.57). □

4.5 Homogenization results

The existence of a corrector solution to (4.19) satisfying suitable properties being established, we now turn to the proof of our homogenization results stated in Theorems 4.2 and 4.3. In Section 4.5.1 we begin by studying the well-posedness of problem (4.1), showing that the first eigenvalue of $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon)$ converges to the first eigenvalue of $-\Delta - \mathcal{M}$. This result will next provide the sufficient condition (4.21) which allows to perform the homogenization of (4.1) and to prove Theorem 4.2. We next use the result of Theorem 4.2 to show the convergence of *all* the eigenvalues of $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon)$ in Proposition 4.6 and we conclude with the proof of Theorem 4.3.

4.5.1 Well-posedness of Problem (4.1)

In order to show, for ε sufficiently small, the well-posedness of problem (4.1), we first need to introduce the following technical lemma :

Lemma 4.9. *Assume $d \geq 2$. Let $(f^\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(B_{4R})$ for some $R > 0$. Assume there exists $p > d$ such that f^ε is bounded in $L^p(B_{4R})$, uniformly with respect to ε, and f^ε weakly converges to 0 in $L^p(B_{4R})$ as ε vanishes. Then there exists a sequence $(\psi^\varepsilon)_{\varepsilon>0}$ of $W^{2,p}(B_{4R})$ such that*

$$\begin{cases} \Delta\psi^\varepsilon = f^\varepsilon & \text{on } B_{4R}, \\ \psi^\varepsilon = 0 & \text{on } \partial B_{4R}, \end{cases}$$

and $\lim_{\varepsilon\to 0} \|\nabla\psi^\varepsilon\|_{L^\infty(B_R)} = 0.$

Proof. We denote by $\psi^\varepsilon \in H_0^1(B_{4R})$ the unique solution (provided by the Lax-Milgram Lemma) to

$$\Delta\psi^\varepsilon = f^\varepsilon \quad \text{on } B_{4R}, \tag{4.58}$$

and we denote by $C_1 > 0$ a constant independent of ε such that

$$\|\psi^\varepsilon\|_{H^1(B_{4R})} \leq C_1 \|f^\varepsilon\|_{L^2(B_{4R})}.$$

Since f^ε is uniformly bounded in $L^2(B_{4R})$ with respect to ε, the sequence ψ^ε is bounded in $H_0^1(B_{4R})$ and, up to an extraction, it therefore weakly converges to a function $v \in H_0^1(B_{4R})$. Passing to the limit in (4.58) when $\varepsilon \to 0$, we obtain

$$\Delta v = 0 \quad \text{on } B_{4R}.$$

Since $v \in H_0^1(B_{4R})$, we deduce that $v = 0$.

We next use elliptic regularity, see for instance [47, Theorem 7.4 p.141], to obtain the existence of a constant $C_2 > 0$, independent of ε, such that :

$$\|D^2\psi^\varepsilon\|_{L^p(B_{4R})} \leq C_2 \|f^\varepsilon\|_{L^p(B_{4R})}.$$

The Morrey inequality (see [42, p.268]) next yields a constant $C_3 > 0$, also independent of ε, such that for every $x, y \in B_{2R}$, $x \neq y$, and $i \in \{1, ..., d\}$:

$$|\partial_i\psi^\varepsilon(x) - \partial_i\psi_\varepsilon(y)| \leq C_3|x-y|^{1-d/p}\|D^2\psi_\varepsilon\|_{L^p(B_{4R})} \leq C_2C_3|x-y|^{1-d/p}\|f^\varepsilon\|_{L^p(B_{4R})}. \tag{4.59}$$

Since f^ε is bounded in $L^p(B_{4R})$, (4.59) shows that $\nabla\psi^\varepsilon$ is bounded in $L^\infty(B_{2R})$ and equicontinuous on B_{2R}, both uniformly with respect to ε. The Arzela-Ascoli theorem therefore shows that the sequence $\nabla\psi^\varepsilon$, up to an extraction, uniformly converges on every compact of B_{2R}. Since $\nabla\psi^\varepsilon$ weakly converges to 0 in $L^2(B_{4R})$, the uniqueness of the limit in $\mathcal{D}'(B_R)$ allows to conclude that $\nabla\psi^\varepsilon$ uniformly converges to 0. Since 0 is the unique adherent value of $\nabla\psi^\varepsilon$ in $L^\infty(B_R)$, a compactness argument ensures that the whole sequence $\nabla\psi^\varepsilon$ converges to 0 in $L^\infty(B_R)$. □

We next establish the convergence of the first eigenvalue of $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon)$ when ε vanishes.

Proposition 4.4. *Let λ_1^ε and μ_1 be respectively the first eigenvalue of $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon)$ and the first eigenvalue of $-\Delta$ on Ω both with homogeneous Dirichlet boundary conditions. Then, under the assumptions of Theorem 4.1,*

$$\lim_{\varepsilon\to 0} \lambda_1^\varepsilon = \mu_1 - \mathcal{M},$$

where $\mathcal{M}$ is the constant given by (4.20) in Theorem 4.1.

Proof. Our proof is adapted from that of the equivalent result [18, Theorem 12.2 p.162] established in the *periodic* case. For $\varepsilon > 0$, we introduce the operator π_ε defined for every $v \in H_0^1(\Omega)$ by :

$$\pi_\varepsilon(v) = \int_\Omega |\nabla v|^2 + \frac{1}{\varepsilon}V(./\varepsilon)v^2.$$

We denote by R the diameter of Ω and we introduce $W_{\varepsilon,\Omega} = W_{\varepsilon,R}$ the corrector given by Theorem 4.1. For every $v \in H_0^1(\Omega)$ and $\varepsilon > 0$ sufficiently small, we define $\phi^\varepsilon \in H_0^1(\Omega)$ by :

$$v = (1 + \varepsilon W_{\varepsilon,\Omega}(./\varepsilon))\phi^\varepsilon.$$

Since $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ uniformly converges to 0 on Ω, ϕ^ε is uniquely defined when ε is sufficiently small.

Since $\Delta W_{\varepsilon,\Omega} = V$ on Ω/ε, we have

$$-\Delta v + \frac{1}{\varepsilon}V(./\varepsilon)v = -(1 + \varepsilon W_{\varepsilon,\Omega}(./\varepsilon))\Delta\phi^\varepsilon - 2\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla\phi^\varepsilon + (VW_{\varepsilon,\Omega})(./\varepsilon)\phi^\varepsilon,$$

and a direct calculation shows that

$$\pi_\varepsilon(v) = I_1^\varepsilon + I_2^\varepsilon,$$

where

$$I_1^\varepsilon = \int_\Omega (1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))^2 |\nabla\phi^\varepsilon(x)|^2 dx,$$
$$I_2 = \int_\Omega (VW_{\varepsilon,\Omega})(x/\varepsilon)(1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2 dx.$$

Our aim is now to study the behavior of I_1^ε and I_2^ε. For $\varepsilon > 0$, we define

$$\delta_1(\varepsilon) = \|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)}$$

and we have

$$I_1^\varepsilon \geq (1 - 2\delta_1(\varepsilon)) \int_\Omega |\nabla\phi^\varepsilon|^2. \tag{4.60}$$

In order to bound I_2^ε from below, we consider $\chi(x) = W_{\varepsilon,\Omega}(x/\varepsilon)(1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2$. Since $\Delta W_{\varepsilon,\Omega}(./\varepsilon) = V(./\varepsilon)$, we have :

$$I_2^\varepsilon = \int_\Omega V(x/\varepsilon)\chi(x)dx = \int_\Omega \Delta W_{\varepsilon,\Omega}(x/\varepsilon)\chi(x)dx = I_{2,1}^\varepsilon + I_{2,2}^\varepsilon, \tag{4.61}$$

where

$$I_{2,1}^\varepsilon = -\int_\Omega |\nabla W_{\varepsilon,\Omega}(x/\varepsilon)|^2 (1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2 dx,$$
$$I_{2,2}^\varepsilon = -\int_\Omega \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon)\nabla W_{\varepsilon,\Omega}(x/\varepsilon).\nabla\left((1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2\right)dx.$$

We note that the rightmost equality in (4.61) is valid since the trace the trace of ϕ^ε vanishes on $\partial\Omega$ (we recall that Ω is assumed to be $\mathcal{C}^1$). Since $|\nabla W_{\varepsilon,\Omega}(./\varepsilon)|^2$ weakly converges to $\mathcal{M}$ in $L^p(\Omega)$, for every $p \in]1, +\infty[$, we can introduce the function $\psi^\varepsilon \in H^1(\Omega)$ given by Lemma 4.9, solution to

$$\Delta\psi^\varepsilon = \mathcal{M} - |\nabla W_{\varepsilon,\Omega}(./\varepsilon)|^2 \quad \text{on } \Omega,$$

and such that $\nabla\psi^\varepsilon$ converges to 0 in $L^\infty(\Omega)$ as ε vanishes. We also define $\delta_2(\varepsilon) = \|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)}$. We split $I^\varepsilon_{2,1}$ as

$$\begin{aligned} I^\varepsilon_{2,1} = &\int_\Omega -\mathcal{M}(1+\varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2 dx \\ &+ \int_\Omega \Delta\psi^\varepsilon(x)(1+\varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2 dx. \end{aligned} \tag{4.62}$$

On the one hand, we have

$$\int_\Omega -\mathcal{M}(1+\varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2 dx \geq -(1+\delta_1(\varepsilon))\mathcal{M}\int_\Omega |\phi^\varepsilon(x)|^2 dx. \tag{4.63}$$

On the other hand, since the trace of ϕ^ε vanishes on $\partial\Omega$, we obtain

$$\int_\Omega \Delta\psi^\varepsilon(1+\varepsilon W_{\varepsilon,\Omega}(./\varepsilon))|\phi^\varepsilon|^2 dx = -\int_\Omega \nabla\psi^\varepsilon.\nabla\left((1+\varepsilon W_{\varepsilon,\Omega}(./\varepsilon))|\phi^\varepsilon|^2\right) dx. \tag{4.64}$$

We remark that $\nabla\left((1+\varepsilon W_{\varepsilon,\Omega}(./\varepsilon))|\phi^\varepsilon|^2\right) = \nabla W_{\varepsilon,\Omega}(./\varepsilon)|\phi^\varepsilon|^2 + (1+\varepsilon W_{\varepsilon,\Omega}(./\varepsilon))\nabla|\phi^\varepsilon|^2$. Since $\nabla|\phi^\varepsilon|^2 = 2\phi^\varepsilon\nabla\phi^\varepsilon$, the Cauchy-Schwarz inequality shows

$$\int_\Omega |\nabla(|\phi^\varepsilon|^2)| \leq 2\|\phi^\varepsilon\|_{L^2(\Omega)}\|\nabla\phi^\varepsilon\|_{L^2(\Omega)} \leq 2\|\phi^\varepsilon\|^2_{H^1(\Omega)}.$$

Using the Hölder inequality and the fact that $\nabla W_{\varepsilon,R}(./\varepsilon)$ is bounded in $L^p(\Omega)$, uniformly with respect to ε and for every $p \in]1,+\infty[$, we have the existence of $C>0$ independent of ε and ϕ^ε such that

$$\int_\Omega |\nabla W_{\varepsilon,\Omega}(./\varepsilon)|\,|\phi^\varepsilon|^2 \leq \begin{cases} \|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^2(\Omega)}\|\phi^\varepsilon\|^2_{L^4(\Omega)} \leq C\|\phi^\varepsilon\|^2_{H^1(\Omega)} & \text{if } d=2, \\ \|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^{\frac{d}{2}}(\Omega)}\|\phi^\varepsilon\|^2_{L^{\frac{2d}{d-2}}(\Omega)} \leq C\|\phi^\varepsilon\|^2_{H^1(\Omega)} & \text{if } d\geq 3. \end{cases} \tag{4.65}$$

The latter inequality above is a consequence of a Sobolev embedding from $H^1_0(\Omega)$ to $L^p(\Omega)$ for every $p<+\infty$ if $d=2$ or every $p \leq \dfrac{2d}{d-2}$ if $d\geq 3$. (4.64) and (4.65) therefore yield the existence of $c_1>0$ such that

$$\int_\Omega \Delta\psi^\varepsilon(1+\varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi^\varepsilon(x)|^2 dx \geq -c_1\delta_2(\varepsilon)(1+\delta_1(\varepsilon))\|\phi^\varepsilon\|^2_{H^1(\Omega)}. \tag{4.66}$$

Finally, using (4.62), (4.63) and (4.66), we show that

$$I^\varepsilon_{2,1} \geq -(1+\delta_1(\varepsilon))\mathcal{M}\int_\Omega |\phi^\varepsilon(x)|^2 dx - c_1\delta_2(\varepsilon)(1+\delta_1(\varepsilon))\|\phi^\varepsilon\|^2_{H^1(\Omega)}.$$

Using that $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ uniformly converges on Ω and $\nabla W_{\varepsilon,\Omega}(./\varepsilon)$ is bounded in $L^p(\Omega)$, we can similarly show the existence of $c_2>0$ such that

$$I^\varepsilon_{2,2} \geq -c_2\delta_1(\varepsilon)(1+\delta_1(\varepsilon))\|\phi^\varepsilon\|^2_{H^1(\Omega)},$$

and we obtain

$$I^\varepsilon_2 \geq -(1+\delta_1(\varepsilon))\mathcal{M}\int_\Omega |\phi^\varepsilon(x)|^2 dx - \left(c_1\delta_2(\varepsilon)+c_2\delta_1(\varepsilon)\right)(1+\delta_1(\varepsilon))\|\phi^\varepsilon\|^2_{H^1(\Omega)}. \tag{4.67}$$

Hence, we use (4.60) and (4.67), and we obtain

$$\pi_\varepsilon(v) \geq (1 - 2\delta_1(\varepsilon) - (c_1\delta_2(\varepsilon) + c_2\delta_1(\varepsilon))(1 + \delta_1(\varepsilon))) \int_\Omega |\nabla\phi^\varepsilon|^2$$
$$- ((1 + \delta_1(\varepsilon))\mathcal{M} - (c_1\delta_2(\varepsilon) + c_2\delta_1(\varepsilon))(1 + \delta_1(\varepsilon))) \int_\Omega |\phi^\varepsilon(x)|^2.$$

The definition of μ_1 gives

$$\int_\Omega |\nabla\phi^\varepsilon|^2 \geq \mu_1 \int_\Omega |\phi^\varepsilon|^2.$$

When ε is sufficiently small, it therefore follows

$$\pi_\varepsilon(v) \geq (\mu_1 - \mathcal{M} - \delta_3(\varepsilon)) \int_\Omega |\phi^\varepsilon|^2,$$

where we have denoted

$$\begin{aligned}\delta_3(\varepsilon) =& \mu_1 (2\delta_1(\varepsilon) + (c_1\delta_2(\varepsilon) + c_2\delta_1(\varepsilon))(1 + \delta_1(\varepsilon))) \\ &+ \delta_1(\varepsilon)\mathcal{M} + (c_1\delta_2(\varepsilon) + c_2\delta_1(\varepsilon))(1 + \delta_1(\varepsilon)).\end{aligned}$$

When ε is sufficiently small, we also have, by definition

$$\phi^\varepsilon = \frac{v}{1 + \varepsilon W_{\varepsilon,R}(./\varepsilon)},$$

and a Taylor expansion provides the existence of $c_3 > 0$ independent of ε such that

$$\|\phi^\varepsilon\|_{L^2(\Omega)} \geq (1 - c_3\delta_1(\varepsilon))\|v\|_{L^2(\Omega)}.$$

So we obtain

$$\pi_\varepsilon(v) \geq (\mu_1 - \mathcal{M} - \delta_3(\varepsilon))(1 - c_3\delta_1(\varepsilon))^2 \int_\Omega |v|^2.$$

Since this inequality holds for every $v \in H_0^1$, it follows :

$$\lambda_1^\varepsilon = \inf_{v \in H_0^1} \frac{\pi_\varepsilon(v)}{\|v\|^2_{L^2(\Omega)}} \geq (\mu_1 - \mathcal{M} - \delta_3(\varepsilon))(1 - c_3\delta_1(\varepsilon))^2. \tag{4.68}$$

We next establish a similar upper bound for $\pi_\varepsilon(v)$. To this aim, we consider $\phi \in H_0^1(\Omega)$ such that $\|\phi\|_{L^2(\Omega)} = 1$ and, for $\varepsilon > 0$, we define $v^\varepsilon = (1 + \varepsilon W_{\varepsilon,\Omega}(./\varepsilon))\phi$. We have again to bound the following two integrals :

$$I_1^\varepsilon = \int_\Omega (1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))^2 |\nabla\phi(x)|^2 dx,$$

and

$$I_2^\varepsilon = \int_\Omega V W_{\varepsilon,\Omega}(x/\varepsilon)(1 + \varepsilon W_{\varepsilon,\Omega}(x/\varepsilon))|\phi(x)|^2 dx.$$

As above, we can show that

$$\begin{aligned}\pi_\varepsilon(v^\varepsilon) &\leq (1 + \delta_4(\varepsilon)) \int_\Omega |\nabla\phi|^2 + (-\mathcal{M} + \delta_5(\varepsilon)) \int_\Omega |\phi|^2 \\ &= (1 + \delta_4(\varepsilon)) \int_\Omega |\nabla\phi|^2 - \mathcal{M} + \delta_5(\varepsilon),\end{aligned}$$

where $\delta_4(\varepsilon)$ and $\delta_5(\varepsilon)$ both depend on $\|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)}$ and $\|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)}$ and converge to 0 when $\varepsilon \to 0$. Since $v^\varepsilon = (1+\varepsilon W_{\varepsilon,\Omega}(./\varepsilon))\phi$, we have

$$\|v^\varepsilon\|_{L^2(\Omega)} \geq (1-\delta_1(\varepsilon))\|\phi\|_{L^2(\Omega)} = 1-\delta_1(\varepsilon),$$

and it follows

$$\lambda_1^\varepsilon = \inf_{v\in H_0^1(\Omega)} \frac{\pi_\varepsilon(v)}{\|v\|_{L^2(\Omega)}^2} \leq \frac{\pi_\varepsilon(v^\varepsilon)}{\|v^\varepsilon\|_{L^2(\Omega)}^2} \leq \frac{1}{(1-\delta_1(\varepsilon))^2}\left((1+\delta_4(\varepsilon))\int_\Omega |\nabla\phi|^2 - \mathcal{M} + \delta_5(\varepsilon)\right).$$

This inequality holds for every $\phi \in H_0^1(\Omega)$ such that $\|\phi\|_{L^2(\Omega)} = 1$. By definition of μ_1, we have $\mu_1 = \inf_{\phi\in H_0^1(\Omega), \|\phi\|_{L^2(\Omega)}=1} \int_\Omega |\nabla\phi|^2$ and since $\delta_1(\varepsilon)$ converges to 0, we obtain the existence of $c_4 > 0$ independent of ε such that

$$\begin{aligned}\lambda_1^\varepsilon &\leq \frac{1}{(1-\delta_1(\varepsilon))^2}\left((1+\delta_4(\varepsilon))\inf_{\phi\in H_0^1(\Omega), \|\phi\|_{L^2(\Omega)}=1}\int_\Omega |\nabla\phi|^2 - \mathcal{M} + \delta_5(\varepsilon)\right) \\ &\leq c_4\left((1+\delta_4(\varepsilon))\mu_1 - \mathcal{M} + \delta_5(\varepsilon)\right).\end{aligned} \tag{4.69}$$

Using (4.68) and (4.69), we have finally shown that

$$(\mu_1 - \mathcal{M} - \delta_3(\varepsilon))\,(1-c_3\delta_1(\varepsilon))^2 \leq \lambda_1^\varepsilon \leq c_4\left((1+\delta_4(\varepsilon))\mu_1 - \mathcal{M} + \delta_5(\varepsilon)\right).$$

Passing to the limit when $\varepsilon \to 0$, we conclude that $\lim_{\varepsilon\to 0}\lambda_1^\varepsilon = \mu_1 - \mathcal{M}$. □

In particular, when (4.21) is satisfied and if ε is sufficiently small, Proposition 4.4 ensures that the quadratic form

$$a^\varepsilon(u,u) = \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega V(./\varepsilon)|u|^2 + \nu\int_\Omega |u|^2,$$

is coercive in $H_0^1(\Omega)$, uniformly with respect to ε. Indeed, given (4.21), a simple adaptation of the proof of Proposition 4.4 shows that there exists $0 < \eta < 1$ such that, for every $\varepsilon > 0$ sufficiently small, the first eigenvalue of $-(1-\eta)\Delta + \frac{1}{\varepsilon}V(./\varepsilon) + \nu$ is positive and, consequently, we have

$$0 \leq (1-\eta)\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega V(./\varepsilon)|u|^2 + \nu\int_\Omega |u|^2.$$

It follows that

$$\begin{aligned}\frac{\eta}{2}\|\nabla u\|_{L^2(\Omega)}^2 &\leq \frac{\eta}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\left((1-\eta)\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega V(./\varepsilon)|u|^2 + \nu\int_\Omega |u|^2\right) \\ &= \frac{1}{2}a^\varepsilon(u,u).\end{aligned} \tag{4.70}$$

Moreover, for ε sufficiently small, Proposition 4.4 also ensures that

$$\frac{\mu_1 - \mathcal{M} + \nu}{2}\|u\|_{L^2(\Omega)}^2 \leq \frac{1}{2}a^\varepsilon(u,u). \tag{4.71}$$

Denoting $C = \min(\eta, (\mu_1 - \mathcal{M} + \nu))$, assumption (4.21) gives $C > 0$ and, using (4.70) and (4.71), we obtain

$$\frac{C}{2}\|u\|^2_{H^1(\Omega)} \leq a^\varepsilon(u,u). \tag{4.72}$$

For such values of ε, problem (4.1) is therefore well-posed in $H^1_0(\Omega)$ as a consequence of the Lax-Milgram lemma.

Remark 4.5. *If the periodic potential $V_{per} = g_{per} + \sum_{k\in\mathbb{Z}^d} \varphi(.-k)$ satisfies assumption (4.5), we remark that assumption (4.21) is not necessarily satisfied by $V = g_{per} + \sum_{k\in\mathbb{Z}^d}\varphi(.-k-Z_k)$. Consider indeed the one-dimensional example for which $g_{per} = 0$ and $V(x) = \sum_{k\in\mathbb{Z}}\psi'(x-k-Z_k)$ where $\psi \in \mathcal{D}(\mathbb{R})$ is nonnegative and has support in $[0,1]$ and $Z_k \in]0,1[$ satisfies*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{R}\sum_{k\in B_R(x_0)/\varepsilon} F(Z_k) = \int_0^1 F(t)dt,$$
$$\forall l \neq 0,\ \lim_{\varepsilon\to 0}\frac{\varepsilon}{R}\sum_{k\in B_R(x_0)/\varepsilon} G(Z_k, Z_{k+l}) = \int_0^1\int_0^1 G(t,u)dtdu,$$

for every $F \in \mathcal{C}^0(\mathbb{R})$, $G \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$. Such a sequence Z indeed exists, as for example shown by the deterministic approximation of random uniform distribution given in Section 4.2.3. In this case, the periodic corrector w_{per} and our adapted corrector w, respectively solution to $w''_{per} = V_{per}$ and solution to $w'' = V$, can both be made explicit and their derivatives are respectively given by $w'_{per} = \sum_{k\in\mathbb{Z}}\psi(.-k)$ and $w' = \sum_{k\in\mathbb{Z}}\psi(.-k-Z_k)$. An explicit calculation using the properties of Z_k shows that

$$\langle |w'_{per}|^2\rangle = \text{weak}\lim_{\varepsilon\to 0}|w'_{per}(./\varepsilon)|^2 = \int_{\mathbb{R}}|\psi(t)|^2dt$$
$$\mathcal{M} = \text{weak}\lim_{\varepsilon\to 0}|w'(./\varepsilon)|^2 = \int_{\mathbb{R}}|\psi(t)|^2dt + \int_{\mathbb{R}}\psi(t)\int_0^1\int_0^1\psi(t+1-u+v)du\,dv\,dt,$$

and $\mathcal{M} > \langle|w'_{per}|^2\rangle$ as soon as $\int_{\mathbb{R}}\psi(t)\int_0^1\int_0^1\psi(t+1-u+v)du\,dv\,dt > 0$. In this case, (4.21) is therefore more restrictive than (4.5).

4.5.2 Proof of Theorem 4.2

In this section, we homogenize (4.1) given the sufficient assumption (4.21). We first introduce the unique solution u^* in $H^1_0(\Omega)$ to (4.22). The existence and uniqueness of u^* is, of course, ensured by (4.21). Our aim is now to show that the sequence $(u^\varepsilon)_{\varepsilon>0}$ of solutions to (4.1), for ε small, indeed converges to u^* and, considering the sequence of remainders R^ε defined by (4.23), to make precise the behavior of u^ε in $H^1(\Omega)$.

To start with, we establish the following proposition :

Proposition 4.5. *Under the assumptions of Theorem 4.1, and if V additionally satisfies (4.21), then the sequence u^ε of solutions to (4.1) converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to the unique solution $u^* \in H^1_0(\Omega)$ to (4.22).*

Proof. When ε is sufficiently small, assumption (4.21) and Proposition 4.4 give the existence of $C > 0$ independent of ε such that

$$C\|u^\varepsilon\|^2_{H^1(\Omega)} \leq \int_\Omega \left(|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} V(./\varepsilon)|u^\varepsilon|^2 + \nu|u^\varepsilon|^2 \right) \leq \|f\|_{L^2(\Omega)}\|u^\varepsilon\|_{H^1(\Omega)}.$$

The function u^ε is therefore bounded in $H^1(\Omega)$, uniformly with respect to ε, and, up to an extraction, it converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to a function $u^* \in H^1_0(\Omega)$. For every $\varepsilon > 0$, we introduce $W_{\varepsilon,\Omega}$, the corrector given by Theorem 4.1 for $R = Diam(\Omega)$ and we define

$$\theta^\varepsilon = 1 + \varepsilon W_{\varepsilon,\Omega}(./\varepsilon).$$

Since $W_{\varepsilon,\Omega}$ satisfies $\Delta W_{\varepsilon,\Omega}(./\varepsilon) = V(./\varepsilon)$ in $\mathcal{D}'(\Omega)$, we have

$$-\Delta\theta^\varepsilon + \frac{1}{\varepsilon}V(./\varepsilon)\theta^\varepsilon = V(./\varepsilon)W_{\varepsilon,\Omega}(./\varepsilon). \tag{4.73}$$

For $\psi \in \mathcal{D}(\Omega)$, we multiply (4.1) and (4.73) respectively by $\theta^\varepsilon\psi$ and $u^\varepsilon\psi$ and we obtain

$$\int_\Omega \nabla u^\varepsilon.\nabla\theta^\varepsilon\psi + \int_\Omega \nabla u^\varepsilon.\nabla\psi\theta^\varepsilon + \int_\Omega \frac{1}{\varepsilon}V(./\varepsilon)u^\varepsilon\theta^\varepsilon\psi + \int_\Omega \nu u^\varepsilon\theta^\varepsilon\psi = \int_\Omega f\theta^\varepsilon\psi,$$

and

$$\int_\Omega \nabla u^\varepsilon.\nabla\theta^\varepsilon\psi + \int_\Omega \nabla\theta^\varepsilon.\nabla\psi u^\varepsilon + \int_\Omega \frac{1}{\varepsilon}V(./\varepsilon)u^\varepsilon\theta^\varepsilon\psi = \int_\Omega V(./\varepsilon)W_{\varepsilon,\Omega}(./\varepsilon)u^\varepsilon\psi.$$

Subtracting the above two equalities, we have

$$\int_\Omega \nabla u^\varepsilon.\nabla\psi\theta^\varepsilon - \int_\Omega \nabla\theta^\varepsilon.\nabla\psi u^\varepsilon + \int_\Omega \nu u^\varepsilon\theta^\varepsilon\psi = \int_\Omega f\theta^\varepsilon\psi - \int_\Omega V(./\varepsilon)W_{\varepsilon,\Omega}(./\varepsilon)u^\varepsilon\psi. \tag{4.74}$$

Since u^ε weakly converges to u^* in $H^1(\Omega)$ and $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ uniformly converges to 0 on Ω, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla u^\varepsilon.\nabla\psi\theta^\varepsilon = \int_\Omega \nabla u^*.\nabla\psi,$$

$$\lim_{\varepsilon\to 0}\int_\Omega \nu u^\varepsilon\theta^\varepsilon\psi = \int_\Omega \nu u^*\psi,$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega f\theta^\varepsilon\psi = \int_\Omega f\psi.$$

Similarly, the weak convergence of $\nabla\theta^\varepsilon = \nabla W_{\varepsilon,\Omega}(./\varepsilon)$ to 0 and the strong convergence of u^ε to u^* in $L^2(\Omega)$ imply

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla\theta^\varepsilon.\nabla\psi u^\varepsilon = 0.$$

Multiplying next the corrector equation (4.19) by $\chi = W_{\varepsilon,\Omega}(./\varepsilon)u^\varepsilon\psi$ and since $u^\varepsilon\psi$ converges to $u^*\psi$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, the convergence properties of the corrector imply

$$\lim_{\varepsilon\to 0}\int_\Omega W_{\varepsilon,\Omega}(./\varepsilon)V(./\varepsilon)u^\varepsilon\psi = \lim_{\varepsilon\to 0} -\int_\Omega |\nabla W_{\varepsilon,\Omega}(./\varepsilon)|^2 u^\varepsilon\psi - \int_\Omega \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\nabla(u^\varepsilon\psi)$$
$$= -\mathcal{M}\int_\Omega u^*\psi.$$

Passing to the limit in (4.74) when $\varepsilon \to 0$, we have shown that for every $\psi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega \nabla u^*.\nabla\psi + (-\mathcal{M}+\nu)\int_\Omega u^*\psi = \int_\Omega f\psi.$$

We have therefore proved that u^* is a solution to (4.22). The limit being independent of the extraction, we can conclude that the sequence u^ε converges to u^*. □

We are now able to study the behavior of u^ε in $H^1(\Omega)$ and to prove Theorem 4.2.

Proof of Theorem 4.2. We first show that $R^\varepsilon = u^\varepsilon - u^* - \varepsilon u^* W_{\varepsilon,\Omega}(./\varepsilon)$ is uniformly bounded in $H^1(\Omega)$ with respect to ε. Indeed, when ε is sufficiently small, assumption (4.21) and Proposition 4.4 ensure that the bilinear form a defined by

$$a^\varepsilon(u,v) = \int_\Omega \nabla u(x).\nabla v(x)dx + \frac{1}{\varepsilon}\int_\Omega V(x/\varepsilon)u(x)v(x)dx + \nu\int_\Omega u(x)v(x)dx,$$

is coercive, uniformly with respect to ε, and that the sequence u^ε is therefore uniformly bounded in $H^1(\Omega)$. Moreover, we know that $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ is uniformly bounded in $L^\infty(\Omega)$ and $\nabla W_{\varepsilon,\Omega}(./\varepsilon)$ is uniformly bounded in $L^p(\Omega)$ for every $p \in [1,+\infty[$. It follows that R^ε is bounded in $H^1(\Omega)$ uniformly with respect to ε. Proposition 4.5 also ensures that R^ε strongly converges to 0 in $L^2(\Omega)$.

For every $\varepsilon > 0$, a simple calculation shows that R^ε is solution in $\mathcal{D}'(\Omega)$ to

$$-\Delta R^\varepsilon + \frac{1}{\varepsilon}V(./\varepsilon)R^\varepsilon + \nu R^\varepsilon = \left(-\mathcal{M} - W_{\varepsilon,\Omega}(./\varepsilon)V(./\varepsilon)\right)u^*$$
$$+ 2\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla u^* + \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\left(-\mathcal{M}u^* - f\right).$$

Since R^ε belongs to $H^1_0(\Omega)$, we have $a(R^\varepsilon, R^\varepsilon) = I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$I_1^\varepsilon = \int_\Omega \left(-\mathcal{M} - W_{\varepsilon,\Omega}(./\varepsilon)V(./\varepsilon)\right)u^*R^\varepsilon,$$
$$I_2^\varepsilon = \int_\Omega 2\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla u^* R^\varepsilon,$$
$$I_3^\varepsilon = \int_\Omega \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\left(-\mathcal{M}u^* - f\right)R^\varepsilon.$$

We next show that the three integrals I_1^ε, I_2^ε and I_3^ε converge to 0 when $\varepsilon \to 0$.

Since u^* and R^ε both belong to $H^1_0(\Omega)$, the function u^*R^ε belong to $W^{1,1}_0(\Omega)$ and since $\Delta W_{\varepsilon,\Omega}(./\varepsilon) = V(./\varepsilon)$, an integration by parts shows that

$$I_1^\varepsilon = \int_\Omega \left(-\mathcal{M} + |\nabla W_{\varepsilon,\Omega}(./\varepsilon)|^2\right)u^*R^\varepsilon + \int_\Omega \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla(u^*R^\varepsilon).$$

Lemma 4.9 next gives the existence of $\psi^\varepsilon \in L^1_{loc}(\Omega)$ such that $\Delta\psi^\varepsilon = -\mathcal{M} + |\nabla W_{\varepsilon,\Omega}(./\varepsilon)|^2$ in $\mathcal{D}'(\Omega)$ and $\lim\limits_{\varepsilon\to 0} \|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)} = 0$. Using the boundedness of R^ε in $H^1(\Omega)$, we therefore have

$$\begin{aligned}
\left|\int_\Omega \left(-\mathcal{M} + |\nabla W_{\varepsilon,\Omega}(./\varepsilon)|^2\right) u^* R^\varepsilon\right| &= \left|\int_\Omega \Delta\psi^\varepsilon u^* R^\varepsilon\right| \\
&= \left|\int_\Omega \nabla\psi^\varepsilon.\nabla u^* R^\varepsilon + \int_\Omega \nabla\psi^\varepsilon.\nabla R^\varepsilon u^*\right| \\
&\leq 2\|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)}\|u^*\|_{H^1(\Omega)}\|R^\varepsilon\|_{H^1(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\left|\int_\Omega \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla(u^* R^\varepsilon)\right| &\leq \|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)} \int_\Omega |\nabla W_{\varepsilon,\Omega}(./\varepsilon)||u^*||\nabla R^\varepsilon| \\
&+ \|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)} \int_\Omega |\nabla W_{\varepsilon,\Omega}(./\varepsilon)||\nabla u^*||R^\varepsilon|.
\end{aligned}$$

The Hölder inequality and a Sobolev embedding give the existence of $C_1 > 0$ independent of ε such that

$$\begin{aligned}
\int_\Omega |u^*||\nabla W_{\varepsilon,\Omega}(./\varepsilon)||\nabla R^\varepsilon| &\leq \left\{\begin{array}{ll} \|u^*\|_{L^4(\Omega)}\|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^4(\Omega)}\|\nabla R^\varepsilon\|_{L^2(\Omega)} & \text{if } d=2, \\ \|u^*\|_{L^{\frac{2d}{d-2}}(\Omega)}\|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^d(\Omega)}\|\nabla R^\varepsilon\|_{L^2(\Omega)} & \text{if } d\geq 3, \end{array}\right. \\
&\leq C_1 \left\{\begin{array}{ll} \|u^*\|_{H^1(\Omega)}\|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^4(\Omega)}\|\nabla R^\varepsilon\|_{L^2(\Omega)} & \text{if } d=2, \\ \|u^*\|_{H^1(\Omega)}\|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^d(\Omega)}\|\nabla R^\varepsilon\|_{L^2(\Omega)} & \text{if } d\geq 3. \end{array}\right.
\end{aligned}$$

We have similarly :

$$\int_\Omega |\nabla u^*||\nabla W_{\varepsilon,\Omega}(./\varepsilon)||R^\varepsilon| \leq C_1 \left\{\begin{array}{ll} \|u^*\|_{H^1(\Omega)}\|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^4(\Omega)}\|R^\varepsilon\|_{H^1(\Omega)} & \text{if } d=2, \\ \|u^*\|_{H^1(\Omega)}\|\nabla W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^d(\Omega)}\|R^\varepsilon\|_{H^1(\Omega)} & \text{if } d\geq 3. \end{array}\right.$$

Since R^ε is bounded in $H^1(\Omega)$ and $\nabla W_{\varepsilon,\Omega}(./\varepsilon)$ is bounded in $L^p(\Omega)$ for every $p\geq 1$, we deduce the existence of $C > 0$ such that

$$\left|\int_\Omega \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla(u^* R^\varepsilon)\right| \leq C\|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

and we have proved that $\lim\limits_{\varepsilon\to 0} I_1^\varepsilon = 0$.

We next study the behavior of I_2^ε. We know that $f \in L^2(\Omega)$ and u^* is solution to (4.22). The elliptic regularity therefore ensures that $u^* \in H^2(\Omega)$. Since $R^\varepsilon \in H^1_0(\Omega)$, an integration by parts shows

$$\begin{aligned}
|I_2^\varepsilon| &= 2\left|\int_\Omega \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\left(\Delta u^* R^\varepsilon + \nabla u^*.\nabla R^\varepsilon\right)\right| \\
&\leq 4\|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)}\|\Delta u^*\|_{L^2(\Omega)}\|R^\varepsilon\|_{H^1(\Omega)}.
\end{aligned}$$

Since $\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)$ uniformly converges to 0 in Ω and R^ε is bounded in $H^1(\Omega)$, we deduce that I_2^ε converges to 0 when $\varepsilon \to 0$.

Similar arguments for I_3^ε give :

$$|I_3^\varepsilon| \leq \|\varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\|_{L^\infty(\Omega)}\| - \mathcal{M}u^* - f\|_{L^2(\Omega)}\|R^\varepsilon\|_{L^2(\Omega)} \xrightarrow{\varepsilon\to 0} 0.$$

We finally conclude that I^ε converges to 0 when ε converges to 0. The uniform coercivity of a^ε in $H_0^1(\Omega)$ next yields a constant $C > 0$ independent of ε such that

$$C\|R^\varepsilon\|^2_{H^1(\Omega)} \leq a^\varepsilon(R^\varepsilon, R^\varepsilon) = I^\varepsilon.$$

We deduce that R^ε strongly converges to 0 in $H^1(\Omega)$ and we can conclude. □

4.5.3 Proof of Theorem 4.3

We are now in position to study the convergence of *all* the eigenvalues of the operator $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon)$ with homogeneous Dirichlet boundary conditions on Ω. This result is established in the following proposition :

Proposition 4.6. *Let λ_l^ε and μ_l be respectively the lower l^{th} eigenvalue (counting multiplicities) of $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon)$ and $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions. Then, under the assumptions of Theorem 4.1,*

$$\lim_{\varepsilon\to 0} \lambda_l^\varepsilon = \mu_l - \mathcal{M}.$$

Proof. Our approach is an adaptation of the method introduced in [61, Section 3] and used in [94, Section 4] for the periodic setting. We fix $\kappa \in \mathbb{R}^*$ such that

$$\mu_1 - \mathcal{M} + \kappa > 0. \tag{4.75}$$

For $f \in L^2(\Omega)$ and $\varepsilon > 0$, we consider $u^{\varepsilon,\kappa}$, solution to

$$\begin{cases} -\Delta u^{\varepsilon,\kappa} + \dfrac{1}{\varepsilon}V(./\varepsilon)u^{\varepsilon,\kappa} + \kappa u^{\varepsilon,\kappa} = f & \text{on } \Omega, \\ u^{\varepsilon,\kappa} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.76}$$

Given (4.75), Proposition 4.4 ensures that $u^{\varepsilon,\kappa}$ is well defined when ε is sufficiently small and Proposition 4.5 shows that $u^{\varepsilon,\kappa}$ strongly converges in $L^2(\Omega)$ to $u^{*,\kappa}$, solution in $H^1(\Omega)$ to

$$\begin{cases} -\Delta u^{*,\kappa} - \mathcal{M}u^{*,\kappa} + \nu u^{*,\kappa} = f & \text{on } \Omega, \\ u^{*,\kappa} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.77}$$

In addition, for $l \in \mathbb{N}^*$, we remark that the eigenvalues $\lambda_l^{\varepsilon,\kappa}$ and $\lambda^{*,\kappa}$, respectively defined as the lower l^{th} eigenvalue of $-\Delta + \dfrac{1}{\varepsilon}V(./\varepsilon) + \kappa$ and $-\Delta - \mathcal{M} + \kappa$ on Ω with homogeneous Dirichlet boundary conditions, are given by $\lambda_l^{\varepsilon,\kappa} = \lambda_l^\varepsilon + \kappa$ and $\lambda_l^{*,\kappa} = \mu_l - \mathcal{M} + \kappa$. For $f \in L^2(\Omega)$, we next denote by $T^{\varepsilon,\kappa}(f)$ and $T^{*,\kappa}(f)$, respectively the unique solution in $H_0^1(\Omega)$ to (4.76) and the unique solution in $H_0^1(\Omega)$ to (4.77). We also denote by $(v_l^{\varepsilon,\kappa})_{l\in\mathbb{N}^*}$ and $(v_l^{*,\kappa})_{l\in\mathbb{N}^*}$ two Hilbert bases of $L^2(\Omega)$ composed of eigenfunctions respectively associated with $(\lambda_l^{\varepsilon,\kappa})_{\mathbb{N}^*}$

and $(\lambda_l^{*,\kappa})_{\mathbb{N}^*}$. An adaptation of the results of [94, Lemma 4.1], which uses the method of [61, Lemma 3.1], next shows that

$$\left|\frac{1}{\lambda_l^{\varepsilon,\kappa}}-\frac{1}{\lambda_l^{*,\kappa}}\right| \leq \max\left\{\max_{\substack{f\perp V_{l-1}^* \\ \|f\|_{L^2(\Omega)}=1}} |\langle (T^{\varepsilon,\kappa}-T^{*,\kappa})f,f\rangle|\,,\ \max_{\substack{f\perp V_{l-1}^\varepsilon \\ \|f\|_{L^2(\Omega)}=1}} |\langle (T^{\varepsilon,\kappa}-T^{*,\kappa})f,f\rangle|\right\}, \quad (4.78)$$

where $V_{l-1}^\varepsilon = Span(v_1^{\varepsilon,\kappa},...,v_{l-1}^{\varepsilon,\kappa})$ and $V_{l-1}^* = Span(v_1^{*,\kappa},...,v_{l-1}^{*,\kappa})$. We note this inequality is actually established in [94, Lemma 4.1] and [61, Lemma 3.1] for a periodic setting. However, the periodicity of the coefficients is only used to ensure the existence of an homogenized operator T^*. Knowing the existence of this homogenized operator, the proof of (4.78) can be easily generalized in our setting since it is only based on a consequence of the min-max principle which uses the fact that both $T^{\varepsilon,\kappa}$ and $T^{*,\kappa}$ are self-adjoint and compact (ensured by the assumption $V \in L^\infty(\mathbb{R}^d)$).

From (4.78), it therefore follows

$$\left|\frac{1}{\lambda_l^{\varepsilon,\kappa}}-\frac{1}{\lambda_l^{*,\kappa}}\right| \leq \|T^{\varepsilon,\kappa}-T^{*,\kappa}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}.$$

Since $\lim\limits_{\varepsilon\to 0}\|T^{\varepsilon,\kappa}-T^{*,\kappa}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}=0$ as a consequence of Proposition 4.5, we deduce that $\dfrac{1}{\lambda_l^{\varepsilon,\kappa}}$ converges to $\dfrac{1}{\lambda_l^{*,\kappa}}$ when $\varepsilon\to 0$. We can therefore conclude that λ_l^ε converges to $\mu_l - \mathcal{M}$ for every $l \in \mathbb{N}^*$. □

We next turn to the proof of Theorem 4.3.

Proof of Theorem 4.3. Given the assumption $\mu_l - \mathcal{M} + \nu \neq 0$ for every $l \in \mathbb{N}^*$, Proposition 4.6 first implies that all the eigenvalues of the operator $-\Delta + \frac{1}{\varepsilon}V(./\varepsilon)+\nu$ are isolated from zero when ε is sufficiently small. The well-posedness of (4.1) is therefore a consequence of the Fredholm alternative. We next claim there exists a constant $C>0$ independent of ε such that

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \quad (4.79)$$

when ε is sufficiently small. We indeed denote by $(v_l^\varepsilon)_{l\in\mathbb{N}^*}\in H_0^1(\Omega)^{\mathbb{N}}$ an Hilbert basis of $L^2(\Omega)$ composed of the eigenfunctions associated with the sequence $(\lambda_l^\varepsilon+\nu)_{\varepsilon>0}$ of the eigenvalues of $-\Delta+\frac{1}{\varepsilon}V(./\varepsilon)+\nu$. We also define $E_+^\varepsilon = \mathrm{Span}\,(v_l^\varepsilon,\ \lambda_l^\varepsilon+\nu>0)$, $E_-^\varepsilon = \mathrm{Span}\,(v_l^\varepsilon,\ \lambda_l^\varepsilon+\nu<0)$ and Π_+^ε, Π_-^ε the orthogonal projections respectively associated with E_+^ε and E_-^ε. For every $u,v\in H_0^1(\Omega)$, we define $a^\varepsilon(u,v) = \displaystyle\int_\Omega \nabla u.\nabla v + \frac{1}{\varepsilon}V(./\varepsilon)uv+\nu uv$. Since u^ε is solution to (4.1) in $H_0^1(\Omega)$, we have

$$a^\varepsilon(\Pi_+^\varepsilon(u^\varepsilon),\Pi_+^\varepsilon(u^\varepsilon)) = a(u^\varepsilon,\Pi_+^\varepsilon(u^\varepsilon)) = \int_\Omega f\Pi_+^\varepsilon(u^\varepsilon) = \int_\Omega \Pi_+^\varepsilon(f)\Pi_+^\varepsilon(u^\varepsilon).$$

We next remark that

$$a^\varepsilon(\Pi_+^\varepsilon(u^\varepsilon),\Pi_+^\varepsilon(u^\varepsilon)) \geq \min_{\lambda_l^\varepsilon+\nu>0}\{\lambda_l^\varepsilon+\nu\}\,\|\Pi_+^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)}^2.$$

We also know that $\min_{\lambda_l^\varepsilon+\nu>0}\{\lambda_l^\varepsilon+\nu\} > \dfrac{\min_{\mu_l-\mathcal{M}+\nu>0}\{\mu_l-\mathcal{M}+\nu\}}{2} > 0$ as a consequence of Proposition 4.6. Exactly as in the proof of (4.72) established in Section 4.5.1, we can therefore show the coercivity of a^ε in $H_0^1(\Omega)\cap E_+^\varepsilon$ and establish the existence of $C_+>0$, independent of ε, such that

$$a^\varepsilon(\Pi_+^\varepsilon(u^\varepsilon),\Pi_+^\varepsilon(u^\varepsilon)) \geq C_+\|\Pi_+^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)}^2.$$

Moreover, the Cauchy-Schwarz inequality gives

$$\left|\int_\Omega \Pi_+^\varepsilon(f)\Pi_+^\varepsilon(u^\varepsilon)\right| \leq \|\Pi_+^\varepsilon(f)\|_{L^2(\Omega)}\|\Pi_+^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}\|\Pi_+^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)},$$

and we obtain

$$\|\Pi_+^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)} \leq \frac{1}{C_+}\|f\|_{L^2(\Omega)}. \tag{4.80}$$

On the other hand, we have

$$-a^\varepsilon(\Pi_-^\varepsilon(u^\varepsilon),\Pi_-^\varepsilon(u^\varepsilon)) = -a^\varepsilon(u^\varepsilon,\Pi_-^\varepsilon(u^\varepsilon)) = -\int_\Omega f\Pi_-^\varepsilon(u^\varepsilon) = -\int_\Omega \Pi_-^\varepsilon(f)\Pi_-^\varepsilon(u^\varepsilon),$$

and

$$-a^\varepsilon(\Pi_-^\varepsilon(u^\varepsilon),\Pi_-^\varepsilon(u^\varepsilon)) \geq \min_{\lambda_l^\varepsilon+\nu<0}\{|\lambda_l^\varepsilon+\nu|\}\,\|\Pi_-^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)}^2.$$

As above, we can also deduce the existence of $C_->0$ such that

$$\|\Pi_-^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)} \leq \frac{1}{C_-}\|f\|_{L^2(\Omega)}. \tag{4.81}$$

We next denote $M=\max\left(\dfrac{1}{C_+},\dfrac{1}{C_-}\right)$. Since $\lambda_l^\varepsilon+\nu\neq 0$ for every $l\in\mathbb{N}$ and for ε sufficiently small, we have $\|u^\varepsilon\|_{H^1(\Omega)} \leq \|\Pi_+^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)}+\|\Pi_-^\varepsilon(u^\varepsilon)\|_{H^1(\Omega)}$. Using (4.80) and (4.81), we obtain

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq 2M\|f\|_{L^2(\Omega)},$$

which yields (4.79). Therefore, up to an extraction, u^ε converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. Repeating step by step the proof of Proposition 4.5, we can then conclude that the sequence u^ε converges, strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, to u^* solution to (4.22). If we next define R^ε by (4.23), R^ε belongs to $H_0^1(\Omega)$, is clearly bounded in $H^1(\Omega)$ uniformly with respect to ε and converges to 0 in $L^2(\Omega)$ as ε vanishes. Moreover, a calculation shows that

$$-\Delta R^\varepsilon + \frac{1}{\varepsilon}V(./\varepsilon)R^\varepsilon + \nu R^\varepsilon = F^\varepsilon,$$

where

$$F^\varepsilon = \left(-\mathcal{M}-W_{\varepsilon,\Omega}(./\varepsilon)V(./\varepsilon)\right)u^* + 2\nabla W_{\varepsilon,\Omega}(./\varepsilon).\nabla u^* + \varepsilon W_{\varepsilon,\Omega}(./\varepsilon)\left(-\mathcal{M}u^*-f\right).$$

Using that $a^\varepsilon(R^\varepsilon,v)=\int_\Omega F^\varepsilon v$ for every $v\in H_0^1(\Omega)$, we can also show that (see the above proofs of estimates (4.79), (4.80) and (4.81)) :

$$\frac{1}{C^2}\|R^\varepsilon\|_{H^1(\Omega)}^2 \leq \left|\int_\Omega F^\varepsilon\Pi_-^\varepsilon(R^\varepsilon)\right| + \left|\int_\Omega F^\varepsilon\Pi_+^\varepsilon(R^\varepsilon)\right|.$$

We finally conclude that both $\left|\int_\Omega F^\varepsilon \Pi_-^\varepsilon(R^\varepsilon)\right|$ and $\left|\int_\Omega F^\varepsilon \Pi_+^\varepsilon(R^\varepsilon)\right|$ converge to 0 as ε vanishes proceeding exactly as in the proof of Theorem 4.2.

□

We conclude this article with a discussion regarding the possibility to extend our homogenization results to a larger class of non periodic potentials V. To this end, we suggest two possible complementary approaches :

1) Extension by density arguments.

We could adapt all of our proofs and establish results similar to those of Theorems 4.1, 4.2 and 4.3 considering a potential of the form (4.6) when φ is no longer compactly supported but decays sufficiently fast at infinity. This is for instance expressed by $|\varphi(x)| \leq \dfrac{C}{1+|x|^\alpha}$ for $\alpha > d$. We indeed remark that a simple adaptation of Lemma 4.7 shows that such a potential is a limit in $L^\infty(\mathbb{R}^d)$ of functions of the class (4.6) that we study in this article. In addition, the proof of Theorem 4.1 is based on the continuity from $L^\infty(\mathbb{R}^d)$ to $BMO(\mathbb{R}^d)$ of $T : f \to \nabla^2 G * f$ (see the proof of Proposition 4.2) and we could easily show that all of our results regarding the corrector equation (the weak-convergence properties of the gradient of our sequence of correctors $W_{\varepsilon,R}$ in particular) could therefore be extended by density arguments. Consequently, since the homogenization results of the present contribution are established only using the specific properties of the adapted corrector and the fact that $V \in L^\infty(\mathbb{R}^d)$, they could be naturally extended to this setting.

2) Extension by algebraic operations.

The homogenization results could be also established considering a potential of the form, say, $V = \sum_{k,l\in\mathbb{Z}^d} \varphi(. - k - Z_k)\psi(. - l - Z_l)$ obtained by multiplying two potentials of the particular class (4.6) that we have studied. For this new setting, although our approach based on the Taylor expansion of V would still be possible to solve the corrector equation, Assumptions (A1) to (A3) would no longer be sufficient to establish the existence of an adapted corrector w since the convergence of $|\nabla w(./\varepsilon)|^2$ would involve the correlations of the sequence Z up to the fourth order. Adapting (A2.a),(A2.b),(A2.c) and (A3) to the fourth order correlations of Z, the method introduced in the present article would again allow to conclude. Iterating this argument and under suitable assumptions for the correlations of Z of any order, our homogenization results could therefore be extended to the whole algebra generated by the potential of the form (4.6).

Bibliographie

[1] Y. Achdou and C. Le Bris, in preparation.

[2] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces*, Second edition, Pure and Applied Mathematics (Amsterdam), Volume 140, Elsevier/Academic Press, Amsterdam, 2003.

[3] G. Allaire, *Homogenization and two-scale convergence*, SIAM Journal on Mathematical Analysis 23, no.6, pp. 1482–1518, 1992.

[4] G. Allaire and A. Piatnitski, *Homogenization of the Schrödinger equation and effective mass theorems*, Communications in Mathematical Physics 258, no.1, pp. 1–22, 2005.

[5] M.A. Al-Gwaiz, *Theory of Distributions*, Pure and Applied Mathematics, CRC Press, 1992.

[6] A. Anantharaman, X. Blanc and F. Legoll, *Asymptotic behaviour of Green functions of divergence form operators with periodic coefficients*, Applied Mathematics Research Express 2013, no.1, pp. 79–101, 2013.

[7] S. Armstrong, A. Gloria and T. Kuusi, *Bounded correctors in almost periodic homogenization*, Archive for Rational Mechanics and Analysis 222, no.1, pp. 393–426, 2016.

[8] S. Armstrong, T. Kuusi and J.-C. Mourrat, *Quantitative stochastic homogenization and large-scale regularity*, Volume 352, Grundlehren der Mathematischen Wissenschaften, Springer, Cham 2019.

[9] S. Armstrong, T. Kuusi and C. Smart, *Large-Scale Analyticity and Unique Continuation for Periodic Elliptic Equations*, Communications on Pure and Applied Mathematics, 2020.

[10] S.N. Armstrong, and Z. Shen, *Lipschitz Estimates in Almost-Periodic Homogenization*, Communications on Pure and Applied Mathematics 69, no.10, pp. 1882–1923, 2016.

[11] M. Avellaneda and F.H. Lin, *Compactness methods in the theory of homogenization*, Communications on Pure and Applied Mathematics 40, no.6, pp. 803–847, 1987.

[12] M. Avellaneda and F.H. Lin, *Compactness methods in the theory of homogenization II : Equations in non-divergence form*, Communications on Pure and Applied Mathematics 42, no.2, pp. 139–172, 1989.

[13] M. Avellaneda and F.H. Lin, *Un théorème de Liouville pour des équations elliptiques à coefficients périodiques*, CR Acad. Sci. Paris Sér. I Math 309, no.5, pp. 245–250, 1989.

[14] M. Avellaneda and F.H. Lin, *L^p bounds on singular integrals in homogenization*, Communications on Pure and Applied Mathematics 44, no.8-9, pp. 897–910, 1991.

[15] G. Bal, *Homogenization with large spatial random potential*, Multiscale Modeling & Simulation 8, no.4, pp. 1484–1510, 2010.

[16] G. Bal, J. Garnier, S. Motsch and V. Perrier, *Random integrals and correctors in homogenization*, Asymptotic Analysis 59, no.1-2, pp. 1–26, 2008.

[17] G. Bal and Y. Gu, *Limiting models for equations with large random potential : A review*, Communications in Mathematical Sciences 13, no.3, pp. 729–748, 2015.

[18] A. Bensoussan, J.-L. Lions and G. Papanicolaou, *Asymptotic analysis for periodic structures*, Studies in Mathematics and its Applications, 5. North-Holland Publishing Co., Amsterdam-New York, 1978.

[19] J. Bergh and J. Löfström, *Interpolation spaces : an introduction*, Springer Science & Business Media, 2012.

[20] X. Blanc, M. Josien and C. Le Bris, *Precised approximations in elliptic homogenization beyond the periodic setting*, Asymptotic Analysis 116, no.2, pp. 93–137, 2020.

[21] X. Blanc and C. Le Bris, *Homogénéisation en milieu périodique... ou non : une introduction*, à paraître dans la collection SMAI-Springer, 2022.

[22] X. Blanc, C. Le Bris and P.-L. Lions, *A definition of the ground state energy for systems composed of infinitely many particles*, Communications in Partial Differential Equations 28, no.1-2, pp. 439–475, 2003.

[23] X. Blanc, C. Le Bris and P.-L. Lions, *Stochastic homogenization and random lattices*, Journal de Mathématiques Pures et Appliquées 88, no.1, pp. 34–63, 2007.

[24] X. Blanc, C. Le Bris and P.-L. Lions, *The energy of some microscopic stochastic lattices*, Archive for Rational Mechanics and Analysis 184, no.2, pp. 303–339, 2007.

[25] X. Blanc, C. Le Bris and P.-L. Lions, *A possible homogenization approach for the numerical simulation of periodic microstructures with defects*, Milan Journal of Mathematics 80, no.2, pp. 351–367, 2012.

[26] X. Blanc, C. Le Bris and P.-L. Lions, *Local profiles for elliptic problems at different scales : defects in, and interfaces between periodic structures*, Communications in Partial Differential Equations 40, no.12, pp. 2173–2236, 2015.

[27] X. Blanc, C. Le Bris and P.-L. Lions, *On correctors for linear elliptic homogenization in the presence of local defects*, Communications in Partial Differential Equations 43, no.6, pp. 965–997, 2018.

[28] X. Blanc, C. Le Bris and P.-L. Lions, *On correctors for linear elliptic homogenization in the presence of local defects : The case of advection-diffusion*, Journal de Mathématiques Pures et Appliquées 124, pp. 106–122, 2019.

[29] X. Blanc and S. Wolf, *Homogenization of the Poisson equation in a non-periodically perforated domain*, Asymptotic Analysis 126, no.1-2, pp. 129–155, 2022.

[30] G. Bourdaud, *Remarques sur certains sous-espaces de* $BMO(\mathbb{R}^n)$ *et de* $bmo(\mathbb{R}^n)$, Annales de l'institut Fourier 52, no.4, pp. 1187–1218, 2002.

[31] A. Braides, *Gamma-Convergence for Beginners*, Volume 22, Oxford University Press, Oxford, 2002.

[32] H. Brezis, *Functional Analysis, Sobolev Space and Partial Differential Equations*, Springer, New-York, 2011.

[33] H. Brezis, *On a conjecture of J. Serrin*, Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali, Rendiconti Lincei Matematica e Applicazioni 19, no.4, pp. 335–338, 2008.

[34] E. Cancès, L. Garrigue and D. Gontier, *Second-order homogenization of periodic Schrödinger operators with highly oscillating potentials*, arXiv preprint arXiv :2112.12008, 2021.

[35] P. Cardaliaguet, C. Le Bris and P.E. Souganidis, *Perturbation problems in homogenization of Hamilton–Jacobi equations*, Journal de Mathématiques Pures et Appliquées 117, pp. 221–262, 2018.

[36] I. Catto, C. Le Bris and P.-L. Lions, *The mathematical theory of thermodynamic limits : Thomas-Fermi type models*, Oxford University Press, 1998.

[37] J. Cigler, *On a theorem of H. Weyl*, Compositio Mathematica 21, no.2, pp. 151–154, 1969.

[38] E. De Giorgi and S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. Ital. (4) 8, pp. 391–411, 1973.

[39] J. Deny and J.-L. Lions, *Les espaces du type de Beppo Levi*, Annales de l'institut Fourier 5, pp. 305–370, 1954.

[40] G. Dolzmann and S. Müller, *Estimates for Green's matrices of elliptic systems by L^p theory*, Manuscripta Mathematica 88, no.1, pp. 261–273, 1995.

[41] M. Drmota and R.F. Tichy, *Sequences, discrepancies and applications*, Springer, 2006.

[42] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[43] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, 2015.

[44] A. Farina, *Liouville-type theorems for elliptic problems*, Handbook of differential equations : stationary partial differential equations 4, pp. 61–116, 2007.

[45] J.N. Franklin, *Deterministic simulation of random processes*, Mathematics of Computation 17, no.81, pp. 28–59, 1963.

[46] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton University Press, 1983.

[47] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Lecture Notes Scuola Normale Superiore di Pisa (New Series), Volume 11, Edizioni della Normale, Pisa, Second edition, 2012.

[48] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Volume 224, no.2, Springer, Berlin, 1977.

[49] A. Gloria and F. Otto, *Quantitative results on the corrector equation in stochastic homogenization*, Journal of the European mathematical society 19, no.11, pp. 3489–3548, 2017.

[50] A. Gloria, S. Neukamm and F. Otto, *A regularity theory for random elliptic operators*, Milan Journal of Mathematics 88, no.1, pp. 99–170, 2020.

[51] M. Gruter and K.O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Mathematica 37, no.3, pp. 303–342, 1982.

[52] Q. Han and F. Lin, *Elliptic Partial Differential Equations*, Volume 1, American Mathematical Soc., 2011.

[53] S. Hofmann and S. Kim, *The Green function estimates for strongly elliptic systems of second order*, Manuscripta Mathematica 124, no.2, pp. 139–172, 2007.

[54] L. Hörmander, *The analysis of linear partial differential operators I : Distribution theory and Fourier analysis*, Springer, Berlin, 2015.

[55] T. Hytönen, J. Van Neerven, M. Veraar and L. Weis, *Analysis in Banach spaces*, Springer, Berlin, 2016.

[56] B. Iftimie, E. Pardoux and A. Piatnitski, *Homogenization of a singular random one-dimensional PDE*, Annales de l'IHP Probabilités et statistiques 44, no.3, pp. 519–543, 2008.

[57] W. Jäger, A. Tambue and J.-L. Woukeng, *Approximation of homogenized coefficients in deterministic homogenization and convergence rates in the asymptotic almost periodic setting*, arXiv Preprint arXiv :1906.11501, 2019.

[58] D.L. Jagerman, *The autocorrelation function of a sequence uniformly distributed modulo 1*, The Annals of Mathematical Statistics 34, no.4, pp. 1243–1252, 1963.

[59] V.V. Jikov, S.M Kozlov and O.A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer Science & Business Media, 2012.

[60] M. Josien, *Étude mathématique et numérique de quelques modèles multi-échelles issus de la mécanique des matériaux*, thèse, 2018.

[61] C.E. Kenig, F. Lin and Z. Shen, *Estimates of eigenvalues and eigenfunctions in periodic homogenization*, Journal of the European Mathematical Society 15, no.5, pp. 1901–1925, 2013.

[62] C.E. Kenig, F. Lin and Z. Shen, *Periodic homogenization of Green and Neumann functions*, Communications on Pure and Applied Mathematics 67, no.8, pp. 1219–1262, 2014.

[63] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience, New-York, 1974.

[64] S.M. Kozlov, *Averaging differential operators with almost periodic, rapidly oscillating coefficients*, Mathematics of the USSR-Sbornik 35, no.4, pp. 481–498, 1979.

[65] S.M. Kozlov, *Averaging of random operators*, Matematicheskii Sbornik 151, no.2, pp. 188–202, 1979.

[66] B. Lawton, *A note on well distributed sequences*, Proceedings of the American Mathematical Society 10, no.6, pp. 891–893, 1959.

[67] C. Le Bris, *Systèmes multi-échelles : modélisation et simulation*, Volume 47, Springer Science & Business Media, 2005.

[68] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, Volume 14, American Mathematical Society, Providence, RI, 1997.

[69] J.-L. Lions, *Some methods in the mathematical analysis of systems and their control*, Science Press, Beijing, 1981.

[70] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Parts 1 & 2*, Ann. Inst. H. Poincaré 1, pp. 109-145 and 223-283, 1984.

[71] P.-L. Lions, G. Papanicolaou and S.R.S. Varadhan, *Homogenization of hamilton-jacobi equations*, unpublished preprint, 1986.

[72] V. Maz'ya, *Sobolev Spaces*, Springer, 2013.

[73] Y. Meyer and R. Coifman, *Wavelets : Calderón-Zygmund and multilinear operators*, Volume 48, Cambridge University Press, 1997.

[74] N.G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 17, no.3, pp. 189–206, 1963.

[75] J. Moser, *On Harnack's theorem for elliptic differential equations*, Communications on Pure and Applied Mathematics 14, no.3, pp. 577–591, 1961.

[76] J. Moser and M. Struwe, *On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus*, Boletim da Sociedade Brasileira de Matemática, Bulletin Brazilian Mathematical Society 23, no.1, pp. 1–20, Springer, 1992.

[77] F. Murat, *Compacité par compensation*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV 5, no.3, pp. 489–507, 1978.

[78] F. Murat and L. Tartar, *Calculus of variations and homogenization* in Topics in the mathematical modelling of composite materials, Progr. Nonlinear Differential Equations Appl., Volume 31, pp. 139–173, Birkhäuser Boston, Boston, 1997.

[79] F. Murat and L. Tartar, *H-convergence* in Topics in the mathematical modelling of composite materials, Progr. Nonlinear Differential Equations Appl., Volume 31, pp. 21–43, Birkhäuser Boston, Boston, 1997.

[80] J. Namazi, *L^∞-BMO boundedness for a singular integral operator*, Proceedings of the American Mathematical Society 108, no.2, pp. 465–470, 1990.

[81] U. Neri, *Singular Integrals*, Lecture Notes in Mathematics, Volume 200, Springer-Verlag, Berlin-New York, 1971.

[82] O.A. Oleinik and V.V. Zhikov, *On the homogenization of elliptic operators with almost-periodic coefficients*, Rendiconti del Seminario Matematico e Fisico di Milano 53, no.1, pp. 149–166, 1982.

[83] C. Ortner and E. Suli, *A Note on Linear Elliptic Systems on $\mathbb{R}^d$*, arXiv preprint, arXiv :1202.3970, 2012.

[84] G.C. Papanicolaou and S.R.S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, In Random fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai, Volume 27, pp. 835–873, North-Holland, Amsterdam-New York, 1981.

[85] G. Pavliotis and A. Stuart, *Multiscale methods : averaging and homogenization*, Springer Science & Business Media, 2008.

[86] L. Schwartz, *Théorie des distributions*, Hermann Paris, 1966.

[87] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, Volume 30, Princeton University Press, Princeton, 1970.

[88] E.M. Stein and T.S. Murphy, *Harmonic analysis : real-variable methods, orthogonality, and oscillatory integrals*, Volume 3, Princeton University Press, 1993.

[89] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, volume 3, Springer Science & Business Media, 2007.

[90] L. Tartar, *The general theory of homogenization : a personalized introduction*, Volume 7, Springer Science & Business Media, 2009.

[91] S. Wolf, *Homogenization of the Stokes system in a non-periodically perforated domain*, Multiscale Modeling & Simulation 20, no.1, pp. 72–106, 2022.

[92] S. Wolf, *Étude de problèmes d'homogénéisation dans un cadre périodique avec défauts*, PhD thesis, in preparation.

[93] V.V. Yurinskii, *Averaging of symmetric diffusion in random medium*, Siberian Mathematical Journal 27, no.4, pp. 603–613, 1986.

[94] Y. Zhang, *Estimates of eigenvalues and eigenfunctions in elliptic homogenization with rapidly oscillating potentials*, Journal of Differential Equations 292, pp. 388–415, Elsevier, 2021.

[95] W.P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, Volume 120, Springer-Verlag, New York, 1989.

Annexe A

Divers résultats associés à l'homogénéisation du problème de diffusion

A.1 Un lemme de type Liouville

Dans cette section on s'intéresse à l'unicité des solutions de $-\operatorname{div}(a\nabla u) = \operatorname{div}(f)$ lorsque ∇u appartient à un espace de Lebesgue $\left(L^q(\mathbb{R}^d)\right)^d$ pour un exposant $q < d$ et lorsque a est un coefficient supposé être uniquement borné et coercif dans le sens de (1.2)-(1.3). Ce résultat d'unicité a été introduit à l'origine dans [27, Lemma 2.2] dans le cadre de l'étude de l'homogénéisation pour une géométrie périodique perturbée par des défauts intégrables sur $\mathbb{R}^d$, plus particulièrement pour établir l'existence d'un correcteur adapté à ce problème. La preuve introduite dans [27, Lemma 2.2] requiert cependant des propriétés de régularité du coefficient et repose fortement sur l'inégalité de De Giorgi-Nash-Moser (voir [75, Theorem 2]), qui n'est vraie que lorsque l'inconnue u est à valeurs scalaires. L'intérêt de la variante de la preuve présentée dans cette section est qu'elle ne fait intervenir que des techniques variationnelles ne nécessitant pas de régularité sur a et s'adapte facilement au cadre des systèmes.

Pour commencer, on établit une proposition de type Liouville lorsque la solution u de $-\operatorname{div}(a\nabla u) = 0$ vérifie $\nabla(|u|^{\frac{p}{2}}) \in \left(L^2(\mathbb{R}^d)\right)^d$ pour $p \geq 2$.

Proposition A.1. *On suppose que $d > 2$ et que a est un coefficient vérifiant les hypothèses de borne uniforme et de coercivité données respectivement par (1.2) et (1.3). Soient $q > 2$ et u solution dans $\mathcal{D}'\left(\mathbb{R}^d\right)$ de :*

$$-\operatorname{div}\left(a\nabla u\right) = 0 \quad \textit{sur } \mathbb{R}^d, \tag{A.1}$$

telle que $\nabla\left(|u|^{\frac{p}{2}}\right) \in \left(L^2(\mathbb{R}^d)\right)^d$ et $|u|^{\frac{p}{2}} \in L^{2^*}(\mathbb{R}^d)$, où $2^* = \dfrac{2d}{d-2}$. Alors $u = 0$.

Démonstration. Soit φ fonction de $\mathcal{D}(\mathbb{R}^d)$ à support dans B_2 et telle que :

$$\varphi = 1 \text{ sur } B_1, \quad \|\varphi\|_{L^\infty(\mathbb{R}^d)} \leq 1.$$

Soit $R > 0$. Dans la suite on définit $\varphi_R = \varphi\left(\frac{\cdot}{R}\right)$ et on va utiliser $\chi = u|u|^{p-2}\varphi_R^2$ comme fonction test dans la formulation variationnelle de (A.1). On obtient alors :

$$(p-1)\int_{\mathbb{R}^d} \left(\nabla u . a\nabla u\right)|u|^{p-2}\varphi_R^2 = -2\int_{\mathbb{R}^d}\left(a\nabla u.\nabla\varphi_R\right)u|u|^{p-2}\varphi_R.$$

On utilise les hypothèses (1.2) et (1.3) et on en déduit :

$$(p-1)\int_{\mathbb{R}^d} |\nabla u|^2 |u|^{p-2}\varphi_R^2 \leq \frac{2M}{\lambda}\int_{\mathbb{R}^d}\left|\left(\nabla u.\nabla\varphi_R\right)|u|^{p-1}\varphi_R\right|, \tag{A.2}$$

où $M = \|a\|_{L^\infty(\mathbb{R}^d)}$.

On remarque alors que $\nabla\left(|u|^{\frac{p}{2}}\right) = \frac{p}{2} u\,|u|^{\frac{p}{2}-2}\,\nabla u$ et on a donc :

$$(p-1)\int_{\mathbb{R}^d} |\nabla u|^2\,|u|^{p-2}\varphi_R^2 = \frac{p-1}{\left(\frac{p}{2}\right)^2}\int_{\mathbb{R}^d}\left|\nabla(|u|^{\frac{p}{2}})\right|^2\varphi_R^2, \tag{A.3}$$

et

$$\int_{\mathbb{R}^d}\left|(\nabla u.\nabla\varphi_R)\,|u|^{p-1}\varphi_R\right| = \frac{2}{p}\int_{\mathbb{R}^d}\left|\left(\nabla(|u|^{\frac{p}{2}}).\nabla\varphi_R\right)|u|^{\frac{p}{2}}\varphi_R\right|. \tag{A.4}$$

On utilise maintenant (A.2), (A.3) et l'inégalité de Cauchy-Schwarz dans (A.4) et on en déduit alors :

$$\int_{\mathbb{R}^d}\left|\nabla\left(|u|^{\frac{p}{2}}\right)\right|^2\varphi_R^2 \le \frac{M}{\lambda}\frac{p}{2(p-1)}\left(\int_{\mathbb{R}^d}|\nabla\varphi_R|^2\,|u|^p\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla\left(|u|^{\frac{p}{2}}\right)\right|^2\varphi_R^2\right)^{1/2}. \tag{A.5}$$

On considère maintenant $\varepsilon > 0$ tel que :

$$\varepsilon\frac{M}{\lambda}\frac{p}{4(p-1)} < 1.$$

En utilisant l'inégalité de Young sur le terme de droite de (A.5), on obtient :

$$\int_{\mathbb{R}^d}\left|\nabla\left(|u|^{\frac{p}{2}}\right)\right|^2\varphi_R^2 \le \frac{M}{\lambda}\frac{p}{4\varepsilon(p-1)}\int_{\mathbb{R}^d}|\nabla\varphi_R|^2|u|^p + \varepsilon\frac{M}{\lambda}\frac{p}{4(p-1)}\int_{\mathbb{R}^d}\left|\nabla\left(|u|^{\frac{p}{2}}\right)\right|^2\varphi_R^2,$$

ce qui est équivalent à

$$\left(1-\varepsilon\frac{M}{\lambda}\frac{p}{4(p-1)}\right)\int_{\mathbb{R}^d}\left|\nabla\left(|u|^{\frac{p}{2}}\right)\right|^2\varphi_R^2 \le \frac{M}{\lambda}\frac{p}{4\varepsilon(p-1)}\int_{\mathbb{R}^d}|\nabla\varphi_R|^2|u|^p. \tag{A.6}$$

On rappelle que $2^* = \frac{2d}{d-2}$ et on vérifie que $\frac{1}{2} = \frac{1}{2^*} + \frac{1}{d}$. En utilisant que $\nabla\varphi_R$ est à support dans $A_{R,2R} = B_{2R}\setminus B_R$, l'inégalité de Hölder nous donne alors :

$$\int_{\mathbb{R}^d}|\nabla\varphi_R|^2|u|^p = \int_{A_{R,2R}}|\nabla\varphi_R|^2|u|^p \le \left(\int_{A_{R,2R}}|\nabla\varphi_R|^d\right)^{2/d}\left(\int_{A_{R,2R}}|u|^{\frac{2^*p}{2}}\right)^{2/2*}. \tag{A.7}$$

Par ailleurs, pour tout $R > 0$ on a $\|\nabla\varphi_R\|_{L^\infty(A_{R,2R})} = \left\|\nabla\left(\varphi\left(\frac{\cdot}{R}\right)\right)\right\|_{L^\infty(A_{R,2R})} \le \frac{1}{R}$ et on en déduit que :

$$\int_{A_{R,2R}}|\nabla\varphi_R|^d \le \frac{|A_{R,2R}|}{R^d} \le C(d), \tag{A.8}$$

où $C(d)$ est une constante positive ne dépendant que de la dimension d. En injectant (A.7) et (A.8) dans (A.6), on obtient alors l'existence d'une contante $C_p > 0$ ne dépendant que de p, $\frac{M}{\lambda}$ et de la dimension, telle que :

$$\|\nabla(|u|^{\frac{p}{2}})\|^2_{L^2(B_R)} \le \int_{\mathbb{R}^d}\varphi_R^2|\nabla(|u|^{\frac{p}{2}})|^2 \le C_p\||u|^{\frac{p}{2}}\|^2_{L^{2^*}(A_{R,2R})}. \tag{A.9}$$

De plus on sait que $|u|^{\frac{p}{2}}$ appartient à $L^{2^*}(\mathbb{R}^d)$. En passant à la limite on a donc :

$$\lim_{R\to\infty} \left\| |u|^{\frac{p}{2}} \right\|_{L^{2^*}(A_{R,2R})} = 0$$

Ainsi, à l'aide de l'inégalité (A.9), on en déduit donc que $\nabla(|u|^{\frac{p}{2}}) = 0$. On conclut alors que $|u|^{\frac{p}{2}}$ est constante. Puisque $|u|^{\frac{p}{2}} \in L^{2^*}(\mathbb{R}^d)$, cette constante est nécessairement nulle et, par conséquent, $u = 0$ l'est également. $\square$

Une conséquence de la Proposition A.1 est ainsi donnée par le résultat d'unicité suivant :

Proposition A.2. *On suppose que $d \geq 2$ et que a vérifie (1.2) et (1.3). On fixe $2 \leq q < d$. Soit u solution dans $\mathcal{D}'(\mathbb{R}^d)$ de :*

$$-\operatorname{div}(a\nabla u) = 0 \quad \textit{sur}\, \mathbb{R}^d,$$

telle que $\nabla u \in \left(L^q(\mathbb{R}^d)\right)^d$. Alors $\nabla u = 0$.

Démonstration. On note $q^* = \dfrac{qd}{d-q}$, l'exposant de Sobolev associé à q et on définit aussi $p := \dfrac{q(d-2)}{(d-q)}$. On a $2 \leq p < +\infty$ et on peut facilement vérifier les égalités suivantes :

$$q = \frac{pd}{p+d-2} \quad \text{et} \quad q^* = \frac{2^*p}{2}.$$

Puisque $\nabla u \in \left(L^q(\mathbb{R}^d)\right)^d$ et $q < d$, une conséquence de l'inégalité de Gagliardo-Nirenberg-Sobolev (voir [42, Section 5.6.1]) nous assure l'existence d'une constante $C > 0$ telle que $u - C \in L^{q^*}(\mathbb{R}^d)$. On définit alors $v := u - C$. On a $\nabla v = \nabla u$ et

$$\left|\nabla |v|^{\frac{p}{2}}\right| = \frac{p}{2}|\nabla v||v|^{\frac{p}{2}-1} = \frac{p}{2}|\nabla u||v|^{\frac{p}{2}-1}.$$

Par ailleurs, on sait que $\nabla u \in \left(L^q(\mathbb{R}^d)\right)^d = \left(L^{pd/(p+d-2)}(\mathbb{R}^d)\right)^d$ et, puisque $|v| \in L^{\frac{2^*p}{2}}(\mathbb{R}^d)$, on a $|v|^{\frac{p}{2}-1} \in L^{2pd/((d-2)(p-2))}(\mathbb{R}^d)$. On a également l'égalité suivante :

$$\frac{d+p-2}{pd} + \frac{(d-2)(p-2)}{2pd} = \frac{2d+2p-4+(d-2)(p-2)}{2pd} = \frac{1}{2}.$$

En utilisant l'inégalité de Hölder, on en déduit donc que $\nabla\left(|v|^{\frac{p}{2}}\right)$ appartient à $\left(L^2(\mathbb{R}^d)\right)^d$. La Proposition A.1 nous permet finalement de conclure que $\nabla v = \nabla u = 0$. $\square$

A.2 Un contre-exemple de continuité

Nous nous intéressons ici aux propriétés de continuité de l'opérateur $\nabla\left(-\operatorname{div}(a\nabla .)\right)^{-1}\operatorname{div}$. La compréhension de cet opérateur est en effet primordial dans l'approche introduite par Blanc, Le Bris et Lions dans [27] et utilisée dans l'étude du problème d'homogénéisation (1.1) pour des coefficients perturbés de la forme $a = a_{per} + \tilde{a}$, approche que nous utilisons fortement dans les **Chapitres** 2 et 3 afin d'établir les résultats d'existence de correcteurs dans des espaces adaptés. Comme nous l'avons précisé dans le chapitre introductif (voir Section 1.3.4 par exemple), pour de tels coefficients perturbés, l'approche naturelle (intuitée à l'aide d'exemples monodimensionnels) est d'établir l'existence d'un correcteur w_p solution de (1.19) sous la forme $w_p = w_{per,p} + \tilde{w}_p$ où ∇w_p possède la même structure "périodique + perturbation" que le coefficient a. En notant $f = \tilde{a}\left(\nabla w_{per,p} + p\right)$, qui appartient alors à l'espace des perturbations, on peut voir que cette approche mène à la résolution d'un problème de la forme

$$-\operatorname{div}(a\nabla u) = \operatorname{div}(f),$$

où $\nabla u = \nabla \tilde{w}_p$ est également cherché dans l'espace des perturbations. Un des principaux objectifs de cette approche est alors de montrer la continuité de l'opérateur $\nabla\left(-\operatorname{div}(a\nabla .)\right)^{-1}\operatorname{div}$ de l'espace des perturbations dans lui même. Les questions sont donc similaires à celles soulevées par la théorie des opérateurs de Calderòn-Zygmund. En particulier, lorsque l'espace des perturbations est un espace de Lebesgue, c'est-à-dire $\tilde{a} \in L^p(\mathbb{R}^d)$ pour $p \in]1, +\infty[$, les résultats de [27] établissent (sous des hypothèses de régularité sur a) la continuité de $\nabla\left(-\operatorname{div}(a\nabla .)\right)^{-1}\operatorname{div}$ de $\left(L^p(\mathbb{R}^d)\right)^d$ dans lui même. Il est alors naturel de se demander si, dans un cadre plus général de perturbations localisées, typiquement lorsque $\tilde{a}$ appartient à l'espace des fonctions continues qui tendent vers 0 à l'infini noté $\mathcal{C}^0_0(\mathbb{R}^d)$, il est possible d'établir des résultats de continuité similaires.

Une question générale reliée à ce problème est donc : l'opérateur $\nabla\left(-\operatorname{div}(a\nabla .)\right)^{-1}\operatorname{div}$ est-il continu de $\left(\mathcal{C}^0_0(\mathbb{R}^d)\right)^d$ dans lui-même ? Nous apportons ici une réponse négative en donnant un contre-exemple dans un cas simple où $d = 2$ et $a = 1$.

On définit, pour $x \in \mathbb{R}^2$:

$$u(x) = -x_1 \ln(\ln(|x|)),$$

et

$$f(x) = \frac{2x_1^2\ln(|x|) - x_2^2}{|x|^2\ln^2(|x|)} e_1 + \frac{2x_1x_2\ln(|x|) + x_1x_2}{|x|^2\ln^2(|x|)} e_2,$$

où (e_1, e_2) est la base canonique de $\mathbb{R}^2$. Lorsque $|x| \to \infty$, on peut alors facilement vérifier que $|\nabla u(x)|$ croît comme $\ln(\ln(|x|))$ et que

$$f(x) = O\left(\frac{1}{\ln(|x|)}\right).$$

De plus, si $|x| > 1$, on a

$$-\Delta u(x) = \operatorname{div}(f)(x).$$

On considère alors une fonction plateau $\chi > 0$ telle que :

$$\chi(x) = 1 \text{ si } |x| \geq 3, \quad \chi(x) = 0 \text{ si } |x| \leq 2.$$

et on introduit également $\phi \in H^1_{loc}(\mathbb{R}^d)$ telle que $\nabla\phi \in \left(L^2(\mathbb{R}^d)\right)^d$, solution de

$$-\Delta\phi = u\Delta\chi + 2\nabla u.\nabla\chi + \nabla\chi.f.$$

Une telle fonction ϕ est effectivement bien définie puisque $u\Delta\chi + 2\nabla u.\nabla\chi + \nabla\chi f$ est à support compact, voir par exemple [25, Lemma 1, step 1] pour la preuve. On remarque alors que :

$$\Delta(u\chi) = u\Delta\chi + 2\nabla u.\nabla\chi + \Delta u\chi$$

et :

$$\operatorname{div}(\chi f) = \nabla\chi . f + \chi \operatorname{div}(f).$$

Finalement en posant $\tilde{u} = u\chi + \phi$ et $\tilde{f} = \chi f$, la fonction $\tilde{u}$ est solution de :

$$-\Delta\tilde{u} = \operatorname{div}(\tilde{f}) \quad \text{sur } \mathbb{R}^d. \tag{A.10}$$

De plus, on remarque que f est régulière sur $\mathbb{R}^d$, bornée et

$$\lim_{|x|\to\infty} \tilde{f}(x) = 0.$$

Cependant, on voit que si $|x| \geq 3$, on a $\tilde{u} = u$ et, puisque ∇u croît vers l'infini comme $\ln(\ln(|x|))$, on a bien établi l'existence d'une solution non bornée de $-\Delta u = \operatorname{div}(f)$ dans un cas où $f \in \left(\mathcal{C}^0_0(\mathbb{R}^d)\right)^d$. Cela assure en fait qu'il n'existe aucune solution v à gradient dans $\left(L^\infty(\mathbb{R}^d)\right)^d$. En effet si une telle solution existait, on aurait

$$-\Delta(\tilde{u} - v) = 0, \quad \text{sur } \mathbb{R}^d,$$

et donc $\tilde{u} - v$ serait un polynôme. Puisque $\nabla v \in \left(L^\infty(\mathbb{R}^d)\right)^d$, le gradient de $\tilde{u} - v$ a exactement une croissance de l'ordre de $\ln(\ln(|x|))$ à l'infini et c'est absurde. On a donc établi qu'il n'existe aucune solution de (A.10) à gradient dans $\left(L^\infty(\mathbb{R}^d)\right)^d$. En particulier, on a montré que $-\nabla\Delta^{-1}\operatorname{div}$ est nécessairement non continu de $\left(\mathcal{C}^0_0(\mathbb{R}^d)\right)^d$ dans $\left(\mathcal{C}^0_0(\mathbb{R}^d)\right)^d$, et plus encore, non continu de $\left(\mathcal{C}^0_0(\mathbb{R}^d)\right)^d$ dans $\left(L^\infty(\mathbb{R}^d)\right)^d$.

A.3 Un cas particulier de petites perturbations

Dans cette section, on expose un travail de recherche effectué pendant cette thèse qui n'a pas complètement abouti. On démontre un résultat partiel d'existence pour l'équation du correcteur (1.19), associée au problème d'homogénéisation (1.1) dans un cadre spécifique de perturbations de la géométrie périodique. En particulier, on s'intéresse à un cas de défauts suffisamment petits, de manière uniforme sur tout l'espace. Dans la suite, cette géométrie sera modélisée par un coefficient de la forme

$$a = a_{per} + \eta\tilde{a}, \tag{A.11}$$

où, comme précédemment, $a_{per} \in L^2_{per}(\mathbb{R}^d)$ est périodique, $\tilde{a}$ représente une perturbation du cadre périodique et $\eta > 0$ est un petit paramètre réel. Pour simplifier, dans la suite on suppose que a est scalaire. On suppose aussi que a et a_{per} vérifient les hypothèses suivantes

$$\lambda \leq a(x), \quad \lambda \leq a_{per}(x), \quad \text{pour } \lambda > 0 \text{ fixé}, \tag{A.12}$$

$$a_{per},\ \tilde{a} \in C^{0,\alpha}(\mathbb{R}^d), \quad \text{pour } \alpha \in]0,1[, \tag{A.13}$$

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy < \infty. \tag{A.14}$$

On note que l'hypothèse (A.14) est une hypothèse technique qui va nous assurer que, pour toute fonction $f \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, il existe une solution u de

$$-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(\tilde{a}f),$$

telle que le gradient de u est borné sur $\mathbb{R}^d$. En notant, G_{per} la fonction de Green associée à l'opérateur $-\operatorname{div}(a_{per}\nabla .)$ sur $\mathbb{R}^d$ (définie par (2.36)), et rappelant l'estimée ponctuelle (2.39) démontrée dans [14, Section 2] qui nous assure que la croissance des dérivées secondes de G_{per} à l'infini est de l'ordre de $\dfrac{1}{|x|^d}$, cette hypothèse (A.14) va en effet nous permettre de définir la convolution $\nabla_x\nabla_y G_{per} * (\tilde{a}f)$ et d'assurer que cette fonction appartient à $\left(L^\infty(\mathbb{R}^d)\right)^d$. Cette hypothèse sert en réalité à pallier le manque de continuité de l'opérateur $\nabla\left(-\operatorname{div}(a_{per}\nabla .)^{-1}\operatorname{div}\right.$ de $\left(L^\infty(\mathbb{R}^d)\right)^d$ dans $\left(L^\infty(\mathbb{R}^d)\right)^d$ sur lequel nous avons insisté dans la section précédente.

Sous cette hypothèse, on va alors montrer que l'équation du correcteur (1.19) admet une solution strictement sous-linéaire à l'infini dès que η est suffisamment petit. On note :

$$\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d) \;\middle|\; \lim_{\varepsilon\to 0} |f(./\varepsilon)| = 0 \;\text{ dans } L^\infty(\mathbb{R}^d) - \star \right\}.$$

On note que $\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)$ est un espace de Banach lorsqu'il est muni de la norme

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x,y\in\mathbb{R}^d,\; x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

On établit alors la proposition suivante :

Proposition A.3. *On suppose $d \geq 2$ et que a est de la forme (A.11) et vérifie (A.12), (A.13) et (A.14). Pour tout $p \in \mathbb{R}^d$, on note $w_{per,p}$ l'unique (à constante additive près) correcteur périodique solution de (1.5). Alors il existe $\eta_0 > 0$ tel que pour tout $0 < \eta < \eta_0$, il existe une unique, à constante additive près, fonction $\tilde{w}_p \in L^1_{loc}(\mathbb{R}^d)$ telle que $\nabla\tilde{w}_p \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ et $w_p = w_{per,p} + \tilde{w}_p$ est une solution de (1.19). De plus $\varepsilon w_p(./\varepsilon)$ converge vers 0 dans $L^\infty_{loc}(\mathbb{R}^d)$ quand ε tend vers 0.*

Comme dans le cadre présenté en Section 1.3.4 de la partie introductive, l'idée est, à nouveau, de chercher un correcteur sous la forme $w_p = w_{per,p} + \tilde{w}_p$, où on veut montrer que $\tilde{w}_p$ est solution de :

$$-\operatorname{div}\left((a_{per} + \eta\tilde{a})\left(\nabla w_{per,p} + p + \nabla\tilde{w}_p\right)\right) = 0 \quad \text{dans } \mathbb{R}^d, \tag{A.15}$$

telle que $\nabla\tilde{w} \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ dès que η est suffisamment petit. En particulier cette équation peut se reformuler par :

$$-\operatorname{div}\left(a_{per}\nabla\tilde{w}_p\right) = \operatorname{div}\left(\eta\tilde{a}\left(\nabla w_{per,p} + p + \nabla\tilde{w}_p\right)\right) \quad \text{dans } \mathbb{R}^d.$$

Ici, l'idée est donc de trouver un point fixe de l'opérateur linéaire ϕ_η qui à $f \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ associe le gradient $\nabla u \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ d'une solution de :

$$-\operatorname{div}\left(a_{per}\nabla u\right) = \operatorname{div}\left(\eta\tilde{a}\left(\nabla w_{per,p} + p + f\right)\right) \quad \text{dans } \mathbb{R}^d.$$

Afin d'utiliser le théorème du point fixe de Banach, l'objectif est donc de montrer que ϕ_η est bien définie et est contractante dès que η est assez petit.

A.3.1 Lemmes préliminaires

On commence tout d'abord par introduire plusieurs lemmes préliminaires qui nous serviront à établir la Proposition A.3. Le Lemme A.1 est un résultat d'unicité, le Lemme A.2 est un résultat assurant la sous-linéarité stricte à l'infini des fonctions u telle que ∇u appartient à $\left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ et, finalement, le Lemme A.3 assure que, sous les hypothèses (A.13) et (A.14), la suite $|\tilde{a}(./\varepsilon)|$ converge faiblement vers 0 pour la topologie $L^\infty(\mathbb{R}^d) - *$, lorsque ε tend vers 0.

Lemma A.1. *Soit u solution dans $\mathcal{D}'(\mathbb{R}^d)$ de :*

$$-\operatorname{div}(a_{per}\nabla u) = 0, \tag{A.16}$$

telle que $\nabla u \in \left(L^\infty(\mathbb{R}^d)\right)^d$. Alors il existe deux constantes $p \in \mathbb{R}^d$, $c \in \mathbb{R}$ telles que

$$u = w_{per,p} + p.x + c,$$

où $w_{per,p}$ est l'unique, à constante additive près, correcteur périodique solution de :

$$-\operatorname{div}\left(a_{per}\left(\nabla w_{per,p} + p\right)\right) = 0 \quad \textit{sur } \mathbb{R}^d.$$

Par conséquent si $\nabla u \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$, alors $\nabla u = 0$.

Démonstration. Pour tout $k \in \mathbb{Z}^d$, en translatant l'équation (A.16), on a par périodicité de a_{per} :

$$-\operatorname{div}(a_{per}\nabla(u - \tau_k u)) = 0 \quad \text{sur } \mathbb{R}^d.$$

Montrons que pour tout $k \in \mathbb{Z}^d$, $u - \tau_k u \in L^\infty(\mathbb{R}^d)$. En effet, on a pour tout $x \in \mathbb{R}^d$:

$$\begin{aligned} |u(x+k) - u(x)| &= \left| \int_0^1 \nabla u(x+tk).k dt \right| \\ &\leq C \|\nabla u\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

Ainsi, d'après [75, Section 6], $u - \tau_k u$ est constante pour tout $k \in \mathbb{Z}^d$. On note $(e_i)_{\{1,\ldots,d\}}$ la base canonique de $\mathbb{R}^d$ et pour i dans $\{1, \ldots, d\}$, on pose p_i la constante telle que, pour tout $x \in \mathbb{R}^d$, $u(x+e_i) - u(x) = p_i$. On note alors $p = (p_i)_{i\in\{1,..,d\}}$. Il en découle que l'application définie par $w(x) := u(x) - p.x$ est $\mathbb{Z}^d$-périodique et puisqu'elle est solution de :

$$-\operatorname{div}\left(a_{per}\left(\nabla w + p\right)\right) = 0 \quad \text{sur } \mathbb{R}^d,$$

c'est donc, à une constante additive près, l'unique correcteur périodique $w_{per,p}$ associé à p. $\square$

Lemma A.2. *Soit $u \in L^1_{loc}(\mathbb{R}^d)$ telle que $\nabla u \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$, alors $\lim\limits_{\varepsilon\to 0} \varepsilon u(./\varepsilon) = 0$ dans $L^\infty_{loc}(\mathbb{R}^d)$. De plus, si on suppose $u(0) = 0$ alors pour tout $p > d$, il existe une constante $C > 0$ telle que pour tout $x \in \mathbb{R}^d \setminus \{0\}$ et $\varepsilon > 0$ on a :*

$$\frac{\varepsilon|u(x/\varepsilon)|}{|x|} \leq C \left(\frac{\varepsilon^d}{|x|^d} \int_{B_{|x|}(x)/\varepsilon} |\nabla u(z)|^p dz \right)^{1/p}.$$

Démonstration. Soient $x, y \in \mathbb{R}^d$ tels que $x \neq y$. On définit $r = |x-y|$. Puisque $\nabla u \in \left(L^\infty(\mathbb{R}^d)\right)^d$, on a $\nabla u \in \left(L^p_{loc}(\mathbb{R}^d)\right)^d$ pour tout $p \geq 1$. On fixe alors $p > d$ et on sait qu'il existe une constante $C > 0$, dépendant uniquement de d, telle que :

$$|u(x) - u(y)| \leq Cr \left(\frac{1}{r^d} \int_{B_r(x)} |\nabla u(z)|^p \, dz\right)^{\frac{1}{p}}.$$

Cette inégalité est établie par exemple dans [42, Remark p.268] comme une conséquence de la preuve de l'inégalité de Morrey ([42, Theorem 4 p.266]). En considérant y = 0, et par une inégalité triangulaire, on en déduit donc

$$\frac{|u(x)|}{r} \leq C\|u\|_{L^\infty(\mathbb{R})}^{(p-1)/p} \left(\frac{1}{r^d} \int_{B_r(x)} |\nabla u(z)| dz\right)^{1/p} + \frac{|u(0)|}{r}.$$

En remplaçant x par x/ε ci-dessus et puisque $\nabla u \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$, on obtient bien le résultat de sous-linéarité et l'inégalité annoncés. ☐

Lemma A.3. *On suppose que $\tilde{a} \in L^\infty(\mathbb{R}^d)$ vérifie (A.14), alors la suite $(|\tilde{a}(./\varepsilon)|)_{\varepsilon>0}$ converge vers 0 pour la topologie faible-$\star$ de $L^\infty(\mathbb{R}^d)$ quand ε tend vers 0.*

Démonstration. Pour $x_0 \in \mathbb{R}^d$ et $R > 0$, on considère dans un premier temps la fonction indicatrice de la boule $B_R(x_0)$, notée $g = 1_{B_R(x_0)}$. Dans la suite, on note $M = R + |x_0| > 0$ et on a clairement $B_R(x_0) \subset B_M$. Ainsi, pour tout $\varepsilon > 0$, :

$$\int_{\mathbb{R}^d} g \, |\tilde{a}(./\varepsilon)| = \int_{B_R(x_0)} |\tilde{a}(./\varepsilon)| \leq \int_{B_M} |\tilde{a}(./\varepsilon)| = \varepsilon^d \int_{B_{\frac{M}{\varepsilon}}} |\tilde{a}|.$$

On effectue maintenant la décomposition suivante :

$$\varepsilon^d \int_{B_{\frac{M}{\varepsilon}}} |\tilde{a}| = \varepsilon^d \int_{B_{\frac{M}{\sqrt{\varepsilon}}}} |\tilde{a}| + \varepsilon^d \int_{B_{\frac{M}{\varepsilon}} \setminus B_{\frac{M}{\sqrt{\varepsilon}}}} |\tilde{a}|.$$

On remarque tout d'abord qu'il existe une constante $C_1 > 0$ ne dépendant que de la dimension d et $M > 0$ telle que

$$\varepsilon^d \int_{B_{\frac{M}{\sqrt{\varepsilon}}}} |\tilde{a}| \leq \varepsilon^d \left|B_{\frac{M}{\sqrt{\varepsilon}}}\right| \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon^{d/2} C_1 \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \overset{\varepsilon \to 0}{\longrightarrow} 0. \tag{A.17}$$

Par ailleurs, pour tout $y \in B_{\frac{M}{\varepsilon}}$, on a clairement $|y|^d \leq \dfrac{M^d}{\varepsilon^d}$ et on en déduit

$$\varepsilon^d \int_{B_{\frac{M}{\varepsilon}} \setminus B_{\frac{M}{\sqrt{\varepsilon}}}} |\tilde{a}| \leq M^d \int_{B_{\frac{M}{\varepsilon}} \setminus B_{\frac{M}{\sqrt{\varepsilon}}}} \frac{|\tilde{a}(y)|}{|y|^d} dy \overset{\varepsilon \to 0}{\longrightarrow} 0, \tag{A.18}$$

La convergence vers 0 dans (A.18) étant une conséquence de l'hypothèse d'intégrabilité (A.14). En utilisant (A.17) et (A.18), on a donc montré que $\displaystyle\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} g \, |\tilde{a}(./\varepsilon)| = 0$. Ce résultat étant vérifié pour toute fonction de la forme $g = 1_{B_R(x_0)}$, on conclut en utilisant la densité des fonctions étagées dans $L^1(\mathbb{R}^d)$. ☐

A.3.2 Existence d'un correcteur

On cherche à construire une solution $\tilde{w}_p$ de (A.15) à gradient dans $\left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$. L'idée est de construire une suite u_n définie par $u_0 = 0$ et u_{n+1} comme la solution de :

$$\begin{cases} -\operatorname{div}(a_{per}\nabla u_{n+1}) = \operatorname{div}(\tilde{a}(\nabla w_{per,p} + p + \nabla u_n)) \quad \text{sur } \mathbb{R}^d, \\ \nabla u_n \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d. \end{cases}$$

On va d'abord montrer qu'à chaque étape n, la solution u_n est bien définie.

Lemma A.4. *On suppose $d \geq 2$ et que a vérifie (A.12), (A.13) et (A.14). Soit $f \in \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, alors il existe une unique solution u (à une constante additive près) dans $L^1_{loc}(\mathbb{R}^d)$ du problème :*

$$-\operatorname{div}\left(a_{per}\nabla u\right) = \operatorname{div}(\tilde{a}f) \quad \textit{sur } \mathbb{R}^d, \tag{A.19}$$

vérifiant $\nabla u \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$. De plus cette solution vérifie :

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C_{\tilde{a}}\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}, \tag{A.20}$$

où $C_{\tilde{a}} = C\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tilde{a}(y)|}{1+|x-y|^d}dy + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right)$ et $C > 0$ est une constante qui ne dépend pas de $\tilde{a}$, f et u.

Démonstration. Dans la suite, et on désignera par la même lettre C toutes les constantes indépendantes de $\tilde{a}$. On veut définir une solution u de (A.19) à l'aide de la fonction de Green G_{per} associée à l'opérateur $-\operatorname{div}(a_{per}\nabla .)$ sur $\mathbb{R}^d$. L'idée est de montrer que la fonction u formellement définie par

$$u = \int_{\mathbb{R}^d} \nabla_y G_{per}(., y)\tilde{a}(y)f(y)dy,$$

est bien définie dans $L^1_{loc}(\mathbb{R}^d)$ et est solution de (A.19). Dans ce but, pour tout $k \in \mathbb{Z}^d$, on note $Q_k = \prod_{i=1^d}]k_i, k_i + 1[$, le cube unité de $\mathbb{R}^d$ translaté de k, et on considère u_k définie par :

$$u_k(x) = \int_{\mathbb{R}^d} \nabla_y G_{per}(x, y)\tilde{a}(y)f(y)1_{Q_k}(y)dy,$$

où 1_{Q_k} désigne la fonction caractéristique de Q_k. Puisque $\tilde{a}f1_{Q_k}$ appartient à $\left(L^2(\mathbb{R}^d)\right)^d$, le résultat de [14, Theorem A] assure que u_k est bien définie dans $H^1_{loc}(\mathbb{R}^d)$ et est solution de :

$$\begin{cases} -\operatorname{div}(a_{per}\nabla u_k) = \operatorname{div}(1_{Q_k}f) \quad \text{sur } \mathbb{R}^d, \\ \nabla u_k \in \left(L^2(\mathbb{R}^d)\right)^d. \end{cases}$$

On sait que $x \mapsto \nabla_y G_{per}(x, y)$ est differentiable en dehors de tout voisinage de y. La fonction u_k est donc régulière en dehors de Q_k et pour tout $x \in \mathbb{R}^d \setminus Q_k$ son gradient est donné par :

$$\nabla u_k(x) = \int_{\mathbb{R}^d} \nabla_x\nabla_y G_{per}(x, y)f(y)\tilde{a}(y)1_{Q_k}(y)dy.$$

Pour tout $n \in \mathbb{N}^*$, on définit alors la fonction $U_n = \sum_{|k|<n} u_k$ et, par linéarité du gradient, on a $\nabla U_n = \sum_{|k|<n} \nabla u_k$. On veut montrer que la suite $(\nabla U_n)_{n\in\mathbb{N}^*}$ est convergente dans $\left(L^1_{loc}(\mathbb{R}^d)\right)^d$. Pour cela, on va montrer que la série $\sum_{k\in\mathbb{Z}^d} \nabla u_k$ est normalement convergente dans $\left(L^1_{loc}(\mathbb{R}^d)\right)^d$, c'est-à-dire, que $\sum_{k\in\mathbb{Z}^d} \|\nabla u_k\|_{L^1(B_R)}$ converge pour tout $R > 0$. On fixe donc $R > 0$ et, en utilisant l'inégalité (2.39), on a :

$$\begin{aligned}
\sum_{|k|>2R} \|\nabla u_k\|_{L^1(B_R)} &= \sum_{|k|>2R} \int_{B_R} \left| \int_{Q_k} \nabla_x \nabla_y G_{per}(x,y)\tilde{a}(y) f(y) dy \right| dx \\
&\leq C \int_{B_R} \sum_{|k|>2R} \int_{Q_k} \frac{|\tilde{a}(y)|}{|x-y|^d} |f(y)| dy dx \\
&\leq C \int_{B_R} \int_{\mathbb{R}^d \setminus B_{2R}} \frac{|\tilde{a}(y)|}{|x-y|^d} |f(y)| dy dx.
\end{aligned}$$

En utilisant maintenant que $f \in \left(L^\infty(\mathbb{R}^d)\right)^d$ et que $\tilde{a}$ vérifie l'hypothèse (A.14), on obtient :

$$\int_{\mathbb{R}^d \setminus B_{2R}} \frac{|\tilde{a}(y)|}{|x-y|^d} |f(y)| dy \leq C \|f\|_{L^\infty(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy.$$

Et on en déduit finalement que :

$$\sum_{|k|>2R} \|\nabla u_k\|_{L^1(B_R)} \leq C\, |B_R|\, \|f\|_{L^\infty(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy < \infty.$$

C'est-à-dire que $\sum_{k\in\mathbb{Z}^d} \|\nabla u_k\|_{L^1(B_R)} < \infty$ pour tout $R > 0$. On a donc la convergence absolue annoncée pour la série dans $L^1(B_R)$. Ce dernier étant un espace de Banach, on en déduit la convergence de la suite des sommes partielles ∇U_n dans $L^1(B_R)$. C'est-à-dire que la suite ∇U_n converge dans $L^1_{loc}(\mathbb{R}^d)$ vers la fonction $T = \sum_{k\in\mathbb{Z}^d} \nabla u_k \in \left(L^1_{loc}(\mathbb{R}^d)\right)^d$. Par ailleurs le Lemme de Schwarz nous assure l'existence de $u \in L^1_{loc}(\mathbb{R}^d)$ telle que $T = \nabla u$. En utilisant la linéarité de l'équation on a pour tout $n \in \mathbb{N}$:

$$-\operatorname{div}(a_{per}\nabla U_n) = \operatorname{div}\left(\tilde{a} f \sum_{|k|\leq n} 1_{Q_k} \right) \quad \text{sur } \mathbb{R}^d. \tag{A.21}$$

On peut alors passer à la limite dans (A.21) et on obtient :

$$-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(\tilde{a} f) \quad \text{sur } \mathbb{R}^d.$$

Montrons maintenant que $\nabla u \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ et qu'il vérifie l'estimation (A.20) annoncée.

On fixe $x_0 \in \mathbb{R}^d$ et on va d'abord montrer que ∇u est bornée dans $\left(\mathcal{C}^{0,\alpha}(B_1(x_0))\right)^d$ indépendamment de x_0. Dans ce but on considère une fonction $\chi \in \mathcal{D}(\mathbb{R}^d)$ positive, à support dans

$B_8(x_0)$ et telle que $\chi \equiv 1$ sur $B_4(x_0)$ et $\|\chi\|_{L^\infty(\mathbb{R}^d)} = 1$. On effectue alors la décomposition $u = u_1 + u_2$, où

$$u_1 := \int_{\mathbb{R}^d} \nabla_y G_{per}(x,y)\tilde{a}(y)f(y)\chi(y)dy, \quad u_2 := u - u_1.$$

On montre comme précédemment que cette décomposition est bien valide et que

$$\nabla u_2(x) = \int_{\mathbb{R}^d} \nabla_x \nabla_y G_{per}(x,y)\tilde{a}(y)f(y)(1-\chi(y))dy, \tag{A.22}$$

pour tout $x \in B_2(x_0)$.

Dans un premier temps, on s'intéresse à u_1. On remarque tout d'abord que u_1 vérifie

$$-\operatorname{div}(a_{per}\nabla u_1) = \operatorname{div}(\tilde{a}f\chi) \quad \text{sur } \mathbb{R}^d. \tag{A.23}$$

En utilisant l'estimation (2.37) et les propriétés de χ, on a pour tout $x \in B_4(x_0)$:

$$\begin{aligned} |u_1(x)| = \left|\int_{\mathbb{R}^d} \nabla_y G_{per}(x,y)\tilde{a}(y)f(y)\chi(y)dy\right| &\leq C\|\tilde{a}f\|_{L^\infty(\mathbb{R}^d)} \int_{B_8(x_0)} \frac{1}{|x-y|^{d-1}}dy \\ &= C\|\tilde{a}f\|_{L^\infty(\mathbb{R}^d)} \int_{B_8(0)} \frac{1}{|y|^{d-1}}dy. \end{aligned}$$

Ainsi $u_1 \in L^\infty(B_1(x_0))$ et on a montré

$$\|u_1\|_{L^\infty(B_1(x_0))} \leq C\|\tilde{a}f\|_{L^\infty(\mathbb{R}^d)}.$$

Une inégalité de régularité elliptique (voir [47, Theorem 4.4 p.63]) appliquée à (A.23) nous donne alors l'existence d'une constante $C > 0$ indépendante de x_0 telle que

$$\|\nabla u_1\|_{L^2(B_2(x_0))} \leq C\left(\|u_1\|_{L^2(B_4(x_0))} + \|\tilde{a}f\|_{L^2(B_4(x_0))}\right). \tag{A.24}$$

On utilise les deux inégalités précédentes et on en déduit :

$$\|\nabla u_1\|_{L^2(B_2(x_0))} \leq C\|\tilde{a}f\|_{L^\infty(\mathbb{R}^d)} \leq C\|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}.$$

On applique le théorème de régularité de type Schauder de [46, Théorème 3.2] et on obtient alors l'existence d'une constante $C > 0$ indépendante de x_0 telle que :

$$\begin{aligned} \|\nabla u_1\|_{C^{0,\alpha}(B_1(x_0))} &\leq C\left(\|\nabla u_1\|_{L^2(B_2(x_0))} + \|\chi f\tilde{a}\|_{C^{0,\alpha}(\mathbb{R}^d)}\right) \\ &\leq C\left(\|\nabla u_1\|_{L^2(B_2(x_0))} + \|f\tilde{a}\|_{C^{0,\alpha}(\mathbb{R}^d)}\right). \end{aligned}$$

Puis en utilisant l'inégalité (A.24) :

$$\|\nabla u_1\|_{C^{0,\alpha}(B_1(x_0))} \leq C\|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}. \tag{A.25}$$

Intéressons nous maintenant à ∇u_2. Pour tout $x \in B_2(x_0)$, (A.22) et (2.39) nous assurent que

$$|\nabla u_2(x)| \leq C\int_{\mathbb{R}^d \setminus B_4(x_0)} \frac{|\tilde{a}(y)|}{|x-y|^d}|f(y)|dy.$$

Par ailleurs, on remarque que pour $x \in B_2(x_0)$ et $y \in \mathbb{R}^d \setminus B_4(x_0)$, on a :

$$\frac{1}{|x-y|^d} \leq \frac{2}{1+|x-y|^d},$$

et on en déduit :

$$|\nabla u_2(x)| \leq C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy. \tag{A.26}$$

Par linéarité, on sait également que u_2 est solution dans $\mathcal{D}'(\mathbb{R}^d)$ de :

$$-\operatorname{div}(a_{per}\nabla u_2) = \operatorname{div}\left(\tilde{a}f(1-\chi(y))\right).$$

On peut donc utiliser à nouveau la régularité de Schauder donnée dans [46, Théorème 3.2] et on en déduit l'existence d'une constante $C > 0$ indépendante de x_0 telle que :

$$\|\nabla u_2\|_{\mathcal{C}^{0,\alpha}(B_1(x_0))} \leq C\left(\|\nabla u_2\|_{L^2(B_2(x_0))} + \|f\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right).$$

Ainsi, en utilisant (A.26), on a finalement :

$$\|\nabla u_2\|_{\mathcal{C}^{0,\alpha}(B_1(x_0))} \leq C\left(\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}. \tag{A.27}$$

En combinant les inégalités (A.25) et (A.27), on en déduit donc l'existence d'une constante $C > 0$ indépendante de x_0 telle que :

$$\begin{aligned}\|\nabla u\|_{\mathcal{C}^{0,\alpha}(B_1(x_0))} &\leq \|\nabla u_1\|_{\mathcal{C}^{0,\alpha}(B_1(x_0))} + \|\nabla u_2\|_{\mathcal{C}^{0,\alpha}(B_1(x_0))} \\ &\leq C\left(\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}.\end{aligned}$$

Cette inégalité étant vraie pour tout $x_0 \in \mathbb{R}^d$, on obtient alors une majoration uniforme sur tout l'espace, c'est-à-dire :

$$\|\nabla u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C\left(\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{a}(y)|}{1+|x-y|^d} dy + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}$$

On a donc établie l'estimation (A.20) annoncée.

Dans la suite on notera $⨍_A := \frac{1}{|A|}\int_A$ pour tout sous ensemble A de $\mathbb{R}^d$. On va montrer que $|\nabla u(./\varepsilon)|$ converge vers 0 pour la topologie faible-$\star$ de $L^\infty(\mathbb{R}^d)$. On considère $R > 0$ et $x_0 \in \mathbb{R}^d$ et on va montrer que $\lim_{\varepsilon\to 0} ⨍_{B_R(x_0)/\varepsilon} |\nabla u| = 0$. Soit $\varepsilon > 0$. De nouveau, on considère une fonction χ_ε à support dans $B_{4R}(x_0)/\varepsilon$ telle que $\chi_\varepsilon \equiv 1$ sur $B_{2R}(x_0)/\varepsilon$ et $\|\chi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} = 1$. On peut alors montrer la décomposition $u = u_1 + u_2$ où

$$u_1 := \int_{\mathbb{R}^d} \nabla_y G_{per}(x,y)\tilde{a}(y)f(y)\chi_\varepsilon(y)dy, \quad u_2 := u - u_1,$$

et pour tout $x \in B_R(x_0)/\varepsilon$,

$$\nabla u_2(x) = \int_{\mathbb{R}^d} \nabla_x\nabla_y G_{per}(x,y)\tilde{a}(y)f(y)(1-\chi_\varepsilon(y))dy.$$

Pour u_1, en utilisant l'estimation (2.37) et l'inégalité de Cauchy-Schwarz, on a :

$$\int_{B_{2R}(x_0)/\varepsilon} |u_1(x)|^2 dx \leq C \left(\int_{B_{2R}(x_0)/\varepsilon} \int_{B_{4R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} |\tilde{a}(y) f(y) \chi_\varepsilon(y)| dy \right)^2 dx$$
$$\leq C \int_{B_{2R}(x_0)/\varepsilon} \left(\int_{B_{4R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} dy \right) \left(\int_{B_{4R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} |\tilde{a}(y) f(y)|^2 dy \right) dx$$
$$\leq C \|f\|^2_{L^\infty(\mathbb{R}^d)} \int_{B_{2R}(x_0)/\varepsilon} \left(\int_{B_{4R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} dy \right) \left(\int_{B_{4R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} |\tilde{a}(y)|^2 dy \right) dx.$$

Or pour tout $x \in B_{2R}(x_0)/\varepsilon$, on a $((B_{4R}(x_0)/\varepsilon) - x) \subset (B_{6R}(x_0)/\varepsilon)$. En notant ensuite $M = 6R + |x_0| > 0$, on obtient :

$$\int_{B_{4R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} dy \leq \int_{B_{M/\varepsilon}} \frac{1}{|y|^{d-1}} dy \leq \frac{C}{\varepsilon}.$$

Par le théorème de Fubini, on a donc :

$$\int_{B_{2R}(x_0)/\varepsilon} |u_1(x)|^2 dx \leq C\varepsilon^{-1} \|f\|^2_{L^\infty(\mathbb{R}^d)} \int_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)|^2 \left(\int_{B_{2R}(x_0)/\varepsilon} \frac{1}{|x-y|^{d-1}} dx \right) dy$$
$$\leq C\varepsilon^{-1} \|f\|^2_{L^\infty(\mathbb{R}^d)} \int_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)|^2 \left(\int_{B_{M/\varepsilon}} \frac{1}{|x|^{d-1}} dx \right) dy$$
$$\leq C \|f\|^2_{L^\infty(\mathbb{R}^d)} \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \varepsilon^{-2} \int_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)| dy.$$

Puisque u_1 est solution de $-\operatorname{div}(a_{per} \nabla u_1) = \operatorname{div}(f \chi_\varepsilon)$, l'inégalité de de régularité elliptique donnée dans [47, Theorem 4.4 p.63] nous fournit l'existence d'une constante $C > 0$ indépendante de ε telle que :

$$\|\nabla u_1\|^2_{L^2(B_R(x_0)/\varepsilon)} \leq C \left(\varepsilon^2 \|u_2\|^2_{L^2(B_{2R}(x_0)/\varepsilon)} + \|f\tilde{a}\|^2_{L^2(B_{2R}(x_0)/\varepsilon)} \right).$$

Ainsi d'après ce qui précède, on a :

$$\|\nabla u_1\|^2_{L^2(B_R)} \leq C \|f\|^2_{L^\infty(\mathbb{R}^d)} \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \int_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)| dy. \tag{A.28}$$

On divise alors (A.28) par $|B_R(x_0)/\varepsilon|$ et on obtient :

$$\frac{\|\nabla u_1\|^2_{L^2(B_R(x_0)/\varepsilon)}}{|B_R(x_0)/\varepsilon|} \leq C \|f\|^2_{L^\infty(\mathbb{R}^d)} \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \frac{1}{|B_{4R}(x_0)/\varepsilon|} \int_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)| dy.$$

On en déduit donc l'existence d'une constante $C > 0$ indépendante de ε telle que :

$$\frac{\|\nabla u_1\|^2_{L^2(B_{4R}(x_0)/\varepsilon)}}{|B_{4R}(x_0)/\varepsilon|} \leq C \fint_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)| dy.$$

En utilisant l'inégalité de Cauchy-Schwarz, on remarque maintenant que :

$$\left(\fint_{B_R(x_0)/\varepsilon} |\nabla u_1(y)| dy \right)^2 \leq \frac{\|\nabla u_1\|^2_{L^2(B_R(x_0)/\varepsilon)}}{|B_R(x_0)/\varepsilon|}.$$

Et finalement on obtient :

$$\fint_{B_R(x_0)/\varepsilon} |\nabla u_1(y)|dy \leq C \left(\fint_{B_{4R}(x_0)/\varepsilon} |\tilde{a}(y)|dy \right)^{1/2}. \tag{A.29}$$

On étudie maintenant le second terme ∇u_2. On a directement à l'aide de l'estimation (2.39) et du théorème de Fubini :

$$\begin{aligned}\int_{B_R(x_0)/\varepsilon} |\nabla u_2(x)|dx &\leq C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{B_R(x_0)/\varepsilon} \left(\int_{\mathbb{R}^d \setminus (B_{2R}(x_0)/\varepsilon)} \frac{|\tilde{a}(y)||1-\chi_\varepsilon|}{|x-y|^d} dy \right) dx \\ &= C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus (B_{2R}(x_0)/\varepsilon)} |\tilde{a}(y)| \left(\int_{B_R(x_0)/\varepsilon} \frac{1}{|x-y|^d} dx \right) dy.\end{aligned}$$

Or pour tout $x \in B_R(x_0)/\varepsilon$ et tout $y \in \mathbb{R}^d \setminus (B_{2R}(x_0)/\varepsilon)$, puisque $B_R(x_0)/\varepsilon = B_{R/\varepsilon}\left(\frac{x_0}{\varepsilon}\right)$ on a $\left|y - \frac{x_0}{\varepsilon}\right| \geq 2\left|x - \frac{x_0}{\varepsilon}\right|$. Par inégalité triangulaire, on obtient donc :

$$|x-y| \geq \left|y - \frac{x_0}{\varepsilon}\right| - \left|x - \frac{x_0}{\varepsilon}\right| \geq \frac{\left|y - \frac{x_0}{\varepsilon}\right|}{2}.$$

D'où :

$$\begin{aligned}\int_{B_R(x_0)/\varepsilon} |\nabla u_2(x)|dx &\leq C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus (B_{2R}(x_0)/\varepsilon)} \frac{|\tilde{a}(y)|}{\left|y - \frac{x_0}{\varepsilon}\right|^d} \int_{B_R(x_0)/\varepsilon} 2^d dx dy \\ &= C\|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus (B_{2R}/\varepsilon)} \frac{|\tilde{a}(y)|}{|y|^d} \int_{B_R(x_0)/\varepsilon} 2^d dx dy \\ &= 2^d \|f\|_{L^\infty(\mathbb{R}^d)} C \frac{|B_R|}{\varepsilon^d} \int_{\mathbb{R}^d \setminus (B_{2R}/\varepsilon)} \frac{|\tilde{a}(y)|}{|y|^d} dy.\end{aligned}$$

Et on a finalement :

$$\fint_{B_R(x_0)/\varepsilon} |\nabla u_2| \leq C \int_{\mathbb{R}^d \setminus (B_{2R}/\varepsilon)} \frac{|\tilde{a}(y)|}{|y|^d} dy. \tag{A.30}$$

En combinant (A.29) et (A.30), et par inégalité triangulaire, on a donc l'existence d'une constante $C > 0$ indépendante de ε telle que :

$$\fint_{B_R(x_0)/\varepsilon} |\nabla u| \leq C \left(\left(\fint_{B_{4R}(x_0)/\varepsilon} |\tilde{a}| \right)^{1/2} + \int_{\mathbb{R}^d \setminus (B_{2R}/\varepsilon)} \frac{|\tilde{a}(y)|}{|y|^d} dy \right).$$

Le Lemme A.3 assure que

$$\lim_{\varepsilon \to 0} \left(\fint_{B_{4R}(x_0)/\varepsilon} |\tilde{a}| \right)^{1/2} = 0,$$

et l'hypothèse (A.14) que

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus (B_{2R}/\varepsilon)} \frac{|\tilde{a}(y)|}{|y|^d} dy = 0.$$

On en déduit donc la convergence vers 0 de $\fint_{B_R(x_0)/\varepsilon} |\nabla u|$. Par densité des fonctions étagées dans L^1, on conclut finalement que $|\nabla u(./\varepsilon)|$ converge vers 0 pour la topologie faible-$\star$ de $L^\infty(\mathbb{R}^d)$ et donc que $\nabla u \in \left(\mathcal{C}_m^{0,\alpha}(\mathbb{R}^d)\right)^d$. □

On peut maintenant démontrer la Proposition A.3 en appliquant une stratégie de point fixe.

Preuve de la Proposition A.3. On définit la suite $(u_n)_{n\in\mathbb{N}}$ par $u_0 = 0$ et pour tout $n \in \mathbb{N}$, u_{n+1} est l'unique solution de :

$$-\operatorname{div}(a_{per}\nabla u_{n+1}) = \operatorname{div}(\eta\tilde{a}(\nabla w_{per,p} + p + \nabla u_n)), \tag{A.31}$$

telle que $\nabla u_n \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$. L'existence de u_{n+1} est en effet assurée par le Lemme A.4. Pour tout $n \in \mathbb{N}^*$, on a donc :

$$-\operatorname{div}\left(a_{per}\left(\nabla u_{n+1} - \nabla u_n\right)\right) = \operatorname{div}\left(\eta\tilde{a}\left(\nabla u_n - \nabla u_{n-1}\right)\right) \quad \text{sur } \mathbb{R}^d.$$

Et puisque $\nabla u_{n+1} - \nabla u_n \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ et $\nabla u_n - \nabla u_{n-1} \in \left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$, le Lemme A.4 nous fournit l'existence d'une constante $C > 0$ telle que pour tout $n \in \mathbb{N}^*$ on a :

$$\|\nabla u_{n+1} - \nabla u_n\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \leq C\eta\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tilde{a}(y)|}{1+|x-y|^d}dy + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right)\|\nabla u_n - \nabla u_{n-1}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}.$$

En choisissant η_0 tel que :

$$C\eta_0\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tilde{a}(y)|}{1+|x-y|^d}dy + \|\tilde{a}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)}\right) < 1,$$

on peut alors montrer que la suite $(\nabla u_n)_n$ est de Cauchy dans $\left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$ et qu'elle admet donc une limite $\nabla\tilde{w}_p$ dans cet espace. On passe alors à la limite dans (A.31) et on obtient alors une solution du problème du correcteur (A.15). De plus, puisque $\nabla\tilde{w}_p$ est dans $\left(\mathcal{C}^{0,\alpha}_m(\mathbb{R}^d)\right)^d$, le Lemme A.2 nous assure que $\varepsilon\tilde{w}_p(./\varepsilon)$ converge vers 0 dans $L^\infty_{loc}(\mathbb{R}^d)$. $\square$